Pseudo-Ehoops

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***Abstract*—This paper presents the concept of pseudo-Ehoops, which serves as an extension of both pseudo-hoop algebras and Ehoops. Some essential properties of pseudo-Ehoops are obtained. In addition, we define ideals, congruences and filters on pseudo-Ehoops. The relations between them are investigated. We find a one-to-one correspondence between ideals and congruences on a pseudo-Ehoop A with the pDN condition. Moreover, our results indicate that every proper pseudo-Ehoop contains at least one maximal filter and each maximal ideal is prime. Prime ideal theorem is also given. Furthermore, using prime state filters, we establish a topology.**

***Index Terms*—pseudo-Ehoop, pseudo-hoop, ideal, filter, congruence, state.**

I. Introduction

HOOPS are naturally ordered commutative residuated integral monoids, first introduced in [15], [16] and further defined in an unpublished paper [14] by Bϋchi and Owens. The concept of hoops was introduced to explore the properties of certain algebraic systems that resemble groups but have distinct characteristics. Hoops, as defined by their associativity and an idempotent operation, allow for a deeper exploration of ordered structures. Researchers sought to expand this notion by investigating pseudo-hoops, which relax some of the assumptions inherent in traditional hoops, leading to a more versatile framework.

Pseudo-hoop algebras, as non-commutative generalizations of hoops, were proposed by Georgescu, Leutean and Preoteasa in [1]. Pseudo-hoops are algebraic structures that generalize the concept of hoops, a specific kind of mathematical structure formed by a set equipped with two binary operations. The study of pseudo-hoops is significant in various branches of mathematics, including lattice theory, universal algebra, and the study of ordered sets, due to their intriguing properties and applications. In [2], it is demonstrated that bounded basic pseudo-hoops and pseudo-BL algebras are point-by-point equivalent. A pseudo-BL algebra with the pDN property is termwise equivalent a pseudo-MV algebra. Notably, pseudo-BL algebras and pseudo-MV algebras represent specific instances of pseudo-hoops. Recent years have seen significant scholarly interest in the theories of ideals, filters, and states on pseudo-hoops, as evidenced by works in [5], [7], [9], [10], [12], [13].

In [11], Dvurečenskij and Zahiri introduced EMV-algebras, extending the concept of MV-algebras. They defined ideals, congruences, and filters on EMV-algebras while examining their interrelations. In [8], introducing EBL-algebras, which extend the concepts of both BL-algebras and EMV-algebras. He defined ideals, filters, and congruences and demonstrated a one-to-one correspondence between ideals and congruences in an EBL-algebra. Xie and Liu investigated Ehoop in [6], further extending the notion of hoops and presenting various properties of Ehoops.

Motivated by these foundational works, we aim to extend the concepts of pseudo-hoop algebras and Ehoops, which we will refer to as pseudo-Ehoops. In Sect.II, we recall some fundamental definitions and properties of pseudo-hoops. Sect.III introduces the definition of pseudo-Ehoops and explores several essential properties of this new structure. In Sect.IV, we give ideals and congruences on pseudo-Ehoops, showing that congruences can be constructed by ideals. We prove that if A is a pseudo-Ehoop satisfying the pDN condition, a one-to-one correspondence between ideals and congruences on A exists. Sect.V presents the concept of filters and discusses the relation between filters and congruence in pseudo-Ehoops, demonstrating that every proper pseudo-Ehoop contains at least one maximal filter. In Sect.VI, we define the notions of prime ideals and maximal ideals, proving that every maximal ideal is prime and presenting a prime ideal theorem for A. In Sect.VII, we introduce implicative filters and positive implicative filters of pseudo-Ehoops, showing that every positive implicative filter that is also normal is an implicative filter. In Sect.VIII, we define internal states of pseudo-Ehoops and establish a topological space through prime state filters on state pseudo-Ehoops.

II. Preliminaries

This section will review essential concepts and results related to pseudo-hoop algebras, which will be pertinent for the discussions that follow in this paper.

Definition 2.1. [1] An algebra $(A, \odot, \to, \rightsquigarrow, 1)$ of type (2,2,2,0) is defined as a pseudo-hoop algebra if: any elements $s, t, w \in A$,
(PH1) $s \odot 1 = 1 \odot s = s$;
(PH2) $s \to s = s \rightsquigarrow s = 1$;
(PH3) $(s \odot t) \to w = s \to (t \to w)$;
(PH4) $(s \odot t) \rightsquigarrow w = t \rightsquigarrow (s \rightsquigarrow w)$;
(PH5) $(s \to t) \odot s = (t \to s) \odot t = s \odot (s \rightsquigarrow t) = t \odot (t \rightsquigarrow s)$.

Let $(A, \odot, \to, \rightsquigarrow, 1)$ denote a pseudo-hoop and $s, t \in A$. According to [1], we define $s \leq t$ as equivalent to $s \to t = 1 \iff s \rightsquigarrow t = 1$. Therefore, $\leq$ constitutes a partial order. If there is $0 \in A$ satisfying $s \geq 0$, we refer to A as a bounded pseudo-hoop. In bounded pseudo-hoops, define $s^- = s \to 0$ and $s^\sim = s \rightsquigarrow 0$. A is good if the condition $s^{-\sim} = s^{\sim -}$ holds. Furthermore, if $s^{-\sim} = s^{\sim -} = s$, then A has the pseudo double negation property (for short pDN). Lastly, if a good pseudo-hoop satisfies $(s \odot t)^{-\sim} = s^{-\sim} \odot t^{-\sim}$, then A is a normal pseudo-hoop.

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Proposition 2.2. [1][7] Set $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ represent a pseudo-hoop. The subsequent statements are valid: any elements $s, t, w \in A$,
(1) $s \odot t \leq w \Longleftrightarrow s \leq t \rightarrow w \Longleftrightarrow t \leq s \rightsquigarrow w$;
(2) $(A, \leq)$ is a meet-semilattice with $s \wedge t = (s \rightarrow t) \odot s = s \odot (s \rightsquigarrow t)$;
(3) $s \odot t \leq s \wedge t \leq s, t$;
(4) $s \leq t$ implies $s \odot w \leq t \odot w$ and $w \odot s \leq w \odot t$;
(5) $s \leq t$ implies $w \rightarrow s \leq w \rightarrow t$ and $w \rightsquigarrow s \leq w \rightsquigarrow t$;
(6) $s \leq t$ implies $t \rightarrow w \leq s \rightarrow w$ and $t \rightsquigarrow w \leq s \rightsquigarrow w$;
(7) $(t \rightarrow w) \odot (s \rightarrow t) \leq s \rightarrow w$, $(s \rightsquigarrow t) \odot (t \rightsquigarrow w) \leq s \rightsquigarrow w$;
(8) $w \rightarrow (s \wedge t) = (w \rightarrow s) \wedge (w \rightarrow t)$;
(9) $s \rightsquigarrow (t \rightarrow w) = t \rightarrow (s \rightsquigarrow w)$, $s \rightarrow (t \rightsquigarrow w) = t \rightsquigarrow (s \rightarrow w)$;
(10) $s \leq (s \rightarrow t) \rightsquigarrow t$, $s \leq (s \rightsquigarrow t) \rightarrow t$;
(11) $s \odot t = s \odot (s \rightsquigarrow (s \odot t)) = (s \rightarrow (s \odot t)) \odot s$.

Proposition 2.3. [2] Set $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-hoop. Any elements $s, t \in A$,
(1) $s \odot 0 = 0 \odot s = 0$;
(2) $s \leq s^{-\sim}, s \leq s^{\sim -}$;
(3) $s^{-\sim -} = s, s^{\sim - \sim} = s$.
If A is good,
(4) $(s^{-\sim} \rightsquigarrow s)^{-} = (s^{-\sim} \rightarrow s)^{\sim} = 0$;
(5) $(s \rightarrow t)^{-\sim} = s^{-\sim} \rightarrow t^{-\sim}, (s \rightsquigarrow t)^{-\sim} = s^{-\sim} \rightsquigarrow t^{-\sim}$;
(6) $(s \wedge t)^{-\sim} = s^{-\sim} \wedge t^{-\sim}$;
(7) $s \rightarrow t^{-} = s^{-\sim} \rightarrow t^{-}, s \rightsquigarrow t^{-} = s^{-\sim} \rightsquigarrow t^{-}$.

Let $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ denote a pseudo-hoop. An element $e \in A$ is termed idempotent if it satisfies the condition $e \odot e = e$. We define $Id(A)$ as the set of all idempotent elements. Any $s \in A$ and $e \in Id(A)$, the equation $s \odot e = s \wedge e = e \odot s$ hold ([3]). $\emptyset \neq S \subseteq A$ is a subalgebra of A if it is closed under $\odot$, $\rightarrow$ and $\rightsquigarrow$. Consider two pseudo-hoops $(A_1, \odot_{A_1}, \rightarrow_{A_1}, \rightsquigarrow_{A_1}, 1_{A_1})$ and $(A_2, \odot_{A_2}, \rightarrow_{A_2}, \rightsquigarrow_{A_2}, 1_{A_2})$. According to [4], if a map $\phi : A_1 \rightarrow A_2$ preserves the operations, then ϕ is a pseudo-hoop homomorphism. Furthermore, if $f : A_1 \rightarrow A_2$ is a pseudo-hoop homomorphism, then $f(1_{A_1}) = 1_{A_2}$. In cases where A_1 and A_2 are bounded, we also have $f(0_{A_1}) = 0_{A_2}$.

For any bounded pseudo-hoop A and $s, t \in A$, we introduce the operations of left addition $\oslash$ and right addition $\otimes$: $s \oslash t = t^{-} \rightsquigarrow s, s \otimes t = s^{\sim} \rightarrow t$.

Definition 2.4. [5] Suppose that A is a bounded pseudo-hoop. $\emptyset \neq I \subseteq A$ is an ideal if it satisfies:
(1) $s, t \in I$ implies $s \oslash t, s \otimes t \in I$;
(2) $s \leq t$ and $t \in I$ imply $s \in I$.

Set A be a pseudo-hoop and $s, t, u, v \in A$. An equivalent relation θ is a congruence if it is compatible with $\odot$, $\rightarrow$ and $\rightsquigarrow$. That is, $(s, t) \in \theta$ and $(u, v) \in \theta$ together imply that $(s \odot u, t \odot v) \in \theta$, $(s \rightarrow u, t \rightarrow v) \in \theta$ and $(s \rightsquigarrow u, t \rightsquigarrow v) \in \theta$ ([5]).

Definition 2.5. [1] Set A be a pseudo-hoop. $\emptyset \neq F \subseteq A$ is a filter if any elements $s, t \in A$,
(1) $s, t \in F \Longrightarrow s \odot t \in F$;
(2) $s \leq t$ and $s \in F$ imply $t \in F$.

III. Pseudo-Ehoops

In this section, we extend the notions of pseudo-hoops and Ehoops and called it pseudo-Ehoops. In addition, some basic properties are given.

Definition 3.1. A pseudo-Ehoop is an algebra $(A, \wedge, \odot)$ of type (2,2) satisfying
(PEH1) $(A, \wedge)$ is a $\wedge$-semilattice;
(PEH2) $(A, \odot)$ is a semigroup;
(PEH3) for all $e \in Id(A)$, set $A_e = \{s \in A | s \leq e\}$, any elements $s, t \in A_e$, the elements $s \rightarrow_e t = \max\{w \in A_e | w \odot s \leq t\}$ and $s \rightsquigarrow_e t = \max\{w \in A_e | s \odot w \leq t\}$ exist, and $(A_e, \odot, \rightarrow_e, \rightsquigarrow_e, e)$ is a pseudo-hoop algebra;
(PEH4) for every $s, t \in A$, there is an element $e \in Id(A)$ satisfying $s, t \leq e$.

A is said to be proper if it has no the largest element.

Remark 3.2. (1) For any $s, t \in A$, denote $s \leq t \Longleftrightarrow s \wedge t = s$. Then $\leq$ is a partial order on A.

(2) For every $e \in Id(A)$, A_e is a pseudo-hoop. Set $s, t \in A_e$. $s \leq_e t \Longleftrightarrow s \rightarrow_e t = e \Longleftrightarrow s \rightsquigarrow_e t = e$ is a partial order on A_e. For any $s, t \in A$ and $e, b \in Id(A)$ satisfying $s, t \leq e \leq b$, we find $s \leq_e t \Longleftrightarrow s \leq t \Longleftrightarrow s \leq_b t$. Indeed, if $s \leq t$, then $e \odot s \leq s \leq t$. It follows $e = \max\{w \in A_e | w \odot s \leq t\} = s \rightarrow_e t$ and so $s \leq_e t$. If $s \leq_e t$, we have $s \rightarrow_e t = e$, which implies $e \odot s \leq t$. Then $s = e \wedge s = e \odot s \leq t$. This proves that $\leq$ and $\leq_e$ are consistent on A_e.

Example 3.3. Pseudo-hoops and pseudo-Ehoops with a top element are termwise equivalent.

Proof: Suppose that $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-hoop. For every $e \in Id(A)$ satisfying $s, t \in A_e$, we obtain $s \odot t \leq e \odot e = e$. It follows $s \odot t \in A_e$. Also, we assert that $(s \rightarrow t) \wedge e = \max\{w \in A_e | w \odot s \leq t\}$. Indeed, $((s \rightarrow t) \wedge e) \odot s = (s \rightarrow t) \odot (e \odot s) = (s \rightarrow t) \odot s \leq t$. Let $w \in A_e$ and $w \odot s \leq t$. We have $w \leq s \rightarrow t$ and so $w \leq (s \rightarrow t) \wedge e$. Thus, $s \rightarrow_e t = \max\{w \in A_e | w \odot s \leq t\} = (s \rightarrow t) \wedge e$. Similarly, we can prove that $s \rightsquigarrow_e t = (s \rightsquigarrow t) \wedge e$. Thus, $s \rightarrow_e t, s \rightsquigarrow_e t \in A_e$.

From $e \in Id(A)$ and $s, t, w \in A_e$, it follows $s \odot e = e \odot s = s$. Also, $s \rightarrow_e s = (s \rightarrow s) \wedge e = e$ and $s \rightsquigarrow_e s = (s \rightsquigarrow s) \wedge e = e$. (PH1) and (PH2) hold.

$$
\begin{aligned}
s \rightarrow_e (t \rightarrow_e w) =& (s \rightarrow ((t \rightarrow w) \wedge e)) \wedge e \\
=& (s \rightarrow (t \rightarrow w)) \wedge (s \rightarrow e) \wedge e \\
& \text{(Proposition 2.2 (8))} \\
=& (s \rightarrow (t \rightarrow w)) \wedge e \\
=& ((s \odot t) \rightarrow w) \wedge e \\
=& (s \odot t) \rightarrow_e w.
\end{aligned}
$$

Analogously, $(s \odot t) \rightsquigarrow_e w = t \rightsquigarrow_e (s \rightsquigarrow_e w)$. This proves (PH3) and (PH4).

$$
\begin{aligned}
(s \rightarrow_e t) \odot s &= ((s \rightarrow t) \wedge e) \odot s \\
&= (e \odot (s \rightarrow t)) \odot s \\
&= e \odot ((s \rightarrow t) \odot s) \\
&= e \odot ((t \rightarrow s) \odot t) \\
&= (e \wedge (t \rightarrow s)) \odot t \\
&= (t \rightarrow_e s) \odot t.
\end{aligned}
$$

Similarly, we show $s \odot (s \rightsquigarrow_e t) = t \odot (t \rightsquigarrow_e s)$. In addition,

$$
\begin{aligned}
(s \rightarrow_e t) \odot s &= ((s \rightarrow t) \wedge e) \odot s \\
&= (e \odot (s \rightarrow t)) \odot s
\end{aligned}
$$

$$\begin{aligned}&= e \odot (s \odot (s \rightsquigarrow t))\\&= (s \odot e) \odot (s \rightsquigarrow t)\\&= s \odot (e \odot (s \rightsquigarrow t))\\&= s \odot (s \rightsquigarrow_e t).\end{aligned}$$

This prove that (PH5) holds. A_e is a pseudo-hoop. For any elements $s, t \in A$, there is $1 \in Id(A)$ that satisfies $s, t \leq 1$. Consequently, we conclude that A is a pseudo-Ehoop.

Conversely, it is obvious. ■

Example 3.4. Let $\{(A_i, \odot, \rightarrow, \rightsquigarrow, 0, 1)\}_{i \in I}$ be a family of bounded pseudo-hoops and $A = \{\phi \in \prod_{i \in I} A_i | supp(\phi)$ is finite$\}$, where $supp(\phi) = \{i \in I | \phi(i) \neq 0\}$. Define $\odot$ and $\wedge$ as follows: for all $\phi = (\phi_i)_{i \in I}, \psi = (\psi_i)_{i \in I} \in A$,

$$\phi \odot \psi = (\phi_i \odot \psi_i)_{i \in I},\ \phi \wedge \psi = (\phi_i \wedge \psi_i)_{i \in I}.$$

Clearly, A is closed under $\odot$ and $\wedge$. Then

$$m = (m_i)_{i \in I} = \begin{cases} 1, & i \in supp(\phi) \cup supp(\psi), \\ 0, & otherwise \end{cases}$$

is an idempotent element and $\phi, \psi \leq m$. Thus, we have that the condition (PEH4) holds and $Id(A) = \{(m_i)_{i \in I} | m_i \in Id(A_i)\}$. Similar to the proof of Example 3.3 in [6], $(A, \wedge, \odot)$ is a pseudo-Ehoop.

Example 3.5. Suppose that A is a pseudo-Ehoop and $X \neq \emptyset$ is finite. A^X represents the set of all functions from X to A. Define $\odot$ and $\wedge$: for all $\phi, \psi \in A^X$ and $s \in X$,

$$(\phi \odot \psi)(s) = \phi(s) \odot \psi(s),\ (\phi \wedge \psi)(s) = \phi(s) \wedge \psi(s).$$

Obviously, (PEH1) and (PEH2) hold. For all $s \in X$, there is $e \in Id(A)$ such that $\phi(s), \psi(s) \leq e$. Let $m_e : X \to e$, we have that m_e is an idempotent of A^X and $\phi, \psi \leq m_e$. A^X has enough idempotent elements. For any $\phi, \psi \in A^X_{m_e} = \{\phi \in A^X | \phi \leq m_e\}$, we define $(\phi \rightarrow_{m_e} \psi)(s) = \phi(s) \rightarrow_{m_e(s)} \psi(s)$ and $(\phi \rightsquigarrow_{m_e} \psi)(s) = \phi(s) \rightsquigarrow_{m_e(s)} \psi(s)$. Then A^X is a pseudo-Ehoop.

Example 3.6. Suppose that A and B are pseudo-Ehoops. Set $A \times B = \{(s_1, s_2) | s_1 \in A, s_2 \in B\}$ and the operations are defined in a pointwise manner. Then $A \times B$ is a pseudo-Ehoop.

Proposition 3.7. $(A, \wedge, \odot)$ is a pseudo-Ehoop if and only if
(PEH1) $(A, \wedge)$ is a $\wedge$-semilattice;
(PEH2) $(A, \odot)$ is a semigroup;
(PEH3′) for any $s, t \in A$, there is an element $e \in Id(A)$ satisfying $s, t \leq e$ and $(A_e, \odot, \rightarrow_e, \rightsquigarrow_e, e)$ is a pseudo-hoop.

Proof: To establish that A is a pseudo-Ehoop, it suffices to demonstrate that condition (PEH3) holds. For each $e \in Id(A)$, there is $b \in Id(A)$ satisfying $e \leq b$ and $(A_b, \odot, \rightarrow_b, \rightsquigarrow_b, b)$ forms a pseudo-hoop. We claim that for any $s, t \in A_e$, the equation $(s \rightarrow_b t) \wedge e = \max\{w \in A_e | w \odot s \leq t\}$ is valid. Similar to the proof of Example 3.3 in [6], we prove that $(A_e, \odot, \rightarrow_e, \rightsquigarrow_e, e)$ is a pseudo-hoop.

Conversely, it is obvious. ■

Proposition 3.8. Set A be a pseudo-Ehoop and $s, t, w \in A$. Let $e, b \in Id(A)$ satisfying $s, t, w \leq e \leq b$, we obtain
(1) $s \rightarrow_e t \leq s \rightarrow_b t$, $s \rightsquigarrow_e t \leq s \rightsquigarrow_b t$;
(2) $s \rightarrow_e t = (s \rightarrow_b t) \wedge e$, $s \rightsquigarrow_e t = (s \rightsquigarrow_b t) \wedge e$;
(3) $(s \rightarrow_e t) \rightarrow_e w \leq (s \rightarrow_b t) \rightarrow_b w$, $(s \rightsquigarrow_e t) \rightsquigarrow_e w \leq (s \rightsquigarrow_b t) \rightsquigarrow_b w$, $(s \rightarrow_e t) \rightsquigarrow_e w \leq (s \rightarrow_b t) \rightsquigarrow_b w$, $(s \rightsquigarrow_e t) \rightarrow_e w \leq (s \rightsquigarrow_b t) \rightarrow_b w$.

Proof: (1) Obviously, $s \rightarrow_e t = \max\{w \in A_e | w \odot s \leq t\} \leq \max\{w \in A_b | w \odot s \leq t\} = s \rightarrow_b t$ by the Definition 3.1. In a similar way, $s \rightsquigarrow_e t \leq s \rightsquigarrow_b t$.

(2) From (1), we obtain $s \rightarrow_e t \leq s \rightarrow_b t$, which implies $s \rightarrow_e t \leq (s \rightarrow_b t) \wedge e$. Moreover,

$$\begin{aligned}((s \rightarrow_b t) \wedge e) \odot s &= ((s \rightarrow_b t) \odot e) \odot s\\&= (s \rightarrow_b t) \odot (e \odot s)\\&= (s \rightarrow_b t) \odot s\\&\leq t.\end{aligned}$$

Thus, we deduce $(s \rightarrow_b t) \wedge e \leq s \rightarrow_e t$. This demonstrates that $s \rightarrow_e t = (s \rightarrow_b t) \wedge e$. Similarly, $s \rightsquigarrow_e t = (s \rightsquigarrow_e t) \wedge e$.

(3) By (2), it follows

$$\begin{aligned}(s \rightarrow_e t) \rightarrow_e w &= (((s \rightarrow_b t) \wedge e) \rightarrow_b w) \wedge e\\&= ((e \odot (s \rightarrow_b t)) \rightarrow_b w) \odot e\\&= (e \rightarrow_b ((s \rightarrow_b t) \rightarrow_b w)) \odot e\\&= e \wedge ((s \rightarrow_b t) \rightarrow_b w)\\&\leq (s \rightarrow_b t) \rightarrow_b w.\end{aligned}$$

Analogously, $(s \rightsquigarrow_e t) \rightsquigarrow_e w \leq (s \rightsquigarrow_b t) \rightsquigarrow_b w$. ■

Remark 3.9. Set A be a pseudo-Ehoop and $s, t \in A$. Any $e, b \in Id(A)$ with $s, t \leq e \leq b$, we conclude $(s \rightarrow_e t) \odot s = ((s \rightarrow_b t) \wedge e) \odot s = (s \rightarrow_b t) \odot (e \odot s) = (s \rightarrow_b t) \odot s$ by Proposition 3.8. Thus $s \wedge t = (s \rightarrow_e t) \odot s = (s \rightarrow_b t) \odot s$.

Let A be a pseudo-Ehoop with the least element 0 and $e \in Id(A)$. For each $s \in A_e$, denote $s^{-_e} = s \rightarrow_e 0$ and $s^{\sim_e} = s \rightsquigarrow_e 0$.

Proposition 3.10. Set A be a pseudo-Ehoop and $s, t, w \in A$. For any $e, b \in Id(A)$ with $s, t, w \leq e, b$, we obtain $(s \rightarrow_e w) \odot t = (s \rightarrow_b w) \odot t$, $t \odot (s \rightarrow_e w) = t \odot (s \rightarrow_b w)$, $(s \rightsquigarrow_e w) \odot t = (s \rightsquigarrow_b w) \odot t$ and $t \odot (s \rightsquigarrow_e w) = t \odot (s \rightsquigarrow_b w)$. In addition, if A has the least element 0, we obtain $s^{-_e} \odot t = s^{-_b} \odot t$, $t \odot s^{-_e} = t \odot s^{-_b}$, $s^{\sim_e} \odot t = s^{\sim_b} \odot t$ and $t \odot s^{\sim_e} = t \odot s^{\sim_b}$.

Proof: Suppose $c \in Id(A)$ satisfying $e, b \leq c$. In A_c,

$$\begin{aligned}(s \rightarrow_e w) \odot t &= ((s \rightarrow_c w) \wedge e) \odot t\\&= ((s \rightarrow_c w) \odot e) \odot t\\&= (s \rightarrow_c w) \odot (e \odot t)\\&= (s \rightarrow_c w) \odot t.\end{aligned}$$

Similarly, we have $(s \rightarrow_b w) \odot t = (s \rightarrow_c w) \odot t$, this means $(s \rightarrow_e w) \odot t = (s \rightarrow_b w) \odot t$. Homoplastically, others are obvious. ■

Proposition 3.11. The Riesz decomposition theorem holds in every pseudo-Ehoop A. That is, if $s, t, w \in A$ and $s \odot t \leq w$, there are two elements $s_1 \geq s$ and $t_1 \geq t$ in A such that $s_1 \odot t_1 = w$.

Proof: For all $s, t, w \in A$, there exists an element $e \in Id(A)$ satisfying $s, t, w \leq e$. Given $s \odot t \leq w$, it follows $s \leq t \rightarrow_e w$. Furthermore, by applying Proposition 2.2, we obtain that $(t \rightarrow_e w) \odot t \leq w$, which leads to $t \leq (t \rightarrow_e w) \rightsquigarrow_e w$. Let us define $s_1 = t \rightarrow_e w$ and $t_1 = (t \rightarrow_e w) \rightsquigarrow_e w$. Then

$s_1 \odot t_1 = (t \to_e w) \odot ((t \to_e w) \rightsquigarrow_e w) = (t \to_e w) \wedge w = w$. ■

IV. Ideals and congruences

This section shall give ideals and congruences of pseudo-Ehoops. Also, the congruences are constructed by ideals. We establish a one-to-one correspondence between congruences and ideals in a pseudo-Ehoop A with the pDN condition.

Let A be a pseudo-Ehoop with the least element 0 and $e \in Id(A)$. For each $s, t \in A_e$, we define $s \oslash_e t = t^{-e} \rightsquigarrow_e s, s \owedge_e t = s^{\sim e} \to_e t$.

Definition 4.1. Set A be a pseudo-Ehoop with the least element 0 and $\emptyset \neq I \subseteq A$. Then I is an ideal if:
(1) for every $e \in Id(A)$ satisfying $s, t \leq e$, $s, t \in I$ implies $s \oslash_e t, s \owedge_e t \in I$;
(2) $s \leq t \in I$ implies $s \in I$.

I is a normal ideal if for each $e \in Id(A)$ with $s, t \leq e$, $s^{-e} \odot t \in I$ implies $t \odot s^{\sim e} \in I$. Define the set of all ideals (normal ideals) of A as $\mathcal{I}(A)$ ($\mathcal{NI}(A)$).

Proposition 4.2. Let A be a pseudo-Ehoop with the least element 0 and $\{0\} \subseteq I \subseteq A$. The next properties are equivalent: Arbitrary elements $s, t \in A$ and $e \in Id(A)$ satisfying $s, t \leq e$,
(1) $I \in \mathcal{I}(A)$;
(2) $s, s^{-e} \odot t \in I$ imply $t \in I$ and $s, t \odot s^{\sim e} \in I$ imply $t \in I$;
(3) $s, (s^{-e} \to_e t^{-e})^{\sim e} \in I$ imply $t \in I$ and $s, (s^{\sim e} \rightsquigarrow_e t^{\sim e})^{-e} \in I$ imply $t \in I$.

Remark 4.3. Let A be a pseudo-Ehoop with the least element 0 and I an ideal. Clearly, $0 \in I$. In addition, for each $s \in A$ and $e \in Id(A)$ with $s \leq e$, $s^{-e\sim e} \in I \iff s \in I \iff s^{\sim e-e} \in I$ holds. In fact, if $s \in I$, from $s^{-e} \odot s^{-e\sim e} = 0 \in I$, we deduce $s^{-e\sim e} \in I$. By Proposition 2.3 (2), it is clear that $s \in I$ can be obtained from $s^{-e\sim e} \in I$. Similarly, $s \in I \iff s^{\sim e-e} \in I$.

A pseudo-Ehoop A with the least element 0 is good if the bounded pseudo-hoop A_e is good for all $e \in Id(A)$. That is, for every $e \in Id(A)$ and $s \in A_e$, we obtain $s^{-e\sim e} = s^{\sim e-e}$. A normal pseudo-Ehoop A is a good pseudo-Ehoop if any elements $s, t \in A$ and $e \in Id(A)$ satisfying $s, t \leq e$, the equation $(s \odot t)^{-e\sim e} = s^{-e\sim e} \odot t^{-e\sim e}$ holds.

In Proposition 4.2, if A is good, we obtain Theorem 4.4 directly.

Theorem 4.4. Set A be a good pseudo-Ehoop and $\{0\} \subseteq I \subseteq A$. The next properties are equivalent: for any $s, t \in A$ and $e \in Id(A)$ with $s, t \leq e$,
(1) $I \in \mathcal{I}(A)$;
(2) $s, s^{-e} \odot t \in I$ implies $t \in I$;
(3) $s, t \odot s^{\sim e} \in I$ implies $t \in I$;
(4) $s, (s^{-e} \to_e t^{-e})^{\sim e} \in I$ implies $t \in I$;
(5) $s, (s^{\sim e} \rightsquigarrow_e t^{\sim e})^{-e} \in I$ implies $t \in I$.

Proposition 4.5. Let A be a pseudo-Ehoop with the least element 0 and $s, t, u, v \in A$.
(1) For all $e \in Id(A)$ satisfying $s, t \leq e$, $s, t \leq s \oslash_e t$ and $s, t \leq s \owedge_e t$.
(2) If $s \leq t$ and $u \leq v$, then for all $e \in Id(A)$ satisfying $s, t, u, v \leq e$, we have $s \oslash_e u \leq t \oslash_e v$ and $s \owedge_e u \leq t \owedge_e v$.
(3) If A is normal, then for each $e \in Id(A)$, $\oslash_e$ and $\owedge_e$ are associative.

Proof: (1) Set $s, t \in A$ and $e \in Id(A)$ satisfy $s, t \leq e$. From $t^{-e} \odot s \leq s$, we deduce $s \leq t^{-e} \rightsquigarrow_e s = s \oslash_e t$. Since $t^{-e} \odot t = 0 \leq s$, $t \leq t^{-e} \rightsquigarrow_e s = s \oslash_e t$. Additionally, given $t \odot s^{\sim e} \leq t$, we obtain $t \leq s^{\sim e} \to_e t = s \owedge_e t$. By $s \odot s^{\sim e} = 0 \leq t$, we obtain $s \leq s^{\sim e} \to_e t = s \owedge_e t$.

(2) Assume $s \leq t$ and $u \leq v$. There is $e \in Id(A)$ such that $s, t, u, v \leq e$. Therefore, $t^{\sim e} \leq s^{\sim e}$ and $v^{-e} \leq u^{-e}$. By Proposition 2.2 (5) and (6), $s \oslash_e u = u^{-e} \rightsquigarrow_e s \leq v^{-e} \rightsquigarrow_e s \leq v^{-e} \rightsquigarrow_e t = t \oslash_e v$ and $s \owedge_e u = s^{\sim e} \to_e u \leq t^{\sim e} \to_e u \leq t^{\sim e} \to_e v = t \owedge_e v$.

(3) By Proposition 3.4 in [5], the proof is clear. ■

Proposition 4.6. Set A be a pseudo-Ehoop with the pDN condition. Then for all $s, t_1, t_2 \in A$ and $e \in Id(A)$ such that $s, t_1, t_2 \leq e$,
(1) $s \oslash_e (t_1 \wedge t_2) = (s \oslash_e t_1) \wedge (s \oslash_e t_2)$;
(2) $(t_1 \wedge t_2) \oslash_e s = (t_1 \oslash_e s) \wedge (t_2 \oslash_e s)$;
(3) $s \wedge (t_1 \oslash_e t_2) \leq (s \wedge t_1) \oslash_e (s \wedge t_2)$.

Proof: (1) From $t_1 \wedge t_2 \leq t_1, t_2$, we have $s \oslash_e (t_1 \wedge t_2) \leq (s \oslash_e t_1) \wedge (s \oslash_e t_2)$ by Proposition 4.5. Assume $u \leq s \oslash_e t_i$ where $i = 1, 2$. Then $t_i^{-e} \odot u \leq s$ and so $t_i^{-e} \leq u \to_e s$. It means $(u \to_e s)^{\sim e} \leq t_i^{-e\sim e} = t_i$. Thus, $(u \to_e s)^{\sim e} \leq t_1 \wedge t_2$. Hence, We conclude $(t_1 \wedge t_2)^{-e} \leq (u \to_e s)^{\sim e-e} = u \to_e s$. From this, we can derive $u \leq (t_1 \wedge t_2)^{-e} \rightsquigarrow_e s = s \oslash_e (t_1 \wedge t_2)$. This proves that $s \oslash_e (t_1 \wedge t_2) = (s \oslash_e t_1) \wedge (s \oslash_e t_2)$.

(2) Clearly, $(t_1 \wedge t_2) \oslash_e s \leq (t_1 \oslash_e s) \wedge (t_2 \oslash_e s)$. Suppose that $u \leq t_i \oslash_e s$ for $i = 1, 2$. We obtain $s^{-e} \odot u \leq t_i$, which means $s^{-e} \odot u \leq (t_1 \wedge t_2)$. Consequently, $u \leq s^{-e} \rightsquigarrow_e (t_1 \wedge t_2) = (t_1 \wedge t_2) \oslash_e s$. Therefore $(t_1 \wedge t_2) \oslash_e s = (t_1 \oslash_e s) \wedge (t_2 \oslash_e s)$.

(3) By (1) and (2), we obtain

$$\begin{aligned}(s \wedge t_1) \oslash_e (s \wedge t_2) &= (s \oslash_e s) \wedge (s \oslash_e t_1) \wedge \\ &\quad (t_1 \oslash_e s) \wedge (t_1 \oslash_e t_2) \\ &\geq s \wedge s \wedge s \wedge (t_1 \oslash_e t_2) \\ &= s \wedge (t_1 \oslash_e t_2).\end{aligned}$$

■

In a similar way, we can get the following statement.

Proposition 4.7. Let A be a pseudo-Ehoop with the pDN condition. For all $s, t_1, t_2 \in A$ and $a \in Id(A)$ such that $s, t_1, t_2 \leq e$,
(1) $s \owedge_e (t_1 \wedge t_2) = (s \owedge_e t_1) \wedge (s \owedge_e t_2)$;
(2) $(t_1 \wedge t_2) \owedge_e s = (t_1 \owedge_e s) \wedge (t_2 \owedge_e s)$;
(3) $s \wedge (t_1 \owedge_e t_2) \leq (s \wedge t_1) \owedge_e (s \wedge t_2)$.

Let A be a pseudo-Ehoop with the least element 0 and $S \subseteq A$. Clearly, the intersection of all ideals containing S forms an ideal, referred to as the ideal generated by S and denoted by $\langle S \rangle$. For any $s \in A$ and $e \in Id(A)$, define $2_e s = s \oslash_e s, \cdots, n_e s = s \oslash_e (n-1)_e s$. Let $\oslash_{e i=1}^{s} s_i = s_1 \oslash_e \cdots \oslash_e s_s$ for each $e \in Id(A)$ such that $s_i \leq e$.

Proposition 4.8. Let A be a normal pseudo-Ehoop, $s \in A$ and $I \subseteq A$.
(1) $\langle I \rangle = \{u \in A | u \leq s_1 \oslash_e s_2 \oslash_e \ldots \oslash_e s_n$, for some $s_1, \ldots, s_n \in I, e \in Id(A), s_1, \ldots, s_n \leq e\}$;
(2) $\langle I \cup \{s\} \rangle = \{u \in A | u \leq \oslash_{e i=1}^{s} (t_i \oslash_e n_{ie} s), t_i \in I, s, n_i \in$

$\mathbb{N}\backslash\{0\}, e \in Id(A), t_i, s \leq e\}$.

Proof: (1) Any element $e \in Id(A)$, we have $\oslash_e$ is associative by Proposition 4.5 (3). Set $H = \{u \in A | u \leq s_1 \oslash_e s_2 \oslash_e \ldots \oslash_e s_n$, for some $s_1, \ldots, s_n \in I, e \in Id(A), s_1, \ldots, s_n \leq e\}$. Obviously, $0 \in H$ and $I \subseteq H$. Let $u, v \in A$ and $e \in Id(A)$ satisfying $u, u^{-_e} \odot v \in H$. There are two elements $b, c \in Id(A)$ that satisfy $s_1, s_2, \ldots, s_m \leq b, t_1, t_2, \ldots, t_n \leq c$, $u \leq s_1 \oslash_e s_2 \oslash_e \ldots \oslash_e s_m$ and $u^{-_e} \odot v \leq t_1 \oslash_e t_2 \oslash_e \ldots \oslash_e t_n$. Choose $d \in Id(A)$ satisfying $e, b, c \leq d$. From Proposition 4.5 (2) and Proposition 3.8 (3), we obtain

$$\begin{aligned} v &\leq u^{-_e} \rightsquigarrow_d (u^{-_e} \odot v) = (u^{-_e} \odot v) \oslash_d u \\ &\leq (t_1 \oslash_e t_2 \oslash_e \ldots \oslash_e t_n) \oslash_d \\ &\quad (s_1 \oslash_e s_2 \oslash_e \ldots \oslash_e s_m) \\ &\leq (t_1 \oslash_d t_2 \oslash_d \ldots \oslash_d t_n) \oslash_d \\ &\quad (s_1 \oslash_d s_2 \oslash_d \ldots \oslash_d s_m) \\ &= t_1 \oslash_d t_2 \oslash_d \ldots \oslash_d t_n \oslash_d s_1 \oslash_d \ldots \oslash_d s_m. \end{aligned}$$

Then $v \in H$. By Theorem 4.4, we get that $H \in \mathcal{I}(A)$.

Assume $K \in \mathcal{I}(A)$ and $I \subseteq K$. For any $u \in H$, there exist $s_1, \ldots, s_n \in I$ and $e \in Id(A)$ such that $s_1, \ldots, s_n \leq e$ and $u \leq s_1 \oslash_e s_2 \oslash_e \ldots \oslash_e s_n$. Thus, $s_1, \ldots, s_n \in K$. We obtain $s_1 \oslash_e s_2 \oslash_e \ldots \oslash_e s_n \in K$, which proves $u \in K$. This means $H \subseteq K$. Therefore, $H = \langle I \rangle$.

Similarly, we can get (2). ∎

Similar to Proposition 4.8, The following statement is straightforward.

Proposition 4.9. Let A be a normal pseudo-Ehoop, $s \in A$ and $I \subseteq A$.
(1) $\langle I \rangle = \{u \in A | u \leq s_1 \obslash_e s_2 \obslash_e \ldots \obslash_e s_n$, for some $s_1, \ldots, s_n \in I, e \in Id(A), s_1, \ldots, s_n \leq e\}$;
(2) $\langle I \cup \{s\} \rangle = \{u \in A | u \leq \obslash_{e_{i=1}}^{s} (t_i \obslash_e n_{ie} s), t_i \in I, s, n_i \in \mathbb{N}\backslash\{0\}, e \in Id(A), t_i, s \leq e\}$.

Definition 4.10. Let A be a pseudo-Ehoop. θ is a congruence if
(1) θ is an equivalence relation;
(2) θ is compatible with $\wedge$ and $\odot$;
(3) $\theta \cap (A_e \times A_e)$ is a congruence on the pseudo-hoop A_e for all $e \in Id(A)$.

Define the set of all congruences of A as $\mathcal{C}(A)$

Proposition 4.11. Set θ be a congruence on a pseudo-Ehoop A. Then, the set $0/\theta = \{s \in A | (s, 0) \in \theta\}$ constitutes an ideal of A. If A is good, then I is normal.

Proof: Obviously, $0 \in 0/\theta$. Set $s, t \in 0/\theta$ and $e \in Id(A)$ with $s, t \leq e$. From $(t, 0) \in \theta$, we derive $(s \oslash_e t, s) = (t^{-_e} \rightsquigarrow_e s, 0^{-_e} \rightsquigarrow_e s) \in \theta$. Consequently, $(s \oslash_e t, 0) \in \theta$, which means $s \oslash_e t \in 0/\theta$. A similar way shows that $s \obslash_e t \in 0/\theta$. If $s \leq t \in 0/\theta$. Since $s \leq t \leq t^{\sim_e -_e}$, we obtain $s \odot t^{\sim_e} = 0$. Hence, $(s, 0) = (s \odot 0^{\sim_e}, s \odot t^{\sim_e}) \in \theta$, which implies $s \in 0/\theta$.

For any $s, t \in A$, there exists $e \in Id(A)$ with $s, t \leq e$. If $s^{-_e} \odot t \in 0/\theta$, we have $(t \rightsquigarrow_e s^{-_e \sim_e}, e) = ((s^{-_e} \odot t)^{\sim_e}, e) \in \theta$. This implies $(t \wedge s^{-_e \sim_e}, t) = (t \odot (s^{-_e} \odot t)^{\sim_e}, t \odot e) \in \theta$. Then $((t \wedge s^{-_e \sim_e}) \odot s^{\sim_e}, t \odot s^{\sim_e}) \in \theta$. Since $(t \wedge s^{-_e \sim_e}) \odot s^{\sim_e} = (s^{-_e \sim_e} \rightarrow_e t) \odot s^{-_e \sim_e} \odot s^{\sim_e} = (s^{-_e \sim_e} \rightarrow_e t) \odot (s^{\sim_e -_e} \odot s^{\sim_e}) = 0$, we get $t \odot s^{\sim_e} \in 0/\theta$. Conversely, it is similar to the proof above. Thus, I is normal. ∎

Proposition 4.12. Suppose that A is a pseudo-Ehoop with the least element 0 and $I \in \mathcal{I}(A)$. θ_I defined by

$$(s, t) \in \theta_I \iff \exists e \in Id(A) \text{ satisfying } s, t \leq e, s^{-_e} \odot t \in I, t^{-_e} \odot s \in I, s \odot t^{\sim_e} \in I, t \odot s^{\sim_e} \in I$$

is an equivalence relation on A.

Proof: Obviously, θ_I is symmetric and reflective. Assume $(s, t) \in \theta_I$ and $(t, w) \in \theta_I$. There are two elements $e, b \in Id(A)$ with $s, t \leq e$, $t, w \leq b$, $s^{-_e} \odot t, t^{-_e} \odot s, s \odot t^{\sim_e}, t \odot s^{\sim_e} \in I$ and $t^{-_b} \odot w, w^{-_b} \odot t, t \odot w^{\sim_b}, w \odot t^{\sim_b} \in I$. Set $c \in Id(A)$ with $e, b \leq c$. By Proposition 3.10, we conclude that $s^{-_c} \odot t, t^{-_c} \odot s, s \odot t^{\sim_c}, t \odot s^{\sim_c}, t^{-_c} \odot w, w^{-_c} \odot t, t \odot w^{\sim_c}, w \odot t^{\sim_c} \in I$. Considering $(w^{-_c} \odot t)^{-_c} \odot (w^{-_c} \odot s) = ((w^{-_c} \rightarrow_c t^{-_c}) \odot w^{-_c}) \odot s \leq t^{-_c} \odot s \in I$, we deduce $(w^{-_c} \odot t)^{-_c} \odot (w^{-_c} \odot s) \in I$. It means $w^{-_c} \odot s \in I$ by Proposition 4.2. Similarly, $s^{-_c} \odot w \in I$. In addition, from $(s \odot w^{\sim_c}) \odot (t \odot w^{\sim_c})^{\sim_c} = s \odot (w^{\sim_c} \odot (w^{\sim_c} \rightsquigarrow_c t^{\sim_c})) \leq s \odot t^{\sim_c} \in I$, we get $(s \odot w^{\sim_c}) \odot (t \odot w^{\sim_c})^{\sim_c} \in I$. This implies $s \odot w^{\sim_c} \in I$. Analogously, $w \odot s^{\sim_c} \in I$. Therefore, $(s, w) \in \theta_I$. ∎

Proposition 4.13. If pseudo-Ehoop A is good and I is normal, θ_I is a congruence on A.

Proof: By Proposition 4.12, θ_I is an equivalence relation. Set $(s, t) \in \theta_I$. There is $e \in Id(A)$ satisfying $s, t \leq e$ and $s^{-_e} \odot t, t^{-_e} \odot s, s \odot t^{\sim_e}, t \odot s^{\sim_e} \in I$. For any $u \in A$ and $b \in Id(A)$ with $s \odot u, t \odot u \leq b$, there is $c \in Id(A)$ satisfying $e, b, u \leq c$. From $(s \odot u) \odot (t \odot u)^{\sim_b} = (s \odot u) \odot (t \odot u)^{\sim_c} = s \odot (u \odot (u \rightsquigarrow_c t^{\sim_c})) \leq s \odot t^{\sim_c} = s \odot t^{\sim_e} \in I$, we have $(s \odot u) \odot (t \odot u)^{\sim_b} \in I$. Similarly, we get $(t \odot u) \odot (s \odot u)^{\sim_b} \in I$. Thus $(t \odot u)^{-_e} \odot (s \odot u) \in I$ and $(s \odot u)^{-_b} \odot (t \odot u) \in I$. This means $(s \odot u, t \odot u) \in \theta_I$. In a similar way, $(u \odot s, u \odot t) \in \theta_I$.

For every $e \in Id(A)$, we shall show that $\theta_I \cap (A_e \times A_e)$ is a congruence on the pseudo-hoop algebra A_e. Assume $(s, t) \in \theta_I \cap (A_e \times A_e)$. An element $b \in Id(A)$ exists that satisfies $s, t \leq b$ and $s^{-_b} \odot t, t^{-_b} \odot s, s \odot t^{\sim_b}, t \odot s^{\sim_b} \in I$. Then $s^{-_e} \odot t, t^{-_e} \odot s, s \odot t^{\sim_e}, t \odot s^{\sim_e} \in I$. Since $(s^{-_e} \odot t)^{-_e} \odot (s^{-_e} \odot t^{-_e \sim_e}) = ((s^{-_e} \rightarrow_e t^{-_e}) \odot s^{-_e}) \odot t^{-_e \sim_e}) \leq t^{-_e} \odot t^{-_e \sim_e} = 0 \in I$, we obtain $(s^{-_e} \odot t)^{-_e} \odot (s^{-_e} \odot t^{-_e \sim_e}) \in I$ and so $s^{-_e} \odot t^{-_e \sim_e} \in I$. Analogously, $t^{-_e} \odot s^{-_e \sim_e} \in I$. Since I is normal, $t^{-_e -_e} \odot s^{-_e} \in I$ and $s^{-_e -_e} \odot t^{-_e} \in I$. Hence, $(s^{-_e}, t^{-_e}) \in \theta_I \cap (A_e \times A_e)$. Similarly, we prove $(s^{\sim_e}, t^{\sim_e}) \in \theta_I \cap (A_e \times A_e)$. Moreover, since $s^{-_e} \odot s^{-_e \sim_e} = 0 \in I$ and $s^{-_e \sim_e -_e} \odot s = s^{-_e} \odot s = 0 \in I$, we have $(s, s^{-_e \sim_e}) \in \theta_I \cap (A_e \times A_e)$. Therefore, $(s, t) \in \theta_I \cap (A_e \times A_e) \iff (s^{-_e \sim_e}, t^{-_e \sim_e}) \in \theta_I \cap (A_e \times A_e)$.

Let $(s, t) \in \theta_I \cap (A_e \times A_e)$ and $u \in A_e$. We have $((s^{-_e \sim_e} \odot u^{\sim_e})^{-_e}, (t^{-_e \sim_e} \odot u^{\sim_e})^{-_e}) \in \theta_I \cap (A_e \times A_e)$. Then $(s^{-_e \sim_e} \rightarrow_e u^{\sim_e -_e}, t^{-_e \sim_e} \rightarrow_e u^{\sim_e -_e}) \in \theta_I \cap (A_e \times A_e)$. Since A is good, we get $(s^{-_e \sim_e} \rightarrow_e u^{-_e \sim_e}, t^{-_e \sim_e} \rightarrow_e u^{-_e \sim_e}) \in \theta_I \cap (A_e \times A_e)$. By Proposition 2.3 (5), $((s \rightarrow_e u)^{-_e \sim_e}, (t \rightarrow_e u)^{-_e \sim_e}) \in \theta_I \cap (A_e \times A_e)$. Then $(s \rightarrow_e u, t \rightarrow_e u) \in \theta_I \cap (A_e \times A_e)$. Similarly, $(s \rightsquigarrow_e u, t \rightsquigarrow_e u) \in \theta_I \cap (A_e \times A_e)$.

Let $(s, t) \in \theta_I \cap (A_e \times A_e)$ and $u \in A_e$. We have $((u^{-_e \sim_e} \odot s^{\sim_e})^{-_e}, (u^{-_e \sim_e} \odot t^{\sim_e})^{-_e}) \in \theta_I \cap (A_e \times A_e)$ and so $(u^{-_e \sim_e} \rightarrow_e s^{\sim_e -_e}, u^{-_e \sim_e} \rightarrow_e t^{\sim_e -_e}) \in \theta_I \cap (A_e \times A_e)$. Since A is good, it means that $((u \rightarrow_e s)^{-_e \sim_e}, (u \rightarrow_e t)^{-_e \sim_e}) \in \theta_I \cap (A_e \times A_e)$, which implies $(u \rightarrow_e s, u \rightarrow_e$

$t) \in \theta_I \cap (A_e \times A_e)$. Analogously, $(u \rightsquigarrow_e s, u \rightsquigarrow_e t) \in \theta_I \cap (A_e \times A_e)$. Therefore, $\theta_I \cap (A_e \times A_e)$ is a congruence of the pseudo-hoop A_e.

For any $(s,t) \in \theta_I$ and $u \in A$, there is $e \in Id(A)$ with $s, t, u \leq e$. It follows from $s \wedge u = s \odot (s \rightsquigarrow_e u)$ that $(s \wedge u, t \wedge u) \in \theta_I$. ■

Remark 4.14. Clearly, by Proposition 3.10, we have

$(s,t) \in \theta_I \Longleftrightarrow \forall e \in Id(A)$ satisfying $s, t \leq e, s^{-e} \odot t \in I, t^{-e} \odot s \in I, s \odot t^{\sim_e} \in I, t \odot s^{\sim_e} \in I$.

Set A be a pseudo-Ehoop with the least element 0. If for each $e \in Id(A)$, A_e satisfies the pDN condition, then A is said to satisfy the pDN condition. That is, for all $s \in A_e$, $s^{-_e\sim_e} = s^{\sim_e -_e} = s$. Obviously, if A has the pDN condition, then A is good and normal.

Proposition 4.15. Let A be a pseudo-Ehoop with the pDN condition. There is a one-to-one correspondence between $\mathcal{NI}(A)$ and $\mathcal{C}(A)$.

Proof: Set $\theta \in \mathcal{C}(A)$. By Proposition 4.11 and Proposition 4.13, $J = 0/\theta$ is a normal ideal of A and θ_J is a congruence on A. Assume $(s,t) \in \theta$ and $e \in Id(A)$ with $s, t \leq e$. Then $(s^{-e} \odot t, t^{-e} \odot t) \in \theta, (t^{-e} \odot s, s^{-e} \odot s) \in \theta, (t \odot s^{\sim_e}, t \odot t^{\sim_e}) \in \theta, (s \odot t^{\sim_e}, s \odot s^{\sim_e}) \in \theta$. We have $s^{-e} \odot t, t^{-e} \odot s, t \odot s^{\sim_e}, s \odot t^{\sim_e} \in 0/\theta$, which implies $(s,t) \in \theta_J$. If $(s,t) \in \theta_J$, then $t \odot s^{\sim_e} \in 0/\theta$. Therefore, we obtain $(t \wedge s^{-_e\sim_e}, t) = ((t \odot s^{\sim_e})^{-e} \odot t, 0^{-e} \odot t) \in \theta$. Thus, $(t \wedge s, t) \in \theta$. Similarly, we deduce $(t \wedge s, s) \in \theta$. This proves $(s,t) \in \theta$.

Let $I \in \mathcal{NI}(A)$. θ_I is a congruence and $0/\theta_I \in \mathcal{NI}(A)$. For any $s \in I$ and $e \in Id(A)$ with $s \leq e$, we get $s^{-e} \odot 0 = 0 \in I, 0^{-e} \odot s = s \in I, s \odot 0^{\sim_e} = s \in I$ and $0 \odot s^{\sim_e} = 0 \in I$. This show that $(s, 0) \in \theta_I$ and so $s \in 0/\theta_I$. If $s \in 0/\theta_I$, $s = s \odot 0^{\sim_e} \in I$. ■

Proposition 4.16. Let A be a pseudo-Ehoop. $S \subseteq A$ is a subalgebra if:
(1) S is closed under $\odot$ and $\wedge$;
(2) for all $e \in Id(A) \cap S$, $S_e = A_e \cap S = \{s \in S | s \leq e\}$ is a subalgebra of the pseudo-hoop $(A_e, \odot, \rightarrow_e, \rightsquigarrow_e, e)$;
(3) for any $s, t \in S$, there exists $b \in Id(S)$ with $s, t \leq b$.

Let A and B be two pseudo-Ehoops. A pseudo-Ehoop homomorphism from A to B is a map $\psi : A \rightarrow B$ which satisfies: (1) ψ preserves $\wedge$ and $\odot$; (2) any elements $s, t \in A$ and $e \in Id(A)$ satisfying $s, t \leq e$, $\psi(s \rightarrow_e t) = \psi(s) \rightarrow_{\psi(e)} \psi(t)$ and $\psi(s \rightsquigarrow_e t) = \psi(s) \rightsquigarrow_{\psi(e)} \psi(t)$. Clearly, $s \leq t$ implies $\psi(s) \leq \psi(t)$.

Proposition 4.17. Let ψ be a pseudo-Ehoop homomorphism from a pseudo-Ehoop A to a pseudo-Ehoop B and S be a subalgebra of A. Then $\psi(S)$ is a subalgebra of B.

Proof: Assume that S is a subalgebra of A. It is obvious that $\psi(S)$ is closed under $\wedge$ and $\odot$. For all $c \in Id(B) \cap \psi(S)$, there exists $e \in S$ that satisfies $\psi(e) = c$. Set $\psi(S)_c = \{s \in \psi(S) | s \leq c\}$. For all $s, t \in \psi(S)_c$, there are $m, n \in S$ that satisfy $\psi(m) = s$ and $\psi(n) = t$. From $s \odot t \leq c \odot c = c$ and $s \odot t = \psi(m) \odot \psi(n) = \psi(m \odot n) \in \psi(S)$, we have $s \odot t \in \psi(S)_c$. Since S is a subalgebra, an element $b \in Id(A) \cap S$ exists that satisfies $m, n, e \leq b$. Therefore, $c = \psi(e) \leq \psi(b)$. We have $s \rightarrow_c t = \psi(m) \rightarrow_{\psi(e)} \psi(n) = (\psi(m) \rightarrow_{\psi(b)} \psi(n)) \wedge \psi(e) = \psi((s \rightarrow_b t) \wedge e) \in \psi(S)_c$ and $s \rightsquigarrow_c t = \psi(m) \rightsquigarrow_{\psi(e)} \psi(n) = (\psi(m) \rightsquigarrow_{\psi(b)} \psi(n)) \wedge \psi(e) = \psi((s \rightsquigarrow_b t) \wedge e) \in \psi(S)_c$. This prove that $\psi(S)_c$ is a subalgebra of B_c.

For all $s, t \in \psi(S)$, there are elements $m, n \in S$ satisfying $\psi(m) = s$ and $\psi(n) = t$. Suppose $d \in Id(A) \cap S$ such that $m, n \leq d$. Then $s, t \leq \psi(d) \in Id(B) \cap \psi(S)$. Therefore, $\psi(S)$ is a subalgebra of B. ■

Proposition 4.18. Let A and B be two pseudo-Ehoops with the least element 0 and $\psi : A \rightarrow B$ a homomorphism.
(1) If $I \in \mathcal{I}(B)$ $(I \in \mathcal{NI}(B))$, then $\psi^{-1}(I) \in \mathcal{I}(A)$ $(\psi^{-1}(I) \in \mathcal{NI}(A))$.
(2) If ψ is a bijection and $I \in \mathcal{I}(A)$ $(I \in \mathcal{NI}(A))$, then $\psi^{-1}(I) \in \mathcal{I}(B)$ $(\psi^{-1}(I) \in \mathcal{NI}(B))$.
(3) $Ker(\psi) = \{s \in A | \psi(s) = 0\}$ is an ideal of A. If B is good, then $\{0\}$ is a normal ideal and so $Ker(\psi)$ is normal.

Proof: (1) From $\psi(0) = 0$, we have $\psi^{-1}(I) \neq \emptyset$. Suppose $s, t \in \psi^{-1}(I)$. For each $e \in Id(A)$ that satisfies $s, t \leq e$, we derive $\psi(s \oslash_e t) = \psi(s) \oslash_{\psi(e)} \psi(t) \in I$, which means $s \oslash_e t \in \psi^{-1}(I)$. Similarly, $s \obslash_e t \in \psi^{-1}(I)$. If $s \leq t \in \psi^{-1}(I)$, then $\psi(s) \leq \psi(t) \in I$. It implies $\psi(s) \in I$ and so $s \in \psi^{-1}(I)$. Thus $\psi^{-1}(I) \in \mathcal{I}(A)$. If I is normal, then $s^{-e} \odot t \in \psi^{-1}(I) \Longleftrightarrow \psi(s)^{-\psi(e)} \odot \psi(t) \in I \Longleftrightarrow \psi(t) \odot \psi(s)^{\sim_{\psi(e)}} \in I \Longleftrightarrow t \odot s^{\sim_e} \in \psi^{-1}(I)$. We have that $\psi^{-1}(I)$ is normal.

(2) Suppose $s, t \in B$ and $s \leq t \in \psi(I)$. There exists $m \in I$ with $\psi(m) = t$. Since ψ is a bijection, an element $n \in A$ exists that satisfies $\psi(n) = s$. Let $e \in Id(A)$ satisfying $m, n \leq e$. Then $\psi(n \rightarrow_e m) = \psi(n) \rightarrow_{\psi(e)} \psi(m) = \psi(e)$ and so $n \rightarrow_e m = e$. This means $n \leq m \in I$. Thus, $n \in I$ and so $s \in \psi(I)$. Suppose $s, t \in \psi(I)$. There are elements $u, v \in I$ that satisfy $s = \psi(u)$ and $t = \psi(v)$. For all $b \in Id(B)$ with $s, t \leq b$, there is $e \in Id(A)$ with $b = \psi(e)$. We obtain that $s \oslash_b t = \psi(u) \oslash_{\psi(e)} \psi(v) = \psi(u \oslash_e v) \in \psi(I)$ and $s \obslash_b t \in \psi(I)$. Hence, $\psi(I) \in \mathcal{I}(B)$. If I is normal, then $\psi(u)^{-\psi(e)} \odot \psi(v) \in \psi(I) \Longleftrightarrow u^{-e} \odot v \in I \Longleftrightarrow v \odot u^{\sim_e} \in I \Longleftrightarrow \psi(v) \odot \psi(u)^{-\psi(e)} \in \psi(I)$. $\psi(I)$ is normal.

(3) Obviously, $\{0\}$ is an ideal of B. By (1), $Ker(\psi) = \psi^{-1}(\{0\}) \in \mathcal{I}(A)$. If B is good, for each $s, t \in B$ and $e \in Id(B)$ that satisfy $s, t \leq e$, $s^{-e} \odot t \in \{0\} \Longleftrightarrow t \leq s^{-_e\sim_e} \Longleftrightarrow t \leq s^{\sim_e -_e} \Longleftrightarrow t \odot s^{\sim_e} \in \{0\}$. This shows that $\{0\}$ is normal. ■

V. Filters

This section will study filters of a pseudo-Ehoop A. Every proper pseudo-Ehoop has at least one maximal filter. Moreover, we construct the congruences and ideals by normal filters on A .

Definition 5.1. Set A be a pseudo-Ehoop. $\emptyset \neq F \subseteq$ is a filter if for any $s, t \in A$,
(1) there exists $e \in Id(A) \cap F$ satisfying $s \leq e$;
(2) $s \leq t$ and $s \in F \Longrightarrow t \in F$;
(3) $s, t \in F$ implies $s \odot t \in F$.

If $F \neq A$, then F is a proper filter. A maximal filter is a proper filter which is not included in any other proper filter. The set of all filters (proper filters, maximal filters) of A is denoted by $\mathcal{F}(A)$ $(\mathcal{PF}(A), \mathcal{MF}(A))$.

Proposition 5.2. F is a filter of a pseudo-Ehoop A if and only if any elements $s,t\in A$ and $e\in Id(A)$ satisfying $s,t\le e$,
(1) there is $b\in Id(A)\cap F$ that satisfies $s\le b$;
(2) $s, s\rightarrow_e t\in F\Longrightarrow t\in F$.

Proof: Let $F\in\mathcal{F}(A)$ and $s,t\in A$. For any $e\in Id(A)$ with $s,t\le e$, we get $t\ge(s\rightarrow_e t)\odot s\in F$ and so $t\in F$.

Conversely, set $s,t\in A$ and $t\ge s\in F$. An element $b\in Id(A)\cap F$ exists that satisfies $s\le t\le b$. It means that $s\rightarrow_b t=b\in F$. Thus, $t\in F$. If $s,t\in F$ and $c\in Id(A)$ satisfying $s,t\le c$, then $s\rightarrow_c(t\rightarrow_c(s\odot t))=(s\odot t)\rightarrow_c(s\odot t)=c\in F$. We have $t\rightarrow_c(s\odot t)\in F$. This together with $t\in F$ implies $s\odot t\in F$. ■

Proposition 5.3. Set A be a pseudo-Ehoop. $F\in\mathcal{F}(A)\Longleftrightarrow$ any elements $s,t\in A$ and $e\in Id(A)$ that satisfy $s,t\le e$,
(1) there is $b\in Id(A)\cap F$ satisfying $s\le b$;
(2) $s, s\rightsquigarrow_e t\in F\Longrightarrow t\in F$.

Proof: By Proposition 5.2, the proof is easy. ■

Set A be a pseudo-Ehoop and $S\subseteq A$. The filter generated by S is defined as $\langle S]$.

Proposition 5.4. Assume that A is a pseudo-Ehoop. If $s\in A$ and $S\subseteq A$, then
(1) $\langle S]=\{u\in A|u\ge s_1\odot s_2\odot\cdots\odot s_m,$ for some $s_1,s_2,\cdots,s_m\in S$ and $m\in\mathbb{N}\backslash\{0\}\}$;
(2) $\langle s]=\{u\in A|u\ge s^m,$ for some $m\in\mathbb{N}\backslash\{0\}\}$.

Proof: (1) Set $G=\{u\in A|u\ge s_1\odot s_2\odot\cdots\odot s_m,$ for some $s_1,s_2,\cdots,s_m\in S$ and $m\in\mathbb{N}\backslash\{0\}\}$. Clearly, $S\ne\emptyset$ and $S\subseteq G$. For all $u\in G$, there exist $e\in Id(A)$ and $s_i\in S$ that satisfy $s_i,u\le e$, $i=1,2,\cdots,m$. Then $e\in G$. Let $u,v\in A$ and $b\in Id(A)$ satisfying $u,v\le b$ and $u,u\rightarrow_b v\in G$. There are elements $m,n\in\mathbb{N}\backslash\{0\}$ and $s_1,s_2,\cdots,s_m,t_1,t_2,\cdots,t_n\in S$ that satisfy $u\ge s_1\odot s_2\odot\cdots\odot s_m$ and $u\rightarrow_b v\ge t_1\odot\cdots\odot t_n$. Then $v\ge(u\rightarrow_b v)\odot u\ge t_1\odot\cdots\odot t_n\odot s_1\odot\cdots\odot s_m$ and so $v\in G$. This shows $G\in\mathcal{F}(A)$ and $S\subseteq G$.

Assume $T\in\mathcal{F}(A)$ and $S\subseteq T$. For all $u\in G$, there are $s_1,s_2,\cdots,s_m\in S$ and $m\in\mathbb{N}\backslash\{0\}$ such that $u\ge s_1\odot\cdots\odot s_m$. Since $s_1,s_2,\cdots,s_m\in T$, we have $s_1\odot\cdots\odot s_m\in T$ and so $u\in T$. Thus, $G\subseteq T$. This means $\langle S]=\{u\in A|u\ge s_1\odot s_2\odot\cdots\odot s_m,$ for some $s_1,s_2,\cdots,s_m\in S$ and $m\in\mathbb{N}\backslash\{0\}\}$.

In particular, if $S=\{s\}$, we can prove $\langle s]=\{u\in A|u\ge s^m,$ for some $m\in\mathbb{N}\backslash\{0\}\}$. ■

Proposition 5.5. Set A be a pseudo-Ehoop and $F\in\mathcal{PF}(A)$. $F\in\mathcal{MF}(A)\Longleftrightarrow$ for each $s\in A\backslash F$ implies $\langle F\cup\{s\}]=A$.

Proof: ($\Longrightarrow$) From $s\notin F$, we obtain $F\subseteq\langle F\cup\{s\}]$ and $F\ne\langle F\cup\{s\}]$. By the maximality of F, we get $\langle F\cup\{s\}]=A$.

($\Longleftarrow$) If $H\in\mathcal{PF}(A)$ with $F\subseteq H$ and $F\ne H$. There exists $s\in H\backslash F$. By (2), we deduce $\langle F\cup\{s\}]=A$. It is derived from $\langle F\cup\{s\}]\subseteq H$ that $H=A$, which is a contradiction. Thus, $F\in\mathcal{MF}(A)$. ■

Set A a pseudo-Ehoop and $F\in\mathcal{F}(A)$. Any elements $s,t\in A$, we define

$$(s,t)\in\theta_{R(F)}\Longleftrightarrow\exists e\in Id(A)\text{ satisfying }s,t\le e\text{ and }s\rightarrow_e t,t\rightarrow_e s\in F$$

and

$$(s,t)\in\theta_{L(F)}\Longleftrightarrow\exists e\in Id(A)\text{ satisfying }s,t\le e\text{ and }s\rightsquigarrow_e t,t\rightsquigarrow_e s\in F.$$

Proposition 5.6. Set A be a pseudo-Ehoop and $F\in\mathcal{F}(A)$. $\theta_{R(F)}$ and $\theta_{L(F)}$ are equivalence relations on A.

Proof: Obviously, $(s,t)\in\theta_{R(F)}\Longleftrightarrow(t,s)\in\theta_{R(F)}$. Moreover, for any $s\in A$, there is an element $e\in Id(A)\cap F$ that satisfies $s\le e$. We have $s\rightarrow_e s=e\in F$, which means that $(s,s)\in\theta_{R(F)}$. Assume that $(s,t)\in\theta_{R(F)}$ and $(t,w)\in\theta_{R(F)}$. There are two elements $e,b\in Id(A)$ that satisfy $s,t\le e$, $t,w\le b$ and $s\rightarrow_e t,t\rightarrow_e s,t\rightarrow_b w,w\rightarrow_b t\in F$. Take $c\in Id(A)$ such that $e,b\le c$. By Proposition 3.8 (1), we obtain $s\rightarrow_c t,t\rightarrow_c s,t\rightarrow_c w,w\rightarrow_c t\in F$. Form Proposition 2.2 (7), $s\rightarrow_c w\ge(t\rightarrow_c w)\odot(s\rightarrow_c t)\in F$ and $w\rightarrow_c s\ge(t\rightarrow_c s)\odot(w\rightarrow_c t)\in F$. Thus, $s\rightarrow_c w,w\rightarrow_c s\in F$, which implies $(s,w)\in\theta_{R(F)}$. We similarly show that $\theta_{L(F)}$ is an equivalence relation. ■

Proposition 5.7. Suppose that A a pseudo-Ehoop and $F\in\mathcal{F}(A)$. For any $s,t\in A$,
(1) $(s,t)\in\theta_{R(F)}\Longleftrightarrow u\odot s=v\odot t$ for some $u,v\in F$;
(2) $(s,t)\in\theta_{L(F)}\Longleftrightarrow s\odot u=t\odot v$ for some $u,v\in F$.

Proof: (1) Assume $(s,t)\in\theta_{R(F)}$. There is $e\in Id(A)$ satisfying $s,t\le e$ and $s\rightarrow_e t,t\rightarrow_e s\in F$. Take $u=s\rightarrow_e t$ and $v=t\rightarrow_e s$. It follows that $u\odot s=(s\rightarrow_e t)\odot s=(t\rightarrow_e s)\odot t=v\odot t$. Conversely, if $u\odot s=v\odot t$ for some $u,v\in F$, there is $b\in Id(A)$ that satisfies $u,v,s,t\le b$. Since $v\rightarrow_b(t\rightarrow_b s)=(v\odot t)\rightarrow_b s=(u\odot s)\rightarrow_b s=b$, we have $t\rightarrow_b s\ge v\in F$ and so $t\rightarrow_b s\in F$. Similarly, we get $s\rightarrow_b t\in F$. Thus, $(s,t)\in\theta_{R(F)}$.

The proof of (2) is similar to (1). ■

Proposition 5.8. Set A be a pseudo-Ehoop and $F\in\mathcal{F}(A)$. For all $s,t,u,v\in A$, there is $c\in Id(A)$ with $s,t,u,v\le c$,
(1) $(s,t)\in\theta_{R(F)}$ and $(u,v)\in\theta_{R(F)}\Longrightarrow s\rightarrow_c u\in F$ iff $t\rightarrow_c v\in F$;
(2) $(s,t)\in\theta_{L(F)}$ and $(u,v)\in\theta_{L(F)}\Longrightarrow s\rightsquigarrow_c u\in F$ iff $t\rightsquigarrow_c v\in F$.

Proof: (1) Assume $(s,t)\in\theta_{R(F)}$ and $(u,v)\in\theta_{R(F)}$. There are $e,b\in Id(A)$ that satisfy $s,t\le e$, $u,v\le b$ and $s\rightarrow_e t,t\rightarrow_e s,u\rightarrow_b v,v\rightarrow_b u\in F$. Let $c\in Id(A)$ with $e,b\le c$. We obtain $s\rightarrow_c t,t\rightarrow_c s,u\rightarrow_c v,v\rightarrow_c u\in F$. By Proposition 2.2 (7), we get that $(u\rightarrow_c v)\odot(s\rightarrow_c u)\odot(t\rightarrow_c s)\le(s\rightarrow_c v)\odot(t\rightarrow_c s)\le t\rightarrow_c v$ and $(v\rightarrow_c u)\odot(t\rightarrow_c v)\odot(s\rightarrow_c t)\le(t\rightarrow_c u)\odot(s\rightarrow_c t)\le s\rightarrow_c u$. Therefore $s\rightarrow_c u\in F\Longleftrightarrow t\rightarrow_c v\in F$.

(2) The proof is similar to (1). ■

A filter F of a pseudo-Ehoop A is said to be normal if for any $s,t\in A$ and $e\in Id(A)$ with $s,t\le e$, $s\rightarrow_e t\in F\Longleftrightarrow s\rightsquigarrow_e t\in F$. The set of all normal filters of A defined by $\mathcal{NF}(A)$. For all $u\in A$, define $u\odot F=\{u\odot s|s\in F\}$ and $F\odot u=\{s\odot u|s\in F\}$.

Proposition 5.9. Suppose that A is a pseudo-Ehoop and F is normal. Then
(1) $u\odot F=F\odot u$ for each $u\in A$;
(2) $\theta_{R(F)}=\theta_{L(F)}$.

Proof: (1) Set $u\in A$ and $t=u\odot s\in u\odot F$. For each $e\in Id(A)$ with $u,s,t\le e$, we obtain $t=t\wedge u=(u\rightarrow_e$

$t) \odot u$. From $s \rightsquigarrow_e (u \rightsquigarrow_e t) = (u \odot s) \rightsquigarrow_e t = t \rightsquigarrow_e t = e$, we get $s \leq u \rightsquigarrow_e t$ and so $u \rightsquigarrow_e t \in F$. Since F is normal, $u \rightarrow_e t \in F$. Hence, $t = (u \rightarrow_e t) \odot u \in F \odot u$, which implies $u \odot F \subseteq F \odot u$. Similarly, $F \odot u \subseteq u \odot F$. Therefore, $u \odot F = F \odot u$.

(2) Set $s, t \in A$ and $(s,t) \in \theta_{R(F)}$. By Proposition 5.7, there are two elements $u, v \in F$ that satisfy $u \odot s = v \odot t$. By (1), there are $m, n \in F$ such that $u \odot s = s \odot m$ and $v \odot t = t \odot n$. This $s \odot m = t \odot n$ for some $m, n \in F$. Applying Proposition 5.7 again, $(s,t) \in \theta_{L(F)}$. Similarly, $(s,t) \in \theta_{L(F)} \Longrightarrow (s,t) \in \theta_{R(F)}$. Hence, $\theta_{R(F)} = \theta_{L(F)}$. ■

Proposition 5.10. Let A be a pseudo-Ehoop, $F \in \mathcal{NF}(A)$ and $s \in A$.

$$\begin{aligned}\langle F \cup \{s\}] =& \{u \in A | u \geq f \odot s^n, \text{ for some } f \in F, \\ & n \in \mathbb{N}\backslash\{0\}\} \\ =& \{u \in A | u \geq s^n \odot f, \text{ for some } f \in F, \\ & n \in \mathbb{N}\backslash\{0\}\}.\end{aligned}$$

Proof: Clearly, $\langle F \cup \{s\}] = \{u \in A | u \geq (f_1 \odot s^{n_1}) \odot \cdots (f_t \odot s^{n_t})$, for some $f_1, \cdots, f_t \in F, n_1, \cdots, n_t \in \mathbb{N}\backslash\{0\}\}$ by Proposition 5.4,

If $t = 1$, then $u \geq f \odot s^{n_1}$.

If $t = 2$, then $u \geq (f_1 \odot s^{n_1}) \odot (f_2 \odot s^{n_2}) = f_1 \odot (s^{n_1} \odot f_2) \odot s^{n_2}$. Since F is normal, we have $s^{n_1} \odot f_2 \in s^{n_1} \odot F = F \odot s^{n_1}$ by Proposition 5.9. There exists $h \in F$ satisfying $s^{n_1} \odot f_2 = h \odot s^{n_1}$. Thus, $u \geq f_1 \odot (s^{n_1} \odot f_2) \odot s^{n_2} = (f_1 \odot h) \odot (s^{n_1} \odot s^{n_2}) = f \odot s^n$, where $f = f_1 \odot h \in F$ and $n = n_1 + n_2$.

Therefore, the final result can be gradually obtained by applying this procedure. ■

Proposition 5.11. Set A be a pseudo-Ehoop and $F \in \mathcal{NF}(A) \cap \mathcal{PF}(A)$. $F \in \mathcal{MF}(A) \Longleftrightarrow$ for all $s \in A\backslash F$ and $t \in A$, there are $m \in \mathbb{N}\backslash\{0\}$ and $e \in Id(A)$ with $s, t \leq e$ that satisfy $s^m \rightarrow_e t \in F$ (or $s^m \rightsquigarrow_e t \in F$).

Proof: ($\Longrightarrow$) Suppose $s \in A\backslash F$. By Proposition 5.5, $\langle F \cup \{s\}] = A$. Let $t \in A$. From Proposition 5.10, there are two elements $f \in F$ and $m \in \mathbb{N}\backslash\{0\}$ that satisfy $f \odot s^m \leq t$. Take $e \in Id(A)$ with $s, t, f \leq e$. We get $f \leq s^m \rightarrow_e t$ and so $s^m \rightarrow_e t \in F$.

($\Longleftarrow$) As $F \in \mathcal{PF}(A)$, there exists $s \in A\backslash F$. By (2), for all $t \in A$, we have $s^m \rightarrow_e t \in F$ for some $m \in \mathbb{N}\backslash\{0\}$ and $e \in Id(A)$ with $s, t \leq e$. From $(s^m \rightarrow_e t) \odot s^m \leq t$, we get $t \in \langle F \cup \{s\}\rangle$. Thus $A \subseteq \langle F \cup \{s\}]$, which means $A = \langle F \cup \{s\}]$. By Proposition 5.5, F is maximal.

Based on the above proof, $F \in \mathcal{MF}(A) \Longleftrightarrow s^m \rightsquigarrow_e t \in F$. ■

If $F \in \mathcal{NF}(A)$, we deduce that $\theta_{R(F)}$ and $\theta_{L(F)}$ are consistent by Proposition 5.9. We define this equivalence relation using θ_F.

Proposition 5.12. Let A be a pseudo-Ehoop and $F \in \mathcal{NF}(A)$. The equivalence relation θ_F is a congruence on A.

Proof: Suppose that $(s,t) \in \theta_F$ and $(u,v) \in \theta_F$. By Proposition 5.7 (1), there exist $p, q \in F$ satisfying $p \odot s = q \odot t$. Then $p \odot (s \odot u) = q \odot (t \odot u)$ and so $(s \odot u, t \odot u) \in \theta_F$. From Proposition 5.7 (2), there are two elements $m, n \in F$ with $u \odot m = v \odot n$. We obtain $(t \odot u) \odot m = (t \odot v) \odot n$ and so $(t \odot u, t \odot v) \in \theta_F$. Therefore $(s \odot u, t \odot v) \in \theta_F$.

Suppose that $e \in Id(A)$ and $(s,t), (u,v) \in \theta_F \cap (A_e \times A_e)$. We need to show that $(s \rightarrow_e u, t \rightarrow_e v) \in \theta_F \cap (A_e \times A_e)$. Since $(s,t), (u,v) \in \theta_F$, there are two elements $b, c \in Id(A)$ that satisfy $s, t \leq b$, $u, v \leq c$ and $s \rightarrow_b t, t \rightarrow_b s, u \rightarrow_c v, v \rightarrow_c u \in F$. Let $d \in Id(A)$ with $e, b, c \leq d$. We deduce $s \rightarrow_d t, t \rightarrow_d s, u \rightarrow_d v, v \rightarrow_d u \in F$. By Proposition 2.2 (7), we get $(u \rightarrow_d v) \odot (s \rightarrow_d u) \leq s \rightarrow_d v$ and so $u \rightarrow_d v \leq (s \rightarrow_d u) \rightarrow_d (s \rightarrow_d v)$. This means that $(s \rightarrow_d u) \rightarrow_d (s \rightarrow_d v) \in F$. In the pseudo-hoop algebra A_d, we have

$$\begin{aligned}&(s \rightarrow_e u) \rightarrow_d (s \rightarrow_e v) \\ =&(s \rightarrow_e u) \rightarrow_d ((s \rightarrow_d v) \wedge e) \\ =&((s \rightarrow_e u) \rightarrow_d (s \rightarrow_d v)) \wedge ((s \rightarrow_e u) \rightarrow_d e) \\ =&((s \rightarrow_e u) \rightarrow_d (s \rightarrow_d v)) \wedge d \\ =&(s \rightarrow_e u) \rightarrow_d (s \rightarrow_d v).\end{aligned}$$

Since $s \rightarrow_e u \leq s \rightarrow_d u$, we have $(s \rightarrow_d u) \rightarrow_d (s \rightarrow_d v) \leq (s \rightarrow_e u) \rightarrow_d (s \rightarrow_d v) = (s \rightarrow_e u) \rightarrow_d (s \rightarrow_e v)$. It follows that $(s \rightarrow_e u) \rightarrow_d (s \rightarrow_e v) \in F$. Similarly, we obtain $(s \rightarrow_e v) \rightarrow_d (s \rightarrow_e u) \in F$. Thus, $(s \rightarrow_e u, s \rightarrow_e v) \in \theta_F$. From $t \rightarrow_d s \leq (s \rightarrow_d v) \rightsquigarrow_d (t \rightarrow_d v)$ and $s \rightarrow_d t \leq (t \rightarrow_d v) \rightsquigarrow_d (s \rightarrow_d v)$, we can prove $(s \rightarrow_e v, t \rightarrow_e v) \in \theta_F$ in a similar way. Thus, $(s \rightarrow_e u, t \rightarrow_e v) \in \theta_F$ and so $(s \rightarrow_e u, t \rightarrow_e v) \in \theta_F \cap (A_e \times A_e)$.

Analogously, $(s,t) \in \theta_F \cap (A_e \times A_e)$ and $(u,v) \in \theta_F \cap (A_e \times A_e)$ imply $(s \rightsquigarrow_e u, t \rightsquigarrow_e v) \in \theta_F \cap (A_e \times A_e)$. This proves that $\theta_F \cap (A_e \times A_e)$ is a congruence on A_e for all $e \in Id(A)$.

For any $s, u \in A$ and $e \in Id(A)$ with $s, u \leq e$, we have $s \wedge u = (s \rightarrow_e u) \odot s$. It means $(s \wedge u, t \wedge v) \in \theta_F$. Therefore, θ_F is a congruence on A. ■

Proposition 5.13. Set A be a pseudo-Ehoop with the least element 0 and $F \in \mathcal{NF}(A)$. Then $I_F = \{s \in A | \exists e \in Id(A)$ that satisfies $s \leq e$ and $s^{-_e} \in F\}$ is an ideal of A.

Proof: Obviously, $0 \in I_F$. Assume that $s, t \in A$ and $e \in Id(A)$ that satisfy $s, t \leq e$ and $s, s^{-_e} \odot t \in I_F$. There are $b, c \in Id(A)$ that satisfy $s \leq b$, $s^{-_e} \odot t \leq c$ and $s^{-_b}, (s^{-_e} \odot t)^{-_c} \in F$. Choose $d \in Id(A)$ with $e, b, c \leq d$. From $s^{-_b} \leq s^{-_d}$, we deduce $s^{-_d} \in F$. By Proposition 3.10, we obtain $(s^{-_e} \odot t)^{-_c} = (s^{-_d} \odot t)^{-_c} \leq (s^{-_d} \odot t)^{-_d}$ and so $(s^{-_d} \odot t)^{-_d} \in F$. Thus $s^{-_d} \rightarrow_d t^{-_d} = (s^{-_d} \odot t)^{-_d} \in F$. By Proposition 5.2, $t^{-_d} \in F$, which implies $t \in I_F$. If $s, t \odot s^{\sim_e} \in I_F$, there are $b, c \in Id(A)$ that satisfy $s \leq b$, $t \odot s^{\sim_e} \leq c$ and $s^{-_b}, (t \odot s^{\sim_e})^{-_c} \in F$. Take $d \in Id(A)$ with $e, b, c \leq d$. We deduce $s^{-_d} \in F$ by $s^{-_b} \leq s^{-_d}$. Because F is normal, $s^{\sim_d} \in F$. Also, we obtain $(t \odot s^{\sim_e})^{-_c} = (t \odot s^{\sim_d})^{-_c} \leq (t \odot s^{\sim_d})^{-_d}$ by Proposition 3.10. This means that $(t \odot s^{\sim_d})^{-_d} \in F$ and so $(t \odot s^{\sim_d})^{\sim_d} \in F$. Hence, $s^{\sim_d} \rightsquigarrow_d t^{\sim_d} \in F$. By Proposition 5.3, we get $t^{\sim_d} \in F$ and so $t^{-_d} \in F$. It follows that $t \in I_F$. Therefore, I_F is an ideal of A by Proposition 4.2. ■

Remark 5.14. (1) Since F is normal, $s \in I_F \Longleftrightarrow \exists e \in Id(A)$ that satisfies $s \leq e$ and $s^{\sim_e} \in F$.

(2)

$$\begin{aligned}s \in I_F \Longleftrightarrow& \exists f \in F, \exists e \in Id(A) \text{ that satisfy } s, f \leq e \\ & \text{and } s \leq f^{-_a} \\ \Longleftrightarrow& \exists f \in F, \exists e \in Id(A) \text{ that satisfy } s, f \leq e\end{aligned}$$

$$\text{and } s \leq f^{\sim_e}.$$

In fact, let $s \in I_F$. There exists $e \in Id(A)$ with $s \leq e$ and $s^{-_e} \in F$. Since F is normal, $s^{\sim_e} \in F$. Set $f = s^{\sim_e}$. It follows $s \leq s^{\sim_e -_e} = f^{-_e}$. Conversely, we have $f \leq f^{-_e \sim_e} \leq s^{\sim_e}$. We have $s^{\sim_e} \in F$ and so $s^{-_e} \in F$. Thus, $s \in I_F$.

Proposition 5.15. Let A be a pseudo-Ehoop with the least element 0 and $F \in \mathcal{NF}(A) \cap \mathcal{PF}(A)$.
(1) $s \in F$ implies $s \notin I_F$.
(2) If F is maximal, for any $e \in Id(A)$ with $e \notin I_F$, we obtain $e \in F$.
(3) If $s \in F$, then for any $e \in Id(A)$ with $s \leq e$, we obtain $s^{-_e} \in I_F$ ($s^{\sim_e} \in I_F$).

Proof: (1) If $s \in I_F$, for any $s \in F$, there are $f \in F$ and $e \in Id(A)$ that satisfy $s, f \leq e$ and $s \leq f^{-_e}$. From $s \in F$, we get $f^{-_e} \in F$. Then $0 = f \odot f^{-_e} \in F$, which is a contradiction.

(2) Suppose $e \notin F$. We have $\langle F \cup \{e\}] = A$. Since $0 \in A$, there exists $f \in F$ and $n \in \mathbb{N}\backslash\{0\}$ that satisfy $0 \geq e^n \odot f$ by Proposition 5.10. Choose $b \in Id(A)$ with $f, e \leq b$. It means $e = e^n \leq f^{-_b}$. By Remark 5.14, $e \in I_F$, which is a contradiction.

The proof of (3) is clear. ■

Proposition 5.16. Let A be a pseudo-Ehoop and $F \in \mathcal{MF}(A) \cap \mathcal{NF}(A)$. For all $e \in Id(A)$, either $F \cap A_e = \emptyset$ or $F \cap A_e$ is a maximal filter of A_e.

Proof: Let $e \in Id(A)$ and $F \cap A_e \neq \emptyset$. There is $s \in A$ that satisfies $s \in F \cap A_e$. Thus $e \in F$. Obviously, $F \cap A_e$ is a normal filter of A_e. Set $s \in A_e \backslash (F \cap A_e)$. We obtain $s \notin F$ and so $\langle F \cup \{s\}] = A$. For all $w \in A_e$, there exist $f \in F$ and $n \in \mathbb{N}\backslash\{0\}$ that satisfy $w \geq f \odot s^n$. Choose $b \in Id(A)$ with $e, f \leq b$. Then $w = e \wedge w \geq e \odot (f \odot s^n) = (e \odot f) \odot s^n$. From $e, f \in F$, we have $e \odot f \in F \cap A_e$. It follows $w \in \langle (F \cap A_e) \cup \{s\}]$ and so $\langle (F \cap A_e) \cup \{s\}] = A_e$, which implies that $F \cap A_e$ is maximal. ■

Theorem 5.17. Set A be a proper pseudo-Ehoop. Then A contains at least one maximal filter.

Proof: Consider the set $\mathcal{F}$ of all proper filters of A. For any element $e \in Id(A)$, it is evident that the subset $\{s \in A | s \geq e\}$ constitutes a proper filter. Thus, $\mathcal{F} \neq \emptyset$. By applying Zorn's Lemma, $\mathcal{F}$ has a maximal element. This means that A has at least one maximal filter. ■

VI. Prime ideals and maximal ideals

This section will investigate prime ideals and maximal ideals of pseudo-Ehoops. If A is a pseudo-Ehoop with the pDN condition, every maximal ideal is prime. Furthermore, we present a prime ideal theorem of A and provide an equivalent form of prime ideals.

Definition 6.1. Set A be a pseudo-Ehoop with the least element 0 and P a proper ideal. P is a prime ideal if for all $s, t \in A$, $s \wedge t \in P \Longrightarrow s \in P$ or $t \in P$.

Theorem 6.2. (Prime ideal theorem) Consider an ideal I of a pseudo-Ehoop A with the pDN condition. $\emptyset \neq S \subseteq A$ and $I \cap S = \emptyset$. If S is closed under $\wedge$, a prime ideal P exists that satisfies $I \subseteq P$ and $P \cap S = \emptyset$.

Proof: Set $K = \{J \in \mathcal{I}(A) | I \subseteq J \text{ and } J \cap S = \emptyset\}$. Clearly, $I \in K$ and $K \neq \emptyset$. By Zorn's Lemma, K has a maximal element P. Consequently, $I \subseteq P$ and $P \cap S = \emptyset$, establishing that P is proper. Assume $s, t \notin P$ and $s \wedge t \in P$. We get $P \subsetneqq \langle P \cup \{s\}\rangle$ and $P \subsetneqq \langle P \cup \{t\}\rangle$. As P is maximal, it means $S \cap \langle P \cup \{s\}\rangle \neq \emptyset$ and $S \cap \langle P \cup \{t\}\rangle \neq \emptyset$. Thus, $u \in S \cap \langle P \cup \{s\}\rangle$ exists. There are $t_i \in P$, $s, m_i \in \mathbb{N}\backslash\{0\}$ and $e \in Id(A)$ such that $t_i, s \leq e$ and $u \leq \oslash_{e_{i=1}}^{s}(t_i \oslash_e m_{ie}s)$. Similarly, we have $v \in S$ and $v \leq \oslash_{b_{i=1}}^{t}(w_i \oslash_b n_{ib}t)$, where $w_i \in P$, $t, n_i \in \mathbb{N}\backslash\{0\}$ and $b \in Id(A)$ such that $w_i, t \leq b$. Let $c \in Id(A)$ with $e, b \leq c$, $w = (\oslash_{c_{i=1}}^{s} t_i) \oslash_c (\oslash_{c_{i=1}}^{t} w_i) \in P$ and $n = \max\{m_1, m_2, \cdots, m_s, n_1, n_2, \cdots, n_t\}$. By Proposition 3.8 and 4.6, we can get

$$\begin{aligned}
u \wedge v &\leq (\oslash_{e_{i=1}}^{s}(t_i \oslash_e m_{ie}s)) \wedge (\oslash_{b_{i=1}}^{t}(w_i \oslash_b n_{ib}t)) \\
&\leq (\oslash_{c_{i=1}}^{s}(t_i \oslash_c m_{ic}s)) \wedge (\oslash_{c_{i=1}}^{t}(w_i \oslash_c n_{ic}t)) \\
&\leq s_c(w \oslash_c n_c s) \wedge t_c(w \oslash_c n_c t) \\
&\leq (st)_c((w \oslash_c n_c s) \wedge (w \oslash_c n_c t)) \\
&= (st)_c(w \oslash_c (n_c s \wedge n_c t)) \\
&\leq (st)_c(w \oslash_c n_c^2(s \wedge t)).
\end{aligned}$$

From $s \wedge t, w \in P$, it follows $u \wedge v \in P$. Since S is closed under $\wedge$, we deduce $u \wedge v \in S$ and so $u \wedge v \in P \cap S$, leading to a contradiction. Thus, P must be a prime ideal. ■

Corollary 6.3. Let A be a pseudo-Ehoop with the pDN condition and $s \notin I \in \mathcal{I}(A)$. There exists a prime ideal P that satisfies $I \subseteq P$ and $s \notin P$.

Proposition 6.4. Consider I to be a proper ideal of a pseudo-Ehoop A with the pDN condition. If the condition holds:
for each $e \in Id(A)\backslash I \Longrightarrow$ for any $b \in Id(A)$ satisfying $e \leq b$, we find $e^{-_b} \in I$. (*)
Then $F_I = \{s \in A | \exists e \in Id(A)\backslash I \text{ satisfying } s \leq e \text{ and } s^{-_e} \in I\}$ is a filter.

Proof: Clearly, there is $s \in A\backslash I$. Let $e \in Id(A)$ with $s \leq e$, implying $e \notin I$. For all $t \in A$, there is $b \in Id(A)$ that satisfies $t \leq b$. Take $c \in Id(A)$ and $e, b \leq c$. Then $c \notin I$. From the condition (*), it means $c^{-_c} \in I$ and so $c \in F_I$. This ensures $F_I \neq \emptyset$ and demonstrates the existence of $c \in Id(A) \cap F_I$ satisfying $t \leq c$ for each $t \in A$.

If $s \leq t$ and $s \in F_I$, there is $e \in Id(A)\backslash I$ that satisfies $s \leq e$ and $s^{-_e} \in I$. For any $b \in Id(A)$ with $s, t, e \leq b$, it follows $e^{-_b} \in I$ by (*), while $b \notin I$. By Proposition 3.8 (2) and Proposition 4.6 (1), we have

$$\begin{aligned}
e^{-_b} \oslash_b s^{-_e} &= e^{-_b} \oslash_b (s^{-_b} \wedge e) \\
&= (e^{-_b} \oslash_b s^{-_b}) \wedge (e^{-_b} \oslash_b e) \\
&= (e^{-_b} \oslash_b s^{-_b}) \wedge b \\
&= e^{-_b} \oslash_b s^{-_b}.
\end{aligned}$$

Then $e^{-_b} \oslash_b s^{-_b} \in I$. From $s^{-_b} \leq e^{-_b} \oslash_b s^{-_b}$, we deduce $s^{-_b} \in I$. Since $t^{-_b} \leq s^{-_b}$, we have $t^{-_b} \in I$. This implies $t \in F_I$.

If $s, t \in F_I$, there are two elements $e, b \in Id(A)\backslash I$ that satisfy $s \leq e$, $t \leq b$ and $s^{-_e}, t^{-_b} \in I$. Let $c \in Id(A)$ with $e, b \leq c$. Thus $c \notin I$. We find $e^{-_c}, b^{-_c} \in I$ and thereby conclude $c \in Id(A)\backslash I$. Similarly, from $s^{-_c} \leq e^{-_c} \oslash_c s^{-_c} = e^{-_c} \oslash_c s^{-_e} \in I$ and $t^{-_c} \leq b^{-_c} \oslash_c t^{-_c} = b^{-_c} \oslash_c t^{-_b} \in I$, we deduce $s^{-_c}, t^{-_c} \in I$. Then, $(s \odot t)^{-_c} = s \rightarrow_c t^{-_c} =$

$s^{-c\sim_c} \to_c t^{-c} = s^{-c} \oslash_c t^{-c} \in I$. This confirms that $s \odot t \in F_I$. Therefore, F_I is a filter. ■

A proper ideal of a pseudo-Ehoop A with the least element 0 is maximal if no other proper ideal of A can strictly contain it. If A is normal, then I is maximal $\Longleftrightarrow$ for any $s \in A\backslash I$, $\langle I \cup \{s\}\rangle = A$.

Proposition 6.5. Set A be a pseudo-Ehoop with the pDN condition and $F \in \mathcal{MF}(A) \cap \mathcal{NF}(A)$. Then $I_F = \{s \in A | \exists f \in F, e \in Id(A)$ that satisfy $s, f \leq e$ and $s \leq f^{-e}\}$ is a maximal ideal.

Proof: By Proposition 5.13 and Remark 5.14, $I_F \in \mathcal{I}(A)$. As $F \in \mathcal{NF}(A)$, there exists $e \in Id(A) \cap F$ that satisfies $s \leq e$ for each $s \in A$. From Proposition 5.15 (1), we get $e \notin I_F$. It follows that I_F is proper.

Let $J \in \mathcal{I}(A)$ and $I_F \subseteq J \neq A$. For all $e \in Id(A)\backslash J$, we have $e \notin I_F$. By Proposition 5.15 (2), we obtain that $e \in F$. Also, for each $b \in Id(A)$ satisfying $e \leq b$, we get $e^{-b} \in I_F \subseteq J$. Hence, F_J is a filter by Proposition 6.4. Let $h \in A\backslash J$. For every $f \in F$, there is $c \in Id(A)$ satisfying $f, h \leq c$. Consequently, $c \notin J$, and since $f^{-c} \in I_F \subseteq J$, we conclude that $f \in F_J$. This proves $F \subseteq F_J$. As F is maximal, we get two cases: either $F_J = F$ or $F_J = A$. If $F_J = A$, then $0 \in F_J$. There is $d \in Id(A)\backslash J$ that meets $d = 0^{-d} \in J$, which creates a contradiction. Therefore, we deduce $F_J = F$.

Let $s \in J$. There is an element $e \in Id(A)\backslash J$ satisfying $s \leq e$. It shows $s^{\sim_e -_e} = s \in J$. Hence, we have $s^{\sim_e} \in F_J = F$. Since $s \leq s^{\sim_e -_e}$, we get $s \in I_F$. This proves that $J \subseteq I_F$ and so $J = I_F$. Therefore, I_F is maximal. ■

Proposition 6.6. Set A be a pseudo-Ehoop with the pDN condition. A maximal ideal I is prime.

Proof: Assume that I is maximal. There is $s \in A$ satisfying $s \notin I$. By Corollary 6.3, a prime ideal P exists that satisfies $I \subseteq P$ and $s \notin P$. It follows $I = P$, confirming that I is prime. ■

Lemma 6.7. Set A be a pseudo-Ehoop with the pDN condition. Any elements $s, t \in A$, we obtain $\langle s \wedge t\rangle = \langle s\rangle \cap \langle t\rangle$.

Proof: Suppose $w \in \langle s \wedge t\rangle$. There are two elements $e \in Id(A)$ and $n \in \mathbb{N}\backslash\{0\}$ that satisfy $s \wedge t \leq e$ and $w \leq n_e(s \wedge t)$. Set $c \in Id(A)$ with $s, t, e \leq c$. By Proposition 4.5 and Proposition 3.8, we obtain $n_e(s \wedge t) \leq n_e s \leq n_c s$ and $n_e(s \wedge t) \leq n_e t \leq n_c t$. Thus, $w \leq n_c s$ and $w \leq n_c t$, which imply $w \in \langle s\rangle \cap \langle t\rangle$.

Conversely, there are $e, b \in Id(A)$ and $m, n \in \mathbb{N}\backslash\{0\}$ that satisfy $s \leq e$, $t \leq b$, $w \leq m_e s$ and $w \leq n_b t$. For any $c \in Id(A)$ with $e, b \leq c$, we obtain $w \leq m_e s \wedge n_b t \leq m_c s \wedge n_c t \leq (mn)_c(s \wedge t)$ by Proposition 4.6. Therefore, $w \in \langle s \wedge t\rangle$. ■

Theorem 6.8. Let P be an ideal of a pseudo-Ehoop A with the pDN condition. Then P is prime if and only if for any ideals I, J, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Proof: To suppose $I, J \nsubseteq P$. We can find elements $s \in I\backslash P$ and $t \in J\backslash P$. Consequently, $s \wedge t \in I \cap J \subseteq P$. We get $s \in P$ or $t \in P$, which is contradictory to our initial assumptions. Thus, it must be that $I \subseteq P$ or $J \subseteq P$.

Conversely, if $s \wedge t \in P$. We obtain $\langle s \wedge t\rangle \subseteq P$. It follows $\langle s\rangle \cap \langle t\rangle \subseteq P$ by Lemma 6.7. Therefore $\langle s\rangle \subseteq P$ or $\langle t\rangle \subseteq P$. This means $s \in P$ or $t \in P$. ■

VII. Implicative filters and positive implicative filters

In this section, we study implicative filters and positive implicative filters of pseudo-Ehoops and the relation between them. It is proved that every positive implicative and normal filter is an implicative filter in a pseudo-Ehoop.

Definition 7.1. Set A be a pseudo-Ehoop. $\emptyset \neq F \subseteq A$ is an implicative filter if for any $s, t, w \in A$ and $e \in Id(A)$ with $s, t, w \leq e$,
(IF1) there is $b \in Id(A) \cap F$ with $s \leq b$;
(IF2) $s \to_e (t \to_e w) \in F$ and $s \rightsquigarrow_e t \in F$ imply that $s \to_e w \in F$;
(IF3) if $s \rightsquigarrow_e (t \rightsquigarrow_e w) \in F$ and $s \to_e t \in F$, then it follows that $s \rightsquigarrow_e w \in F$.

The set of all implicative filters of A is defined by $\mathcal{IF}(A)$.

Proposition 7.2. Any implicative filter of a pseudo-Ehoop A is a filter.

Proof: To demonstrate this, let $s, t \in A$ and $e \in Id(A)$ that satisfy $s, t \leq e$ and $s, s \rightsquigarrow_e t \in F$. Thus, we obtain $e \rightsquigarrow_e (s \rightsquigarrow_e t) = s \rightsquigarrow_e t \in F$ and $e \to_e s = s \in F$. It follows $t = e \rightsquigarrow_e t \in F$. By Proposition 5.3, $F \in \mathcal{F}(A)$. ■

Proposition 7.3. Set A be a pseudo-Ehoop and $F \in \mathcal{IF}(A)$. Then for any $s, t \in A$, there is $e \in Id(A)$ with $s, t \leq e$. We obtain
(1) if $s \rightsquigarrow_e (s \rightsquigarrow_e t) \in F$, then $s \rightsquigarrow_e t \in F$;
(2) if $s \to_e (s \to_e t) \in F$, then $s \to_e t \in F$.

Proof: (1) Let $s, t \in A$. There is $e \in Id(A) \cap F$ with $s, t \leq e$. It follows $s \rightsquigarrow_e s = e \in F$. According to Definition 7.1, we have $s \to_e t \in F$.

The proof of (2) is similar to (1). ■

Proposition 7.4. Let A be a pseudo-Ehoop, $F \in \mathcal{NF}(A)$ and $s, t \in A$. For each $e \in Id(A)$ with $s, t \leq e$, if $s \to_e (s \to_e t) \in F$ implies $s \to_e t \in F$ and $s \rightsquigarrow_e (s \rightsquigarrow_e t) \in F$ implies $s \rightsquigarrow_e t \in F$, then $F \in \mathcal{IF}(A)$.

Proof: Set $s, t, w \in A$ and $e \in Id(A)$ that satisfy $s, t, w \leq e$, $s \rightsquigarrow_e (t \rightsquigarrow_e w) \in F$ and $s \to_e t \in F$. As F is normal, we can conclude that $s \to_e (t \rightsquigarrow_e w) \in F$. By Proposition 2.2 (9), this implies $t \rightsquigarrow_e (s \to_e w) \in F$, which further leads to $t \to_e (s \to_e w) \in F$. We have $(t \to_e (s \to_e w)) \odot (s \to_e t) \in F$. According to Proposition 2.2 (7), it shows $(t \to_e (s \to_e w)) \odot (s \to_e t) \leq s \to_e (s \to_e w)$, leading to $s \to_e (s \to_e w) \in F$. Therefore, we obtain $s \to_e w \in F$ and therefore $s \rightsquigarrow_e w \in F$. Similarly, if $s \to_e (t \to_e w) \in F$ and $s \rightsquigarrow_e t \in F$, we deduce $s \to_e w \in F$. ■

Proposition 7.5. Set A be a pseudo-Ehoop and $F \in \mathcal{NF}(A)$. Then $F \in \mathcal{IF}(A) \Longleftrightarrow$ For each $s \in A$, there is an element $e \in Id(A)$ satisfying $s \leq e$ and $s \to_e s^2 \in F$ ($s \rightsquigarrow_e s^2 \in F$).

Proof: ($\Longrightarrow$) Suppose $s \in A$. There is $e \in Id(A) \cap F$ with $s \leq e$. From $s \to_e (s \to_e s \odot s) = (s \odot s) \to_e (s \odot s) = e \in F$ and $s \rightsquigarrow_e s = e \in F$, we deduce that $s \to_e s \odot s \in F$.

($\Longleftarrow$) Let $s, t, w \in A$ and $e \in Id(A)$ that satisfy $s, t, w \leq e$. If $s \to_e (t \to_e w) \in F$ and $s \rightsquigarrow_e t \in F$. We obtain

$s \rightsquigarrow_e (t \to_e w) \in F$ as well as $s \to_e t \in F$. Consequently, we get $(s \rightsquigarrow_e (t \to_e w)) \odot (s \to_e t) \in F$. Using the relation $s \odot (s \rightsquigarrow_e (t \to_e w)) \odot (s \to_e t) \odot s = (s \wedge (t \to_e w)) \odot (s \wedge t) \leq (t \to_e w) \odot t \leq w$, we conclude that $(s \rightsquigarrow_e (t \to_e w)) \odot (s \to_e t) \leq s \rightsquigarrow_e (s \to_e w)$ and so $s \rightsquigarrow_e (s \to_e w) \in F$. Thus $s^2 \to_e w = s \to_e (s \to_e w) \in F$. By (2), there exists $b \in Id(A)$ satisfying $s \leq b$ and $s \to_b s^2 \in F$. Choose $c \in Id(A)$ with $e, b \leq c$. Then $s \to_c s^2 \in F$. By Proposition 3.10, we obtain $(s^2 \to_e w) \odot (s \to_e s^2) = (s^2 \to_e w) \odot (s \to_c s^2) \in F$. It is derived from Proposition 2.2 (7) that $(s^2 \to_e w) \odot (s \to_e s^2) \leq s \to_e w$. Then $s \to_e w \in F$. Similarly, (IF3) holds. ■

Proposition 7.6. Set A be a pseudo-Ehoop and $F \in \mathcal{IF}(A) \cap \mathcal{NF}(A)$. For every $u \in A$, the set $A^u = \{s \in A | \exists e \in Id(A)$ satisfying $u, s \leq e$ and $u \to_e s \in F\}$ is a filter.

Proof: For each $s \in A$, there exists an element $e \in Id(A) \cap F$ satisfying $s \leq e$. Let $b \in Id(A)$ be chosen so that $u, e \leq b$. According to Proposition 7.2, F is a filter. Then $u \to_b b = b \in F$ and so $b \in A^u$.

Let $s, t \in A$ and $e \in Id(A)$ satisfying $s, t \leq e$. Assume $s \in A^u$ and $s \to_e t \in A^u$. There exists $b \in Id(A)$ for which $u, s \leq b$ and $u \to_b s \in F$. Furthermore, there exists $c \in Id(A)$ that satisfies $u, s \to_e t \leq c$ and $u \to_c (s \to_e t) \in F$. Take $d \in Id(A)$ with $e, b, c \leq d$. From $u \to_b s \leq u \to_d s$ and $u \to_c (s \to_e t) \leq u \to_d (s \to_e t) \leq u \to_d (s \to_d t)$, we have $u \to_d s \in F$ and $u \to_d (s \to_d t) \in F$. As F is normal, it follows $u \rightsquigarrow_d s \in F$ and so $u \to_d t \in F$. Hence we deduce $t \in A^u$. By Proposition 5.2, A^u is a filter. ■

Definition 7.7. Set A be a pseudo-Ehoop. $\emptyset \neq F \subseteq A$ is a positive implicative filter if $s, t, w \in A$ and $e \in Id(A)$ satisfying $s, t, w \leq e$,
(1) there exists $b \in Id(A) \cap F$ with $s \leq b$;
(2) $s \to_e ((t \to_e w) \rightsquigarrow_e t) \in F$ and $s \in F$ imply $t \in F$;
(3) $s \rightsquigarrow_e ((t \rightsquigarrow_e w) \to_e t) \in F$ and $s \in F$ imply $t \in F$.

We denote $\mathcal{PIF}(A)$ by the set of all positive implicative filters of A. By $s \to_e ((t \to_e e) \to_e t) = s \to_e t$ and Proposition 5.2, we obtain $F \in \mathcal{PIF}(A) \Longrightarrow F \in \mathcal{F}(A)$.

Proposition 7.8. Set A a pseudo-Ehoop and $F \in \mathcal{F}(A)$. Then $F \in \mathcal{PIF}(A) \Longleftrightarrow$ for any $s, t \in A$ and $e \in Id(A)$ with $s, t \leq e$, $(s \to_e t) \rightsquigarrow_e s \in F$ implies $s \in F$ and $(s \rightsquigarrow_e t) \to_e s \in F$ implies $s \in F$.

Proof: ($\Longrightarrow$) Set $s, t \in A$ and $e \in Id(A)$ with $s, t \leq e$. If $(s \to_e t) \rightsquigarrow_e s \in F$, then $e \to_e ((s \to_e t) \rightsquigarrow_e s) = (s \to_e t) \rightsquigarrow_e s \in F$. As $(s \to_e t) \rightsquigarrow_e s \leq e$, It means $e \in F$. Hence, $s \in F$. In a similar way, if $(s \rightsquigarrow_e t) \to_e s \in F$, then $s \in F$.

($\Longleftarrow$) Consider $s, t, w \in A$ and $e \in Id(A)$ satisfying $s, t, w \leq e$. Suppose that $s \in F$ and $s \to_e ((t \to_e w) \rightsquigarrow_e t) \in F$. By Proposition 5.2, we find $(t \to_e w) \rightsquigarrow_e t \in F$. According to the assumption, we have $t \in F$. Similarly, if $s \in F$ and $s \rightsquigarrow_e ((t \rightsquigarrow_e w) \to_e t) \in F$, we obtain $t \in F$. ■

Proposition 7.9. Set A be a pseudo-Ehoop and $F \in \mathcal{NF}(A)$. Then $F \in \mathcal{PIF}(A) \Longrightarrow F \in \mathcal{IF}(A)$.

Proof: Set $s \in A$. Since $F \in \mathcal{NF}(A)$, there is $e \in Id(A) \cap F$ satisfying $s \leq e$. Then

$$\begin{aligned}
& e \to_e (((s \to_e s^2) \to_e s^2) \rightsquigarrow_e (s \to_e s^2)) \\
=& ((s \to_e s^2) \to_e s^2) \rightsquigarrow_e (s \to_e s^2) \\
=& s \to_e (((s \to_e s^2) \to_e s^2) \rightsquigarrow_e s^2) \text{ (Proposition 2.2 (9))} \\
\geq& s \to_e (s \to_e s^2) \text{ (Proposition 2.2 (10))} \\
=& s^2 \to_e s^2 = e \in F.
\end{aligned}$$

Thus, $e \to_e (((s \to_e s^2) \to_e s^2) \rightsquigarrow_e (s \to_e s^2)) \in F$. By Definition 7.7, $s \to_e s^2 \in F$. Consequently, from Proposition 7.5, we determined that $F \in \mathcal{IF}(A)$. ■

Proposition 7.10. Set A be a pseudo-Ehoop and $F \in \mathcal{NF}(A)$. The next statements are equivalent:
(1) $F \in \mathcal{MF}(A) \cap \mathcal{PIF}(A)$;
(2) $F \in \mathcal{MF}(A) \cap \mathcal{IF}(A)$;
(3) for any $s, t \in A \backslash F$, there is an element $e \in Id(A)$ that satisfies $s, t \leq e$ and $t \to_e s \in F$.

Proof: (1) $\Longrightarrow$ (2) By Proposition 7.9, the proof is obvious.

(2) $\Longrightarrow$ (3) Suppose $s, t \notin F$. By Proposition 7.6, $A^t = \{w \in A | \exists b \in Id(A)$ satisfying $t, w \leq b$ and $t \to_b w \in F\}$ is a filter. Take $u \in F$. There is $c \in Id(A)$ satisfying $t, u \leq c$. From $u \leq t \to_c u$, we obtain $t \to_c u \in F$. It follows $u \in A^t$. Thus, $F \subseteq A^t$. Since $F \in \mathcal{F}(A)$, there is $d \in Id(A) \cap F$ that satisfies $t \leq d$. We obtain $t \to_d t = d \in F$ and so $t \in A^t$. Then $F \cup \{t\} \subseteq A^t$. By the maximality of F, $A^t = A$. Hence, $s \in A^t$. There is an element $e \in Id(A)$ that satisfies $t, s \leq e$ and $t \to_e s \in F$.

(3) $\Longrightarrow$ (1) Assume that $F \notin \mathcal{PIF}(A)$. According to Proposition 7.8, there are $s, t \in A$ and $e \in Id(A)$ that satisfy $s, t \leq e$. Also, $(s \to_e t) \rightsquigarrow_e s \in F$ but $s \notin F$ or $(s \rightsquigarrow_e t) \to_e s \in F$ but $s \notin F$. Without loss of generality, we consider the first case. If $t \in F$, from $t \leq s \to_e t$, we derive $s \to_e t \in F$. By Proposition 5.3, we have $s \in F$, which leads to a contradiction. If $t \notin F$, from (3), there is $b \in Id(A)$ satisfying $s, t \leq b$ and $s \to_e t \in F$. Choose $c \in Id(A)$ with $e, b \leq c$. We obtain $s \to_c t \in F$ and

$$\begin{aligned}
(s \to_e t) \rightsquigarrow_e s &\leq ((s \to_c t) \wedge e) \rightsquigarrow_c s \\
&= (e \odot (s \to_c t)) \rightsquigarrow_c s \\
&= (s \to_c t) \rightsquigarrow_c (e \rightsquigarrow_c s).
\end{aligned}$$

Thus $(s \to_c t) \rightsquigarrow_c (e \rightsquigarrow_c s) \in F$, it follows $e \rightsquigarrow_c s \in F$. Since $e \in F$, we deduce $s \in F$, resulting once more in a contradiction.

Now, we prove that F is maximal. Suppose $u \notin F$. From the proof of (2) $\Longrightarrow$ (3), we have $F \cup \{u\} \subseteq A^u$. Let T be any filter of A and $F \cup \{u\} \subseteq T$. If $s \in A^u$, there is an element $e \in Id(A)$ that satisfies $u, s \leq e$ and $u \to_e s \in F \subseteq T$. Since $u \in T$, then $s \in T$. Thus we conclude that $A^u \subseteq T$ and $A^u = \langle F \cup \{u\}]$. Consider any element $v \in A$. If $v \in F$, we get $v \in A^u$. If $v \notin F$, there is $b \in Id(A)$ satisfying $u, v \leq b$ and $u \to_b v \in F$ by (3), which means $v \in A^u$. Consequently, $A = A^u$. ■

VIII. Internal States

In this section, we define internal states of pseudo-Ehoops and further study state filters. Furthermore, it is proved that PSF$[A]$ constitutes a topological space.

Definition 8.1. An internal state on a pseudo-Ehoop A with the least element 0 is a mapping $\tau : A \to A$ satisfying: any elements $s, t \in A$,
(S1) $\tau(0) = 0$;
(S2) for any $e \in Id(A)$ such that $s, t \le e$, $\tau(s \to_e t) = \tau(s) \to_{\tau(e)} \tau(s \wedge t)$ and $\tau(s \rightsquigarrow_e t) = \tau(s) \rightsquigarrow_{\tau(e)} \tau(s \wedge t)$;
(S3) for any $e \in Id(A)$ such that $s, t \le e$, $\tau(s \odot t) = \tau(s) \odot \tau(s \rightsquigarrow_e (s \odot t)) = \tau(t \to_e (s \odot t)) \odot \tau(t)$;
(S4) $\tau(\tau(s) \odot \tau(t)) = \tau(s) \odot \tau(t)$;
(S5) $\tau(\tau(s) \wedge \tau(t)) = \tau(s) \wedge \tau(t)$.

Let A be a pseudo-Ehoop with the least element 0 and τ be an internal state on A. The pair (A, τ) is called a state pseudo-Ehoop.

Example 8.2. Let A be a pseudo-Ehoop with the least element 0. From Proposition 2.2 (11), it is easy to check that the identity $1_A : A \to A$ is an internal state on A.

Example 8.3. Suppose that A and B are two pseudo-Ehoops with the least element 0. From Example 3.6, $A \times B$ is a pseudo-Ehoop with the least element 0. For all $(s_1, s_2) \in A\times B$, define the function $\tau : A\times B \to A\times B, (s_1, s_2) \mapsto (s_1, 0)$. By Proposition 2.2 (8) and (11), we have that τ is an internal state on $A \times B$.

Proposition 8.4. Let (A, τ) be a state pseudo-Ehoop. Then for all $s, t \in A$,
(1) if $e \in Id(A)$, $\tau(e) \in Id(A)$;
(2) $s \le t$ implies $\tau(s) \le \tau(t)$;
(3) any element $e \in Id(A)$ with $s \le e$, $\tau(s^{-_e}) = (\tau(s))^{-_{\tau(e)}}$ and $\tau(s^{\sim_e}) = (\tau(s))^{\sim_{\tau(e)}}$;
(4) $\tau(s \odot t) \ge \tau(s) \odot \tau(t)$;
(5) for all $e \in Id(A)$ such that $s, t \le e$, $\tau(s \to_e t) \le \tau(s) \to_{\tau(e)} \tau(t)$ and $\tau(s \rightsquigarrow_e t) \le \tau(s) \rightsquigarrow_{\tau(e)} \tau(t)$. If s and t are comparable (i.e. $s \le t$ or $t \le s$), $\tau(s \to_e t) = \tau(s) \to_{\tau(e)} \tau(t)$ and $\tau(s \rightsquigarrow_e t) = \tau(s) \rightsquigarrow_{\tau(e)} \tau(t)$;
(6) $\tau^2(s) = \tau(s)$;
(7) $\tau(A) = \{s \in A|\tau(s) = s\}$.

Proof: (1) Let $e \in Id(A)$. From (S3), it is clear.

(2) Suppose $s \le t$. For any $e \in Id(A)$ with $s, t \le e$, we get $s = s\wedge t = (t \to_e s)\odot t$. Applying (S3), $\tau(s) = \tau((t \to_e s) \odot t) = \tau(t \to_e ((t \to_e s) \odot t)) \odot \tau(t) \le \tau(t)$.

(3) For all $e \in Id(A)$ satisfying $s \le e$, by (S1) and (S2), $\tau(s^{-_e}) = \tau(s \to_e 0) = \tau(s) \to_{\tau(e)} \tau(s \wedge 0) = \tau(s) \to_{\tau(e)} \tau(0) = \tau(s) \to_{\tau(e)} 0 = (\tau(s))^{-_{\tau(e)}}$. Similarly, $\tau(s^{\sim_e}) = (\tau(s))^{\sim_{\tau(e)}}$.

(4) Any element $e \in Id(A)$ with $s, t \le e$. We obtain $t \le s \rightsquigarrow_e (s \odot t)$. It is derived from (2) that $\tau(t) \le \tau(s \rightsquigarrow_e (s \odot t))$. By (S3), $\tau(s) \odot \tau(t) \le \tau(s) \odot \tau(s \rightsquigarrow_e (s \odot t)) = \tau(s \odot t)$.

(5) Let $e \in Id(A)$ satisfying $s, t \le e$. As $\tau(s \wedge t) \le \tau(t)$, we obtain $\tau(s \to_e t) = \tau(s) \to_{\tau(e)} \tau(s \wedge t) \le \tau(s) \to_{\tau(e)} \tau(t)$. If $s \le t$, we have $\tau(s) \le \tau(t)$. That is $\tau(s) \to_{\tau(e)} \tau(t) = \tau(e)$. On the other hand, we get $\tau(s \to_e t) = \tau(s) \to_{\tau(e)} \tau(s\wedge t) = \tau(s) \to_{\tau(e)} \tau(s) = \tau(e)$. Thus, $\tau(s \to_e t) = \tau(s) \to_{\tau(e)} \tau(t)$. If $t \le s$, then $\tau(s \to_e t) = \tau(s) \to_{\tau(e)} \tau(s \wedge t) = \tau(s) \to_{\tau(e)} \tau(t)$.

(6) Set $e \in Id(A)$ with $s \le e$. It follows $\tau^2(s) = \tau(\tau(s)) = \tau(\tau(s) \wedge \tau(e)) = \tau(s) \wedge \tau(e) = \tau(s)$ by (S5).

(7) Suppose $t \in \tau(A)$. There is $s \in A$ satisfying $\tau(s) = t$. It means that $\tau(t) = \tau(\tau(s)) = \tau(s) = t$ and so $t \in \{s \in A|\tau(s) = s\}$. Therefore, $\tau(A) \subseteq \{s \in A|\tau(s) = s\}$. The other direction is obvious. ■

Proposition 8.5. Set (A, τ) be a state pseudo-Ehoop. $\tau(A)$ is a subalgebra of A.

Proof: Assuming $\tau(s), \tau(t) \in \tau(A)$. By (S4) and (S5), $\tau(A)$ is closed under the operations $\wedge$ and $\odot$. Let $e \in Id(A)$ with $s, t \le e$. We deduce that $\tau(s), \tau(t) \le \tau(e) \in Id(\tau(A))$ and $\tau(e) \in Id(A) \cap \tau(A)$. By Proposition 3.12 in [10], $\tau(A_e) = \{\tau(s) \in \tau(A)|s \le e\}$ is a subalgebra of A_e. Therefore, $\tau(A)$ is a subalgebra of A. ■

Definition 8.6. Suppose that (A, τ) is a state pseudo-Ehoop. A subset $\emptyset \ne F \subseteq A$ is a state filter of (A, τ) if
(1) $F \in \mathcal{F}(A)$;
(2) $s \in F \Longrightarrow \tau(s) \in F$ for each $s \in A$.

Let $\emptyset \ne S \subseteq A$. It is easy to see that the intersection of all state filters of (A, τ) is a state filter. We denote the smallest state filter of (A, τ) containing S by $\lfloor S\rceil_\tau$, which is the state filter generated by S. If $S = \{s\}$, we use $\lfloor s\rceil_\tau$ instead of $\lfloor\{s\}\rceil_\tau$.

Theorem 8.7. Set $S \ne \emptyset$ be a subset of a state pseudo-Ehoop (A, τ) and F a normal state filter of (A, τ) and $s \notin F$.
(1) $\lfloor S\rceil_\tau = \{u \in A|u \ge (s_1 \odot \tau(s_1)) \odot \cdots \odot (s_n \odot \tau(s_n)), s_1, \cdots, s_n \in S, n \ge 1\}$;
(2) $\lfloor F\cup\{s\}\rceil_\tau = \{u \in A|u \ge f\odot(s\odot\tau(s))^n, f \in F, n \ge 1\}$.

Proof: (1) Set $G = \{u \in A|u \ge (s_1 \odot \tau(s_1)) \odot \cdots \odot (s_n \odot \tau(s_n)), s_1, \cdots, s_n \in S, n \ge 1\}$. Assume $s \in S$. It follows from $u \ge u \odot \tau(u)$ that $u \in G$ and so $S \subseteq G$. For all $u \in A$ and $s_1, \cdots, s_n \in S$, there is $e \in Id(A)$ that satisfies $u, s_1, \cdots, s_n, \tau(s), \tau(s_1), \cdots, \tau(s_n) \le e$. Thus $e \ge (s_1 \odot \tau(s_1)) \odot \cdots \odot (s_n \odot \tau(s_n))$, which means $e \in G$. Let $b \in Id(A)$ satisfying $u, v \le b$ and $u, u \rightsquigarrow_b v \in S$. There exist $m, n \ge 1$ and $s_1, \cdots, s_m, t_1, \cdots, t_n \in S$ such that $u \ge (s_1 \odot \tau(s_1)) \odot \cdots \odot (s_m \odot \tau(s_m))$ and $u \rightsquigarrow_b v \ge (t_1\odot\tau(t_1))\odot\cdots\odot(t_n\odot\tau(t_n))$. Then $(s_1\odot\tau(s_1))\odot\cdots\odot(s_m\odot \tau(s_m))\odot(t_1\odot\tau(t_1))\odot\cdots\odot(t_n\odot\tau(t_n)) \le u\odot(u \rightsquigarrow_b v) \le t$. So $v \in S$. By Proposition 5.3, S is a filter of A containing S.

Let $u \in G$. There exist $m \ge 1$ and $s_1, \cdots, s_m \in S$ such that $u \ge (s_1 \odot \tau(s_1)) \odot \cdots \odot (s_m \odot \tau(s_m))$. For all $i = 1, \cdots, m$, we have $\tau(s_i), s_i \ge s_i \odot \tau(s_i)$ and so $\tau(s_i), s_i \in G$. Since G is a filter, $\tau(s_i) \odot s_i \in G$. By Proposition 8.4 (4),

$$\begin{aligned}\tau(u) &\ge \tau((s_1 \odot \tau(s_1)) \odot \cdots \odot (s_m \odot \tau(s_m)))\\ &\ge \tau((s_1 \odot \tau(s_1))) \odot \cdots \odot \tau((s_m \odot \tau(s_m)))\\ &\ge (\tau(s_1) \odot s_1) \odot \cdots \odot (\tau(s_m \odot s_m).\end{aligned}$$

Then $\tau(u) \in G$. This proves that G is a state filter of (A, τ) containing S. If T is a state filter of (A, τ) containing S. For all $u \in G$, there exist $m \ge 1$ and $s_1, \cdots, s_m \in S$ such that $u \ge (s_1 \odot \tau(s_1)) \odot \cdots \odot (s_m \odot \tau(s_m))$. We get $\tau(s_1), \cdots, \tau(s_m) \in T$. Hence $(s_1 \odot \tau(s_1)) \odot \cdots \odot (s_m \odot \tau(s_m)) \in T$. This implies $u \in T$ and so $G \subseteq T$. Therefore, $\lfloor S\rceil_\tau = \{u \in A|u \ge (s_1 \odot \tau(s_1)) \odot \cdots \odot (s_n \odot \tau(s_n)), s_1, \cdots, s_n \in S, n \ge 1\}$.

(2) By the proof of (1) and Proposition 5.9 (1), the proof is straightforward. ■

Remark 8.8. By Theorem 8.7, for all $s \in A$, $\lfloor s\rceil_\tau = \{v \in$

$A | v \geq (s \odot \tau(s))^n, n \geq 1\}$

Proposition 8.9. Set (A, τ) be a state pseudo-Ehoop and $s, t \in A$.
(1) $s \leq t \Longrightarrow \lfloor t \rceil_\tau \subseteq \lfloor s \rceil_\tau$;
(2) $\lfloor \tau(s) \rceil_\tau \subseteq \lfloor s \rceil_\tau = \lfloor s \odot \tau(s) \rceil_\tau$.

Proof: (1) Let $s \leq t$. We have $\tau(s) \leq \tau(t)$. If $u \in \lfloor t \rceil_\tau$, there is $m \geq 1$ with $u \geq (t \odot \tau(t))^m$. Hence $u \geq (t \odot \tau(t))^m \geq (s \odot \tau(s))^m$. It means $u \in \lfloor s \rceil_\tau$.

(2) For any $u \in \lfloor \tau(s) \rceil_\tau$, there exists $m \geq 1$ with $u \geq (\tau(s) \odot \tau^2(s))^m = (\tau(s) \odot \tau(s))^m$. Since $\tau(s) \geq s \odot \tau(s)$, $\tau(s) \in \lfloor s \rceil_\tau$. Then $(\tau(s) \odot \tau(s))^m \in \lfloor s \rceil_\tau$ and so $u \in \lfloor s \rceil_\tau$.

Suppose $u \in \lfloor s \odot \tau(s) \rceil_\tau$. There exists $m \geq 1$ satisfying $u \geq ((s \odot \tau(s)) \odot \tau(s \odot \tau(s)))^m$. From Proposition 8.4 (4), we derive that $u \geq (s \odot \tau(s))^m$, it means $u \in \lfloor s \rceil_\tau$. By (1), the other direction is clear. ■

Definition 8.10. Set (A, τ) be a state pseudo-Ehoop. A proper state filter F is a prime state filter if $F_1 \cap F_2 \subseteq F$ implies $F_1 \subseteq F$ or $F_2 \subseteq F$, where F_1, F_2 are arbitrary state filters.

Define the set of all prime state filters of (A, τ) by PSF$[A]$.

Definition 8.11. A proper state filter F of a state pseudo-Ehoop (A, τ) is said to be maximal if no proper state filters of (A, τ) can strictly contain it.

Lemma 8.12. Set (A, τ) be a state pseudo-Ehoop and F a normal state filter. The statements listed below are equivalent:
(1) F is maximal;
(2) for each $s \in A \backslash F$, $\lfloor F \cup \{s\} \rceil_\tau = A$;
(3) any element $s \in A \backslash F$, there are $e \in Id(A)$ and $n \in \mathbb{N} \backslash \{0\}$ that satisfy $s \leq e$ and $((\tau(s))^n)^{-_{\tau(e)}} \in F$.

Proof: (1) $\Longleftrightarrow$ (2) Similar to Proposition 5.5, the proof is straightforward.

(2) $\Longrightarrow$ (3) Set $s \in A \backslash F$. We have $0 \in A = \lfloor F \cup \{s\} \rceil_\tau$. There exist $f \in F$ and $m \in \mathbb{N}^*$ that satisfy $0 \geq f \odot (s \odot (\tau(s)))^m$. Thus $0 = \tau(0) \geq \tau(f) \odot \tau((s \odot (\tau(s)))^m) \geq \tau(f) \odot (\tau(s))^{2m}$. There exists $e \in Id(A)$ satisfying $s, f \leq e$. So $\tau(f) \leq ((\tau(s))^{2m})^{-_{\tau(e)}}$. From $\tau(f) \in F$, we obtain $((\tau(s))^{2n})^{-_{\tau(e)}} \in F$.

(3) $\Longrightarrow$ (2) Suppose $s \in A \backslash F$. Take $n \geq 1$ and $e \in Id(A)$ with $s \leq e$ and $((\tau(s))^n)^{-_{\tau(e)}} \in F$. Therefore, $((\tau(s))^n)^{-_{\tau(e)}} \odot (s \odot \tau(s))^n \leq ((\tau(s))^n)^{-_{\tau(e)}} \odot (\tau(s))^n = 0$. This together with $((\tau(s))^n)^{-_{\tau(e)}} \in F$ implies $0 \in \lfloor F \cup \{s\} \rceil_\tau$. Thus, $\lfloor F \cup \{s\} \rceil_\tau = A$. ■

Let (A, τ) be a state pseudo-Ehoop and $S \subseteq A$. We define $[S] = \{F \in \text{PSF}[A] | S \nsubseteq F\}$, which is a subset of PSF$[A]$. If $S = \{s\}$, let $[s] = [\{s\}] = \{F \in \text{PSF}[A] | s \notin F\}$.

Proposition 8.13. Suppose that (A, τ) is a state pseudo-Ehoop and $S, T \subseteq A$. Set $\{S_i\}_{i \in I}$ be family subsets of A. We have
(1) $S \subseteq T \Longrightarrow [S] \subseteq [T]$;
(2) $[0] = \text{PSF}[A]$, $[\emptyset] = \emptyset$;
(3) $[S] \cap [T] = [\lfloor S \rceil_\tau \cap \lfloor T \rceil_\tau]$;
(4) $\bigcup_{i \in I} [S_i] = [\bigcup_{i \in I} S_i]$;
(5) $[S] = [\lfloor S \rceil_\tau]$.

Proof: (1) Let $S \subseteq T$ and $F \in [S]$. We have $S \nsubseteq F$. It means $T \nsubseteq F$ and so $F \in [T]$.

(2) The proof is straightforward.

(3) Let $F \in [S] \cap [T]$. We get $S, T \nsubseteq F$. It follows $\lfloor S \rceil_\tau, \lfloor T \rceil_\tau \nsubseteq F$. Therefore $\lfloor S \rceil_\tau \cap \lfloor T \rceil_\tau \nsubseteq F$. This means $F \in [\lfloor S \rceil_\tau \cap \lfloor T \rceil_\tau]$. Hence, $[S] \cap [T] \subseteq [\lfloor S \rceil_\tau \cap \lfloor T \rceil_\tau]$. Conversely, if $F \in [\lfloor S \rceil_\tau \cap \lfloor T \rceil_\tau]$, we have $\lfloor S \rceil_\tau \cap \lfloor T \rceil_\tau \nsubseteq F$. Then $S \subseteq \lfloor S \rceil_\tau \nsubseteq F$ and $T \subseteq \lfloor T \rceil_\tau \nsubseteq F$. Thus, $F \in [S] \cap [T]$. This proves $[\lfloor S \rceil_\tau \cap \lfloor T \rceil_\tau] \subseteq [S] \cap [T]$.

(4) Any $i \in I$, we get $S_i \subseteq \bigcup_{i \in I} S_i$, which means $[S_i] \subseteq [\bigcup_{i \in I} S_i]$. Then $\bigcup_{i \in I} [S_i] \subseteq [\bigcup_{i \in I} S_i]$. Conversely, let $F \in [\bigcup_{i \in I} S_i]$. There is $j \in I$ with $S_j \nsubseteq F$, which means that $F \in [S_j]$ and $F \in \bigcup_{i \in I} [S_i]$. Therefore, $[\bigcup_{i \in I} S_i] \subseteq \bigcup_{i \in I} [S_i]$.

(5) From $S \subseteq \lfloor S \rceil_\tau$ and (1), we obtain $[S] \subseteq [\lfloor S \rceil_\tau]$. Suppose $F \in [\lfloor S \rceil_\tau]$, that is, $\lfloor S \rceil_\tau \nsubseteq F$. We deduce $S \nsubseteq F$. In fact, let $S \subseteq F$. Then $\lfloor S \rceil_\tau \subseteq F$, which is a contradiction. Hence, $F \in [S]$. ■

Proposition 8.14. Assume that (A, τ) is a state pseudo-Ehoop and $s, t \in A$. Then
(1) $[s] = [\lfloor s \rceil_\tau]$;
(2) $s \leq t \Longrightarrow [t] \subseteq [s]$;
(3) $[s] \cup [t] = [s \odot t]$.

Proof: By Proposition 8.13, the proofs of (1) and (2) are clear.

(3) From $s \odot t \leq s, t$, we have $[s], [t] \subseteq [s \odot t]$ by (2) and so $[s] \cup [t] \subseteq [s \odot t]$. Suppose $F \in [s \odot t]$. Then $s \odot t \notin F$. This follows $s \notin F$ or $t \notin F$. Therefore, $F \in [s]$ or $F \in [t]$. This proves that $F \in [s] \cup [t]$. ■

Theorem 8.15. Let (A, τ) be a state pseudo-Ehoop and $\mathcal{T} = \{[S] | S \subseteq A\}$ a subset of the power set of PSF$[A]$. Then $\mathcal{T}$ is a topology on PSF$[A]$ and $\{[s] | s \in A\}$ is a basis for $\mathcal{T}$.

Proof: Clearly, $\mathcal{T}$ is a topology on PSF$[A]$ by Proposition 8.13 (2), (3) and (4). For each $S \subseteq A$, we obtain $[S] = [\bigcup_{s \in S} \{s\}] = \bigcup_{s \in S} [s]$ by Proposition 8.13 (4), which implies that the open set of $\mathcal{T}$ is the union of some elements of $\{[s] | s \in A\}$. Therefore, $\{[s] | s \in A\}$ is a basis for $\mathcal{T}$. ■

IX. Conclusion

Pseudo-hoops are noncommutative generalizations of hoops. Ehoops are unbounded extensions of hoops. In this paper, we introduce pseudo-Ehoops, which are noncommutative and unbounded extensions of hoops. Some basic properties of pseudo-Ehoops are given. We mainly investigate ideals, congruences and filters on peudo-Ehoops and obtain some important conclusions. If A is a pseudo-Ehoop with the pDN condition, we establish a one-to-one correspondence between ideals and congruences on A. In addition, a prime ideal theorem of A is obtained. It is demonstrated that every proper pseudo-Ehoop A contains at least one maximal filter, and every maximal ideal is prime. Moreover, we construct a topology on PSF$[A]$.

In future work, the following topics can be studied. (1) Can we investigate the notions of state ideals and prime state ideals of pseudo-Ehoops? (2) We can study Bosbach states, Riečan states and state-morphisms on pseudo-Ehoops.

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