Shared Carpooling Queue Model with Variable Matching Mechanism

Xiuli Xu, Yitong Zhang*, Yuting Tan and Mingxin Liu

***Abstract*—The carpooling queue means that the e-hailing can continue to match with another passenger when it serves the first matched passenger. If the matching is successful, two passengers will share the carpooling service, otherwise, the first passenger will enjoy the express taxi service. When the carpooling probability increases, the expected sojourn time of passengers reduces and the benefits of the e-hailing drivers can be significantly improved. Motivated by carpooling strategies in sharing platforms, this paper constructs the M/M/c+m queueing model of ride-sharing matching with variable matching rates. The steady-state equilibrium condition, the steady-state probability distribution, and the main performance indexes are derived by using the matrix geometric solution and iterative methods. Furthermore, the sensitivity of the expected queue length, the expected length of successfully matched customers, and the matching probability are illustrated through sufficient numerical experiments. The revenue functions are established from the view of the customers, the e-hailing drivers, and the e-hailing system, respectively, and the tripartite benefits are explicitly analyzed by setting different simulation parameters. Finally, the maximal benefits and the optimal arrival rates are obtained by using the seagull optimization algorithm. The research results can reveal the operation mechanism of the sharing platform and provide decisions and implications for customers and e-hailing drivers, which have theoretical and practical significance.**

***Index Terms*—M/M/c+m queue, matrix geometric solution, e-hailing, shared carpooling, variable matching rate.**

I. Introduction

Queuing theory is a powerful mathematical tool to study the congestion problems in stochastic service systems, which has been widely applied in communication networks [1,2], reliability facilities [3,4], transportation systems [5,6], and so on. The performance structure of complex service systems can be evaluated and optimized by solving the statistical rules of the main performance indicators in the queueing model. There are many double-ended queueing models in practice, which means that when the number of customers is greater than the number of servers, the customers form a one-end queue; Instead, the servers form a one-end queue. For example, taxi drivers can choose to unconstrained setup, and the passengers can wait for a taxi randomly as well. Based on the randomness of the queueing process, numerous scholars incorporated the Markov chain into the comprehensive urban transportation system. Wong et al. [7] established a mathematical model to describe the daily operation of taxis and analyzed the relationship between customers and taxis using the absorbing Markov chains. Considering the limit behaviors in the light-traffic and heavy-traffic scenarios, Liu et al. [8] established a priority polling system consisting of three M/M/1 queues. Wang and Yan [9] constructed a taxi-dispatching model for simulating the operation scenario of the taxi, and the numerical results can be used in station engineering and system management. Wang et al. [10] studied the optimal queueing strategy of airport taxis under different numbers of drive lanes. Based on the Markovian performance analysis methods, Ahmadi et al. [11] calculated the queueing model with impatient customers and processor sharing.

Accomplishing the development of information technology and the popularization of smartphones, online car-hailing has gradually become the mainstream for passengers in recent times. In the online car-hailing platform, the system can centrally manage and control the number of e-hailing and acts as an intermediary for matching between customers and e-hailing drivers. The continuous improvement of the online car-hailing system means that the dynamics and operations of the taxi markets become more visible and competitive. Based on the vehicle trajectory data, Cai et al. [12] studied the differences in travel patterns between the taxis and the customers. Cui et al. [13] proposed a spatial point model for random vehicle locations and accurately described diverse spatial point patterns in different cities. Król and Król [14] established a simplified simulation model for the operation of the taxi system, and the optimal number of taxis was determined in daily operation. Ooi et al. [15] reviewed the results from the perspective of passengers and analyzed the operation characteristics and current status of e-hailing.

In the era of the sharing economy, the rapid development of online car-hailing has caused a considerable impact on the traditional taxi industry. There has been an upsurge of interest in the operation mechanism of online car-hailing, and how to balance the tripartite game among passengers, drivers, and platforms has received significant attention. Qian et al. [16] investigated the incentive strategy of taxi operations and analyzed the system benefits when different incentive levels were applied to customers and drivers. Braverman et al. [17] considered a closed queueing network model with a shared system and studied an empty-car routing strategy to optimize the utility. Considering the dynamic pricing mechanism, Wu

Manuscript received August 8, 2024; revised November 21, 2024.

This work was supported in part by the National Natural Science Foundation under Grant 62171143 and Natural Science Foundation of Hebei Province under Grant G2024203008.

Xiuli Xu is a professor of the School of Science, Yanshan University, Hebei Qinhuangdao, 066004, China. (e-mail: xxl-ysu@163.com).

Yitong Zhang is a lecturer of the School of Science, Tianjin University of Commerce, Tianjin, 300134, China. (corresponding author to provide phone: 17302236019; e-mail: zhangyitong@tjcu.edu.cn).

Yuting Tan is a postgraduate student of the School of Science, Yanshan University, Hebei Qinhuangdao, 066004, China. (e-mail: 1176878197@qq.com)

Mingxin Liu is a professor of the School of Electronics and Information Engineering, Guangdong Ocean University, Guangdong, Zhanjiang, 524000, China. (e-mail: liumx@gdou.edu.cn)

et al. [18] conducted the pricing and revenue distribution of the e-hailing system, and the maximal benefits can provide the references for the regulation and decision-making of the e-hailing market. In an on-demand mobile service platform, Sayarshad and Chow [19], Zhang et al. [20], and Xiao and Shen [21] applied the queueing theory for urban transportation to reduce operation costs. Moreover, Wang and Liu [22] considered a dynamic taxi control strategy and discussed the relationship between the individual equilibrium and socially optimal strategies under the different information levels. Recently, Zhang et al. [23] proposed an innovative modeling structure for the competitive taxi market and constructed a queueing model between the e-hailing and urban systems, which feeds back the performance of the road network to the e-hailing platforms. Based on the game group theory, Nguyen and Tuan [24] analyzed the joining probabilities of the taxi queueing system in unobservable cases. By constructing a complex queuing system, the reasonable configuration of the e-hailing resources and the optimization of system efficiency can be achieved by exploring the operation mechanism of the online car-hailing platform.

In a fixed area, the online car-hailing system randomly assigns e-hailing riders to customers. If a customer chooses the carpooling order, the driver will serve this customer after successful matching, and the driver may match another customer during the service. If the carpooling is successful, two passengers will share the carpooling price. Otherwise, the first rider will be charged for the express taxi at the end of the service. Based on the above practical background, this paper establishes a double-ended queueing model with the variable matching strategy, which reflects the carpooling process of shared systems realistically. The main performance measures of the stochastic service system are derived from the quasi-birth-death process and matrix geometric solution method, and the comparisons of different system parameters on the performance indicators are illustrated by numerical examples. Besides, this paper constructs utility functions from the perspectives of the customers, e-hailing drivers, and e-hailing systems respectively, and the graphical findings can be used to evaluate the operation and management of online car-hailing systems.

The remainder of this paper is described as follows. Section 2 constructs the shared carpooling system with the variable matching mechanism. Section 3 explicitly describes the M/M/c+m queueing model. Section 4 is devoted to the steady-state analysis of the system and obtains the main performance indicators. Section 5 illustrates the effects of the diverse system parameters on the expected queue length and the benefits through sufficient numerical comparisons. Finally, in Section 6, we briefly conclude the paper.

II. Model assumption

In this paper, we introduce the shared matching mechanism and variable matching probability in the M/M/c queueing model. When the servers stay idle, they can serve the customers who joined the system at any time. Meanwhile, the server can still match new customers while serving the first customer. If the match is successful, the first customer will share the service with the matched customer. If there is no matched customer at the end of the first service, the sharing will be considered failed and the first service will be performed as the express taxi service mechanism. The carpooling successfully probability can be increased when the expected queue length of customers increases. The variable matching rate based on the shared carpooling queueing model is described as follows:

(1) Customers' interarrival times follow an exponential distribution with parameter λ, and the service rule is first-come first-served.

(2) Assume that the server (e-hailing) serves a maximum of two customers at the same time. When the idle server accepts a shared carpooling order, the server can still match another customer while serving the first customer. If the second customer is not successfully matched before the first customer's service ends, the server will be idle. If the matching is successful, the server will serve two customers and no more orders will be accepted. Considering that two carpooling customers will have different drop-off locations, after serving one of them, the server will continue to serve the other and will not match the other customers during the service. When the service is finished, the server will serve customers again if customers are waiting in line.

(3) Assume that the service time follows the exponential distribution with parameter μ_1 when the server is serving one customer, and follows the exponential distribution with parameter μ_2 $(\mu_2 < \mu_1)$ when the server is sharing the service between two customers.

(4) Assume states 1 and 2 indicate that the server has successfully matched one and two customers in a carpooling order, respectively. Assume state 3 represents that the server has completed the service of one of the two customers who successfully carpooled, but the server cannot match other customers until the residual customer completes the service. In a fixed area, assuming that there are c servers in the system, which means that the system has at most c servers in state 1 or 2 at the same time. In addition, assume that the system has at most m servers in state 3 at the same time.

(5) Carpooling matching is an independent process that follows an exponential distribution. The success rate of carpooling matching η_n varies dynamically depending on the number of waiting customers n. Assume that $\eta_n = \alpha_n \eta$, where

$$\alpha_n = \begin{cases} 1, & 0 \le n \le n_1 \\ 1 + \frac{1}{2}\left(\ln \frac{N}{n_1}\right)\ln \frac{n_1}{n}, & n_1 < n \le N. \\ 1.5, & n > N \end{cases}$$

(6) If a customer arrives at the system without idle servers or carpooling marching servers with matching capability, the customer will join the queue with probability b or leave the system with probability $1-b$, that is, the efficient arrival rate is $b\lambda$ in this case. The customers may be impatient and leave the system when they wait for a long time in the queue, then the departure time follows an exponential distribution with parameter σ_n, which varies dynamically depending on the number of waiting customers n, assume that

$$\sigma_n = \begin{cases} n\sigma, & 1 \le n \le N, \\ N\sigma, & n > N. \end{cases}$$

(7) When there are queues in the system, the idle servers will join the system, and the joining process follows an exponential distribution with parameter q.

III. MODEL DESCRIPTION

Based on the above assumptions, let $N(t)$ be the number of customers waiting in the queue at time t, let $K(t)$, $J(t)$ and $S(t)$ be the number of servers in state 1, 2 and 3 at time t, respectively. The four-dimensional Markov chain $\{(K(t), J(t), S(t), N(t)), t \ge 0\}$ has state space $\Omega = \{(k, j, s, n) \mid 0 \le k + j \le c, 0 \le s \le m, n \ge 0\}$. According to the lexicographical order of states, the infinitesimal generator Q of the quasi-birth-death (QBD) process has the following form

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{C}_0 & & & & & \\ \boldsymbol{B}_1 & \boldsymbol{A}_1 & \boldsymbol{C}_0 & & & & \\ & \boldsymbol{B}_2 & \boldsymbol{A}_2 & \boldsymbol{C}_0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \boldsymbol{B}_N & \boldsymbol{A}_N & \boldsymbol{C}_0 & \\ & & & & \boldsymbol{B}_N & \boldsymbol{A}_N & \boldsymbol{C}_0 \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Its submatrices are as follows:

- $\boldsymbol{A}_0$ is a square matrix of order $m(c+1)(c+2)/2$, and

$$\boldsymbol{A}_0 = \begin{bmatrix} \boldsymbol{Y}_0^0 & \boldsymbol{X}_0^0 & & & & \\ \boldsymbol{Z}_1^0 & \boldsymbol{Y}_1^0 & \boldsymbol{X}_1^0 & & & \\ & \boldsymbol{Z}_2^0 & \boldsymbol{Y}_2^0 & \boldsymbol{X}_2^0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \boldsymbol{Z}_{m-1}^0 & \boldsymbol{Y}_{m-1}^0 & \boldsymbol{X}_{m-1}^0 \\ & & & & \boldsymbol{Z}_m^0 & \boldsymbol{Y}_m^0 \end{bmatrix},$$

where $\boldsymbol{X}_s^0 (0 \le s \le m-1)$, $\boldsymbol{Y}_s^0 (0 \le s \le m)$, $\boldsymbol{Z}_s^0 (1 \le s \le m)$ are square matrices of order $(c+1)(c+2)/2$.

(1) $\boldsymbol{Y}_s^0 (0 \le s \le m)$ are partitioned as

$$\boldsymbol{Y}_s^0 = \begin{bmatrix} D_{0,s}^0 & \boldsymbol{S}_0 & & & & \\ \boldsymbol{F}_1 & \boldsymbol{D}_{1,s}^0 & \boldsymbol{S}_1 & & & \\ & \boldsymbol{F}_2 & \boldsymbol{D}_{2,s}^0 & \boldsymbol{S}_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \boldsymbol{F}_{c-1} & \boldsymbol{D}_{c-1,s}^0 & \boldsymbol{S}_{c-1} \\ & & & & \boldsymbol{F}_c & \boldsymbol{D}_{c,s}^0 \end{bmatrix},$$

a) $\boldsymbol{D}_{k,s}^0 (0 \le k \le c)$ are square matrices of order $(k+1)$, and

$$D_{0,s}^0 = -(\lambda + b\lambda + s\mu_1),$$

$$\boldsymbol{D}_{k,s}^0 = \begin{bmatrix} h_{0,s}^0 & l_0 & & & \\ & h_{1,s}^0 & l_1 & & \\ & & \ddots & \ddots & \\ & & & h_{k-1,s}^0 & l_{k-1} \\ & & & & h_{k,s}^0 \end{bmatrix}, 1 \le k \le c,$$

when $1 \le k \le c-1, 0 \le s \le m-1$,

$$h_{i,s}^0 = \begin{cases} -[\lambda + (k-i)\mu_1 + i\mu_2 + (k-i)\eta + s\mu_1], & 0 \le i \le k-1, \\ -[\lambda + b\lambda + (k-i)\mu_1 + i\mu_2 + (k-i)\eta + s\mu_1], & i = k, \end{cases}$$

$$l_i = (k-i)\eta, \quad 0 \le i \le k-1,$$

when $1 \le k \le c-1, s = m$,

$$h_{i,s}^0 = \begin{cases} -[\lambda + (k-i)\mu_1 + (k-i)\eta + s\mu_1], & 0 \le i \le k-1, \\ -[\lambda + b\lambda + (k-i)\mu_1 + (k-i)\eta + s\mu_1], & i = k, \end{cases}$$

$$l_i = (k-i)\eta, \quad 0 \le i \le k-1,$$

when $k = c, 0 \le s \le m-1$,

$$h_{i,0} = \begin{cases} -[(k-i)\mu_1 + i\mu_2 + (k-i)\eta + s\mu_1], & 0 \le i \le k-1, \\ -[b\lambda + (k-i)\mu_1 + i\mu_2 + (k-i)\eta + s\mu_1], & i = k, \end{cases}$$

$$l_i = (k-i)\eta, \quad 0 \le i \le k-1,$$

when $k = c, s = m$,

$$h_{i,0} \begin{cases} -[(k-i)\mu_1 + (k-i)\eta + s\mu_1], & 0 \le i \le k-1, \\ -[b\lambda + (k-i)\mu_1 + (k-i)\eta + s\mu_1], & i = k, \end{cases}$$

$$l_i = (k-i)\eta, \quad 0 \le i \le k-1.$$

b) $\boldsymbol{S}_k (0 \le k \le c)$ are k-by-$(k+1)$ matrices, and

$$\boldsymbol{S}_0 = [\lambda \ \ 0], \quad \boldsymbol{S}_k = \left[\mathrm{diag}(\lambda, \lambda, ..., \lambda), \boldsymbol{0}_{k \times 1}\right], \quad 1 \le k \le c-1.$$

c) $\boldsymbol{F}_k (1 \le k \le c)$ are $(k+1)$-by-k matrices, and

$$\boldsymbol{F}_1 = \begin{bmatrix} \mu_1 \\ 0 \end{bmatrix}, \quad \boldsymbol{F}_k = \begin{bmatrix} \mathrm{diag}(k\mu_1, (k-1)\mu_1, ..., \mu_1) \\ \boldsymbol{0}_{1 \times k} \end{bmatrix}, \quad 2 \le k \le c-1.$$

(2) $\boldsymbol{Z}_s^0 (0 \le s \le m)$ are partitioned as

$$\boldsymbol{Z}_s^0 = \begin{bmatrix} \boldsymbol{z}_{0,s}^0 & & & \\ & \boldsymbol{z}_{1,s}^0 & & \\ & & \ddots & \\ & & & \boldsymbol{z}_{c,s}^0 \end{bmatrix},$$

where $\boldsymbol{z}_{i,s}^0 = \mathrm{diag}(s\mu_1, s\mu_1, ..., s\mu_1), 0 \le i \le c$ are square matrices of order $(i+1)$.

(3) $\boldsymbol{X}_s^0 (0 \le s \le m-1)$ are partitioned as

$$\boldsymbol{X}_s^0 = \begin{bmatrix} \boldsymbol{0} & & & & & \\ \boldsymbol{L}_1 & \boldsymbol{0} & & & & \\ & \boldsymbol{L}_2 & \boldsymbol{0} & & & \\ & & \ddots & \ddots & & \\ & & & \boldsymbol{L}_{c-1} & \boldsymbol{0} & \\ & & & & \boldsymbol{L}_c & \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{L}_i (1 \le i \le c)$ are $(i+1)$-by-i matrices, and

$$\boldsymbol{L}_1 = \begin{bmatrix} 0 \\ \mu_2 \end{bmatrix}, \quad \boldsymbol{L}_i = \begin{bmatrix} 0 & & & \\ \mu_2 & 0 & & \\ & 2\mu_2 & 0 & \\ & & \ddots & \ddots \\ & & & i\mu_2 & 0 \end{bmatrix}, 2 \le i \le c.$$

- $\boldsymbol{C}_0$ is a square matrix of order $m(c+1)(c+2)/2$, and

$$\boldsymbol{C}_0 = \begin{bmatrix} \boldsymbol{C}_{0,0} & & & \\ & \boldsymbol{C}_{0,0} & & \\ & & \ddots & \\ & & & \boldsymbol{C}_{0,0} \end{bmatrix},$$

where

$$\boldsymbol{C}_{0,0} = \begin{bmatrix} \boldsymbol{c}_{0,0} & & & & \\ & \boldsymbol{c}_{0,1} & & & \\ & & \boldsymbol{c}_{0,2} & & \\ & & & \ddots & \\ & & & & \boldsymbol{c}_{0,c} \end{bmatrix},$$

$\boldsymbol{c}_{0,i} (1 \le i \le c)$ are square matrices of order $(i+1)$, and $\boldsymbol{c}_{0,i} = \mathrm{diag}(0, ..., 0, b\lambda, 0, ..., 0).$

- $\boldsymbol{A}_n (1 \le n \le N)$ are square matrices of order $m(c+1)(c+2)/2$, and

$$\boldsymbol{A}_n = \begin{pmatrix} \boldsymbol{Y}_0^n & \boldsymbol{X}_0^0 & & & & \\ \boldsymbol{Z}_1^0 & \boldsymbol{Y}_1^n & \boldsymbol{X}_1^0 & & & \\ & \boldsymbol{Z}_2^0 & \boldsymbol{Y}_2^n & \boldsymbol{X}_2^0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \boldsymbol{Z}_{m-1}^0 & \boldsymbol{Y}_{m-1}^n & \boldsymbol{X}_{m-1}^0 \\ & & & & \boldsymbol{Z}_m^0 & \boldsymbol{Y}_m^n \end{pmatrix},$$

where $\boldsymbol{Y}_s^n (0 \le s \le m)$ are square matrices of order $m(c+1)(c+2)/2$,

$$\boldsymbol{Y}_s^n = \begin{bmatrix} D_{0,s}^n & \boldsymbol{S}_0 & & & & \\ \boldsymbol{F}_1 & \boldsymbol{D}_{1,s}^n & \boldsymbol{S}_1 & & & \\ & \boldsymbol{F}_2 & \boldsymbol{D}_{2,s}^n & \boldsymbol{S}_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \boldsymbol{F}_{c-1} & \boldsymbol{D}_{c-1,s}^n & \boldsymbol{S}_{c-1} \\ & & & & \boldsymbol{F}_c & \boldsymbol{D}_{c,s}^n \end{bmatrix},$$

and $\boldsymbol{D}_{k,s}^n (0 \le k \le c)$ are $(k+1)$th -order square matrices

$$D_{0,s}^n = -(\lambda + b\lambda + n\sigma + q + s\mu_1),$$

$$\boldsymbol{D}_{k,s}^n = \begin{bmatrix} h_{0,s}^n & l_0 & & & \\ & h_{1,s}^n & l_1 & & \\ & & \ddots & \ddots & \\ & & & h_{k-1,s}^n & l_{k-1} \\ & & & & h_{k,s}^n \end{bmatrix}, 1 \le k \le c,$$

when $1 \le k \le c-1, 0 \le s \le m-1$,

$$h_{i,s}^n = \begin{cases} -[\lambda + (k-i)\mu_1 + i\mu_2 + 2(k-i)\eta + s\mu_1 + q], 0 \le i \le k-1, \\ -[\lambda + b\lambda + (k-i)\mu_1 + i\mu_2 + s\mu_1 + n\sigma + q], \ \ i = k, \end{cases}$$

$$l_i = (k-i)\eta, \ \ 0 \le i \le k-1,$$

when $1 \le k \le c-1, s = m$,

$$h_{i,s}^0 = \begin{cases} -[\lambda + (k-i)\mu_1 + 2(k-i)\eta + s\mu_1 + q], 0 \le i \le k-1, \\ -[\lambda + b\lambda + (k-i)\mu_1 + s\mu_1 + n\sigma + q], i = k, \end{cases}$$

$$l_i = (k-i)\eta, \ \ 0 \le i \le k-1,$$

when $k = c, 0 \le s \le m-1$,

$$h_{i,0} = \begin{cases} -[(k-i)\mu_1 + i\mu_2 + 2(k-i)\eta + s\mu_1 + q], 0 \le i \le k-1, \\ -[b\lambda + (k-i)\mu_1 + i\mu_2 + s\mu_1 + n\sigma + q], \ \ i = k, \end{cases}$$

$$l_i = (k-i)\eta, \ \ 0 \le i \le k-1,$$

when $k = c, s = m$,

$$h_{i,0} = \begin{cases} -[(k-i)\mu_1 + 2(k-i)\eta + s\mu_1 + q], 0 \le i \le k-1, \\ -[b\lambda + (k-i)\mu_1 + s\mu_1 + n\sigma + q], \ \ i = k, \end{cases}$$

$$l_i = (k-i)\eta, \ \ 0 \le i \le k-1.$$

- $\boldsymbol{B}_n$ $(1 \le n \le N)$ are square matrices of order $m(c+1)(c+,$ $2)/2$, and

$$\boldsymbol{B}_n = \begin{bmatrix} \boldsymbol{G}_n & & \\ & \boldsymbol{G}_n & \ddots \\ & & \ddots \ \boldsymbol{G}_n \end{bmatrix},$$

where $\boldsymbol{G}_n$ is square matrix of order $(c+1)(c+2)/2$, and

$$\boldsymbol{G}_n = \begin{bmatrix} r_{0,n} & \boldsymbol{v}_0 & & & \\ & \boldsymbol{r}_{1,n} & \boldsymbol{v}_1 & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{r}_{c-1,n} & \boldsymbol{v}_{c-1} \\ & & & & \boldsymbol{r}_{c,n} \end{bmatrix}.$$

(1) $\boldsymbol{r}_{k,n} (0 \le k \le c)$ are $(k+1)$ -order square matrices

$$r_{0,n} = n\sigma, \ \ \boldsymbol{r}_{k,n} = \begin{bmatrix} 0 & k\eta_n & & & \\ & 0 & (k-1)\eta_n & & \\ & & \ddots & \ddots & \\ & & & 0 & \eta_n \\ & & & & n\sigma \end{bmatrix}, \ \ 1 \le k \le c.$$

(2) $\boldsymbol{v}_k (0 \le k \le c-1)$ are $(k+1)$-by-$(k+2)$ matrices

$$\boldsymbol{v}_0 = [q \ \ 0], \ \ \boldsymbol{v}_k = \left[\operatorname{diag}(q, q, ..., q), \boldsymbol{0}_{(k+1)\times 1}\right], \ \ 1 \le k \le c-1.$$

IV. STEADY-STATE ANALYSIS

Considering that the Markov process $\{(K(t), S(t), J(t), N(t)), t \ge 0\}$ is a QBD process, the steady-state distribution of the system exists when the Markov process is positive recurrent.

The sufficient and necessary condition that the Markov process is positive recurrent is that the matrix quadratic equation

$$\boldsymbol{R}^2 \boldsymbol{B}_N + \boldsymbol{R}\boldsymbol{A}_N + \boldsymbol{C}_0 = \boldsymbol{0} \tag{1}$$

has a minimal non-negative solution $\boldsymbol{R}$, and the spectral radius $sp(\boldsymbol{R}) < 1$. The $m(N+1)(c+1)(c+2)/2$ -order square matrix

$$B[\boldsymbol{R}] = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{C}_0 & & & & \\ \boldsymbol{B}_1 & \boldsymbol{A}_1 & \boldsymbol{C}_0 & & & \\ & \boldsymbol{B}_2 & \boldsymbol{A}_2 & \boldsymbol{C}_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \boldsymbol{B}_{N-1} & \boldsymbol{A}_{N-1} & \boldsymbol{C}_0 \\ & & & & \boldsymbol{B}_N & \boldsymbol{R}\boldsymbol{B}_N + \boldsymbol{A}_N \end{pmatrix}$$

has a left zero vector $\boldsymbol{x}$, that is, $\boldsymbol{x}B[\boldsymbol{R}] = \boldsymbol{0}$ has a positive solution.

The steady state distribution is denoted by

$$\pi_{k,j,s}^n = \lim_{t\to\infty} P\{K(t) = k, J(t) = j, S(t) = s, N(t) = n\},$$
$$(k, j, s, n) \in \Omega,$$

the steady-state probability vector is

$$\boldsymbol{\Pi}_n = \left[\boldsymbol{\pi}_0^n, \boldsymbol{\pi}_1^n, \cdots, \boldsymbol{\pi}_m^n\right], \ \ n \ge 0,$$

$$\boldsymbol{\pi}_s^n = \left[\pi_{0,0,s}^n, \pi_{1,0,s}^n, \pi_{0,1,s}^n, \pi_{2,0,s}^n, \pi_{1,1,s}^n, \pi_{0,2,s}^n, \cdots, \pi_{c,0,s}^n, \pi_{c-1,1,s}^n, \cdots, \pi_{1,c-1,s}^n, \pi_{0,c,s}^n\right], 0 \le s \le m.$$

Hereafter, $\boldsymbol{e}$ is an appropriate dimension column vector with all element ones, $\boldsymbol{I}$ is an appropriate order unit matrix. Based on the matrix geometric solution method in [25], we get

$$(\boldsymbol{\Pi}_0, \boldsymbol{\Pi}_1, \boldsymbol{\Pi}_2, \cdots, \boldsymbol{\Pi}_N) B[\boldsymbol{R}] = \boldsymbol{0}, \tag{2}$$

$$\boldsymbol{\Pi}_n = \boldsymbol{\Pi}_N \boldsymbol{R}^{n-c}, n \ge c, \tag{3}$$

$$\sum_{n=0}^{N-1} \boldsymbol{\Pi}_n \boldsymbol{e} + \boldsymbol{\Pi}_N (\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{e} = 1. \tag{4}$$

In order to derive the steady-state distribution and obtain the performance indices of the system, the expression of the matrix $\boldsymbol{R}$ needs to be solved. However, the analytical solution of the matrix $\boldsymbol{R}$ cannot be directly accomplished by ordinary mathematical methods because the equation (1) is arduous to be solved. The iterative method is used to approximate $\boldsymbol{R}$ by using [26], and the specific iterative procedure is as follows:

Step 1 Determine system parameters $c, m, n_1, N, \lambda, \mu_1, \mu_2, \eta, \sigma, q, b$, and the error precision $\varepsilon_1 = 10^{-10}$.

Step 2 Input $\boldsymbol{C}_0, \boldsymbol{B}_0, \boldsymbol{A}_N$. Define $\boldsymbol{R}(n) = \boldsymbol{R}$, $\boldsymbol{R} = -(\boldsymbol{C} + \boldsymbol{R}^2\boldsymbol{B})\boldsymbol{A}_N^{-1}$, $\boldsymbol{R}(n+1) = \boldsymbol{R}$, and perform iterations.

Step 3 If $\|\boldsymbol{R}(n+1) - \boldsymbol{R}(n)\| < \varepsilon_1$, $\boldsymbol{R} = \boldsymbol{R}(n+1)$, the iteration ends, else go to step 2.

After solving the matrix $\boldsymbol{R}$ by the above algorithm, the following recursive equations can be derived from equations (2)-(4).

$$\boldsymbol{\Pi}_0 = \boldsymbol{\Pi}_1 \boldsymbol{B}_1 (-\boldsymbol{A}_0)^{-1} = \boldsymbol{\Pi}_1 \boldsymbol{\omega}_1, \tag{5}$$

$$\boldsymbol{\Pi}_n = \boldsymbol{\Pi}_{n+1} \boldsymbol{B}_{n+1} [-(\boldsymbol{\omega}_n \boldsymbol{C}_0 + \boldsymbol{A}_n)]^{-1} = \boldsymbol{\Pi}_{n+1} \boldsymbol{\omega}_{n+1},\ 1 \le n \le N-1, \tag{6}$$

$$\boldsymbol{\Pi}_N \boldsymbol{\omega}_N \boldsymbol{C} + \boldsymbol{\Pi}_N \boldsymbol{A}_N + \boldsymbol{\Pi}_N \boldsymbol{R}\boldsymbol{B} = \boldsymbol{0}, \tag{7}$$

$$\sum_{n=0}^{\infty} \boldsymbol{\Pi}_n \boldsymbol{e} = \boldsymbol{\Pi}_N \left[\sum_{n=1}^{N} \prod_{i=n}^{N} \boldsymbol{\omega}_i + (\boldsymbol{I} - \boldsymbol{R})^{-1} \right] \boldsymbol{e} = 1. \tag{8}$$

Then the steady-state probability vector $\boldsymbol{\Pi} = (\boldsymbol{\Pi}_0, \boldsymbol{\Pi}_1, \boldsymbol{\Pi}_2, \cdots, \boldsymbol{\Pi}_N, \cdots)$ can be obtained by (5)-(8).

The performance indices of the system can be further obtained as follows.

(1) The expected queue length of customers waiting in the queue is

$$L_q = \sum_{j=1}^{N-1} j \boldsymbol{\Pi}_j \boldsymbol{e} + \sum_{j=N}^{\infty} \boldsymbol{\Pi}_N j \boldsymbol{R}^{j-N+1} \boldsymbol{e}$$
$$= \boldsymbol{\Pi}_N \left[\sum_{j=2}^{N} (j-1) \prod_{i=j}^{N} \boldsymbol{\omega}_j + N(\boldsymbol{I} - \boldsymbol{R})^{-1} + \boldsymbol{R}(\boldsymbol{I} - \boldsymbol{R})^{-2} \right] \boldsymbol{e},$$

(2) The expected number of servers successfully matching one customer is

$$E(K) = \sum_{j=0}^{\infty} \boldsymbol{\Pi}_j \boldsymbol{t}_1 = \boldsymbol{\Pi}_N \left[\sum_{n=1}^{N} \prod_{i=n}^{N} \boldsymbol{\omega}_i + (\boldsymbol{I} - \boldsymbol{R})^{-1} \right] \boldsymbol{t}_1,$$

where $\boldsymbol{t}_1 \left(\boldsymbol{t}_1 = [\boldsymbol{t}_{11}, \boldsymbol{t}_{11}, \cdots, \boldsymbol{t}_{11}]^{\mathrm{T}} \right)$ is $m(c+1)(c+2)/2$- dimensional column vector, and $\boldsymbol{t}_{11} \left(\boldsymbol{t}_{11} = [0,1,0,2,1,0,\cdots,c,c-1,\cdots,1,0]^{\mathrm{T}} \right)$ is $(c+1)(c+2)/2$- dimensional column vector.

(3) The expected number of servers successfully carpooling matching two customers is

$$E(D) = \sum_{j=0}^{\infty} \boldsymbol{\Pi}_j \boldsymbol{t}_2 = \boldsymbol{\Pi}_N \left[\sum_{n=1}^{N} \prod_{i=n}^{N} \boldsymbol{\omega}_i + (\boldsymbol{I} - \boldsymbol{R})^{-1} \right] \boldsymbol{t}_2,$$

where $\boldsymbol{t}_2 \left(\boldsymbol{t}_2 = [\boldsymbol{t}_{22}, \boldsymbol{t}_{22}, \cdots, \boldsymbol{t}_{22}]^{\mathrm{T}} \right)$ is $m(c+1)(c+2)/2$- dimensional column vector, and $\boldsymbol{t}_{22} \left(\boldsymbol{t}_{22} = [0,0,1,0,1,2,\cdots,0,1,2,\cdots,c,c-1,]^{\mathrm{T}} \right)$ is $(c+1)(c+2)/2$- dimensional row vector.

(4) The expected number of servers that have finished serving one of two carpooling customers is

$$E(S) = \sum_{j=0}^{\infty} \boldsymbol{\Pi}_j \boldsymbol{t}_3 = \boldsymbol{\Pi}_N \left[\sum_{n=1}^{N} \prod_{i=n}^{N} \boldsymbol{\omega}_i + (\boldsymbol{I} - \boldsymbol{R})^{-1} \right] \boldsymbol{t}_3,$$

where $\boldsymbol{t}_3 = \left[\boldsymbol{t}_{03}, \boldsymbol{t}_{13}, \cdots, \boldsymbol{t}_{m3} \right]^{\mathrm{T}}$, $\boldsymbol{t}_{s3} = s\boldsymbol{e}, 0 \le s \le m$.

(5) The probability of the successfully carpooling is

$$P_s = \frac{E(D)}{E(D) + E(K) + E(S)}.$$

V. Numerical analysis

In the actual operation of the carpooling system, the order commission percentage can greatly affect the variation of the system parameters. Considering that the high demand for travel during holidays or peak commuting periods, the customers prefer to carpool in higher arrival rate to reduce the waiting time and commuting costs. This section first analyzes the impacts of different parameters on the system performance indicators through numerical examples, then establishes the functions to analyze the effects of various parameters on the benefits of customers, e-hailing taxis, and the system, respectively.

A. *Sensitivity analysis*

In order to intuitively discuss the influence of system parameters on the performance indexes, assuming that $c = 3, s = 3, N = 10, b = 0.5$ and $n_1 = 5$ in this section.

Assuming that $\sigma = 0.3, \mu_1 = 0.5, \mu_2 = 0.3$, Figure 1 indicates that L_q increases with the arrival rate λ when η_n is constant. If λ is given, L_q also increases as η_n increases. This is because the carpooling matching success rate is low and the probability of successful carpooling is small when η_n is relatively small. If there are customers waiting in the queue, the new servers will join the system and accept the service directly. When the carpooling matching success rate is larger, more servers can serve two customers at the same time, which can prevent some servers from joining the system. Furthermore, the carpooling service rate μ_2 is slower than the rate μ_1, which leads to the increase of L_q. However, the increase of L_q has slight fluctuations with parameter η_n. This is because when η_n reaches a certain level, the probability of successful carpooling is large enough, and the growth of the successful probability and the expected queue length L_q grow slowly.

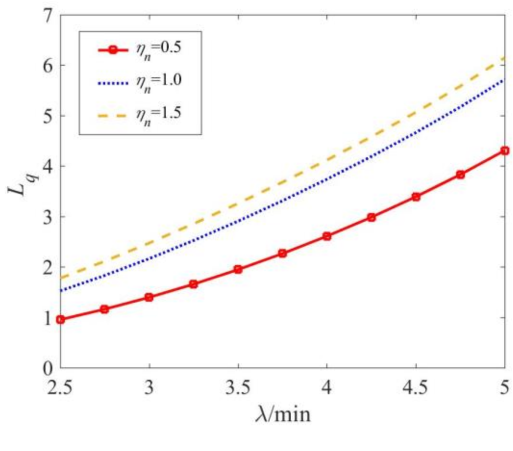

Fig. 1. L_q vs. λ and η_n

Assuming that $\eta = 0.5, \sigma = 0.3, \mu_1 = 0.5, \mu_2 = 0.3$, Figure 2 reflects the relation between $E(K), E(D), E(S)$ with λ.

With the increasing of λ, $E(K), E(D)$ and $E(S)$ all increase in different degrees. This is because when a new server joins the system, the probability that the server is in state 2 is increasing due to the increase of λ. Besides, the expected number of $E(S)$ increases with the arrival rate λ. This is because the server state entering the system can only be transferred from state 1 to state 2, and state 2 to state 3. Therefore, $E(S)$ will continue to increase until it is greater than $E(D)$ when the server states are continuous.

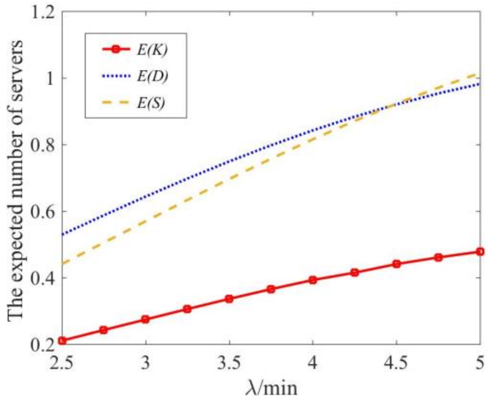

Fig. 2. $E(K), E(D), E(S)$ vs. λ

Due to the actual background of online car-hailing, customers may choose to leave the system at any time and choose other modes of transportation. In the following, we discuss the impact of the arrival rate λ and the departure rate σ_n on the system performance.

Figure 3 demonstrates that L_q decreases as σ_n increases, and the decrement of L_q increases as λ increases. This indicates that more customers join the system when λ increases, and L_q increases accordingly, which makes more customers choose to leave the system. In this case, the system should take some incentives to make the idle servers join the system. When the service rates μ_1 and μ_2 increase, the expected sojourn time of customers is reduced. Therefore, the system may consider its own benefits, and some incentives should be taken to make the idle server outside the system. Besides, the expected waiting time of customers can be reduced when improving the service rate. The system can also take discounts and other strategies for orders when customers wait longer, which make customers more willing to wait for service within the system.

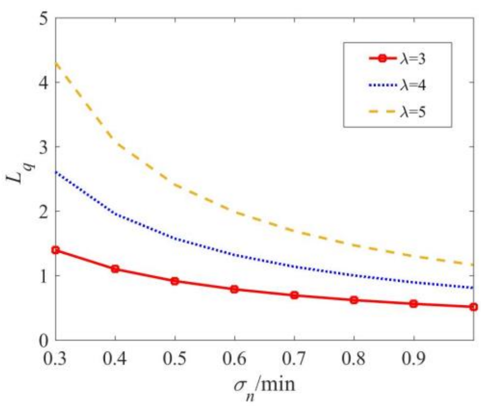

Fig. 3. L_q vs. σ_n and λ

Figure 4 means that $E(K)$, $E(D)$ and $E(S)$ decrease as σ_n increases, but the decreasing rate of $E(S)$ is significantly larger than the $E(D)$. The reason is that the service rate of the servers in state 3 is larger than the servers in state 2, and the server in state 3 will serve the customers faster. Therefore, when there are more customers in the system, the system can increase the carpooling matching probability η_n to reduce the number of reneging customers. When there are fewer customers in the system, the system can decrease the joining rate q of the servers to reduce the operation costs and increase the overall profits.

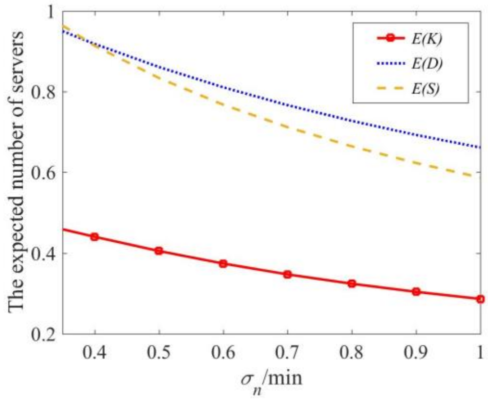

Fig. 4. $E(K), E(D), E(S)$ vs. σ_n

B. Benefit analysis

In the analysis of customer benefit, the customers are interested in the price factors and waiting costs. Customers will choose the optimal way to travel by considering their time costs and price preferences. There is a certain difference between the price of an express taxi service and a carpooling service. If the carpooling benefit is significantly smaller than the express taxi benefit, it will affect the enthusiasm of the e-hailing to join the carpooling system, and the benefits of the system may be reduced. Therefore, the system can make reasonable pricing standards, and dynamically adjust the commission ratio of the e-hailing.

Customer benefit analysis

Assume that the customer benefit function

$$R_c = C_1 - WC_2 - \left[C_{00} P_s \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} + C_{01}(1 - P_s)\frac{1}{\mu_1} \right], \quad (9)$$

where C_{00} is the price of the carpooling service, C_{01} is the price of the express taxi service, C_1 is the reward of the customer received after service, and C_2 is the expected waiting cost of the customer, W is the expected waiting time of the customer.

Assuming that $C_{00} = 1.5, C_{01} = 4, C_1 = 10, C_2 = 4, c = 3,$ $s = 3, N = 10, b = 0.5, n_1 = 5$, the effects of the arrival rate λ, carpooling matching success rate η_n, service rates μ_1, μ_2, and the joining rate q on the customer's benefit R_c are given in Figures 5 and 6. When $\sigma = 0.3$, $\mu_1 = 0.5$ and $\mu_2 = 0.3$, Figure 5 shows the relationship of R_c with q and λ, and the benefit R_c decreases as λ increases. When the probability q is larger, the decreasing rate of R_c decreases with the arrival rate λ increases. The arrival rate λ increases can extend the waiting time of customers, but the probability q can attract more servers to join the system faster and reduce the overall sojourn time customers.

Assuming that $q=3, \sigma=0.3, \mu_1=0.5, \mu_2=0.3$, Figure 6 indicates that R_c keeps increasing with the increasing of η_n when $\lambda=2$, but the variation is slight. The benefit R_c decreases with the increasing of η_n when λ becomes larger, but the decreasing rate of R_c changes sharply. This is because the expected waiting time of customers is lower when λ is small, then the corresponding waiting cost is lower. Therefore, if the success rate of carpool matching η_n increases, the probability of carpooling increases and the traffic cost of the customer will be reduced, then R_c will keep increasing. However, the expected waiting time of customers will become longer when λ is larger, and the service time will be longer when customers choose carpooling services. Then the time cost of customers will continue to increase and the benefit R_c keeps decreasing. Besides, the increasing of η_n can reduce the traffic cost of customers and lower the decreasing rate of R_c. Therefore, when λ is relatively small, the customers prefer carpooling services. When λ is relatively large, if the increment of the waiting cost is greater than the price discounts resulting from carpooling, the customers may choose the express taxi service to reduce their waiting cost and improve the individual benefits.

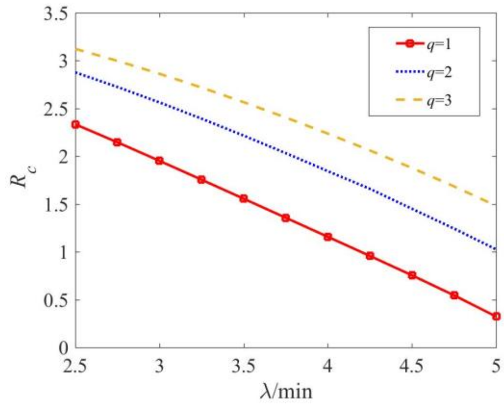

Fig. 5. R_c vs. λ and q

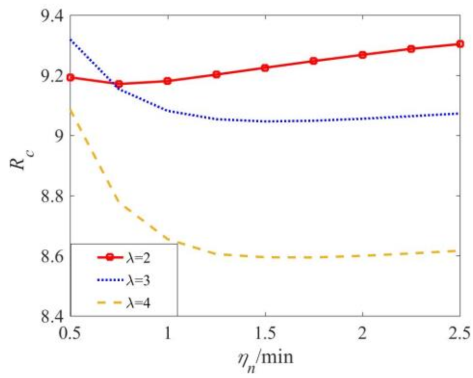

Fig. 6. R_c vs. η_n and λ

E-hailing driver benefit analysis

If an e-hailing accepts a carpooling order, the success of carpool matching will affect the benefits of drivers. If there is a significant difference in the benefits of these two types of orders, it will greatly affect the operation of the online car-hailing system. Therefore, it is significant to balance the benefits between the two types of orders.

In the e-hailing queueing system with the success rate η_n of carpool matching, the expected profit R_d per unit time of the server (driver) is defined by

$$R_d=\frac{\mu_1}{\mu_1+\eta_n}\left(C_3-C_4\frac{1}{\mu_1}\right)+\frac{\eta_n}{\mu_1+\eta_n}\left[C_5-C_4\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\right], \quad (10)$$

where C_3 is the server's benefit when the carpooling is unsuccessful, C_4 is the cost per unit time of the server in state 1, C_5 is the server's benefit when the carpooling is successful.

Theorem 5.1. R_d is a monotone function of η_n.

Proof. The first-order derivative of the function R_d with respect to η_n is given by

$$\frac{\mathrm{d}R}{\mathrm{d}\eta_n}=\frac{\mu_1}{(\mu_1+\eta_n)^2}\left(C_5-C_4\frac{1}{\mu_2}-C_3\right).$$

R_d is a monotonically increasing (decreasing) function if $C_5-C_4/\mu_2-C_3>0\ (<0)$.

In the following, we discuss how to adjust the system parameter η_n to balance the benefits of two types of orders under different situations.

If the order profit of successful carpooling is significantly greater (less) than that of unsuccessful carpooling in the online car-hailing system, then

Case1 If $C_5-C_4/\mu_2-C_3>0$, the system should reduce (increase) the success rate η_n of carpooling matching, and reduce (increase) the carpooling order benefit of e-hailing taxi drivers.

Case2 If $C_5-C_4/\mu_2-C_3<0$, the system should increase (reduce) the success rate η_n of carpooling matching and reduce (increase) the carpooling order benefit of e-hailing taxi drivers.

System benefit analysis

The benefits of the e-hailing system are obtained by taking a percentage of the order price, and the system can set different commission percentages for different types of orders to adjust the choices of customers and drivers.

Assume that the expected benefit function per unit time of the online car-hailing system is

$$R_s=C_6E(K)+C_7E(D)+C_8E(S), \quad (11)$$

where C_6, C_7, C_8 respectively represents the benefit per unit time of the server in state 1, 2 or 3.

Assuming that $C_6=10, C_7=14, C_8=8, q=3, c=3, s=3, N=10, b=0.5$, the effects of arrival rate λ, carpooling matching success rate η_n, service rate μ_1, μ_2 and server joining rate q on the expected system benefit per unit time R_s are demonstrates in Figures 7 and 8, respectively.

Assuming that $\sigma=0.3$, $\mu_1=0.5$ and $\mu_2=0.3$, Figure 7 indicates that R_s increases as λ increases, that is, the more customers, the greater the system profit. Especially, when $\eta_n=1$, R_s increases slowly with the increase of the arrival rate λ. This is because if the success rate η_n of carpooling matching is large, more customers will choose to carpool, and the lower overall price of carpooling orders will have a certain impact on the system profits. Therefore, the platform will consider reducing the dispatch rate of carpooling orders if there is no excessive customer waiting in the system, and the success rate η_n of carpool matching decreases accordingly. When the number of customers gradually increases, the platform may increase the dispatch rate of carpooling orders to reduce customer losses, and the success rate of carpool matching η_n continuously increases to ensure the optimal operation of the system.

Assuming that $\sigma = 0.3, \eta = 0.5$ and $\mu_2 = 0.3$, Figure 8 shows that R_s decreases slowly as μ_1 increases when $\lambda = 3$ and $\lambda = 5$. When the arrival rate is $\lambda = 7$, R_s increases slowly as μ_1 increases. This is because when λ is relatively small, the increasing of μ_1 enables the server to serve new customers faster. However, the benefits of the system are lower due to the lower price of carpooling. Consequently, the increase of μ_1 may have a negative impact on the system benefit. Therefore, the system should reasonably adjust the express taxi service rate μ_1 according to the expected queue length to maximize the system benefit.

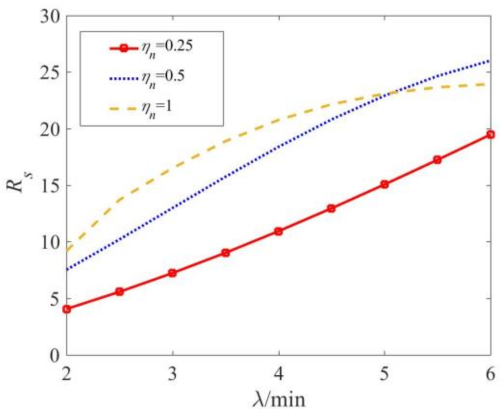

Fig. 7. R_s vs. λ and η_n

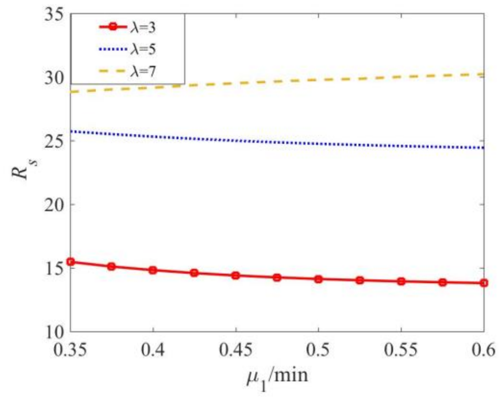

Fig. 8. R_s vs. λ and μ_1

Benefit strategy based on seagull optimization algorithm

This section mainly uses the seagull optimization algorithm to compare and analyze the maximal benefit of customers and the servers.

The Seagull Optimization Algorithm (SOA) was proposed by Gaurav Dhiman and Vijay Kumar in [27]. The basic criterion is to perform multiple iterations to find the optimal value by simulating the migration and attack behavior of seagulls in nature. According to the change of seasons, seagulls always migrate to the most suitable location for survival, and the migration behavior affects the global exploration ability of SOA algorithm. When attacking prey, the seagulls shows a spiral motion pattern, and the attack behavior affects the local development ability of the SOA algorithm. As a swarm intelligence optimization algorithm, the simple algorithm structure of the seagull algorithm can bring lower algorithm complexity and efficient computing power. The seagull optimization algorithm is used to optimize the shared carpooling matching queueing model based on variable matching rate, which can ensure the maximization of benefits in the carpooling queuing system. The algorithm parameters are set as follows: the population size is 60, and the threshold boundary is 10. In order to balance the quality and efficiency of the solution, the accuracy of $\theta = 10^{-10}$ is selected as the termination condition of the algorithm.

According to sections 5.2.1 and 5.2.2, the optimal arrival rate λ_c^* that maximizes customer's benefit R_c^* and the optimal arrival rate λ_d^* that maximizes server's benefit R_d^* are obtained respectively.

$$\lambda_c^* = \arg\max_{\lambda>0}\left\{C_1 - WC_2 - \left[C_{00}P_s\frac{\mu_1+\mu_2}{\mu_1\mu_2} + C_{01}(1-P_s)\frac{1}{\mu_1}\right]\right\},$$

$$\lambda_d^* = \arg\max_{\lambda>0}\left\{\frac{\mu_1}{\mu_1+\eta_n}\left(C_3 - C_4\frac{1}{\mu_1}\right) + \frac{\eta_n}{\mu_1+\eta_n}\left[C_5 - C_4\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\right]\right\}.$$

The sensitivity analysis of the maximal benefit and the optimal arrival rate of customers under different carpool matching success rates is shown in Table 1. The maximal benefit of customers is gradually reduced with the increase of the success rate of carpool matching. This is because when the success rate of carpooling matching increases, the expected sojourn time of customers and the service time when customers choose carpooling service will increase, and the benefit of customers will continue to decrease, but the downward trend is weak. In addition, with the increase of the success rate of carpooling matching, customers may choose to reduce the individual arrival rates in order to maximize their own benefits. When the threshold boundary of the seagull optimization algorithm is 10, the maximal benefit of customers and the optimal arrival rate are obtained as $R_c^* = 1.9011$ and $\lambda_c^* = 10$ when the carpool matching success rate is $\eta_n = 0.5$.

Table 1 The maximal benefit of customers R_c^* and the optimal arrival rate λ_c^* under different carpool matching success rates

η_n	0.5	1.0	1.5	2.0	2.5
R_c^*	**1.9011**	1.3653	1.1369	1.0161	0.9423
λ_c^*	**10**	8.4077	7.2484	6.7569	6.4874

Table 2 shows the sensitivity analysis of the maximal benefit of servers and the optimal arrival rate of customers under different carpool matching success rates. The maximal benefit of online car-hailing is gradually reduced with the increase of the success rate of carpooling matching, which is consistent with the results of Table 1. However, with the increase of the success rate of carpooling matching, online car-hailing will choose to increase the individual arrival rate of customers in order to maximize their own benefits. When the threshold boundary of the seagull optimization algorithm is 10, the maximal benefit of servers is $R_d^* = 6.3141$ when the carpool matching success rate is $\eta_n = 0.5$, and the optimal customer arrival rate is $\lambda_d^* = 8.4650$ when the carpool matching success rate is $\eta_n = 2.5$.

Table 2 The maximal benefit of servers R_d^* and the optimal arrival rate λ_d^* under different carpool matching success rates

η_n	0.5	1.0	1.5	2.0	2.5
R_d^*	**6.3141**	5.2315	4.6196	4.2262	3.9520
λ_d^*	4.1627	5.9416	5.4980	7.3400	**8.4650**

Based on the numerical analysis and the main research content of this paper, the research results and corresponding suggestions are summarized as follows:

(1). In order to comply with the development of the sharing economy, carpooling has become one of the main ways of fast-paced travel in medium and large cities. From the perspective of customers, the original intention of customers to choose carpooling is to save time and get more benefits. However, the results show that the greater the success rate of carpool matching, the lower the service rate of customers, resulting in longer waiting time and reduced benefits for customers. From the point of view of the carpooling system, the higher the success rate of carpooling, the higher the system benefits. Therefore, in order to encourage customers to choose carpooling, the platform should offer carpooling coupons and corresponding discounts for customers.

(2). The greater the arrival rate of customers, the greater the waiting time of customers. From the perspective of the non-cooperative game, the arrival rate of the customer group will reduce its own benefit. As a result, customers often choose to carpool in areas with less passenger traffic. On the contrary, from the perspective of social benefit, the arrival rate of customers will increase the individual revenue of the server (driver). This creates a contradiction between the distribution of servers (drivers) and the concentration of customers. Significantly improving the work efficiency of the servers (drivers) can effectively alleviate this contradiction.

(3). With the increase in service rates, the overall revenue of the system shows a decreasing trend, but the decrease rate is slow. This shows that although improving the service rate may cause a certain loss in system costs, it can also effectively increase the willingness of customers to join the queue, which can fully offset the cost pressure. Considering the benefits of customers comprehensively, it is suggested that the system managers appropriately increase the service rate of the servers (drivers), which can be exchanged for a larger system benefit with a smaller cost loss to a certain extent.

(4). The numerical results show that the greater the matching rate, the smaller the benefits for customers and servers (drivers), but the greater the benefits for the ridesharing system. In order to obtain higher benefits, servers (drivers) usually choose places with higher matching rates and arrival rates. This will make remote areas no one to take orders, resulting in an imbalance in vehicle resources. Therefore, it is suggested that the platform increase some subsidy measures for servers (drivers) in places with low passenger flow, so that the shared carpooling resources are evenly distributed.

VI. Conclusions

This paper focuses on the online car-hailing queueing model with variable carpooling matching and shared matching strategies. Different from the classical double-end queueing model, the server in the e-hailing carpooling system can also match with the second customer when the first customer is served. When the matching is successful, the cost and the expected sojourn time of customers are decreased, and the benefit of the platform can increase accordingly. The main performance indexes of the system are investigated using the matrix geometric solution and iterative methods. Furthermore, this paper establishes the corresponding benefit functions from different perspectives, and the effects of various parameters on profits are explicitly illustrated by numerical comparisons. The research results provide a performance analysis method and useful insights for the simulation and control of shared car-hailing systems. Further extension of this work may explore the spatial heterogeneity and information diversity in complex carpooling scenarios. The circular carpooling in shared systems for multi-arrivals could be constructed and analyzed reasonably based on the varying incentive levels, which is also an intriguing and challenging direction.

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Xiuli Xu received her Ph.D. degree from Yanshan University, Qinhuangdao, China, in 2006. Currently, she is a full professor in the School of Science at Yanshan University. She has published more than 70 papers in international leading journals in the areas of operations research and applied mathematics. She has been involved in several projects supported by the National Natural Science Foundation of China, the National Natural Science Foundation of Hebei Province, and other foundations. Her research interests include the economics of queues, fluid queues and stochastic process theory.

Yitong Zhang received her B.Sc. degree from Hebei Normal University, Shijiazhuang, and her Ph.D. degree from Yanshan University, Qinhuangdao, China, in 2018 and 2024, respectively. Currently, she is a full lecturer in the School of Science at Tianjin University of Commerce. She has been involved in several projects supported by the National Natural Science Foundation of China, the National Natural Science Foundation of Hebei province, and other foundations. Her research interests include game theory, the economics of queues, and fluid queues.

Yuting Tan received his master's degree from Yanshan University in 2023. His research interests are queueing theory models and their applications. He has published several papers on queueing models of online taxi systems.

Mingxin Liu received his Ph.D. degree from Yanshan University, Qinhuangdao, China, in 2006. Currently, he is a full professor in the School of Electronics and Information Engineering at Guangdong Ocean University. He has been involved in several projects supported by the National Natural Science Foundation of China, the National Natural Science Foundation of Guangdong Province, and other foundations. His research interests are Marine and ocean engineering and electronic information engineering.