Markov Chain Monte Carlo Based Bayesian Estimation of Weibull-Burr XII Distribution for Censored Data

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Abstract- **This study examines the Bayesian estimation of Weibull-Burr XII distribution in the context of censored data. The analysis involves three algorithms namely Metropolis-Hastings (M-H), Slice sampling and Hamiltonian Monte Carlo (HMC) algorithm for estimating the parameters. A comparative evaluation is performed using both simulated and real datasets. Results indicate that HMC maintains stability across all five parameters of the distribution as evidenced by metrics such as Effective Sample Size (ESS), Integrated Autocorrelation Time (IAT), and Monte Carlo Standard Error (MCSE). Additionally, Monte Carlo simulations demonstrate that HMC outperforms both M-H and slice sampling methods, particularly exhibiting lower mean squared error (MSE) at lower levels of censorship for both Type-I and Type-II censoring techniques.**

Index terms: **Weibull Burr XII distribution, Right censoring, Bayesian estimation, M-H algorithm, Slice sampling, Hamiltonian Monte Carlo simulation.**

I. Introduction

The Weibull distribution is a widely utilized parametric survival model with extensive applications across various fields. To better capture real-world complexities, extended versions of classical distributions have been developed. Based on the Weibull distribution, several new families of distributions have emerged. Bourguignon et al. [3] introduced the Weibull-G family of distributions by employing the T-G family generating function. Additionally, developed distributions such as the Weibull-Weibull and Weibull-Burr XII distributions with their fundamental properties and parameters estimated using the maximum likelihood method with real datasets. Guerras et al. [7] analyzed the shapes of hazard functions, derived key properties and estimated the parameters of the five-parameter Weibull-Burr XII distribution using maximum likelihood estimation. Censored observations are common in survival analysis, often arise due to factors such as loss to follow-up, unexpected events affecting the observation and study duration limits. Parametric models commonly utilize the exponential, Weibull and gamma distributions to handle such censored data. Traditional methods such as Maximum Likelihood Estimation (MLE) are commonly used for estimating these parameters. However, with the growing complexity of real- world data more advanced techniques like Bayesian inference have gained popularity for parameter estimation. In Bayesian inference, sampling-based methods such as Markov Chain Monte Carlo (MCMC) algorithms are often employed to estimate the posterior distributions of parameters if the posterior distribution doesn't attain closed form. Among these, the Metropolis-Hastings (M-H) algorithm, Slice sampling and Hamiltonian Monte Carlo (HMC) have emerged as powerful tools for efficient sampling. Each of these algorithms has its own strengths and weaknesses, particularly in terms of convergence speed, computational complexity and accuracy. Sundaram et al. [20] estimated the parameters of the Weibull-G-Weibull distribution using the Metropolis-Hastings (M-H) algorithm, incorporating both SELF and LINEX loss functions under Type-I and Type-II censoring schemes. Neal [14] introduced the Slice sampling algorithm as an alternative to the M-H algorithm. Saraiva et al. [18] estimated the parameters of the right-censored Weibull distribution using independent M-H, random walk M-H and Slice sampling methods. The statistical application of Hamiltonian Monte Carlo (HMC) simulation method is provided by Neal [15] and Betancourt [2]. Thach TT et.al.[21] employed Hamiltonian Monte Carlo simulation to estimate the parameters of Improved new modified Weibull distribution.

This paper focuses on parameter estimation for the Weibull-Burr XII distribution under a Bayesian framework, utilizing MCMC techniques specifically, the Metropolis-Hastings algorithm, Slice sampling and Hamiltonian Monte Carlo simulation for both Type-I and Type-II censored data.

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II. Methodology

The probability density function (p.d.f) and cumulative distribution function (c.d.f) of family of Weibull-G is given by [3]

$$F(x,\alpha,\beta,\xi)=\int_0^{\frac{G(x;\xi)}{1-G(x;\xi)}}\alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}dt$$

$$=1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{\beta}\right\} \tag{1}$$

$$f(x;\alpha,\beta,\xi)=\alpha\beta g(x;\xi)\frac{G(x;\xi)^{\beta-1}}{\bar{G}(x;\xi)^{\beta+1}}\exp\left\{-\alpha\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{\beta}\right\} \tag{2}$$

Where $\bar{G}(x)=1-G(x)$.

The c.d.f and p.d.f of the Weibull-Burr XII distribution are derived by substituting the densities of the parent Burr XII distribution into equations (1) and (2). The p.d.f and c.d.f of the three-parameter Burr-XII distribution are provided by [16].

$$f(x;\theta,\gamma,\eta)=\gamma\eta\theta^{-\gamma}x^{\gamma-1}\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{-\eta-1};\theta,\gamma,\delta>0 \quad (3)$$

$$F(x;\theta,\gamma,\eta)=1-\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{-\eta} \quad (4)$$

If $X_1,X_2,\dots,X_n$ are the random sample of size n which follows Weibull-Burr XII distribution then its p.d.f is given by [7]

$$f(x;\alpha,\beta,\theta,\gamma,\eta)=\alpha\beta\gamma\eta\theta^{-\gamma}x^{\gamma-1}\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{\eta-1}$$
$$\times\left[\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta-1}\exp\left\{-\alpha\left[\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\}$$
$$x>0,\alpha,\beta,\theta,\gamma,\eta>0 \quad (5)$$

here α and β are the shape parameters of the generating family of Weibull-G and θ is the scale parameter, γ and η are the shape parameters of Burr XII distribution.

The corresponding c.d.f is

$$F(x;\alpha,\beta,\theta,\gamma,\eta)=$$
$$1-\exp\left\{-\alpha\left[\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\},x>0 \quad (6)$$

The survival and Hazard function of this distribution is given by

$$S(x;\alpha,\beta,\theta,\gamma,\eta)=\exp\left\{-\alpha\left[\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\}$$
$$,x>0 \quad (7)$$

$$H(x;\alpha,\beta,\theta,\gamma,\eta)=\alpha\beta\gamma\eta\theta^{-\gamma}x^{\gamma-1}\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{\eta-1}$$
$$\times\left[\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1} \quad (8)$$

A. Right Censoring

Right censoring occurs when an observation is included in a study with the last known survival time serving as the lower bound [11]. Type-I, Type-II and random censoring are common right censoring methods in parametric survival analysis. This paper discusses the Weibull-Burr XII distribution under Type-I and Type-II censoring.

1) Type-I censoring:

If $x_1,x_2,\dots,x_n$ are independently and identically distributed survival times with n sample observations then the survival time x_i for the ith observation is observed only if $x_i<T\,;i=1,2,\dots,n$ where T is the pre fixed time period. Let δ_i denote the indicator function given by [11]

$$\delta_i=\begin{cases}1\,;\ x_i\le T\\0\,;\ x_i>T\end{cases} \quad (9)$$

The likelihood function for Type-I censoring given by

$$L(\Theta;x)=\prod_{i=1}^{n}[f(x_i)]^{\delta_i}[S(x_i)]^{1-\delta_i} \quad (10)$$

On substituting equation (5) and (6) in equation (10) the likelihood function of Weibull Burr XII distribution for Type-I censoring is:

$$L(\alpha,\beta,\gamma,\theta;x)=\prod_{i=1}^{n}\left[\alpha\beta\gamma\eta\theta^{-\gamma}x_i^{\gamma-1}\left[1\right.\right.$$
$$\left.\left.+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta-1}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1}\right]^{\delta_i}$$
$$\times\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\}^{\delta_i}$$
$$\times\left[\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\}\right]^{1-\delta_i}$$
$$=\prod_{i=1}^{n}\left[\alpha\beta\gamma\eta\theta^{-\gamma}x_i^{\gamma-1}\right]^{\delta_i}$$
$$\times\left[\left[1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta-1}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1}\right]^{\delta_i}$$
$$\times\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\} \quad (11)$$

2) Type-II censoring:

For Type-II censoring scheme let $x_1\le x_2\le x_3\dots\le x_r\le\cdots\le x_{n-1}\le x_n$ be the ordered sample observations time, n be the total number of observations in the sample and $r<n$ be the pre-defined number such that the study ends after reaching the r events and remaining observations are considered as censored ,

The likelihood function for the Type-II censoring is [11]

$$L(\Theta;x)=\prod_{i=1}^{r}[f(x_i)][S(x_{(r)})]^{n-r} \quad (12)$$

Where x_r is the r th observed order failure observations time.

on substituting (5) and (6) in (12) the likelihood function for Weibull-Burr XII distribution is given by

$$L(\alpha,\beta,\gamma,\theta;x)=\prod_{i=1}^{r}\alpha\beta\gamma\eta\theta^{-\gamma}x_i^{\gamma-1}\left[1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta-1}$$
$$\times\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1}\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\}$$
$$\times\exp\left\{-\alpha\left[\left(1+\left(\frac{x_{(r)}}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\} \quad (13)$$

B. Maximum Likelihood Estimation (MLE)

The maximum likelihood estimation for the Weibull-Burr XII distribution is discussed in this section.

For Type-I censored Weibull-Burr XII distribution the log likelihood function is obtained by taking logarithm for the equation (11)

$$\log\big(L(\Theta;x)\big)=\sum_{i=1}^{n}\delta_i\log(\alpha\beta\theta\gamma\eta)-\gamma\sum_{i=1}^{n}\delta_i\log(\theta)$$
$$+(\gamma-1)\sum_{i=1}^{n}\delta_i\log(x_i)$$
$$+(\eta-1)\sum_{i=1}^{n}\delta_i\log\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)$$
$$+(\beta-1)\sum_{i=1}^{n}\left[\delta_i\log\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)-1\right]$$
$$-\alpha\sum_{i=1}^{n}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right] \quad (14)$$

Similarly, for Type II censoring the log likelihood function obtained from equation (13) is:

$$\log(L(\Theta;x) = r\log(\alpha\beta\gamma\eta) - r\gamma\log(\theta) + (\gamma-1)\sum_{i=1}^{r}\log(x_i) + (\eta-1)\sum_{i=1}^{r}\log\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right) + (\beta-1)\sum_{i=1}^{r}\log\left(\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right) - \alpha\sum_{i=1}^{r}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta} - \alpha\left[\left(1+\left(\frac{x_{(r)}}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta} \quad (15)$$

The parameters of Type-I and Type-II censored Weibull-Burr XII are estimated by solving equation (10) and (11) using Newton Raphson optimization technique.

C. Bayesian Estimation

In this section the parameters of Weibull-Burr XII distribution under Type-I and Type-II censoring are estimated using Bayesian approach. For both Type-I and Type-II censoring methods the parameters $\alpha, \beta, \gamma, \eta$ and θ are assumed to follow independent prior distributions as shown in TABLE I.

The joint prior distribution for the Weibull-Burr XII distribution is given by

$$g(\alpha,\beta,\gamma,\eta,\theta) \propto \frac{c_1^{b_1}c_2^{b_2}c_3^{b_3}c_4^{b_4}}{a_1\Gamma b_1\Gamma b_2\Gamma b_3\Gamma b_4}e^{-(\gamma c_2+\eta c_3+\theta c_4)}\beta^{b_1-1}\gamma^{b_2-1}\eta^{b_3-1}\theta^{b_4-1} \quad (16)$$

TABLE. I
PRIOR DISTRIBUTIONS FOR WEIBULL-BURR XII DISTRIBUTION

PARAMETERS	PRIORS	P.D.F
α	$U(0,a_1)$	$\frac{1}{a_1}$
β	$Gamma(b_1,c_1)$	$\frac{c_1^{b_1}}{\Gamma b_1}e^{-\gamma c_1}\gamma^{b_1-1}$
γ	$Gamma(b_2,c_2)$	$\frac{c_2^{b_2}}{\Gamma b_2}e^{-\theta c_2}\theta^{b_2-1}$
η	$Gamma(b_3,c_3)$	$\frac{c_3^{b_3}}{\Gamma b_3}e^{-\theta c_3}\theta^{b_3-1}$
θ	$Gamma(b_4,c_4)$	$\frac{c_4^{b_4}}{\Gamma b_4}e^{-\theta c_4}\theta^{b_4-1}$

The posterior density is defined by [7]

$$\pi(\Theta|X) = \frac{L(\Theta)g(X|\Theta)}{\int L(\Theta)g(X|\Theta)d\Theta} \quad (17)$$

1) Estimation of Type-I censoring:

For Weibull-Burr XII distribution under Type-I censoring the posterior density are obtained by substituting (11) and (16) in the equation (17). Hence the posterior density is given in equation (18).

2) Estimation of Type-II censoring:

For Weibull-Burr XII distribution under Type-II censoring the posterior density obtained on substituting (13) and (16) in the equation (17). Hence the posterior density for Weibull-Burr XII distribution under Type-II censoring is given in equation (19)

For both Type-I and Type-II censoring the parameter estimates of posterior densities from equation (18) and (19) do not have explicit expression. Hence the MCMC techniques are adopted.

$$\pi(\alpha,\beta,\gamma,\eta,\theta|x) = \frac{\prod_{i=1}^{n}\left[\alpha\beta\gamma\eta\theta^{-\gamma}x_i^{\gamma-1}\left[1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta-1}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1}\right]^{\delta_i}\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\} \times \frac{c_1^{b_1}c_2^{b_2}c_3^{b_3}c_4^{b_4}}{a_1\Gamma b_1\Gamma b_2\Gamma b_3\Gamma b_4}e^{-(\gamma c_2+\eta c_3+\theta c_4)}\beta^{b_1-1}\gamma^{b_2-1}\eta^{b_3-1}\theta^{b_4-1}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\prod_{i=1}^{n}\left[\alpha\beta\gamma\eta\theta^{-\gamma}x_i^{\gamma-1}\left[1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta-1}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1}\right]^{\delta_i}\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\} \times \frac{c_1^{b_1}c_2^{b_2}c_3^{b_3}c_4^{b_4}}{a_1\Gamma b_1\Gamma b_2\Gamma b_3\Gamma b_4}e^{-(\gamma c_2+\eta c_3+\theta c_4)}\beta^{b_1-1}\gamma^{b_2-1}\eta^{b_3-1}\theta^{b_4-1}d\alpha d\beta d\gamma d\eta d\theta} \quad (18)$$

$$\pi(\alpha,\beta,\gamma,\eta,\theta|x) = \frac{\prod_{i=1}^{r}\alpha\beta\gamma\eta\theta^{-\gamma}x_i^{\gamma-1}\left[1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta-1}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1}\times\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\} \times\exp\left\{-\alpha\left[\left(1+\left(\frac{x_{(r)}}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\}\times\frac{c_1^{b_1}c_2^{b_2}c_3^{b_3}c_4^{b_4}}{a_1\Gamma b_1\Gamma b_2\Gamma b_3\Gamma b_4}e^{-(\gamma c_2+\eta c_3+\theta c_4)}\beta^{b_1-1}\gamma^{b_2-1}\eta^{b_3-1}\theta^{b_4-1}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\prod_{i=1}^{r}\alpha\beta\gamma\eta\theta^{-\gamma}x_i^{\gamma-1}\left[1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta-1}\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right]^{\eta}-1\right]^{\beta-1}\times\exp\left\{-\alpha\left[\left(1+\left(\frac{x_i}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\} \times\exp\left\{-\alpha\left[\left(1+\left(\frac{x_{(r)}}{\theta}\right)^{\gamma}\right)^{\eta}-1\right]^{\beta}\right\}\times\frac{c_1^{b_1}c_2^{b_2}c_3^{b_3}c_4^{b_4}}{a_1\Gamma b_1\Gamma b_2\Gamma b_3\Gamma b_4}e^{-(\gamma c_2+\eta c_3+\theta c_4)}\beta^{b_1-1}\gamma^{b_2-1}\eta^{b_3-1}\theta^{b_4-1}d\alpha d\beta d\gamma d\eta d\theta} \quad (19)$$

3) Metropolis-Hastings (M-H) Algorithm:

In 1970, H.K Hastings [8] extended the Metropolis algorithm to a generalized form. Initially, define proposal distribution with initial values for the unknown parameters $\alpha, \beta, \gamma, \eta$ and θ. The step of M-H algorithm to draw samples from the joint posterior densities are as follows [13]:

Step 1: Fix sampler specifications, iterations and burn-ins draw.

Step 2: Set the initial values for parameters $\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)}$, $\eta^{(0)}$ and $\theta^{(0)}$ through MLE and denote the values of the ith iteration as $\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}$and $\theta^{(i)}$,then the values of $(i+1)$th iteration $\alpha^{(i+1)}, \beta^{(i+1)}, \gamma^{(i+1)}$ $\eta^{(i+1)}$and $\theta^{(i+1)}$obtained by the following steps.

Step 3: Let ε be the random perturbation from normal distribution. Generate ε from the normal distribution with mean 0 and standard deviation σ_K(i.e. $\varepsilon \sim N(0, \sigma_K)$. The σ_K can be chosen independently for the parameters $\alpha, \beta, \gamma, \eta, \theta$.
Set $(\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*) = (\alpha^{(i)} + \varepsilon, \beta^{(i)} + \varepsilon,$ $\gamma^{(i)} + \varepsilon, \eta^{(i)} + \varepsilon, \theta^{(i)} + \varepsilon)$.

Step 4: Calculate

$$P_\Theta = \frac{L(\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*|x)\pi(\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*) \times g\left(\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}, \theta^{(i)}|(\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*)\right)}{L(\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}, \theta^{(i)}|x)\pi(\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}, \theta^{(i)}) \times g((\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*)|\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}, \theta^{(i)})}$$

and φ be the accepted probability of $(\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*)$ such that
$\varphi\left((\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*)\middle|\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}, \theta^{(i)}\right)$
$= \min(1, P_\Theta)$

Step 5: Generate $u \sim U(0,1)$,if

$$u \leq \varphi\left((\alpha^*, \beta^*, \gamma^*, \eta^*, \theta^*)\middle|\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}, \theta^{(i)}\right)$$

accept $(\alpha^*, \beta^*, \gamma^*, \theta^*)$
$= \left(\alpha^{(i+1)}, \beta^{(i+1)}, \gamma^{(i+1)}, \eta^{(i+1)}, \theta^{(i+1)}\right)$
otherwise keep

$$\left(\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \eta^{(i)}, \theta^{(i)}\right) = \left(\alpha^{(i+1)}, \beta^{(i+1)}, \gamma^{(i+1)}, \eta^{(i+1)}, \theta^{(i+1)}\right)$$

Step 6: Repeat the steps (3-6) until the fixed number of iterations reached.

Step 7: Remove the initial samples for fixed burn-in periods.

The estimated parameters $\hat{\alpha}_M, \hat{\beta}_M, \hat{\gamma}_M, \hat{\eta}_M$ and $\hat{\theta}_M$ are given by

$$\left.\begin{array}{c} \hat{\alpha}_M = \frac{1}{N-M}\sum_{i=M+1}^{N} \alpha^{(i)}, \hat{\beta}_M = \frac{1}{N-M}\sum_{i=M+1}^{N} \beta^{(i)}, \\ \hat{\gamma}_M = \frac{1}{N-M}\sum_{i=M+1}^{N} \gamma^{(i)}, \hat{\eta}_M = \frac{1}{N-M}\sum_{i=M+1}^{N} \eta^{(i)} \\ \hat{\theta}_M = \frac{1}{N-M}\sum_{i=M+1}^{N} \theta^{(i)} \end{array}\right\} \quad (20)$$

Where N is the fixed number of iterations and M is the number of burn-in periods.

4) Slice Sampling:

Slice sampling algorithm, proposed by Neal (2003) [14] is an alternative approach to the M-H algorithm. The key challenge in the M-H algorithm is determining the step size ϵ, as it involves an accept-reject proposal. If the step size is too small, the algorithm requires large number of iterations to attain the stationarity. On the other hand, if the ϵ is large most of the samples are rejected [1]. Slice sampling act as an alternative, functioning as a Gibbs sampler that simulates uniform random variables and automatically adjusts the step size w.

The Slice sampling for Weibull-Bur XII distribution is carried out using following steps [5]

Step 1: Draw random sample $u \sim U[0, \pi(\Theta^0|x)]$,Where $\Theta^0 = (\alpha^0, \beta^0, \gamma^0, \eta^0, \theta^0)$ is the initial guess of the parameters satisfies $P\left(\pi(\Theta^0|x)\right) > 0$.The Maximum likelihood estimators are considered as initial guess.

Step 2: Set the threshold $u^* \sim U[0,1]$ for each parameter.

Step 3: initialize the upper and lower bound for the initial slice $\Theta_{min} = \Theta^i - u^* w$ and $\Theta_{max} = \Theta^i + w$

Step 4: If $u < P(\Theta_{min})$ do $\Theta_{min}(i) = \Theta_{min}(i) - w(i)$ and $u > P(\Theta_{max})$ do $\Theta_{min}(i) = \Theta_{max}(i) + w(i)$

Step 5: Then draw a new sample uniformly from the slice $\Theta^*(i) = \Theta_{min} + u^*\left(\Theta_{max}(i) - \Theta_{min}(i)\right)$

Step 6: $u < P(\Theta^*)$ then $\Theta(i+1) = \Theta^*$ repeat the steps for N number iterations.

After removing the M initial burn-ins then the estimated parameters are given by

$$\left.\begin{array}{c} \hat{\alpha}_S = \frac{1}{N-M}\sum_{i=M+1}^{N} \alpha^{(i)}, \hat{\beta}_S = \frac{1}{N-M}\sum_{i=M+1}^{N} \beta^{(i)}, \\ \hat{\gamma}_S = \frac{1}{N-M}\sum_{i=M+1}^{N} \gamma^{(i)}, \hat{\eta}_S = \frac{1}{N-M}\sum_{i=M+1}^{N} \eta^{(i)} \\ \hat{\theta}_S = \frac{1}{N-M}\sum_{i=M+1}^{N} \theta^{(i)} \end{array}\right\} \quad (21)$$

5) Hamiltonian Monte Carlo:

Hamiltonian Monte Carlo (HMC) is an advanced sampling technique based on Hamiltonian dynamics, primarily used for parameter estimation in Bayesian statistics by Neal [15] and Betancourt [2]. This method simulates particle dynamics within a potential energy landscape, facilitating the exploration of complex parameter spaces. By leveraging gradient information, HMC improves sampling efficiency and reduces the correlation between samples relative to traditional Markov Chain Monte Carlo (MCMC) methods.

The HMC algorithm consists of several essential steps for effective sampling from a target distribution. It integrates Hamiltonian dynamics to generate proposals by introducing auxiliary momentum variables, leading to an auxiliary probability distribution that includes the target distribution as a marginal.

Define the Hamiltonian $H(\Theta, \phi)$ as the total energy:

$$H(\Theta, \phi) = U(\Theta) + K(\phi)$$

where $U(\Theta)$ is the potential energy and $K(\phi)$ is the kinetic energy. In this framework, the potential energy $U(\Theta)$ corresponds to the posterior density of the target distribution, while the kinetic energy $K(\phi)$ is typically modelled as a multivariate normal distribution $N(\mu, \Sigma)$.The dynamics are governed by the Hamiltonian equations:

$$\frac{d\Theta_i}{dt} = \frac{\partial H}{\partial \phi_i} \quad (22)$$

$$\frac{d\phi_i}{dt} = \frac{\partial H}{\partial \Theta_i} \quad (23)$$

Using the equations (22) and (23), starting with the initial conditions Θ_0 and ϕ_0 at time t_0, the new values Θ_i and ϕ_i can be predicted at a later time $t = t_0 + T$ by simulating the system over the duration T. The Leapfrog algorithm is employed to iteratively update Θ and ϕ at discrete time intervals defined by a step size ϵ. The Leapfrog method updates Θ and ϕ in an alternating manner, ensuring a stable numerical approximation. By applying this algorithm iteratively for a total of L steps, the updated values of Θ and ϕ can be obtained at each increment. Steps to obtain samples from joint posterior distribution of the Weibull-Bur-XII distribution are given bellow:

Step 1: Define the target distribution $U(\Theta) = \pi(\alpha,\beta,\gamma,\eta,\theta|x)$ and initialize parameters $\Theta^0 = (\alpha^0,\beta^0,\gamma^0,\eta^0,\theta^0)$.

Step 2: Introduce momentum variables $K(\phi) \sim N(0,\ \Sigma)$, where Σ is the covariance matrix of multivariate normal distribution.

Step 3: fix the parameters ϵ(step size) and L (number of leapfrog steps) for the leapfrog algorithm.

Step 4: Execute the leapfrog steps for the log-likelihood function of the target distribution using Θ^i and ϕ^i for the predetermined number L (from Step 3) and obtain updated $\Theta^*$and ϕ^*.

Step 5: Establish the acceptance criterion using them M-H algorithm

$$p = \frac{U(x|\Theta^*)U(\Theta^*)K(\phi^*)}{U(x|\Theta^i)U(\Theta^i)K(\phi^i)}$$

Generate $u \sim U(0,1)$ and update the parameter values:

$$\Theta^{i+1} = \begin{cases} \Theta^* \ if\ u \le p \\ \Theta^i \ otherwise \end{cases}$$

Note it is not necessary to store the ϕ^i values. The new ϕ is automatically generated from $N(0,\Sigma)$ after applying the acceptance criterion.

Step 6: Run the above steps for fixed Niterations and remove the initial burn-in samples.

The HMC performance is depends on the number of Leapfrog steps L and step size ϵ.The L and ϵ are automatically updated using No-U-Turn sampler (NUTS) [9][10].

The estimated parameters by HMC are given by:

$$\left.\begin{aligned} \hat{\alpha}_H = \frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(i)}, \hat{\beta}_H = \frac{1}{N-M}\sum_{i=M+1}^{N}\beta^{(i)}, \\ \hat{\gamma}_H = \frac{1}{N-M}\sum_{i=M+1}^{N}\gamma^{(i)}, \hat{\eta}_H = \frac{1}{N-M}\sum_{i=M+1}^{N}\eta^{(i)} \\ \hat{\theta}_H = \frac{1}{N-M}\sum_{i=M+1}^{N}\theta^{(i)} \end{aligned}\right\} \quad (24)$$

D. Comparative Measures

To compare different types of MCMC sampling algorithms, several measures are used to assess convergence numerically, while plots are employed for visual inspection. In this study, the convergence and comparison of the algorithms are evaluated using three criteria namely, effective sample size (ESS), integrated autocorrelation time (IAT), and Monte Carlo standard errors (MCSE). In the MCMC process, successive iterations produce correlated samples if the process has not reached stationarity. The effective sample size and integrated autocorrelation time helps to verify that the samples are independent. ESS indicates the amount of independent information present in autocorrelated chains [8]. It is defined as

$$ESS = \frac{N}{1 + 2\sum_{t=1}^{\infty}\rho_k} \quad (25)$$

Where N is the size of the samples and ρ_k is the correlation at lag k.

The greater the ESS value is considered as better algorithm. The integrated autocorrelation time is the estimate of the number of iterations on average, for an independent sample to be drawn from the algorithm chain. The algorithm gives lower IAT considered as better [4].

The Monte Carlo standard error which is found with the basis of ESS which shows the amount of noise in the samples drawn in the algorithm. The MCSE is given as [8]

$$MCSE = \frac{\sqrt{Var(\Theta)}}{\sqrt{ESS}} \quad (26)$$

The time efficiency of each algorithm varies, and while one algorithm may produce a higher Effective Sample Size (ESS), it may also require more time to do so compared to others. To account for this trade-off, the metric ESS per second defined as follows:

$$ESS/s = \frac{Total\ ESS}{time\ taken\ in\ seconds} \quad (27)$$

This metric provides a balanced comparison of algorithm performance by normalizing the ESS against the time taken for computation, thus highlighting the relationship between ESS and execution duration.

The trace plot and ACF plots are employed to check the convergence visually.

III. Simulation Study

In this study, the algorithms are compared using Monte Carlo simulation with sample sizes of n=10, 30, 50 and 100 representing small, medium and large samples. The samples are generated from the Weibull-Burr XII distribution with initial parameter values $\alpha = 1, \beta = 0.5, \gamma = 1.5, \eta = 1$ and $\theta = 1$. Data is generated under both Type-I and Type-II censoring methods with censorship levels of 5%, 10% and 20%. The hyperparameter values are fixed as $a_1 = b_1 = b_2 = b_3 = b_4 = 1, c_1 = c_2 = c_3 = c_4 = 2$. The MCMC techniques are used for parameter estimation with $N = 100000$ iterations and $M = 10000$ Burn in periods. The comparison of algorithms is done in terms of MSE, given by

$$\hat{\alpha}_{MSE} = \frac{1}{N-M}\sum_{i=N-M}^{N}\left(\alpha^{(i)}-\alpha\right)^2, \hat{\beta}_{MSE} = \frac{1}{N-M}\sum_{i=N-M}^{N}\left(\beta^{(i)}-\beta\right)^2$$

$$\hat{\theta}_{MSE} = \frac{1}{N-M}\sum_{i=N-M}^{N}\left(\theta^{(i)}-\theta\right)^2, \hat{\gamma}_{MSE} = \frac{1}{N-M}\sum_{i=N-M}^{N}\left(\gamma^{(i)}-\gamma\right)^2$$

$$\hat{\eta}_{MSE} = \frac{1}{N-M}\sum_{i=N-M}^{N}\left(\eta^{(i)}-\eta\right)^2$$

The smaller MSE indicates better overall quality of the estimates. All the calculations are performed through R software (version-4.4.0).The HMC is performed using Rstan [19]. The results are shown in Table II and III. MSE are given in bold.

TABLE II
PARAMETER ESTIMATES, MSE OF M-H ALGORITHM, SLICE SAMPLING AND HAMILTONIAN MONTE CARLO AT $\alpha = 1, \beta = 0.5, \gamma = 1.5, \eta = 1$ AND $\theta = 1$ UNDER TYPE-I CENSORING WITH FIXED TIME T=3,2.5,1.5.

Algorithm	n	10			30			50			100		
	Censoring	5%	10%	20%	5%	10%	20%	5%	10%	20%	5%	10%	20%
MH	α_M	0.8299	0.8628	0.9115	0.8870	0.9637	1.0585	0.9655	1.0424	1.1142	0.9257	0.9871	1.1492
		0.1364	**0.1639**	**0.1846**	**0.0998**	**0.1241**	**0.1582**	**0.1014**	**0.1218**	**0.1422**	**0.0916**	**0.1102**	**0.1245**
	β_M	0.6620	0.7886	0.8849	0.5343	0.7743	0.8705	0.4687	0.7157	0.8261	0.5906	0.8936	1.0129
		0.5606	**0.5551**	**0.7460**	**0.2955**	**0.4547**	**0.7022**	**0.2151**	**0.4140**	**0.4696**	**0.1454**	**0.3477**	**0.4255**
	γ_M	1.1349	1.0921	1.2128	0.9598	1.0901	1.6241	0.9687	1.2327	1.7243	0.7453	0.8848	1.2590
		0.7751	**0.8041**	**1.5416**	**0.5527**	**0.6830**	**1.5052**	**0.5060**	**0.5715**	**0.8947**	**0.2876**	**0.3226**	**0.4218**
	η_M	1.4246	1.1302	0.8538	1.1701	0.8529	0.5130	1.1000	0.8086	0.4556	1.2436	1.0708	0.6375
		0.6401	**0.6766**	**1.1859**	**0.4198**	**0.6020**	**1.0951**	**0.3972**	**0.5877**	**0.8323**	**0.3214**	**0.3525**	**0.8071**
	θ_M	1.1124	0.9033	1.1534	1.0401	0.8190	0.7853	0.8878	0.6637	0.5021	0.8796	0.7837	0.5089
		0.7545	**0.9198**	**1.5573**	**0.5144**	**0.6027**	**0.8030**	**0.4638**	**0.5319**	**0.7022**	**0.4308**	**0.5220**	**0.6708**
SS	α_S	0.8494	0.8692	0.9067	0.8931	0.9754	1.0379	0.9609	1.0211	1.1274	0.9309	0.9816	1.1224
		0.1337	**0.1599**	**0.1666**	**0.1024**	**0.1235**	**0.1511**	**0.0923**	**0.1147**	**0.1409**	**0.0908**	**0.1046**	**0.1226**
	β_S	0.6664	0.7944	0.8122	0.5496	0.7612	0.8617	0.4647	0.7463	0.8281	0.6483	0.8973	0.9580
		0.3493	**0.5455**	**0.6379**	**0.2129**	**0.4264**	**0.4968**	**0.1782**	**0.3437**	**0.4702**	**0.1276**	**0.3162**	**0.4127**
	γ_S	1.1455	1.1065	1.1540	1.0132	1.1555	1.5633	1.0044	1.1355	1.7289	0.8071	0.8523	1.1805
		0.6974	**0.7345**	**1.5331**	**0.5359**	**0.6515**	**1.3928**	**0.4657**	**0.5079**	**0.7362**	**0.2862**	**0.2980**	**0.3823**
	η_S	1.3895	1.1805	0.8146	1.1460	0.8639	0.5481	1.0505	0.8390	0.4450	1.2164	1.0401	0.6710
		0.6196	**0.6761**	**1.1772**	**0.4139**	**0.5882**	**1.0561**	**0.3877**	**0.5544**	**0.8153**	**0.2685**	**0.3337**	**0.7684**
	θ_S	1.0404	0.9904	0.9589	0.9830	0.8733	0.7766	0.7966	0.6833	0.5212	0.8644	0.7322	0.5353
		0.7287	**0.7570**	**0.8181**	**0.5028**	**0.5171**	**0.7004**	**0.4455**	**0.5070**	**0.6000**	**0.4162**	**0.4978**	**0.5774**
HMC	α_H	0.9167	0.9116	0.8941	0.9683	0.9651	0.9617	0.9800	0.9788	0.9769	0.9898	0.9898	0.9887
		0.0127	**0.0142**	**0.0201**	**0.0019**	**0.0024**	**0.0028**	**0.0008**	**0.0009**	**0.0010**	**0.0002**	**0.0002**	**0.0003**
	β_H	0.6308	0.7463	0.7870	0.4998	0.7325	0.8204	0.4753	0.7449	0.8172	0.6403	0.8927	0.9002
		0.2711	**0.3753**	**0.5606**	**0.1891**	**0.3560**	**0.4530**	**0.1495**	**0.3310**	**0.4488**	**0.1250**	**0.3053**	**0.3550**
	γ_H	1.1957	1.1603	1.2124	1.0807	1.1938	1.6516	1.0116	1.1230	1.6976	0.8413	0.8625	1.1476
		0.6531	**0.6859**	**1.4369**	**0.4696**	**0.6483**	**1.3673**	**0.4481**	**0.6210**	**0.7810**	**0.2629**	**0.2777**	**0.3492**
	η_H	1.3088	1.0866	0.7481	1.0560	0.8279	0.5365	1.0235	0.8279	0.4885	1.1646	1.0198	0.7121
		0.5867	**0.5823**	**1.0861**	**0.3884**	**0.5668**	**1.0243**	**0.3848**	**0.5267**	**0.8134**	**0.2404**	**0.3327**	**0.6881**
	θ_H	1.1186	1.0418	0.9755	1.0352	0.8577	0.7451	0.8058	0.6333	0.4410	0.8779	0.7184	0.4589
		0.5356	**0.7568**	**0.7734**	**0.4698**	**0.4970**	**0.6223**	**0.4243**	**0.4684**	**0.5087**	**0.3691**	**0.4626**	**0.5082**

TABLE III
PARAMETER ESTIMATES, MSE OF M-H ALGORITHM, SLICE SAMPLING AND HAMILTONIAN MONTE CARLO AT $\alpha = 1, \beta = 0.5, \gamma = 1.5, \eta = 1$ AND $\theta = 1$ UNDER TYPE-II CENSORING WITH 5%, 10%, 20% CENSORSHIPS.

Algorithm	N	10			30			50			100		
	Censoring	5%	10%	20%	5%	10%	20%	5%	10%	20%	5%	10%	20%
MH	α_M	0.8281	0.8629	0.8554	0.8777	0.8873	0.8929	0.9466	0.9475	0.9985	0.9120	0.8951	0.9580
		0.2666	**0.2726**	**0.2903**	**0.2633**	**0.2694**	**0.2702**	**0.2633**	**0.2664**	**0.2681**	**0.2554**	**0.2629**	**0.2662**
	β_M	0.7549	0.7102	0.8377	0.8604	0.8181	0.8743	0.7869	0.8678	0.8604	0.8567	0.8553	0.8523
		0.4928	**0.5765**	**0.6124**	**0.4281**	**0.4546**	**0.4865**	**0.3741**	**0.3910**	**0.4787**	**0.2832**	**0.3726**	**0.4163**
	γ_M	1.0675	1.2006	1.0798	1.0145	1.0526	0.9629	0.8954	0.9677	1.0743	0.8910	0.8315	0.8723
		0.6398	**0.6528**	**0.7583**	**0.5635**	**0.6092**	**0.6087**	**0.5251**	**0.5792**	**0.6109**	**0.2792**	**0.2818**	**0.2873**
	η_M	1.4764	1.2663	1.1494	1.3360	1.1710	1.0735	1.2434	1.3126	0.9606	1.4422	1.3291	1.3046
		0.6853	**0.7499**	**0.8574**	**0.5046**	**0.5638**	**0.7617**	**0.4290**	**0.5072**	**0.6381**	**0.3338**	**0.3597**	**0.4691**
	θ_M	1.0533	0.9738	0.9509	1.0916	0.9557	0.9316	0.9669	1.0330	0.7736	1.0909	0.8920	0.9928
		0.9141	**0.9172**	**1.1676**	**0.8275**	**0.8963**	**0.9073**	**0.7633**	**0.7987**	**0.8863**	**0.6702**	**0.7685**	**0.8293**
SS	α_S	0.8494	0.8631	0.8823	0.8667	0.8799	0.9113	0.9295	0.9525	0.9935	0.8915	0.9405	0.9419
		0.1593	**0.1661**	**0.1749**	**0.1499**	**0.1555**	**0.1594**	**0.1307**	**0.1313**	**0.1530**	**0.1190**	**0.1275**	**0.1328**
	β_S	0.8122	0.7899	0.7511	0.8320	0.8305	0.7869	0.8358	0.7921	0.7757	0.8646	0.8573	0.8251
		0.4597	**0.4944**	**0.4979**	**0.4353**	**0.4702**	**0.4829**	**0.4199**	**0.4410**	**0.4623**	**0.3907**	**0.4271**	**0.4559**
	γ_S	1.1455	1.1306	1.1238	1.0599	1.0694	1.1044	1.0087	1.0828	1.1142	0.8464	0.8803	0.9132
		0.6194	**0.6503**	**0.6531**	**0.5586**	**0.6001**	**0.6312**	**0.4961**	**0.5487**	**0.6044**	**0.2761**	**0.2791**	**0.2861**
	η_S	1.3895	1.2810	1.0866	1.2601	1.2023	1.0719	1.2490	1.2581	0.9441	1.4541	1.3501	1.3151
		0.6761	**0.7358**	**0.7545**	**0.4808**	**0.5277**	**0.6860**	**0.4227**	**0.4768**	**0.5476**	**0.3186**	**0.3497**	**0.4078**
	θ_S	1.0404	1.0179	0.9867	1.0223	0.9915	0.9680	0.9356	0.9467	0.7564	0.9918	0.9496	0.9007
		0.7545	**0.8539**	**0.9631**	**0.6977**	**0.7012**	**0.8900**	**0.6345**	**0.6529**	**0.6980**	**0.5889**	**0.6203**	**0.6578**
HMC	α_H	1.0655	1.0859	1.1067	1.0880	1.1264	1.1589	1.1690	1.1853	1.2709	1.1357	1.1593	1.1979
		0.1518	**0.1622**	**0.1666**	**0.1432**	**0.1535**	**0.1552**	**0.1288**	**0.1308**	**0.1443**	**0.1116**	**0.1254**	**0.1284**
	β_H	0.7792	0.8004	0.8239	0.7851	0.8176	0.8456	0.7436	0.8108	0.8445	0.8214	0.8582	0.8482
		0.4630	**0.4720**	**0.5262**	**0.4313**	**0.4619**	**0.5121**	**0.4166**	**0.4571**	**0.4898**	**0.3570**	**0.4528**	**0.4693**
	γ_H	1.1474	1.1431	1.1055	1.0650	1.1001	1.1220	1.0197	1.0709	1.1484	0.8996	0.8877	0.9270
		0.5230	**0.6086**	**0.6370**	**0.4889**	**0.5374**	**0.5379**	**0.3234**	**0.3809**	**0.4032**	**0.2593**	**0.2793**	**0.2855**
	η_H	1.2892	1.1664	0.9726	1.1617	1.0846	0.9477	1.1669	1.1437	0.8307	1.3565	1.2655	1.2088
		0.5515	**0.7017**	**0.7038**	**0.4617**	**0.4832**	**0.5811**	**0.3729**	**0.4450**	**0.5447**	**0.2771**	**0.3357**	**0.3560**
	θ_H	1.0856	1.1204	1.1648	1.0985	1.1666	1.1840	0.8962	1.0692	1.1034	1.0823	1.1107	1.1802
		0.6987	**0.7762**	**0.7841**	**0.6581**	**0.6774**	**0.7280**	**0.5901**	**0.6437**	**0.6696**	**0.4983**	**0.5789**	**0.6110**

From the results obtained in TABLE II and III it is observed that:

1. The mean squared error (MSE) decreases as the sample size increases for all three methods, demonstrating enhanced precision in parameter estimates for both Type-I and Type-II censoring.
2. For a given sample size n, an increase in the percentage of censorship correlates with an increase in the MSE of the estimates for both censoring methods.
3. Across all sample sizes and levels of censorship, the Hamiltonian Monte Carlo algorithm consistently produces lower mean squared error (MSE) than the Metropolis-Hastings algorithm and Slice sampling, indicating improved accuracy in the parameter estimates for both Type-I and Type-II censoring.

IV. Real Life Applications

This section examines the performance of the Metropolis-Hastings (M-H), Slice sampling and Hamiltonian Monte Carlo (HMC) algorithm across two datasets. The measures used for comparison include Effective Sample Size (ESS), Integrated Autocorrelation Time (IAT), Monte Carlo Standard Error (MCSE) and effective sample size per second (ESS/s). All the calculations are carried out using R-Software.

A. Dataset 1

This study employs the veteran dataset from the survival package in R, which includes survival data for 137 male patients with advanced lung cancer [17]. A Kolmogorov-Smirnov (K-S) test was conducted to assess the fit of the Weibull-Burr XII distribution, yielding a test statistic of $D = 0.0937$ and a p-value of 0.6272 confirming a suitable fit to the data. Parameters were estimated using the Metropolis-Hastings (MH) algorithm, Slice Sampling and Hamiltonian Monte Carlo (HMC) with $N = 100{,}000$ iterations and a burn-in of $M = 10{,}000$. The estimated parameters under Type-I censoring (at time $T = 35, 25, 14$) and Type-II censoring (at $5\%, 10\%$ and 20% censorship levels) along with comparison metrics such as Effective Sample Size (ESS), Integrated Autocorrelation Time (IAT), Monte Carlo Standard Error (MCSE) and ESS/s are reported in TABLE IV and V.

B. Dataset 2

The second dataset consists of 44 patients diagnosed with head and neck cancer, who were treated with radiotherapy as reported by Efron [6]. The K-S statistic yielded a value of D=0.113 and a p-value of 0.7521, indicating that the data fits the distribution well. For parameter estimation using MCMC algorithms, $N = 100{,}000$ iterations and $M = 10{,}000$ burn-in are fixed. The parameter estimates of the Weibull-Burr distribution under Type-I censoring (with $T = 800{,}500{,}300$) and Type-II censoring (at $5\%, 10\%$ and 20% censoring levels) along with comparison metrics such as Effective Sample Size (ESS), Integrated Autocorrelation Time (IAT), Monte Carlo Standard Error (MCSE) and ESS/s are provided in TABLE VI and VII.

C. Summary

TABLE VI-VII shows the estimates, ESS, IAT, MCSE and ESS/s values for M-H algorithm, Slice sampling and HMC algorithm of Type-I and Type-II censoring for different censoring time and censorship percentages. Figure 1-4 shows trace and ACF plots for two different datasets. From the TABLE VI-VII and Fig 1-4 it is observed that:

1. The Effective Sample Size (ESS) values for Hamiltonian Monte Carlo are consistently higher than the other algorithms across all parameters and datasets, with the exception of the Slice Sampling estimate of θ in Dataset 2.
2. Integrated Autocorrelation Time (IAT) values for Slice Sampling and Hamiltonian Monte Carlo are significantly lower than those for the M-H algorithm, suggesting that the samples produced by the M-H algorithm are highly correlated across all five parameters.
3. In general, Hamiltonian Monte Carlo exhibits lower correlations compared to Slice sampling.
4. The Monte Carlo Standard Error (MCSE) for Hamiltonian Monte Carlo is less than the other two algorithms for all parameters across both datasets, except for the Slice sampling estimate of θ in Dataset 2.
5. Slice Sampling demonstrates superior performance for the parameter θ in dataset 2 but it requires more computational time than the other two algorithms, which results in a lower Effective Sample Size per second (ESS/s).
6. The M-H algorithm is efficient in terms of computational time but produces more correlated samples, whereas Slice sampling generates less correlated samples and achieves a higher ESS, though it requires more time.
7. Hamiltonian Monte Carlo successfully balances computational efficiency and effective sample size, resulting in a higher ESS/s compared to both the M-H algorithm and Slice Sampling.
8. Fig 1 and 3 present the autocorrelation function (ACF) plots for the M-H algorithm, Slice sampling and HMC for all parameters of the Weibull-Burr XII distribution in Datasets 1 and 2 respectively. It is observed that the ACF values for HMC approaches zero rapidly at successive lags, in contrast the slower convergence of the ACF values for both Slice sampling and the M-H algorithm across all five parameters.
9. Fig 2 and 4 shows the trace plots for the M-H algorithm, Slice sampling and HMC for all five parameters of the Weibull-Burr XII distribution. The trace plots for Slice Sampling and HMC demonstrate effective mixing, indicating a thorough exploration of the parameter space. Conversely the trace plot for the M-H algorithm reveals a lack of mixing for some parameters, suggesting potential issues with convergence.

TABLE IV
PARAMETER ESTIMATES AND CORRESPONDING IAT, ESS, MCSE AND ESS/S FOR DIFFERENT ALGORITHMS UNDER TYPE-I CENSORING WITH TIME $T = 35,25$ AND 14 FOR DATASET I.

Censoring Time		35					25					14				
	par	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s
M-H	α_M	1.4608	152.8	675	0.0138	14.7	1.5743	93.3	1078	0.0107	23.3	1.6491	59.3	1706	0.0076	35.8
	β_M	0.6822	663.0	178	0.0111	3.9	0.5178	482.5	160	0.0090	3.5	0.2998	741.5	159	0.0057	3.3
	γ_M	1.7215	676.0	133	0.0307	2.9	2.2904	589.7	139	0.0389	3.0	3.7751	689.5	133	0.0564	2.8
	η_M	0.4772	246.6	398	0.0058	8.7	0.3326	143.6	704	0.0041	15.2	0.1573	94.9	1198	0.0027	25.1
	θ_M	6.2333	233.8	533	0.0864	11.6	6.4950	325.3	448	0.0936	9.7	7.4188	376.1	352	0.0859	7.4
SS	α_S	1.4596	20.2	4381	0.0076	18.4	1.5811	13.7	6884	0.0052	27.5	1.6452	7.4	12196	0.0031	43.3
	β_S	0.6980	77.9	1176	0.0102	4.9	0.5089	56.6	1644	0.0065	6.6	0.3052	44.9	2011	0.0037	7.1
	γ_S	1.7177	72.1	1346	0.0241	5.7	2.3315	51.9	1731	0.0280	6.9	3.7088	44.8	2153	0.0372	7.6
	η_S	0.4802	26.9	3587	0.0034	15.1	0.3308	16.3	5527	0.0021	22.1	0.1588	12.5	7467	0.0012	26.5
	θ_S	6.2434	22.7	4534	0.0488	19.0	6.5726	30.1	3131	0.0553	12.5	7.3238	31.7	2798	0.0529	9.9
HMC	α_H	1.9840	12.5	7174	0.0002	64.3	1.9831	13.7	6585	0.0002	58.4	1.9805	14.2	6335	0.0002	56.2
	β_H	0.7555	20.8	4320	0.0045	38.7	0.5085	25.4	3549	0.0034	31.5	0.3101	30.5	2947	0.0023	26.1
	γ_H	1.5713	19.3	4653	0.0088	41.7	2.3343	25.0	3593	0.0152	31.9	3.6443	29.6	3045	0.0225	27.0
	η_H	0.3873	15.2	5903	0.0013	52.9	0.2567	14.4	6231	0.0010	55.3	0.1063	21.2	4243	0.0008	37.6
	θ_H	7.0811	15.2	5927	0.0318	53.1	7.4671	20.7	4350	0.0378	38.6	7.7701	22.5	3999	0.0333	35.5

TABLE V
PARAMETER ESTIMATES AND CORRESPONDING IAT, ESS, MCSE AND ESS/S FOR DIFFERENT ALGORITHMS UNDER TYPE-II CENSORING WITH 5%, 10%, 20% CENSORSHIPS FOR DATASET I.

censoring		5%					10%					20%				
	par	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s
M-H	α_M	1.2542	353.1	342	0.0182	7.9	1.2810	222.8	421	0.0176	9.9	1.2810	222.8	421	0.0176	10.7
	β_M	1.1041	1155.9	66	0.0296	1.5	0.9308	838.2	118	0.0203	2.8	0.9308	838.2	118	0.0203	3.0
	γ_M	1.1721	1033.6	82	0.0307	1.9	1.3487	910.5	90	0.0335	2.1	1.3487	910.5	90	0.0335	2.3
	η_M	0.7276	638.9	198	0.0101	4.5	0.6752	411.8	249	0.0099	5.8	0.6752	411.8	249	0.0099	6.3
	θ_M	6.7059	255.8	459	0.0962	10.5	6.4975	203.0	430	0.0991	10.1	6.4975	203.0	430	0.0991	10.9
SS	α_S	1.2726	48.0	2079	0.0125	8.9	1.2847	38.1	2360	0.0115	10.6	1.1863	47.5	2295	0.0132	11.0
	β_S	0.9908	131.7	709	0.0215	3.0	0.9552	127.8	704	0.0215	3.2	1.0502	117.3	776	0.0254	3.7
	γ_S	1.2761	123.7	772	0.0262	3.3	1.3534	101.0	869	0.0278	3.9	1.3015	92.3	960	0.0286	4.6
	η_S	0.7029	79.6	1309	0.0076	5.6	0.6838	49.2	1838	0.0071	8.3	0.8405	54.7	1702	0.0104	8.2
	θ_S	6.6733	21.8	3982	0.0531	17.1	6.6388	21.1	4355	0.0520	19.6	7.0873	22.1	3996	0.0570	19.2
HMC	α_H	1.2598	10.8	8347	0.0047	44.3	1.2946	10.6	8521	0.0046	61.8	1.1887	12.0	7496	0.0053	58.2
	β_H	0.9815	7.4	12096	0.0050	64.1	0.9430	9.6	9378	0.0055	68.0	1.0361	46.4	1939	0.0147	15.1
	γ_H	1.3016	5.5	16338	0.0056	86.6	1.3667	7.0	12768	0.0066	92.6	1.3118	7.8	11517	0.0073	89.4
	η_H	0.7057	8.4	10678	0.0022	56.6	0.6791	8.4	10769	0.0023	78.1	0.8375	8.3	10799	0.0033	83.9
	θ_H	6.6573	5.4	16819	0.0190	89.2	6.6878	5.8	15501	0.0197	112.4	7.0484	5.2	17338	0.0199	134.6

TABLE VI
PARAMETER ESTIMATES AND CORRESPONDING ESS, IAT AND MCSE USING M-H ALGORITHM AND SLICE SAMPLING UNDER TYPE-I CENSORING WITH FIXED TIME $T = 800,500,300$ For Dataset II

Censoring Time		800					500					300				
	par	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s
M-H	α_M	1.1916	174.2	619	0.0179	24.2	1.2968	114.5	955	0.0153	37.2	1.3432	71.8	1310	0.0136	51.3
	β_M	0.9589	233.3	388	0.0120	15.1	0.8686	114.5	679	0.0085	26.5	0.7600	70.6	1352	0.0057	52.9
	γ_M	0.2469	449.2	160	0.0057	6.2	0.2115	343.9	276	0.0046	10.8	0.1605	171.6	523	0.0031	20.4
	η_M	2.1505	214.1	512	0.0318	20.0	2.3697	125.0	761	0.0294	29.7	2.6872	86.5	1197	0.0300	46.8
	θ_M	52.2702	612.2	123	0.2971	4.8	51.1420	848.2	107	0.3188	4.2	51.4638	680.1	112	0.3114	4.4
SS	α_S	1.2421	21.9	4485	0.0097	15.5	1.3093	14.4	6410	0.0075	20.5	1.3466	10.7	8330	0.0062	25.8
	β_S	0.9611	26.5	3604	0.0060	12.5	0.8656	28.8	3208	0.0056	10.3	0.7638	10.0	9351	0.0027	29.0
	γ_S	0.2506	36.6	2431	0.0031	8.4	0.2029	35.4	3447	0.0024	11.0	0.1605	18.8	5151	0.0016	16.0
	η_S	2.1537	23.9	3939	0.0172	13.6	2.4176	15.8	5659	0.0149	18.1	2.6829	10.9	8369	0.0131	25.9
	θ_S	50.4018	2.4	37397	0.0458	129.2	50.8618	2.2	40631	0.0436	130.0	51.3199	2.2	42965	0.0435	133.2
HMC	α_H	1.9543	13.6	6625	0.0006	132.3	1.9512	12.6	7121	0.0006	235.1	0.9760	12.6	7132	0.0003	231.6
	β_H	1.0946	33.6	2682	0.0050	53.6	0.9477	18.6	4839	0.0030	159.7	0.8226	17.3	5202	0.0028	168.9
	γ_H	0.1834	34.0	2647	0.0014	52.8	0.1448	19.9	4516	0.0008	149.1	0.2263	21.0	4290	0.0011	139.3
	η_H	1.9406	26.2	3438	0.0113	68.6	2.2572	16.6	5417	0.0097	178.8	2.4840	17.3	5202	0.0112	168.9
	θ_H	50.6856	11.6	7756	0.0814	154.9	51.2462	9.6	9410	0.0743	310.6	50.7335	9.8	9159	0.0736	297.4

TABLE VII
PARAMETER ESTIMATES AND CORRESPONDING IAT, ESS, MCSE AND ESS/S FOR DIFFERENT ALGORITHMS UNDER TYPE-II CENSORING WITH 5%, 10%, 20% CENSORSHIPS FOR DATASET II

censoring		5%					10%					20%				
	par	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s	EST	IAT	ESS	MCSE	ESS/s
M-H	α_M	1.1110	214.8	468	0.0194	17.9	1.1097	217.5	546	0.0195	11.5	0.9999	200.5	495	0.0204	10.5
	β_M	1.0579	217.4	390	0.0140	14.9	1.1249	309.7	324	0.0170	6.9	1.2615	416.1	249	0.0237	5.3
	γ_M	1.9763	212.0	448	0.0337	17.1	1.8886	271.6	448	0.0338	9.5	1.7111	256.9	408	0.0330	8.6
	η_M	0.3209	655.7	179	0.0069	6.8	0.3375	521.4	178	0.0074	3.8	0.4208	396.7	195	0.0097	4.1
	θ_M	50.2264	577.1	132	0.2841	5.0	50.2016	1074.2	105	0.3202	2.2	49.6953	615.6	120	0.3007	2.5
SS	α_S	0.9316	41.9	2016	0.0145	6.9	0.9230	40.7	2067	0.0147	7.3	0.7770	50.1	1797	0.0159	6.8
	β_S	1.1004	72.3	1155	0.0165	3.9	1.1662	66.6	1255	0.0181	4.4	1.3367	68.8	1248	0.0239	4.7
	γ_S	1.8044	53.4	1562	0.0275	5.3	1.7079	60.2	1692	0.0272	5.9	1.5370	60.6	1478	0.0282	5.6
	η_S	0.4209	83.0	1080	0.0082	3.7	0.4484	70.7	1134	0.0083	4.0	0.5918	89.7	950	0.0139	3.6
	θ_S	50.4999	2.8	31742	0.0506	107.9	50.4874	2.6	34118	0.0485	119.9	50.5673	2.8	31789	0.0504	120.0
HMC	α_H	1.0731	10.7	8398	0.0054	201.8	1.0622	10.8	8311	0.0055	187.7	0.9907	11.9	7532	0.0061	185.6
	β_H	1.0874	12.3	7329	0.0047	176.1	1.1180	11.9	7543	0.0050	170.3	1.2770	19.8	4538	0.0086	111.9
	γ_H	1.9121	6.7	13455	0.0072	323.4	1.8773	6.5	13821	0.0072	312.1	1.6909	7.8	11472	0.0078	282.7
	η_H	0.3426	9.1	9940	0.0017	238.9	0.3574	8.2	10971	0.0018	247.8	0.4385	8.9	10080	0.0025	248.4
	θ_H	50.2375	4.5	19817	0.0504	476.3	50.1778	5.1	17752	0.0531	400.9	50.2740	5.3	17069	0.0543	420.7

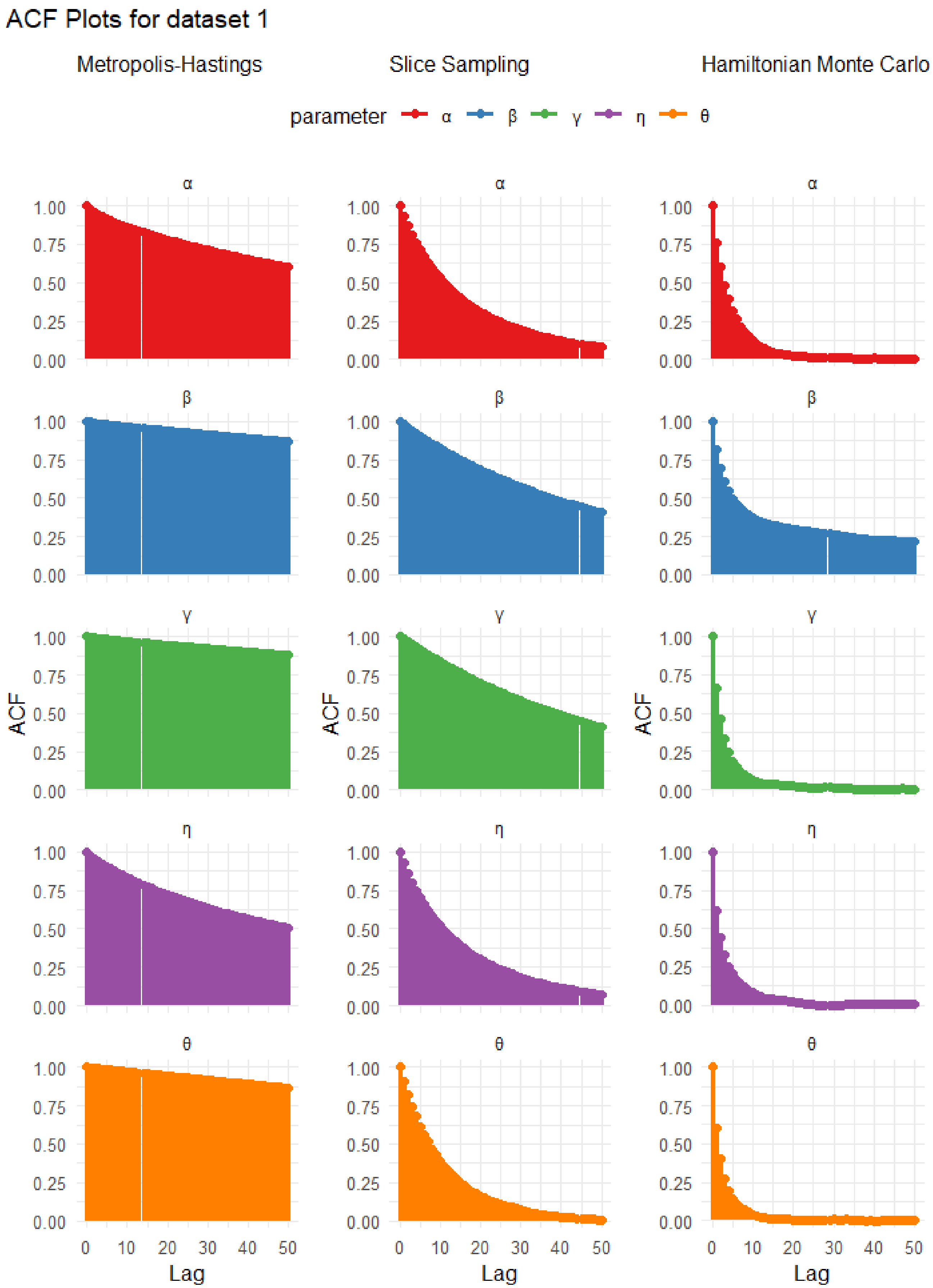

Fig.1 ACF plots of parameter estimates using Metropolis-Hastings, Slice Sampling and Hamiltonian Monte Carlo algorithms for Dataset 1.

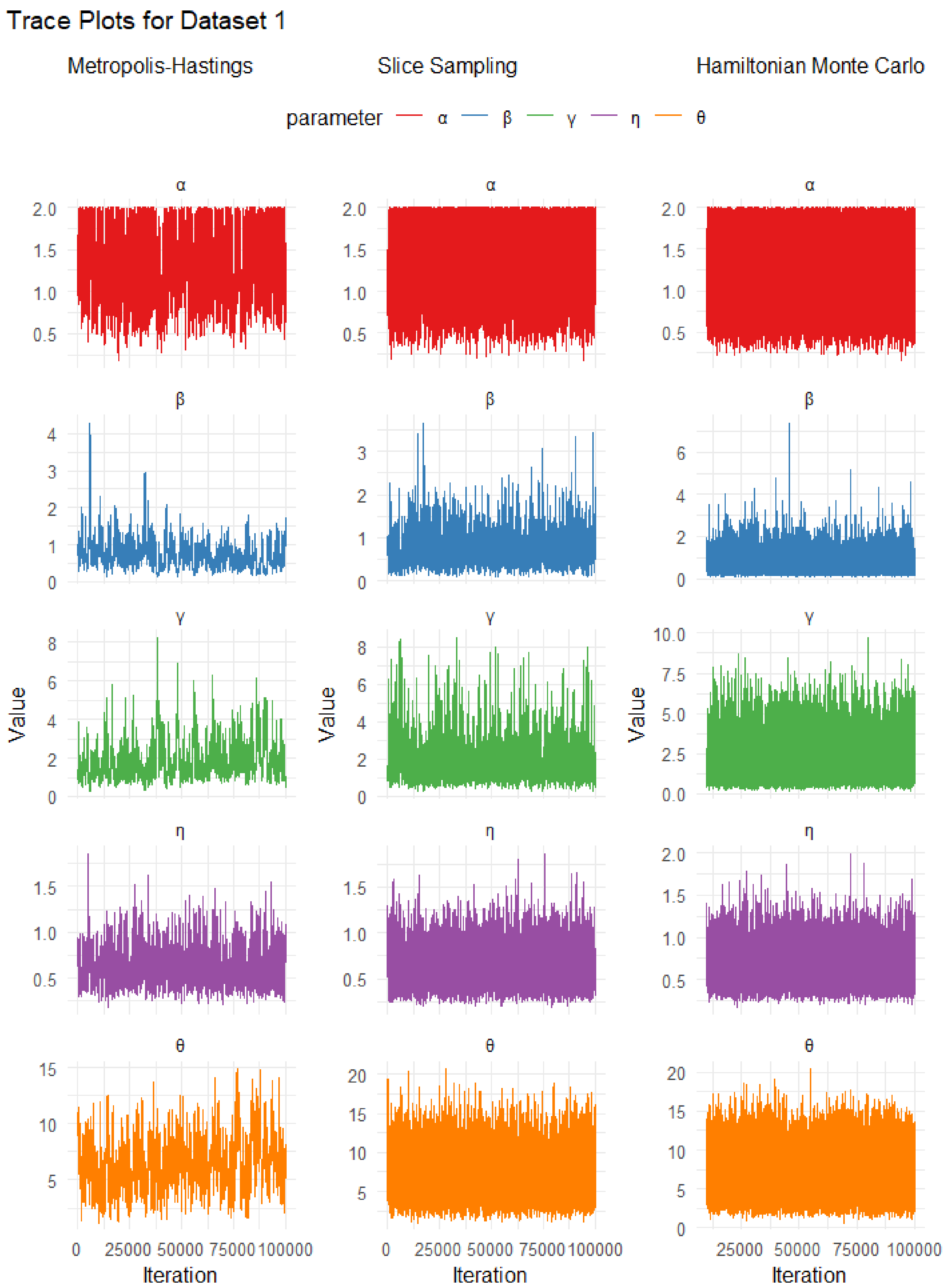

Fig.2 Trace plots of parameter estimates using Metropolis-Hastings, Slice Sampling and Hamiltonian Monte Carlo algorithms for Dataset 1.

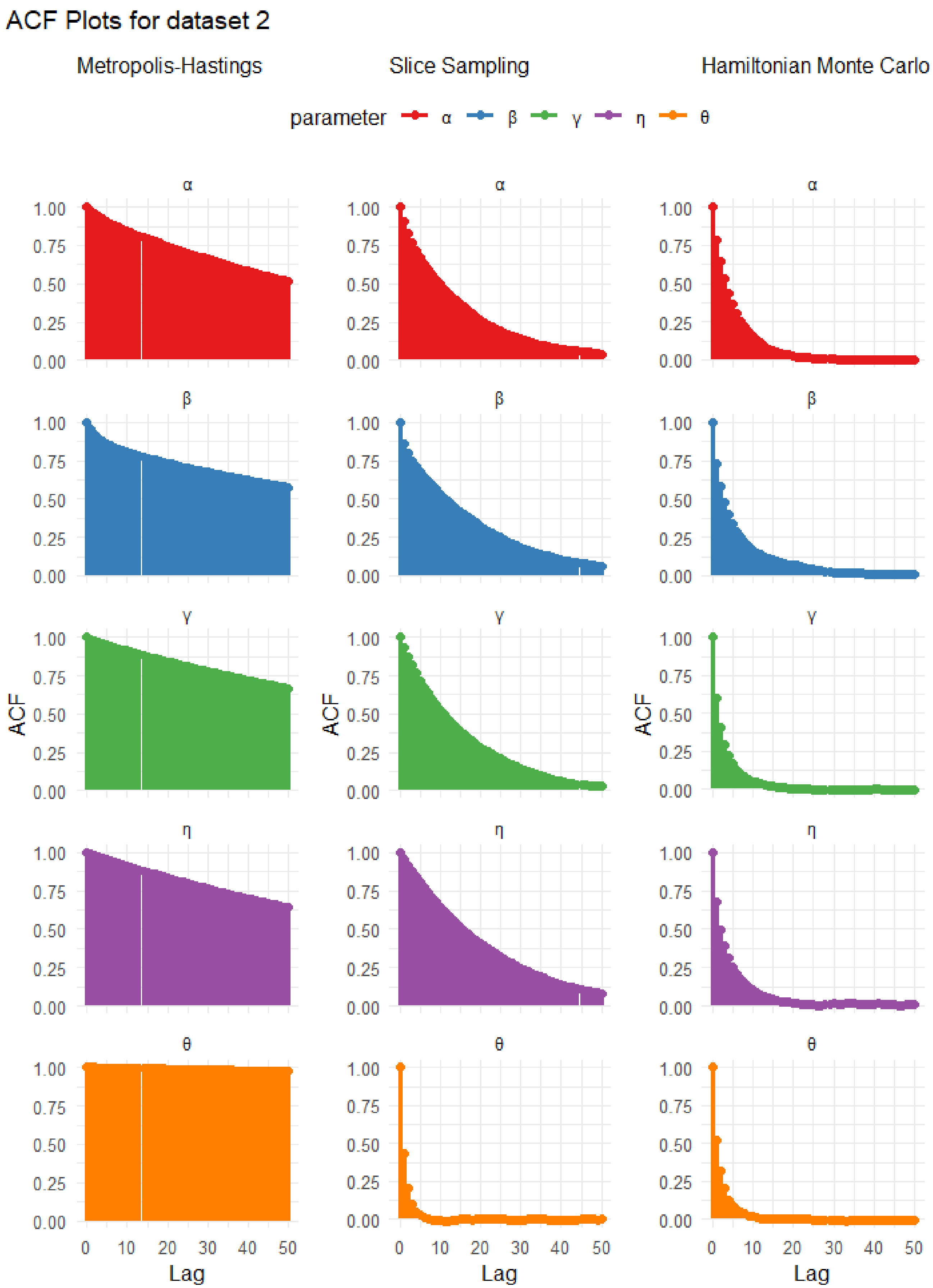

Fig.3 ACF plots of parameter estimates using Metropolis-Hastings, Slice Sampling and Hamiltonian Monte Carlo algorithms for Dataset 2.

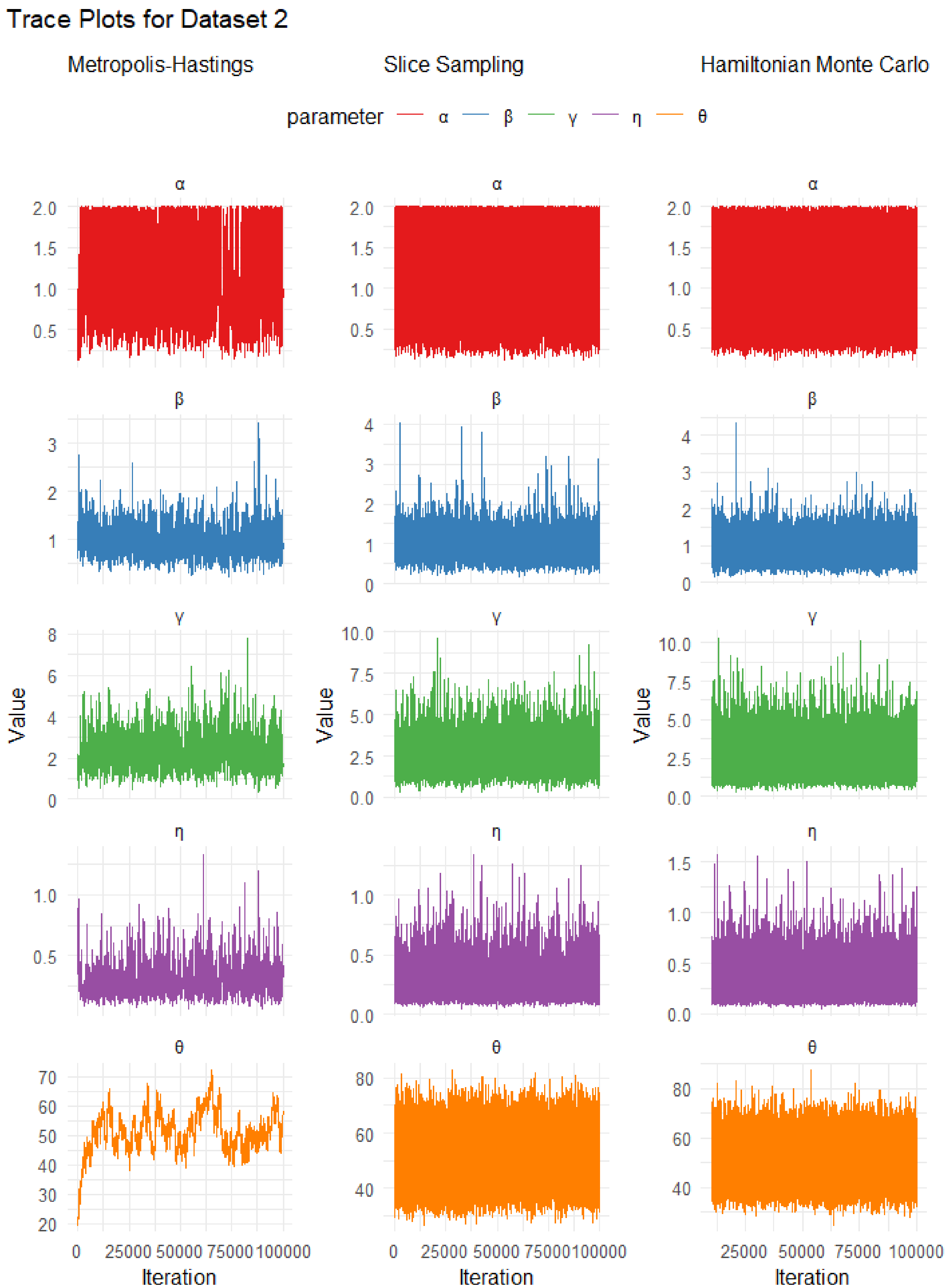

Fig 4: Trace plots of parameter estimates using Metropolis-Hastings, Slice Sampling and Hamiltonian Monte Carlo algorithms for Dataset 2.

IV. Conclusion

This study conducted a detailed evaluation of the performance of three MCMC algorithms namely Metropolis-Hastings (M-H), Slice sampling and Hamiltonian Monte Carlo (HMC) for estimating the Weibull-Burr XII distribution, specifically focusing on the effects of Type-I and Type-II censoring. From simulation and real data applications it is concluded that:

1. The Hamiltonian Monte Carlo (HMC) consistently demonstrated superior performance over both the M-H algorithm and Slice Sampling, as evidenced by lower mean squared error (MSE) in simulation study and higher effective sample size per second (ESS/s) values across all parameters and datasets. This underscores its effectiveness in providing accurate parameter estimates.
2. The lower Integrated Autocorrelation Time (IAT) values for HMC and Slice Sampling suggest that these methods produce less correlated samples, enhancing the reliability of the estimates. In contrast, the M-H algorithm exhibited higher correlation among samples, which may restrict accurate inference.
3. The analysis highlighted that as censorship levels increased, the MSE of parameter estimates also increased for both censoring types. This finding emphasizes the importance of considering censorship effects in survival analysis.
4. The slice sampling occasionally yielded better estimates for specific parameters, it often required more computational time resulting in a lower Effective Sample Size per second (ESS/s) compared to HMC. The M-H algorithm is computationally efficient but it produced less stable estimates due to its higher autocorrelation.

Among the algorithms, the Hamiltonian Monte Carlo is most preferred approach for estimating parameters of the Weibull-Burr XII distribution under censoring, where HMC provides strong balance of accuracy, efficiency and effective sample. Future work is needed to examine how advanced MCMC sampling methods could further refine parameter estimation, particularly under varying censorship conditions and complex real-world situations.

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