Willow Algorithm for Consumption-Investment under Stochastic Volatility Model with Jump Diffusion

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Abstract–We mainly focus on the willow algorithm which can be used to solve the optimal dynamic multi-cycle investment and consumption problem under the jump-diffusion stochastic volatility model. First we obtain the moment generating function of the risky asset and then coalescing the Johnson-Curve transformation theory to generate the willow algorithm, which solving the multi-cycle dynamic investment and consumption problem is designed based on the two-dimensional willow framework. Moreover, through comparing our proposed solution with the optimal investment and consumption display solutions under the geometric Brownian motion model, we further discuss the analysis of the willow algorithm sensitivity. The willow algorithm for optimal investment and consumption proposed in this paper is able to extend the willow method which is effective from the field of option pricing to the field of investment portfolio, and it also provides a new idea for numerically solving the multi-period optimal investment and consumption decision.

Index Terms–stochastic volatility model, the willow algorithm, dynamic optimization, consumption-investment, multi-cycle portfolio.

I. Introduction

THE following two parameter index are employed by Markowitz [1] to measure the return and risk: the expected return rate of risky assets and the variance of return rate. The study of portfolio, utilizing mathematical models and operational optimization theoretical methods, opened the research of modern portfolio theory and considerably promoted the development of modern finance. Portfolio research including portfolio selection, pricing and evaluation has been an influential topic in financial risk management and corporate finance management for decades.

On the basis of the above study, Mao [2] primarily focused on single-cycle static portfolios, in which an investor sets an asset allocation at the beginning of the period and holds it until the end, while investment behavior particularly by the institutional investors tends to be long-term. Long-term investors adjust their portfolio positions in time as the investment environment changes, rather than locking into an investment account after the initial construction of the portfolio. This dynamic portfolio selection behavior led to the realization that traditional one-cycle portfolio theory was no longer suitable for flexible and dynamic investment decisions, prompting the emergence of multi-cycle dynamic portfolio theory at a pivotal historical moment. The works of Merton [3][4][5], Samuelson [6], Fama [7], and Hakansson [8] marked the beginning of the study of multi-period dynamic portfolio.

The multi-asset dynamic portfolio problem in continuous time was formulated by Merton [3][4][5]. Assuming that the assets comply with geometric Brownian motion, Merton derived dynamical equations for optimal portfolios with more than one asset and investigated the portfolio problem for two assets with constant relative risk aversion or equal elastic marginal utility in detail. The general techniques employed to study uncertainty in a wide range of intertemporal economic problems can be utilized. In contrast, dynamic programming methods were proposed by Jiang et al. [9], SengPun and Ye [10], Chen et al. [11] for solving dynamic portfolio problems in discrete time. In the optimal consumption and security selection problem, it is assumed in the paper that there are two optional classes of assets in the investment market: one is a risk-free asset and the other is a risky asset. To develop more practical and efficient optimal investment consumption strategies, simulations of risky assets are particularly crucial. In the studies of Merton [3] and Samuelson [6], risky assets were considered to follow geometric Brownian motion, known as the constant volatility model. But in the real financial markets, it is often hard to accurately describe the market. For example, in real markets where asset prices obey peaks and thick tails, there is difficulty for conventional models of geometric Brownian motion in describing. In this case, the stochastic volatility (SV) model was born due to stochastic trends.

The stochastic volatility model assumes that the fluctuation ratio of asset prices is a stochastic process dominated by factors like asset prices, the mean reversion trend of volatility and the variance of volatility. It has the capacity of effectively characterizing and capturing the features of the peaks and thick tails of the price distribution in finan-

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cial markets. Heston [12] proposed the stochastic volatility model (Heston model) in which the fluctuation ratio of assets follows the Cox-Ingersoll-Ross (CIR) process. Owing to its exceptional analytical properties, the Heston model has been extensively studied and applied. Furthermore, Heston [13] presented a fresh inverse CIR process called the 3/2 model. Unlike the CIR process, the instantaneous fluctuation ratio of this model exhibits a relatively rapid recovery when asset prices are highly elevated or severely depressed, aligning more closely with real-world conditions. Grasselli [14] brought forward the idea of combining the Heston model and 3/2 model. The fluctuation ratio of the two models is linearly superimposed by $\left(a\sqrt{V_t}+\frac{b}{\sqrt{V_t}}\right)$, where a and b are greater than 0, and V_t is the CIR process. This model was named the $4/2(=1/2+3/2)$ stochastic volatility model. Moreover, it is shown in documents of Jacod [15], Barndorff-Nielsen [16], Hu and Wang [17], Han and Wang [18], Chang and Li [19] that there are uninterrupted jumps happening in the price of risky assets.

In this paper, the willow method will be employed to model risky assets under the jump-diffusion model. Considered an optimized binary tree algorithm, the willow method differs from the binary tree algorithm in that it fixes the number of willow nodes at each step once the number of nodes is determined. Initially proposed by Curran [20], the willow method was subsequently enhanced by Xu et al. [21], who improved the point-taking strategy and extended its application to the pricing problem of path-dependent options. Intuitive, efficient, accurate, and widely applicable, the willow method provides a novel tree structure for simulating Brownian motion based on the concept of discrete Markov chains. Rich research results in the field of option pricing have been achieved using the willow method. A method for solving multi-order moments was provided by Wang et al. [22], which utilizes corresponding stochastic processes during the pricing of the stochastic interest rate model in the form of non-geometric Brownian motion. Furthermore, in conjunction with the content concerning the Johnson curve in the research by Xu et al. [23], a willow frame constructed for any continuous random variable. Additionally, Ma et al. [24] developed a general two-dimensional willow framework and studied the pricing of American options and singular options under the existing stochastic volatility model. However, it is important to note that the approximation of the trapezoidal formula in the stochastic integral of the fluctuation ratio introduces significant errors and does not encompass the study of the jump diffusion model as well.

The aim of this paper is to investigate the willow method under the stochastic volatility model of jump diffusion. Taking the Heston model as an example, an efficient point selection strategy for sample points is designed, and the multi-period dynamic optimal investment consumption problem is addressed based on the two-dimensional willow framework. This will significantly broaden the scope of applications of the willow algorithm. The main contributions of this paper are as follows:

- The traditional optimal investment consumption problem is predicated on the assumption that the risky asset follows a constant volatility and non-jump diffusion process. The characteristic function of the underlying asset with stochastic volatility and jump diffusion is derived in this paper. Through the application of the moment matching method and Johnson curve transformation theory, we achieved precise sampling of the underlying asset process, thereby overcoming the approximations made in the differential integration approach outlined in [24]. This has resulted in an enhancement of the accuracy and efficiency of the sampling process. Subsequently, we successfully constructed a two-dimensional willow framework for the asset indexed by a jump-diffusion stochastic volatility model.

- In traditional research methods, the most commonly used approach to address dynamic optimal investment and consumption decision problems is the stochastic control method. However, there are two limitations to its application: on one hand, stochastic control methods can only be employed when the value function is continuous and differentiable. On the other hand, solving for the optimal solution requires addressing nonlinear HJB (Hamilton-Jacobi-Bellman) equation. In practice, obtaining a closed-form or numerical solution to this equation poses significant challenges. In this paper, an optimization algorithm is proposed for solving the multi-period dynamic optimal investment and consumption problem under a discrete state framework within a two-dimensional risk asset setting. This algorithm avoids the complex process of solving the HJB equation and the closed-form utility function, and is applicable to any stochastic volatility model and complex utility functions. Furthermore, the computational complexity of this method increases linearly with the number of time discretization steps, making it superior to various numerical methods.

- Based on the multi-period investment and consumption framework for two assets proposed by Samuelson [6], this paper introduces the Willow algorithm for solving multi-period investment and consumption decisions. This algorithm meticulously considers the risk assets and risk accounts during each segment and optimizes the consumption and investment ratios. A novel numerical method is provided for solving the multi-period optimal investment and consumption decision problem in discrete cases. Additionally, this approach offers insights for solving other optimal investment and consumption problems, such as the optimal investment and consumption problem with dividend reinvestment.

The paper is structured as follows: In Section II., the characteristic function of the logarithmic asset price is derived using the 4/2 model as an example based on affine structure theory. Section III. presents the construction of the two-dimensional willow frame. In Section IV., the optimal investment consumption willow algorithm is designed based on the two-dimensional willow structure of the prices of risky assets. Section V. concerns numerical experiments so as to validate the effectiveness of the proposed algorithm and analyze its sensitivity. Section VI. is devoted to a summary.

II. Stochastic volatility model and moment generation function

Consider a complete domain manifold and probability space $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{Q})$, where the risky asset process S_t is a measurable process adapted to the domain manifold $\{\mathcal{F}_t\}_{t\geq 0}$. The information flow $\{\mathcal{F}_t\}_{0\leq t<\infty}$ satisfies the general assumptions that $\mathcal{F}_0$ contains all $\mathcal{Q}$-null subsets of $\mathcal{F}$, $\mathcal{F}$ is right-continuous and the $\mathcal{Q}$ is a risk-neutral pricing measure. The following stochastic process is defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathcal{Q})$.

For the classical Heston model with jump diffusion. Asset prices and volatility is represented by

$$\begin{cases} \frac{\mathrm{d}S_t}{S_t} = (r-\lambda\mu)\mathrm{d}t + \sqrt{V_t}\mathrm{d}B_t^1 + \left(e^J - 1\right)\mathrm{d}N_t, \\ \mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \sigma\sqrt{V_t}\mathrm{d}B_t, \end{cases} \tag{1}$$

where B_t^1 and B_t are Wiener processes with correlation ρ, N_t is a Poisson process with frequency λ, J is a random jump in the price of risky assets, N_t and J are independent of B_t^1 and B_t, $\rho \in [-1, 1]$, r is the risk-free rate and $\mu = \mathrm{E}\left[e^J\right] - 1$ is the compensation for jumping. Moreover, σ is the volatility corresponding to the stochastic state variable V_t, κ and θ represent the regression speed and mean of V_t, respectively.

To transform

$$X(t) = \ln\left(\frac{S(t)}{S(0)}\right) - \frac{\rho}{\sigma}(v(t) - v(0)) - \left(r - \lambda\kappa - \frac{\rho\eta\theta}{\sigma}\right)t, \tag{2}$$

the correlation between the two Brownian motions is eliminated. According to the Itô's lemma, the Equation (1) is rewritten as

$$\begin{cases} dX(t) = \left(\frac{\rho\eta}{\sigma_v} - \frac{1}{2}\right)v(t)dt + \sqrt{(1-\rho^2)\,v(t)}dB_t^{\perp} + JdN_t, \\ dV(t) = \eta(\theta - V(t))dt + \sigma\sqrt{V(t)}dB_t, \end{cases} \tag{3}$$

where $dB_t^{\perp}$ and dB_t are independent of each other and both are standard Brownian motions. Integrating both sides of t_n from t_{n+1} to $X(t)$ yields

$$\begin{aligned} X(t_{n+1}) =& X(t_n) + \left(\frac{\rho\eta}{\sigma} - \frac{1}{2}\right)\int_{t_n}^{t_{n+1}} V(s)ds \\ &+ \sqrt{1-\rho^2}\int_{t_n}^{t_{n+1}}\sqrt{V(s)}dB_s^{\perp} + \sum_{i=N(t_n)+1}^{N(t_{n+1})}\ln y_i, \end{aligned} \tag{4}$$

where $\int_{t_n}^{t_{n+1}}\sqrt{v(s)}dB_s^{\perp}$ follows normal distribution $N\left(0, \int_{t_n}^{t_{n+1}} v(s)ds\right)$. Affine processes have extremely pleasant properties with

Lemma 1. *If the conditional characteristic function of $\boldsymbol{Y}(T)$, with respect to the domain σ of $\mathcal{F}_t$, is an exponential affine form of $\boldsymbol{Y}(t)$, then it is an existential function*

$$\phi(t, \boldsymbol{u}) : R_+ \times \mathrm{i}R^d \to C, \boldsymbol{\psi}(t, \boldsymbol{u}) : R_+ \times \mathrm{i}R^d \to C^d,$$

where $\phi(t, \boldsymbol{u}), \boldsymbol{\psi}(t, \boldsymbol{u})$ are continuously distinguishable with respect to t. Such that the diffusion process $\boldsymbol{Y}(t)$ for any $\boldsymbol{u} \in \mathrm{i}R^d, t \leq T, \boldsymbol{y} \in \chi$ and the state space $\chi \subset R^d$ satisfies

$$\begin{aligned} \mathrm{E}\left[\exp\left(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{Y}(T)\right)|\mathcal{F}_t\right] = \exp(&\phi(T-t, \boldsymbol{u}) \\ &+ \boldsymbol{\psi}(T-t, \boldsymbol{u})^{\mathrm{T}}\boldsymbol{Y}(t)), \end{aligned} \tag{5}$$

then $Y(t)$ is called an affine process.

Let $\mathrm{d}Y_1(t) = v(t), \mathrm{d}Y_2(t) = \ln S(t)$, then Heston's model is written as

$$\begin{cases} \mathrm{d}Y_1(t) &= \eta\left(\theta - Y_1(t)\right)\mathrm{d}t + \sigma_v\sqrt{Y_1(t)}\mathrm{d}B_t, \\ \mathrm{d}Y_2(t) &= \left(r - \frac{1}{2}Y_1(t)\right)\mathrm{d}t \\ & \quad + \sqrt{Y_1(t)}\left(\rho\mathrm{d}B_t + \sqrt{1-\rho^2}\mathrm{d}B_t^{\perp}\right), \end{cases}$$

write it as a vector, then it is

$$\begin{cases} \mathrm{d}\boldsymbol{Y}(t) = \boldsymbol{b}(\boldsymbol{Y}(t))\mathrm{d}t + \boldsymbol{\rho}(\boldsymbol{Y}(t))\mathrm{d}\boldsymbol{B}(t), \\ \boldsymbol{Y}(0) = \boldsymbol{y}, \end{cases} \tag{6}$$

where $\boldsymbol{b}(\boldsymbol{y}) = \left(\eta\theta - \eta y_1, r - \frac{1}{2}y_1\right)^{\mathrm{T}}$, $\boldsymbol{\rho}(\boldsymbol{y}) = \begin{pmatrix} \sigma_v\sqrt{y_1} & 0 \\ \sqrt{y_1}\rho & \sqrt{y_1(1-\rho^2)} \end{pmatrix}$, then we have

$$\begin{aligned} &\boldsymbol{b}(\boldsymbol{y}) = \boldsymbol{b} + y_1\boldsymbol{\beta}_1 + y_2\boldsymbol{\beta}_2, \\ &\boldsymbol{a}(\boldsymbol{y}) = \boldsymbol{\rho}(\boldsymbol{y})\boldsymbol{\rho}(\boldsymbol{y})^{\mathrm{T}} = \boldsymbol{a} + y_1\boldsymbol{\alpha}_1 + y_2\boldsymbol{\alpha}_2, \end{aligned} \tag{7}$$

where $\boldsymbol{a}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ is the null matrix of 2×2: $\boldsymbol{b} = (\eta\theta, 0)^{\mathrm{T}}$, $\boldsymbol{\alpha}_1 = \begin{pmatrix} \sigma_v^2 & \rho\sigma_v \\ \rho\sigma_v & 1 \end{pmatrix}$.

Using the definition of Lemma 1, we can obtain the following theorem.

Theorem 1. *The conditional characteristic function of $X(t)$ of the Heston-jump diffusion model after substitution of Equation (2) is*

$$\begin{aligned} \mathrm{E}\left[e^{zX(t)}|\mathcal{F}_0\right] = &\exp\{g(-\frac{\rho}{\sigma}z) + f(-\frac{\rho}{\sigma}z)v_0 \\ &+ z\left[\frac{\rho}{\sigma}V_0 + \frac{\rho\eta\theta}{\sigma}t\right] + \lambda t\left[e^{\alpha_J z + \frac{1}{2}\sigma_J^2 z^2} - 1\right]\}, \end{aligned}$$

where $z = i\omega, \omega \in R$ and

$$\begin{aligned} g(\zeta) &= \frac{\eta\theta}{A}\ln\frac{2\delta e^{\frac{1}{2}(\delta - B)t}}{(\delta - B - 2A\zeta)e^{\delta t} + B + \delta + 2A\zeta}; \\ f(\zeta) &= \frac{[2C + (B+\delta)\zeta]e^{\delta t} + \zeta(\delta - B) - 2C}{(\delta - B - 2A\zeta)e^{\delta t} + B + \delta + 2A\zeta}; \\ A &= \frac{1}{2}\sigma^2; \\ B &= \rho\sigma z - \eta; \\ C &= \frac{1}{2}(z^2 - z); \\ \delta &= \sqrt{B^2 - 4AC}. \end{aligned}$$

Proof. According to Equation (7), $\boldsymbol{a}(\boldsymbol{y})$ and $\boldsymbol{b}(\boldsymbol{y})$ have affine form, i.e.

$$\boldsymbol{b}(\boldsymbol{y}) = \boldsymbol{b} + \sum_{i=1}^{d} y_i \boldsymbol{\beta}_i, \quad \boldsymbol{a}(\boldsymbol{y}) = \boldsymbol{\rho}(\boldsymbol{y})\boldsymbol{\rho}(\boldsymbol{y})^{\mathrm{T}} = \boldsymbol{a} + \sum_{i=1}^{d} y_i \boldsymbol{\alpha}_i,$$

where $\boldsymbol{a}, \boldsymbol{\alpha_i}$ is a $d \times d$-dimensional constant matrix and $\boldsymbol{b}, \boldsymbol{\beta_i}$ is a $d \times 1$-dimensional constant column vector.

The Ricatti equations for $\phi(t, \boldsymbol{u})$ and $\boldsymbol{\psi}(t, \boldsymbol{u}) = (\psi_1(t, \boldsymbol{u}), \psi_2(t, \boldsymbol{u}), \ldots, \psi_d(t, \boldsymbol{u}))^{\mathrm{T}}$ are

$$\begin{cases} \frac{\partial \phi(t,\boldsymbol{u})}{\partial t} = \frac{1}{2}\boldsymbol{\psi}(t, \boldsymbol{u})^{\mathrm{T}} \boldsymbol{a} \boldsymbol{\psi}(t, \boldsymbol{u}) + \boldsymbol{b}^{\mathrm{T}} \boldsymbol{\psi}(t, \boldsymbol{u}), \\ \phi(0, \boldsymbol{u}) = 0, \\ \frac{\partial \psi_i(t,\boldsymbol{u})}{\partial t} = \frac{1}{2}\boldsymbol{\psi}(t, \boldsymbol{u})^{\mathrm{T}} \boldsymbol{\alpha}_i \boldsymbol{\psi}(t, \boldsymbol{u}) + \boldsymbol{\beta}_i^{\mathrm{T}} \boldsymbol{\psi}(t, \boldsymbol{u}), \quad 1 \le i \le d, \\ \boldsymbol{\psi}(0, \boldsymbol{u}) = \boldsymbol{u} = (u_1, u_2, \ldots, u_d)^{\mathrm{T}}. \end{cases}$$

Li et al. [26] pointed out that there exists a distinct solution (ϕ, ψ) to this system of equations, where

$$\begin{aligned} \phi(t, \boldsymbol{u}) &= \frac{\eta\theta}{A} \ln\left(\frac{2\delta \mathrm{e}^{\frac{1}{2}(\delta - B)t}}{(\delta - B - 2Au_1)\,\mathrm{e}^{\delta t} + B + \delta + 2Au_1}\right), \\ &\quad + rtu_2, \\ \psi_1(t, \boldsymbol{u}) &= \frac{(2C + (B + \delta)u_1)\,\mathrm{e}^{\delta t} + u_1(\delta - B) - 2C}{(\delta - B - 2Au_1)\,\mathrm{e}^{\delta t} + B + \delta + 2Au_1}, \\ \psi_2(t, \boldsymbol{u}) &= u_2, \end{aligned}$$

where $A = \frac{1}{2}\sigma^2$, $B = \rho\sigma u_2 - \eta$, $C = \frac{1}{2}\left(u_2^2 - u_2\right)$, $\delta = \sqrt{B^2 - 4AC}$ and this particular solution is non-negative for all real parts of $t > 0$, $\phi(t, u) + \psi(t, u)^{\mathrm{T}} y$, thus Equation (6) is an affine process.

Letting $u_2 = z$, $u_1 = -\frac{\rho}{\sigma}z$ and Multiplying both sides of Equation (5) by $\exp\left\{z\left[\frac{\rho}{\sigma_v}v_0 - (r - \frac{\rho\eta\theta}{\sigma})T\right] + \lambda T\left[\mathrm{e}^{\alpha_J z + \frac{1}{2}\sigma_J^2 z^2} - 1\right]\right\}$, Equation (5) is derived as

$$\begin{aligned} \mathrm{E}\left[e^{zX(T)}|\mathcal{F}_t\right] &= \exp\{g(-\frac{\rho}{\sigma}z) + f(-\frac{\rho}{\sigma}z)v_t \\ &\quad + z\left[\frac{\rho}{\sigma}v_t + \frac{\rho\eta\theta}{\sigma}T\right] + \lambda T\left[e^{\alpha_J z + \frac{1}{2}\sigma_J^2 z^2} - 1\right]\}, \end{aligned} \tag{8}$$

and then, if we take $t = 0$ and rewrite T as t, the Equation (8) is rewritten as

$$\begin{aligned} \mathrm{E}\left[e^{zX(t)}|\mathcal{F}_0\right] &= \exp\{g(-\frac{\rho}{\sigma}z) + f(-\frac{\rho}{\sigma}z)v_0 \\ &\quad + z\left[\frac{\rho}{\sigma}v_0 + \frac{\rho\eta\theta}{\sigma}t\right] + \lambda t\left[e^{\alpha_J z + \frac{1}{2}\sigma_J^2 z^2} - 1\right]\}, \end{aligned}$$

So the theorem is proven. □

Since the 3/2 model does not meet the requirements of affine structure, we treat the 3/2 model under the more general 4/2 model. Asset prices and volatility in the 4/2 model that similar to the parameter setting of Equation (9) is represented by

$$\begin{cases} \frac{\mathrm{d}S_t}{S_t} &= (r - \lambda\mu)\mathrm{d}t + m\left(V_t\right)\left(\rho \mathrm{d}B_t^{\perp} + \sqrt{(1 - \rho^2)}\mathrm{d}B_t\right) \\ &\quad + \left(e^J - 1\right)\mathrm{d}N_t, \\ \mathrm{d}V_t &= \alpha(V_t)(\theta - V_t)\mathrm{d}t + \beta(V_t)\mathrm{d}B_t, \end{cases} \tag{9}$$

where $m\left(V_t\right)$ represents the volatility of S_t, $\alpha(V_t)$ represents the drift term of V_t, $\beta(V_t)$ represents the volatility of V_t and we take $\alpha(v) = \kappa(\theta - v)$, $\beta(v) = \sigma(v)\sqrt{(v)}$, $m(v) = a\sqrt{(v)} + \frac{b}{\sqrt{(v)}}$. It is essential to note that $a = 1$, $b = 0$ corresponds to Heston model and $a = 0$, $b = 1$ corresponds to 3/2 model.

According to Theorem 1 and Zeng et al. [25], We can directly compute the fourth moment of $X(t)$ at time t and then use the transformation theory of Johnson [29] to directly generate $X(t)$-nodes by exact sampling. This method is more accurate because it does not approximate the integral $\int_0^{t_n} v(s)ds$.

As the calculation of the transition probability of $X(t)$ still relies on the value of $v(t)$, we cannot simply discard the willow tree of volatility $v(t)$ and utilize the one-dimensional willow tree of $X(t)$ for the computation. This is because there is no apparent relationship between $X(t)$ and $v(t)$ during node generation. Therefore, it is necessary to couple the $X(t)$ generated in this manner with $v(t)$. We adopt a direct "splicing" approach for coupling (refer to section III. for details).

III. Build a two-dimensional willow architecture

This section will describe how to use the two-dimensional willow method to simulate the price of risky assets when the price of risky assets follows the 4/2 stochastic volatility model with Merton jump.

The stochastic volatility model consists of two main parts: the first part is a stochastic process of the price of the risky asset $S(t)$, which follows a geometric Brownian motion when the volatility value is given. the second part is the stochastic process followed by volatility V_t. Under the model of 4/2, volatility follows the square root process. Based on the characteristics of the above two parts, the two-dimensional willow method under the stochastic volatility model is constructed according to the following steps:

- The correlation between asset price S_t and volatility V_t is removed to facilitate later calculations;
- Construct a one-dimensional willow of random volatility V_t;
- Put the one-dimensional willow of volatility into the one-dimensional willow structure of price S_t, and then form the two-dimensional willow of S_t;
- Finally, after constructing all nodes of the two-dimensional willow tree, the optimal investment and consumption strategies are solved.

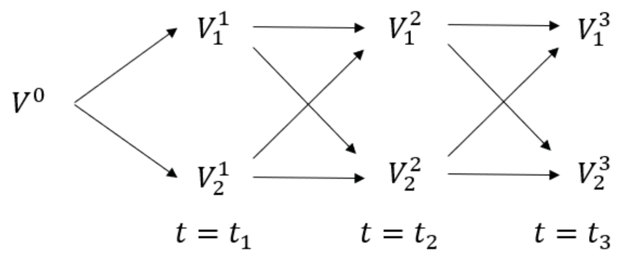

Figure 1: Volatility one-dimensional willow node diagram.

The 4/2 model under jump diffusion is expressed as

$$\begin{cases} \frac{\mathrm{d}S_t}{S_t} & = \left(a\sqrt{V_t} + \frac{b}{\sqrt{V_t}}\right)\left(\rho \mathrm{d}B_t + \sqrt{1-\rho^2}\ \mathrm{d}B_t^{\perp}\right) \\ & \quad +(r-\lambda\mu)\mathrm{d}t + \left(e^J - 1\right)\mathrm{d}N_t, \\ \mathrm{d}V_t & = \kappa\left(\theta - V_t\right)\mathrm{d}t + \sigma\sqrt{V_t}\ \mathrm{d}B_t, \end{cases} \tag{10}$$

where $a, b, r, \kappa, \theta, \sigma \in \mathcal{R}_+$, $a = 1, b = 0$ corresponds to Heston model and $a = 0, b = 1$ corresponds to 3/2 model. Under the 4/2 model, assets with the elimination of correlation are denoted as X_t and the expression of X_t is derived by

$$X_t = \ln\left(\frac{S_t}{S_0}\right) - \frac{\rho}{\sigma}\left(a\left(V_t - V_0\right) + b\ln(\frac{V_t}{V_0})\right) - (r-\lambda\mu)t. \tag{11}$$

There are two key points in the construction of the fluctuating willow, one is the generation of the willow nodes and the other is the computation of transition probabilities between nodes. Assume that the current time is 0, the investment time limit is T and divide the time T into N segments on average. The fluctuating willow tree has m_v distinct time nodes at each time node. The fluctuating willow tree constructed using $N = 3, m_v = 2$ as an example is shown in 1.

In Figure 1, v_b^a represents the b volatility value under the a time node (as the node of the willow tree) and v^0 is given as a parameter. The arrows indicate transferability, so that from $t = 0$ to $t = t_1$ one needs to compute a probability transition vector for $1 \times m_v$ and from $t = t_1$ to $t = t_2$ one needs to compute a probability transition matrix for $m_v \times m_v$. Going from $t = t_2$ to $t = t_3$ is also a probability transition matrix of $m_v \times m_v$.

In contrast to the binary tree method, the number of willow nodes at each step remains fixed when the parameter m_v is given. Moreover, the total number of nodes in the willow tree method does not increase at each step, but grows linearly instead of quadratically as in the binary tree, significantly improving the efficiency of the numerical method. Additionally, it is important to note that the generation of the willow nodes at each step is not directly related to the node values from the previous step, but rather depends solely on the model parameters and moments at the current step, as detailed in the followings.

At time t, we will adopt the Johnson Curve-moment matching method to generate m_v nodes. According to Wang and Xu [22], we calculate the fourth moment of V_t.

After the fourth moment of each step is obtained (by substituting $t = t_i, i = 1, 2, \cdots N$), the transformation theory of Johnson [29] transformation theory is considered. According to Wang and Xu [22], the conditional probability density function of v^{i+1} under v^i is

$$p\left(\nu\left(t_{i+1}\right) \mid v\left(t_i\right)\right) = \frac{1}{\sqrt{2\alpha\sigma_v^2 v\left(t_i\right)\Delta t}} \times e^{-\frac{\left(v\left(t_{i+1}\right) - v\left(t_i\right) - \kappa\left(\theta - v\left(t_i\right)\right)\Delta t\right)^2}{2\sigma_v v\left(t_i\right)\Delta t}}, \tag{12}$$

where $\Delta t = t_{i+1} - t_i = T/N$ is the step size. Then, we get one-dimensional willow structure of price S_t based on Johnson [29] and Equation (12).

According to Equation (11), under the 4/2 jump diffusion model, the expression of asset price eliminating correlation is

$$X_t = Y_t - Y_u - \frac{\rho}{\sigma}\left(a\left(V_t - v_u\right) + b\ln(\frac{V_t}{v_u})\right) - (r-\lambda\mu)(t-u), \tag{13}$$

and just to make it easier, we take $K_1 = -Y_u - \frac{\rho}{\sigma}\left(a\left(V_t - v_u\right) + b\ln(\frac{V_t}{v_u})\right) - (r - \lambda\mu)(t - u)$. K_1 is constant given V_t. In the case of unknown V_t, it can be viewed as a function of V_t denoted $K_1(v')$, $K_2(v') = e^{z(r-\lambda\mu)(t-u)+z\rho\left[f\left(v'\right)-f(v)\right]+(\lambda\mu_z+C)(t-u)}$. Using the conditional expectation formula, we have $E\left[e^{zX_t} \mid x_u = x, v_u = v\right] = \int_R e^{zK_1(v')+zK_2(v')}\frac{\phi(v)}{\phi(v')}p^{\hat{Q}}\left(t-u, v, v'\right)dv'$, the transfer density $p^{\hat{\mathbf{Q}}}\left(t-u, v, v'\right)$ is

$$p^{\hat{\mathbf{Q}}}\left(t-u, v, v'\right) = \frac{2\hat{\kappa}e^{\frac{\hat{\kappa}(t-u)}{2}\left(\frac{2\hat{\kappa}\hat{\theta}}{\sigma^2}+1\right)}}{\sigma^2\left[e^{\hat{\kappa}(t-u)} - 1\right]}\left(\frac{v'}{v}\right)^{\frac{1}{2}\left(\frac{2\hat{\kappa}\hat{\theta}}{\sigma^2}-1\right)} \times e^{-\frac{2\hat{\kappa}\left(v+e^{\hat{\kappa}(t-u)}v'\right)}{\sigma^2\left(e^{\hat{\kappa}(t-u)}-1\right)}} I_{\frac{2\hat{\kappa}\hat{\theta}}{\sigma^2}-1}\left(\frac{2\hat{\kappa}\sqrt{vv'}}{\sigma^2\sinh\left(\frac{\hat{\kappa}(t-u)}{2}\right)}\right),$$

assuming the form $\phi(x) = e^{-w_1x+w_2}$, then $\hat{\kappa} = \kappa + \sigma^2 w_1$, $\hat{\theta} = \frac{\kappa\theta+\sigma^2 w_2}{\kappa+\sigma^2 w_1}$. See Zeng et al. [25] for a concrete analytical solution of $E\left[e^{zX_t} \mid x_u = x, v_u = v\right]$.

Based on Theorem 1 and Zeng et al. [25], we obtain the first four moments of X_t. For the 4/2 jump diffusion model, it is impractical and analytical solutions of the derivatives can only be obtained from moment generating functions under a special equation. The first four moments are solved by numerical inversion with adaptive modification of the moment generating function proposed by Choudhury et al. [30]. Then, the simulation of X_t at each time node can be completed by using the Johonson curve method [29] mentioned.

Next, we need to couple the nodes of X_t and V_t together. Define a two-dimensional willow tree ($m_v = 3$,$m_x = 3$ as an example). The $v_p^{t_n}$ indicates that this is the p volatility(obtained from the volatility willow) under time node t_n, $n = 1, 2 \ldots N, p = 1, 2, \ldots m_v$, where $N = 3, m_v = 3$; $S_{pq}^{t_n}$ implies that this is the q that X-point of the p fluctuation at the node of time t_n. $k = pq$, S_{pq}^{k} is called $S_k^{t_n}$.

$$S_k^{t_n} = \exp\{X_q^{t_n} + lnS_u + \frac{\rho}{\sigma}\left(a\left(V_p^{t_n} - V_u\right) + b\log(\frac{V_p^{t_n}}{V_u})\right) + (r - \lambda\mu)(t-u)\},$$

the above expression yields the two-dimensional willow node value of the asset price.

IV. Optimization algorithm design for optimal investment-consumption problem

The financial investment market is assumed to be complete with two types of assets available to investors. One is a riskless asset (bank deposits) whose price R_t at time $t(t \geq 0)$ satisfies

$$\frac{\mathrm{d}R_t}{R_t} = r\ \mathrm{d}t,$$

where $r > 0$ is the bank yield. The other is a risky asset (stock) whose price S_t at time t follows Equation (10). The wealth of the investor W_t at the moment is t, the consumption process is C_t and the proportion invested in risky assets is $\alpha_t \in [0, 1]$. That is, $\alpha_t W_t$ invests in risky assets and $(1 - \alpha_t) W_t$ invests in riskless assets, then the process of shifting the wealth of the investor is

$$\mathrm{d}W_t = (\frac{\mathrm{d}S_t}{S_t})\alpha_t W_t + r\left(1 - \alpha_t\right) W_t\ \mathrm{d}t - C_t \mathrm{d}t, \tag{14}$$

substituting Equation (10) into Equation (14), then Equation (14) is rewritten as

$$\mathrm{d}W_t = \left(rW_t - \alpha_t W_t \lambda\mu - C_t\right)\mathrm{d}t + \alpha_t W_t((e^J - 1)\mathrm{d}N_t + (a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})(\rho \mathrm{d}B_t + \sqrt{1-\rho^2}\ \mathrm{d}B_t^{\perp})),$$

the initial conditions $W_{t_0} = w$, t_0 are the initial times; The value $w \geq 0$ is the initial wealth.

Samuelson [6] is used to discretize Equation (14) on $[t_{n-1}, t_n]$, where $n > 1$,$t_n = \frac{T}{N} * n$, can be obtained as follows

$$W_{t_n} - W_{t_{n-1}} = \alpha_{t_{n-1}} W_{t_{n-1}} \frac{S_{t_n}}{S_{t_{n-1}}} + r\left(1 - \alpha_{t_{n-1}}\right) W_{t_{n-1}}\left(t_n - t_{n-1}\right) - C_{t_{n-1}},$$

where $C_{t_{n-1}} = \left[W_{t_{n-1}} - \frac{W_{t_n}}{\left[(1-\alpha_{n-1})(1+r) + \alpha_{n-1}\frac{S_{t_n}}{S_{t_{n-1}}}\right]}\right]$. An investor's investment strategy (α_t, C_t) at time t satisfies $W_t \geq 0$. This strategy is called an admissible strategy. Let us call all allowed policies $\Sigma = \{(\alpha, C) : \alpha = \{\alpha_t, t \geq t_0\}, C = \{C_t, t \geq t_0\}\}$, the purpose of investors is to choose the optimal investment and consumption strategy, so as to maximize the expected consumption utility function(called objective function) in the whole investment period, namely

$$J = \max_C E\left(\int_0^T \mathrm{e}^{-\rho t} U\left(C_t\right)\mathrm{d}t\right), \tag{15}$$

where E denotes the mathematical expectation, the interest rate $\rho > 0$ is the discount rate for future consumption, $U\left(C_t\right) = \frac{C_t^\gamma}{\gamma}$ is the utility function of consumption and $\gamma \in (0, 1)$ is the relative risk aversion parameter.

This paper adopts the framework of Samuelson [6] to discretize the problem:

$$\begin{aligned} J &= \max_C E\left(\int_0^T \mathrm{e}^{-\rho t} U\left(C_t\right)\mathrm{d}t\right) \\ &= \max_C E\left(\sum_{t=0}^{T} \mathrm{e}^{-\rho t} U\left(C_t\right)\right), \\ \textbf{s.t.} \quad C_t &= \left[W_t - \frac{W_{t+\Delta_t}}{\left[\left(1-\alpha_t\right)\left(1+r\right) + \alpha_t \frac{S_{t+\Delta_t}}{S_t}\right]}\right], \end{aligned} \tag{16}$$

during the investment period $[0, T]$, the time interval for policy adjustment between periods is chosen to be consistent with the time step of the two-dimensional willow method in the previous sections. Namely, $\Delta_t = T/N$ and N denote the time steps in the willow algorithm, $t_i = \Delta_t * i$, $i = 0, 1, 2, 3, \cdots, N$. The optimal investment consumption problem is solved by a forward recursion in the framework of the willow tree.

Based on the above results, Algorithm A for solving the optimal investment strategies and consumption using the willow tree method is designed, with the pseudocode provided in Appendix as shown. It is worth noting that the final investment-consumption strategy obtained by the algorithm is a top-dimensional array of $M * K * (N + 1)$, where $C(m, k, n)$ represents the optimal consumption value at time $t = t_n$ when the wealth is $W_{t_n,k}$, and the corresponding investment value is $W_{t_n,k} - C(m, k, n)$ when the price of risky assets is $S_{t_n,k}$. $\alpha(M, K, N + 1)$ represents the proportion of investment in risky assets under the above conditions. In particular, at time $t = t_0$, when the risky asset has only one price S_0 and the wealth account has only one value W_0, then the values of $C(M, K, 1)$ and $\alpha(M, K, 1)$ reduce to a single value W_0. When $t = t_N$, the end of the investment, $\alpha(M, K, N + 1)$ is a zero-matrix. While at the end of the investment, $C(M, K, N + 1)$ be a matrix such that each column is the same when we run out of accounts.

V. Numerical experiment

In this section, the analysis utilizing Algorithm A (Solving the optimal investment-consumption algorithm generation algorithm) based on the two-dimensional willow method is conducted, commonly denoted as WT.

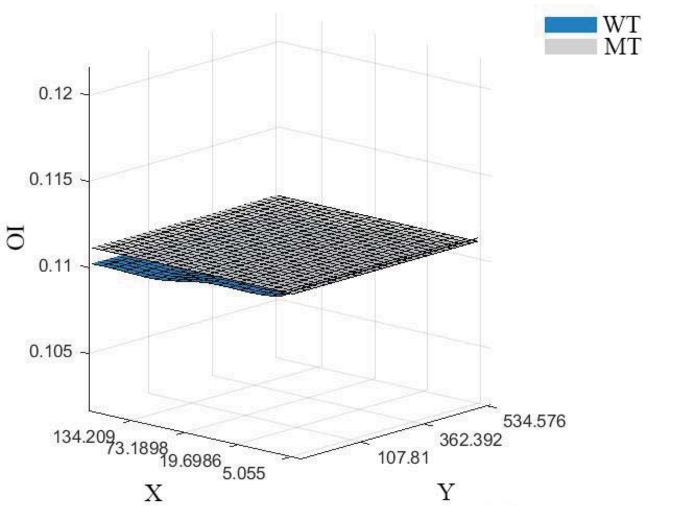

Figure 2: Difference surface graph of the optimal investment strategy with WT and Merton's analytical solution under model GBM.

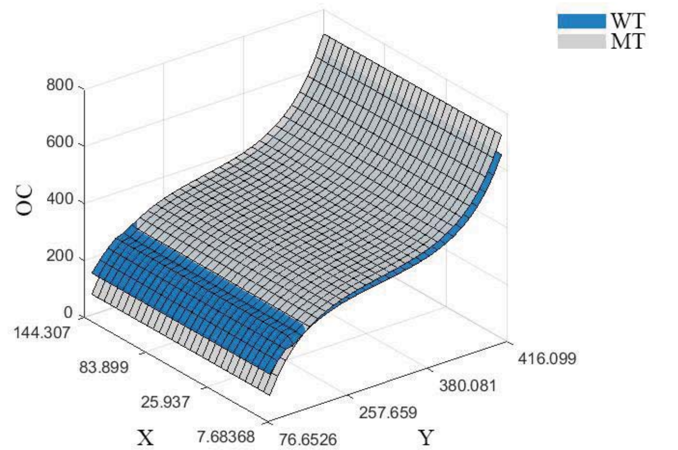

Figure 3: Difference surface plots of the optimal consumption strategy with WT and the Merton solution under the GBM model.

To validate the effectiveness of Algorithm A, the subsequent analysis considers the scenario where the risky asset follows a geometric Brownian motion (GBM), and compares the results of Algorithm A with the analytical solution from Merton's literature. Subsequently, an extension of the risky asset model is conducted to explore the optimal investment-consumption strategy when the risky asset follows the Heston model, the 3/2 model, and the 4/2 jump-diffusion model, respectively. Finally, a numerical analysis is performed to examine the sensitivity of this strategy to parameters.

The difference between the optimal investment consumption strategy and Merton's analytical solution is investi-

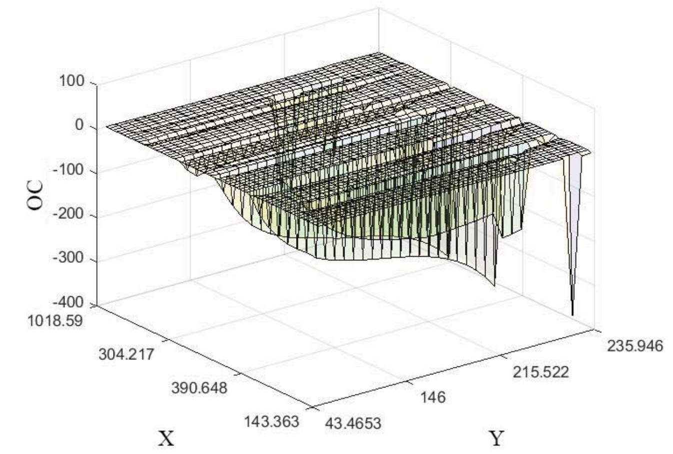

Figure 4: The interpolation between the optimal consumption in the Heston Mmodel at $r = 0.02$ and $r = 0.01$ risk-free interest rates.

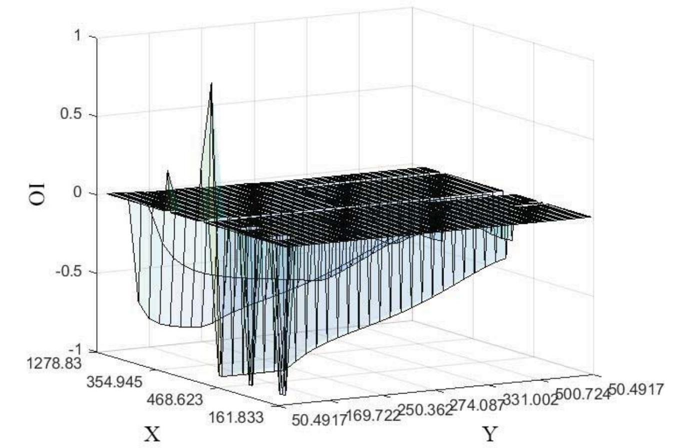

Figure 5: The interpolation between the optimal investment in the Heston model at $r = 0.02$ and $r = 0.01$ risk-free interest rates.

gated using a 3D graph. The method in this paper is commonly denoted as WT. The axis X represents the nodes of risky asset prices, the axis Y represents the node of the account, the axis OI and OC denote optimal investment and optimal consumption in the following figures, respectively. The parameters are chosen as follows $S_0 = 100$; $\mu = 0.02$; $\sigma = 0.3$; $T = 10$; $M = 30$; $\gamma = 0.5$; $\rho = 0.8$; $K = 30$; $W_0 = 100$; $r = 0.015$.

In Figure 2, the WT represents the willow tree method value and the MT represents Merton's analytical solution respectively. The optimal investment strategy obtained by the willow algorithm is in excellent agreement with the Merton solution, with a relative error within 4%. Similarly, in Figure 3, the WT represents the willow tree method value and the MT represents Merton's analytical solution respectively. The optimal consumption strategy proposed

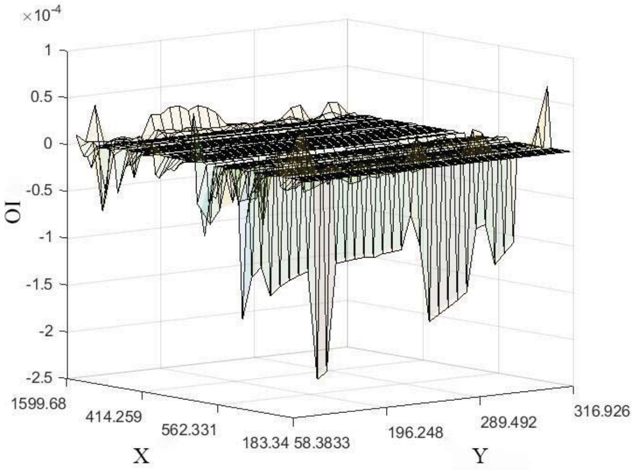

Figure 6: The interpolation in optimal investment for the Heston model with consumption discounting coefficients of $\rho = 0.8$ and $\rho = 0.2$.

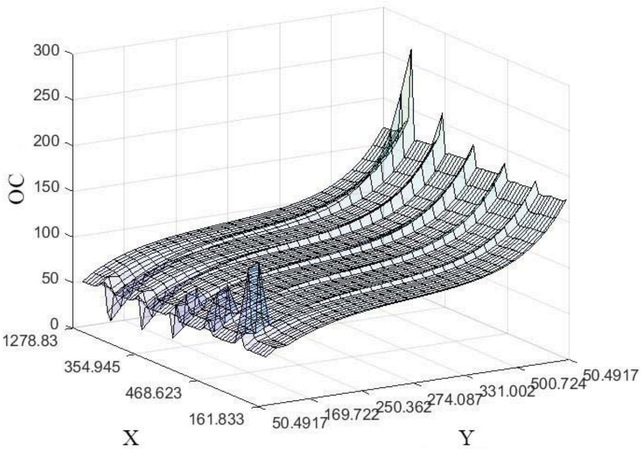

Figure 7: The interpolation in optimal comsumption for the Heston model with consumption discounting coefficients of $\rho = 0.8$ and $\rho = 0.2$.

by the Willow algorithm differs slightly from the Merton solution, with the relative error essentially maintained within 10%.

Through the analysis of Figure 2 and Figure 3, the effectiveness of using the WT method to solve the optimal investment consumption problem can be observed. Generally, optimal control theory combined with the HJB equation is used to solve the optimal investment consumption problem with stochastic volatility and jump-diffusion processes. However, this method usually cannot derive exact solutions for the corresponding optimal investment strategy and optimal consumption. In this paper, the WT method based on the asset price moment-generating function and the willow tree method can be used to simulate this type of problem. Next, we will use the WT method to analyze the optimal investment consumption problem

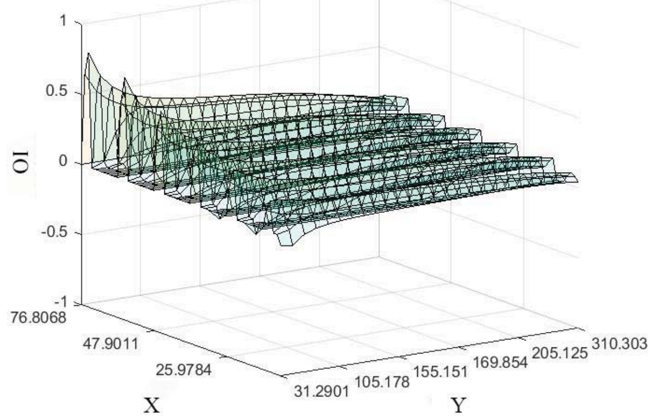

Figure 8: The interpolation between the optimal investment in the 3/2 model with absolute risk aversion coefficients of $\gamma = 0.7$ and $\gamma = 0.3$.

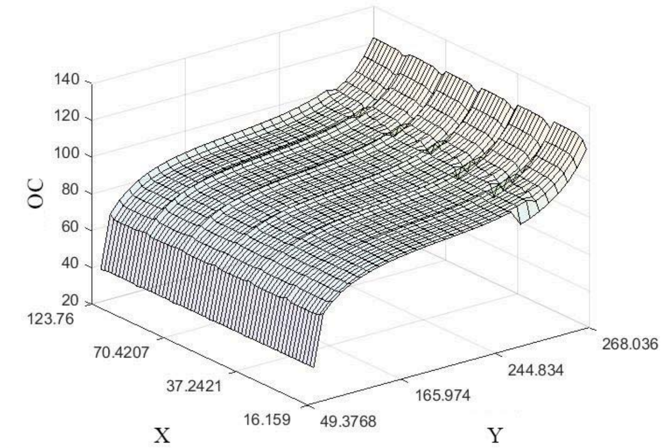

Figure 9: The interpolation between the optimal consumption in the 3/2 model with absolute risk aversion coefficients of $\gamma = 0.7$ and $\gamma = 0.3$.

for assets with stochastic volatility and jump-diffusion processes.

By examining Figures 5 through 12, the variations in the sensitivity of the optimal investment and consumption to the risk-free interest rates r, the discounting coefficients of ρ, the absolute risk aversion coefficients of γ and the jump intensity λ were correspondingly obtained. A numerical analysis of the simulation results was then conducted, incorporating insights from behavioral economics.

Firstly, using the Heston model as an example, we investigate the variations in optimal investment and consumption strategies for $r = 0.01$ and $r = 0.02$. In order to better illustrate the differences, we plot the net value of the optimal strategy for $r = 0.02$ subtracted by the strategy value for $r = 0.01$. The parameters for the Heston model are as follows: $T = 10; V_0 = 0.03; S_0 = 100; \mu = 0.05; \rho =$

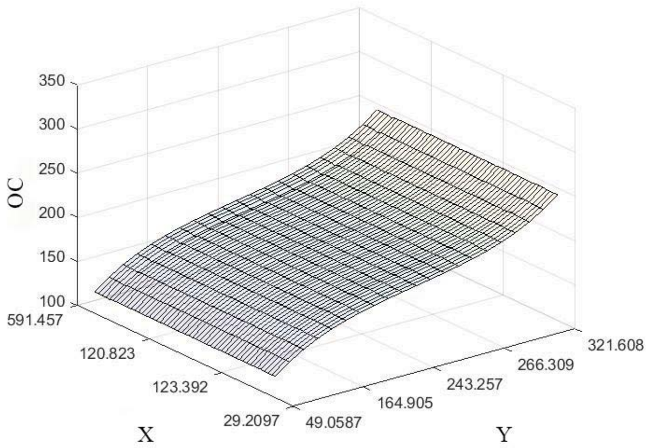

Figure 10: Optimal consumption in Heston's jump diffusion model with jump intensity $\lambda = 1$.

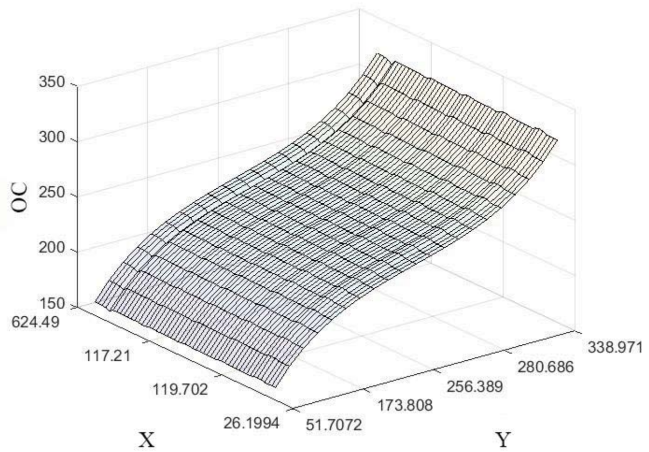

Figure 11: Optimal consumption in Heston's jump diffusion model with jump intensity $\lambda = 2$.

$-0.3; \kappa = 2; \theta = 0.04; \sigma = 0.12; a = 1; b = 0$. By analyzing Figure 4 and Figure 5, we can see that when the risk-free rate increases, the overall optimal consumption decreases and the optimal proportion of investments in risky assets decreases.

Similarly, we plot the net value of the optimal strategy for $\rho = 0.8$ subtracted from the strategy value for $\rho = 0.2$. As can be seen in Figure 6 and Figure 7, when the discount factor of consumption increases, consumption increases considerably during the intermediate period of investment. However, while the risk of the investment will be reduced depending on the value of the asset, the proportion of the investment in risky assets will not be modified much overall.

Next, the 3/2 model is taken as an example to investigate the sensitivity of the optimal portfolio in the proposed algorithm to different absolute risk aversion coefficients. Here, the parameter values of the 3/2 model are: $T = 10; V_0 = 0.04; S_0 = 100; \mu = 0.04; \rho = -0.3; \kappa = 1.8; \theta = 0.04; \sigma = 0.2; a = 0; b = 1;$, where γ is 0.3 and 0.7 respectively, and the remaining portfolio parameters are the same as above. Figure 8 and Figure 9 is $\gamma = 0.7$ subtracted from the strategy value corresponding to $\gamma = 0.3$. The surface reflects that when the risk aversion coefficient increases, the proportion of investments in risky assets increases, and the optimal consumption increases.

When the jump strength is inconsistent, the risk asset prices at the willow price nodes are also inconsistent. Consequently, the corresponding price values and account values at each point do not align during the graph plotting process. Therefore, they are plotted separately.

The sensitivity of the optimal investment strategy to jump intensity is investigated under the Heston model of jump diffusion. The parameters are as follows: $T = 10; V_0 = 0.03; S_0 = 100; \mu = 0.05; \rho = -0.3; \kappa = 2; \theta = 0.04; \sigma = 0.12; a = 1; b = 0$. The relevant parameters of the jump are as follows: the mean value of the jump is $\alpha_J = -0.05$, the standard deviation of the jump is $\sigma_J = 0.05$, and the strength of the jump is $\lambda = 1$ and $\lambda = 2$, respectively.

Based on Figure 10 and Figure 11, the variations in optimal consumption of the Merton jump-diffusion model are presented for jump intensities λ of 1 and 2, respectively. The consumption value under the optimal strategy increases as the jump strength of risky asset prices increases. This suggests that with increased uncertainty from risky assets, the optimal strategy leans more towards spending and less towards investing.

VI. Conclusion

This study derives the moment generating function of risky asset prices based on Ito's lemma and the measure transformation formula. By combining the properties of the willow structure with transition probabilities, a two-dimensional willow framework is constructed to simulate the wealth process. In the case of asset prices exhibiting stochastic volatility and jump diffusion models, integrating the two-dimensional willow framework with optimal control theory yields the willow method for tackling complex optimal investment and consumption problems.

The paper constructs nodes of a wealth account based on the willow algorithm for pricing risky assets, enabling the derivation of optimal investment and consumption strategies at each time node. The effectiveness of the willow algorithm is verified by comparing it with Merton's analytical solution. Subsequently, the paper conducts numerical analysis with different parameters to demonstrate that the willow algorithm can adjust strategies based on current market and account conditions, making it a practical method for addressing dynamic programming problems. While the willow algorithm has traditionally been used as a beneficial tool for studying derivative pricing in past research, this paper contributes by extending its application to optimal investment and consumption problems in a multi-period discrete setting. Additionally, the time-discrete structure of the willow algorithm provides advantages by helping to avoid the need to solve the HJB equation for complex asset price models and utility functions.

To better validate the effectiveness of the proposed algorithm, future research can involve parameter calibration of the risk asset model based on market conditions. Further substantiation of the innovative and practical value of the willow algorithm can be achieved through analyzing investment returns in real-world applications.

Appendices

Algorithm A: Solving the optimal investment consumption algorithm. The algorithm for solving the optimal investment strategies and consumption using the willow tree method is presented as follows.

Input: Given the number of willow nodes M, the number of account nodes K, the number of time steps N, and the investment period T; The input risk asset willow node$nodes_S_t$, which is a $M * N$ matrix.
Output: $[\alpha, C]$, where α represents the proportion of investment in risky assets and C represents the optimal consumption.
for $i = 1\ n$ **do**
end for
$\alpha = zeros(M, K, N+1)$, $C = zeros(M, K, N+1)$;
$J\_points = zeros(K, 2, M)$;
for n=N:-1:1 **do**
  $t_n = \frac{T}{N} * n$;
  $S_{t_n,max} = max(nodes\_S_t(:, t_n))$;
  $W_{t_n,max} = W_0 \cdot \frac{S_{t_n,max}}{S_0}$;
  for i=1:M **do**
    for k=1:K **do**
      $W_{t_n,k} = W_0 \cdot k \cdot \frac{W_{t_n,max}}{K}$;
      if $n = N$ **then**
        $C^*_{(t_n,ik)} = W_{t_n,k}$;
        $\alpha^*_{(t_n,ik)} = 0$;
        if n=N-1 **then**

$$
\begin{aligned}
J_1^i(W_{t_n,k}) = & \max_{\{C_{(t_n,ik)};\alpha_{(t_n,ik)}\}} U[C_{t_n,ik}] \\
& + \sum_{j=1}^{m} P_{ij}(1+\rho)^{-\Delta_t} U[(W_{t_n,k} - C_{t_n,ik}) \\
& \times \{(1-\alpha_{t_n,ik})(1+r) + \alpha_{t_n,ik}\frac{S_{t_{n+1},j}}{S_{t_n,i}}\}];
\end{aligned}
$$

          $Points_list(k, i) = [W_{t_n,k}, J_1^i(W_{t_n,k})]$;
        end if
      else
        for j=1:M **do**
          $x = Points\_list(:, 1, j)$;
          $J^j = Points\_list(1, :, j)$;
          $J^j(x)$ is obtained by cubic spline interpolation of K points;
        end for

$$
\begin{aligned}
J^i_{(N-n)}(W_{t_n,k}) = & \max_{\{C_{(t_n,ik)};\alpha_{(t_n,ik)}\}} U[C_{t_n,ik}] \\
& + \sum_{j=1}^{M} P_{ij}(1+\rho)^{-\Delta_t} J^j_{(N-n+1)} \\
& \times [(W_{t_n,k} - C_{t_n,ik})\{(1-\alpha_{t_n,ik})(1+r) \\
& + \alpha_{t_n,ik}\frac{S_{t_{n+1},j}}{S_{t_n,i}}\}];
\end{aligned}
$$

        $Points_list(k, i) = [W_{t_n,k}, J^i_{N-n}(W_{t_n,k})]$;
      end if
      $C(i, k, n+1) = C^*_{(t_n,ik)}$;
      $\alpha(i, k, i+1) = \alpha^*_{(t_n,ik)}$;
    end for
  end for
end for
return $[\alpha, C]$.

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