Packing [2,8] Coloring of Planar Graphs with Maximum Degree 4

Xiao Wei, Lei Sun* and Wei Zheng

***Abstract*—For a graph G and a sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers with $s_1 \leq s_2 \leq \cdots \leq s_k$, we say that G is packing S-colorable or G admits a packing S-coloring, if $V(G)$ can be partitioned into k subsets V_1, V_2, ..., V_k such that for each $1 \leq i \leq k$ and any $x, y \in V_i$, $x \neq y$, there is $d(x, y) \geq s_i + 1$, where $d(x, y)$ is the distance of vertices x and y in G. For simplicity, we define the packing S-coloring as packing $[2, 8]$ coloring while $S = (1, 1, 2, 2, 2, 2, 2, 2, 2, 2)$. In this paper, we proved that every planar graph G with maximum degree $\Delta \leq 4$ is packing $[2, 8]$ colorable.**

***Index Terms*—planar graphs, maximum degree, packing coloring, discharging.**

I. Introduction

In this paper, all graphs we considered are finite and simple. Let G be a graph. We denote the set of vertices, edges and faces of G by $V(G)$, $E(G)$ and $F(G)$, respectively. Let $d_G(u)$ ($d_G(f)$) denote the degree of a vertex u (respectively, a face f) in G, which is the number of edges of G incident with u (respectively, the number of edges incident with f). A k-vertex (k^--vertex, k^+-vertex) is a vertex of degree k (respectively, at most k, at least k). A k (k^- or k^+)-face and a k (k^- or k^+)-neighbor is defined similarly. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree of G, respectively. A $(x_1, x_2, \cdots, x_k)$-face ($[u_1 u_2 \cdots u_k]$-face) is a k-face with vertices of degree x_1, x_2, $\cdots$, x_k on its boundary (respectively, with vertices u_1, u_2, $\cdots$, u_k on its boundary). For a vertex u in a graph G, denote $N_i(u) = \{v \in V(G) | d(u, v) = i\}$, $i \geq 1$ and $N_1(u) = N(u)$, where $d(u, v)$ is the distance between vertices u, v.

For a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, a vertex coloring of a graph G is called a packing S coloring if $V(G)$ can be partitioned into k subsets V_i such that for each $1 \leq i \leq k$ the distance between any two distinct vertices x, y in V_i is at least $s_i + 1$. The packing chromatic number of G, denoted by $\chi_\rho(G)$, is the smallest k such that G has a packing $(1, 2, \ldots, k)$ coloring. Goddard et al. [1] first introduced the concept of packing $(1, 2, \ldots, k)$-coloring, termed broadcast coloring, and showed that deciding whether $\chi_\rho(G) \leq 4$ is NP-hard. Goddard and Xu [2] later expanded broadcast coloring to packing coloring for any non-decreasing sequence S. In recent years, the packing S coloring received a lot of attention, and some new results emerged. Gastineau and Togni [3] proved that subcubic graphs are (1, 2, 2, 2, 2, 2, 2)-packing colorable and (1, 1, 2, 2, 2)-packing colorable. J. Balogh, A. Kostochka and X. Liu [12] proved that the subdivision of any subcubic graph has packing chromatic number at most 8. Recently, X. Liu, X. Zhang and Y. Zhang [13] proved that every subcubic graph is (1, 1, 2, 2, 3)-packing colorable and $\chi_\rho(D(G)) \leq 6$, improving the previous bound. There is also some research on subclasses of subcubic graphs in [4, 5], such as outerplanar subcubic graphs and i-saturated subcubic graphs, etc. Future research issues and existing results on packing coloring of subcubic graphs were summarized by M. Mortada and O. Togni [14]. Due to the complexity of packing coloring, current research on packing coloring mostly focuses on subcubic graphs and subclasses of subcubic graphs. In this paper, we study the packing S coloring on planar graphs with maximum degree at most 4 by discharge method.

A 2-distance coloring of a graph G is a mapping $\Phi : V \to \{1, 2, \ldots, k\}$, satisfying: $\Phi(x) \neq \Phi(y)$ for any two distinct vertices x, y in V with $d(x, y) \leq 2$. The 2-distance chromatic number of G, denote by $\chi_2(G)$, is the smallest number k such that G has a 2-distance k-coloring. 2-distance coloring and proper coloring are two special kinds of packing S coloring. In fact, the packing S coloring is proper coloring of G, if $s_i = 1$, $i = 1, 2, \ldots, k$. The packing S coloring is 2-distance coloring of G, if $s_i = 2$, $i = 1, 2, \ldots, k$.

In 1977, Wegner [11] proposed the following conjecture:

Conjecture 1.1 ([11]). If G is a planar graph, then $\chi_2(G) \leq 7$ if $\Delta(G) = 3$, $\chi_2(G) \leq \Delta(G) + 5$ if $4 \leq \Delta(G) \leq 7$ and $\chi_2(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 1$ if $\Delta(G) \geq 8$.

In 2018, Thomassen [15] proved that every planar graph with $\Delta(G) = 3$ has $\chi_2(G) \leq 7$. However, Conjecture 1.1 remains open for planar graphs with $\Delta(G) \geq 4$. Some scholars [6, 7] study the 2-distance chromatic number of planar graphs by imposing constraints on the maximum degree. Recently, Aoki [8] and Zou et al. [9] independently proved that $\chi_2(G) = 17$ if G is a planar graph with $\Delta(G) \leq 5$. Deniz [10] proved that every planar graph with $\Delta(G) \leq 5$ has $\chi_2(G) = 16$. With regard to planar graphs with $\Delta(G) = 4$, N. Bousquet, L. de Meyer, et al. [16] proved that $\chi_2(G) \leq 12$.

A color set is a set of vertices of the same color. In this paper, we relax the condition of 2-distance coloring. We color the graph using ten colors such that for two of the color sets, any pair of vertices within the same color set has a distance of at least 2, while for the remaining eight color sets, the distance between any pair of vertices within the same color set is at least 3. It is easy to check that this coloring

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is a packing S coloring with $S = (1, 1, 2, 2, 2, 2, 2, 2, 2, 2)$. For convenience, we define this coloring as packing $[2, 8]$ coloring.

Our main result is as following.

Theorem 1.2. If G is a planar graph with maximum degree $\Delta \leq 4$, then G is packing $[2, 8]$ colorable.

We name the ten colors of packing $[2, 8]$ coloring as 1_a, 1_b, 2_a, 2_b, 2_c, 2_d, 2_e, 2_f, 2_g, 2_h. We say that 1_a, 1_b are 1-colors and 2_a, 2_b, 2_c, 2_d, 2_e, 2_f, 2_g, 2_h are 2-colors. For a vertex u, let $C_2(u)$ denote the set of 2-colors which are assigned to the vertices with distance 2 to u and $A_2(u) = \{2_a, 2_b, 2_c, 2_d, 2_e, 2_f, 2_g, 2_h\} \setminus C_2(u)$. By the definitions, any two vertices in the same 1-color set have distance greater than 1, while any two vertices in the same 2-color set have distance greater than 2.

II. Reducible configurations

Let G be a counterexample of Theorem 1.2 with minimum $|V(G)| + |E(G)|$. More specifically, G has no packing $[2, 8]$ coloring, but any subgraph $G^{'}$ of G has a packing $[2, 8]$ coloring. By minimality, the graph G is connected. In this section, we will give some reducible configurations of G.

Our main ideas are as follows. Let $G^{'}$ be the graph obtained from G by deleting a vertex u and adding some necessary edges satisfied $\Delta(G^{'}) \leq 4$ and $\left|V(G^{'})\right| + \left|E(G^{'})\right| < |V(G)| + |E(G)|$. Then $G^{'}$ has a packing $[2, 8]$ coloring $\pi^{'}$. Next we extend $\pi^{'}$ to a packing $[2, 8]$ coloring π of G. Obviously, G has no cut-vertex.

Let $G^{'}(a, b)$ denote the subgraph of $G^{'}$ induced by the vertices of colors 1_a and 1_b. And we denote by $\pi_1^{'}(u)$ the set of colors of the neighbors of u in $G^{'}$.

It is an obvious fact that the higher the vertex degree, the greater the complexity of graph coloring. Therefore, it is sufficient to prove the worst case, that is, the degree of a vertex without special specification is 4. In the following, we will analysis some needed vertices under this worst assumption.

Lemma 2.1 In graph G, there is no cut edge.

Proof: On the contrary, suppose that there is a cut edge uv in graph G with $N(u) = \{v, u_1, u_2, u_3\}$ and $N(v) = \{u, v_1, v_2, v_3\}$. Let $G^{'} = G - uv$, we can get two components G_1, G_2. By minimality, both G_1 and G_2 have packing $[2, 8]$ coloring π_1 and π_2, respectively. Then we extend to a packing $[2, 8]$ coloring π of G. We can permute two 1-colors or two of the eight 2-colors in either G_1 or G_2 to ensure that u and v are not in the same 1-color set, that any two vertices in $N(u)$ are not in the same 2-color set, and that any two vertices in $N(v)$ are not in the same 2-color set. Then we can get a packing $[2, 8]$ coloring π of G, a contradiction. Take an example, if u and v are in the same 1-color set, then we only need to permutation 1_a, 1_b in G_1 or G_2. Thus, we get a packing $[2, 8]$ coloring π of G. Other cases can be easily checked. □

Lemma 2.2 $\delta(G) \geq 3$.

Proof: On the contrary, since there is no cut edge, we can assume that there is a 2-vertex u and $N(u) = \{v, w\}$. Let $N(v) = \{u, v_1, v_2, v_3\}$, $N(w) = \{u, w_1, w_2, w_3\}$.

Let $G^{'} = G - u + \{vw\}$, if $vw \notin E(G)$; otherwise, let $G^{'} = G - u$. By the minimality of G, $G^{'}$ has a packing $[2, 8]$ coloring $\pi^{'}$. Then we extend to a packing $[2, 8]$ coloring π of G. Since $d_{G^{'}}(v, w) \leq 2$, we only need to color the vertex u.

If $\{1_a, 1_b\} \not\subseteq \pi_1^{'}(u)$, then color u with 1_a or 1_b, G has a packing $[2, 8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1^{'}(u)$. Without loss of generality, suppose $\pi^{'}(v) = 1_a$, $\pi^{'}(w) = 1_b$. Besides, v has a neighbor of color 1_b and w has a neighbor of color 1_a; otherwise, recolor v by 1_b and color u with 1_a (or recolor w by 1_a and color u with 1_b), then G has a packing $[2, 8]$ coloring π, a contradiction. Since $|N_2(u)| \leq 6$, $|C_2(u)| \leq 6 - 2 = 4$ and $|A_2(u)| \geq 4$. Color u with an available 2-color, G has a packing $[2, 8]$ coloring π, a contradiction. □

Lemma 2.3 In graph G, 3-vertices are independent.

Proof: Let u be a 3-vertex with $N(u) = \{u_1, u_2, u_3\}$. On the contrary, assume u_1 be a 3-vertex. Let $N(u_1) = \{u, u_{1,1}, u_{1,2}\}$.

If the edges $u_1u_2 \notin E(G)$ or $u_1u_3 \notin E(G)$, then $G^{'}$ is obtained by removing the vertex u from G and adding u_1u_2 or u_1u_3; otherwise, u_1u_2 or u_1u_3 is not added (see Fig. 1). By the minimality of G, $G^{'}$ has a packing $[2, 8]$ coloring $\pi^{'}$. Then we extend to a packing $[2, 8]$ coloring π of G. Since $d_{G^{'}}(u_i, u_j) \leq 2$, where $i, j \in \{1, 2, 3\}$, we only need to color the vertex u.

If $\{1_a, 1_b\} \not\subseteq \pi_1^{'}(u)$, then color u with 1_a or 1_b, G has a packing $[2, 8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1^{'}(u)$.

Case 1: $\pi_1^{'}(u) = \{1_a, 1_b\}$.

Without loss of generality, suppose $\pi^{'}(u_1) = 1_a$, $\pi^{'}(u_2) = \pi^{'}(u_3) = 1_b$. Similar to the discussion of Lemma 2.2, u_1 has a neighbor of color 1_b and u_2 or u_3 has a neighbor of color 1_a. Since $|N_2(u)| \leq 8$, $|C_2(u)| \leq 8 - 2 = 6$ and $|A_2(u)| \geq 2$. Color u with an available 2-color, G has a packing $[2, 8]$ coloring π, a contradiction.

Case 2: $\pi_1^{'}(u) = \{1_a, 1_b, 2_a\}$.

If $\pi^{'}(u_1) = 2_a$, then $N_2(u)$ has no vertex colored 2_a in $G^{'}$. If $\{1_a, 1_b\} \neq \{\pi^{'}(u_{1,1}), \pi^{'}(u_{1,2})\}$, then recolor u_1 by the rest 1-color and set $\pi(u) = \pi^{'}(u_1) = 2_a$, G has a packing $[2, 8]$ coloring π. Therefore, $\{1_a, 1_b\} = \{\pi^{'}(u_{1,1}), \pi^{'}(u_{1,2})\}$. Similar to the discussion of Lemma 2.2, both u_2 and u_3 have a neighbor of 1-color, respectively. Thus, $|C_2(u)| \leq 8 + 1 - 2 - 2 = 5$ and $|A_2(u)| \geq 3$. Color u with an available 2-color, G has a packing $[2, 8]$ coloring π, a contradiction.

Suppose $\pi^{'}(u_2) = 2_a$ (symmetry, $\pi^{'}(u_3) = 2_a$). Suppose $\pi^{'}(u_1) = 1_a$, $\pi^{'}(u_3) = 1_b$. If $1_b \notin \pi_1^{'}(u_2)$, then recolor u_2 by 1_b. By Case 1, G has a packing $[2, 8]$ coloring π. Therefore, $1_b \in \pi_1^{'}(u_2)$. Similar to the discussion of Lemma 2.2, both u_1 and u_3 have a neighbor of 1-color, respectively. Thus, $|C_2(u)| \leq 8 + 1 - 2 - 1 = 6$ and $|A_2(u)| \geq 2$. Color u with an available 2-color, G has a packing $[2, 8]$ coloring π, a contradiction. □

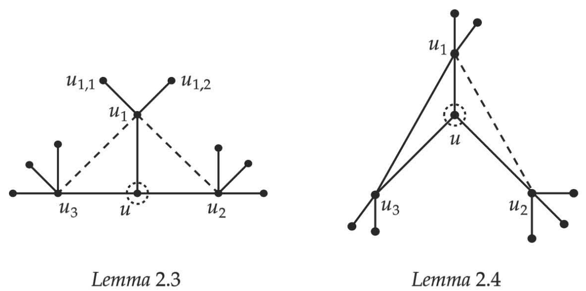

Fig. 1: Illustrations of Lemma 2.3-2.4

Lemma 2.4 In graph G, no 3-vertex is incident with any 3-faces.

Proof: Let u be a 3-vertex with $N(u) = \{u_1, u_2, u_3\}$. On the contrary, assume u is incident with a 3-face. Without loss of generality, suppose this 3-face is $[uu_1u_3]$.

Let $G' = G - u + \{u_1u_2\}$ (see Fig. 1). By the minimality of G, G' has a packing $[2,8]$ coloring π'. Then we extend to a packing $[2,8]$ coloring π of G. Since $d_{G'}(u_i, u_j) \leq 2$, where $i, j \in \{1,2,3\}$, we only need to color the vertex u.

If $\{1_a, 1_b\} \not\subseteq \pi_1'(u)$, then color u with 1_a or 1_b, G has a packing $[2,8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1'(u)$.

Case 1: $\pi_1'(u) = \{1_a, 1_b\}$.

Without loss of generality, suppose $\pi'(u_1) = 1_a$, $\pi'(u_2) = \pi'(u_3) = 1_b$. Since $|N_2(u)| \leq 7$, $|C_2(u)| \leq 7$ and $|A_2(u)| \geq 1$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2: $\pi_1'(u) = \{1_a, 1_b, 2_a\}$.

If $\pi'(u_1) = 2_a$, then $N_2(u)$ has no vertex colored 2_a in G'. Suppose $\pi'(u_2) = 1_a$, $\pi'(u_3) = 1_b$. Similar to the discussion in Lemma 2.2, u_2 has a neighbor of color 1_b and u_3 has a neighbor of color 1_a. If $1_a \notin \pi_1'(u_1)$, then recolor u_1 with 1_a and set $\pi(u) = \pi'(u_1) = 2_a$, G has a packing $[2,8]$ coloring π. Therefore, $1_a \in \pi_1'(u_1)$. Since $|N_2(u)| \leq 7$, $|C_2(u)| \leq 7 + 1 - 3 = 5$ and $|A_2(u)| \geq 3$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction.

Suppose $\pi'(u_2) = 2_a$. Suppose $\pi'(u_1) = 1_a$, $\pi'(u_3) = 1_b$. If $1_b \notin \pi_1'(u_2)$, then recolor u_2 with 1_b. By Case 1, G has a packing $[2,8]$ coloring π. Therefore, $1_b \in \pi_1'(u_2)$. Since $|N_2(u)| \leq 7$, $|C_2(u)| \leq 7 + 1 - 1 = 7$ and $|A_2(u)| \geq 1$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction.

Suppose $\pi'(u_3) = 2_a$. Suppose $\pi'(u_1) = 1_a$, $\pi'(u_2) = 1_b$. Similar to the discussion in Lemma 2.2, u_1 has a neighbor of color 1_b and u_2 has a neighbor of color 1_a. Since $|N_2(u)| \leq 7$, $|C_2(u)| \leq 7 + 1 - 2 = 6$ and $|A_2(u)| \geq 2$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction. □

Lemma 2.5 In graph G, no 3-vertex is incident with any 4-faces.

Proof: Let u be a 3-vertex with $N(u) = \{u_1, u_2, u_3\}$. On the contrary, assume u is incident with a 4-face. Without loss of generality, suppose this 4-face is $[uu_2xu_3]$. Let $N(u_1) = \{u, u_{1,1}, u_{1,2}, u_{1,3}\}$, $N(u_2) = \{u, x, u_{2,1}, u_{2,2}\}$, $N(u_3) = \{u, x, u_{3,1}, u_{3,2}\}$, $N(x) = \{u_2, u_3, x_1, x_2\}$.

Let $G' = G - u + \{u_1u_3\}$ (see Fig. 2). By the minimality of G, G' has a packing $[2,8]$ coloring π'. Then we extend to a packing $[2,8]$ coloring π of G. Since $d_{G'}(u_1, u_2) = 3$, u_1 and u_2 may have the same 2-color in G'. Let's discuss this situation first.

Case 1: $\pi'(u_1)$ and $\pi'(u_2)$ are the same 2-color.

Without loss of generality, suppose $\pi'(u_1) = \pi'(u_2) = 2_a$.

Case 1.1: $\pi'(u_3)$ is 1-color, suppose $\pi'(u_3) = 1_a$.

If $1_a \notin \{\pi'(u_{1,1}), \pi'(u_{1,2}), \pi'(u_{1,3})\}$ or $1_a \notin \pi_1'(u_2)$, then recolor u_1 or u_2 with 1_a and color u with 1_b, G has a packing $[2,8]$ coloring π. Therefore, $1_a \in \{\pi'(u_{1,1}), \pi'(u_{1,2}), \pi'(u_{1,3})\}$ and $1_a \in \pi_1'(u_2)$. Similarly, if $1_b \notin \{\pi'(u_{1,1}), \pi'(u_{1,2}), \pi'(u_{1,3})\}$ or $1_b \notin \pi_1'(u_2)$, then recolor u_1 or u_2 with 1_b. Since $|C_2(u)| \leq 8 + 1 - 2 = 7$ and $|A_2(u)| \geq 1$, color u with an available 2-color, G has a packing $[2,8]$ coloring π. Therefore, $1_b \in \{\pi'(u_{1,1}), \pi'(u_{1,2}), \pi'(u_{1,3})\}$ and $1_b \in \pi_1'(u_2)$. Hence, $\{1_a, 1_b\} \subset \{\pi'(u_{1,1}), \pi'(u_{1,2}), \pi'(u_{1,3})\}$ and $\{1_a, 1_b\} \subset \pi_1'(u_2)$.

We first erase the color of vertex u_2.

If $\pi'(x)$ is 1-color, then $\pi'(x) = 1_b$ and $|C_2(u_2)| \leq 2 + 2 + 3 + 1 = 8$. If $|C_2(u_2)| \leq 7$, then $|A_2(u_2)| \geq 1$. Color u_2 with an available 2-color and u with 1_b, G has a packing $[2,8]$ coloring π. If $|C_2(u_2)| = 8$, then both x_1 and x_2 have neighbors of color 1_a; otherwise, recolor x_1 or x_2 with 1_a and then $|C_2(u_2)| \leq 7$, a contradiction. We erase the color of vertex x. If $1_b \notin \pi_1'(x_1)$, then recolor x_1 with 1_b; otherwise, keep x_1 unchanged. x_2 is done in the similar way. Thus, $|C_2(x)| \leq 2+2+1+1 = 6$ and $|A_2(x)| \geq 2$. Recolor x with an available 2-color and color u_2 with 1_b. Since $|C_2(u)| \leq 8 + 1 - 3 = 6$ and $|A_2(u)| \geq 2$, color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction.

If $\pi'(x)$ is 2-color, then $\{1_a, 1_b\} = \{\pi'(u_{2,1}), \pi'(u_{2,2})\}$. Besides, $u_{2,1}$ and $u_{2,2}$ are in the same connected component in $G'(a,b)$. Suppose $\pi'(u_{2,1}) = 1_a$, $\pi'(u_{2,2}) = 1_b$. Hence, $u_{2,1}$ has a neighbor of color 1_b and $u_{2,2}$ has a neighbor of color 1_a. If $\pi'(x_1) \neq 1_b$ and $\pi'(x_2) \neq 1_b$, then recolor x with 1_b; otherwise keep x unchanged. Thus, $|C_2(u_2)| \leq 2+2+2+1 = 7$ and $|A_2(u_2)| \geq 1$. Color u_2 with an available 2-color and u with 1_b, G has a packing $[2,8]$ coloring π, a contradiction.

Case 1.2: $\pi'(u_3)$ is 2-color, suppose $\pi'(u_3) = 2_b$.

If $\{1_a, 1_b\} \not\subseteq \pi_1'(u_3)$, then recolor u_3 with 1_a or 1_b. By Case 1.1, G has a packing $[2,8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1'(u_3)$. If $\{1_a, 1_b\} \not\subseteq \pi_1'(u_1)$, then recolor u_1 with 1_a (or 1_b) and color u with 1_b (or 1_a), G has a packing $[2,8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1'(u_1)$. Similarly, $\{1_a, 1_b\} \subset \pi_1'(u_2)$.

Since u_1, u_2 and u_3 are colored 2-color, color u with 1_a or 1_b. The key is to recolor the vertex u_2. Let $N(u_{2,1}) = \{u_2, t_1, t_2, t_3\}$. We erase the color of vertex u_2.

If $\pi'(x)$ is 2-color, then $\{1_a, 1_b\} = \{\pi'(u_{2,1}), \pi'(u_{2,2})\}$. Suppose $\pi'(u_{2,1}) = 1_a$, $\pi'(u_{2,2}) = 1_b$. Hence, $u_{2,1}$ has a neighbor of color 1_b and $u_{2,2}$ has a neighbor of color 1_a. If $\pi'(x_1)$ and $\pi'(x_2)$ are 2-color, then recolor x with 1_b. Now, $|C_2(u_2)| \leq 2 + 2 + 2 + 1 + 1 = 8$. If $|C_2(u_2)| \leq 7$, then $|A_2(u_2)| \geq 1$. Color u_2 with an available 2-color and u with 1_a or 1_b, G has a packing $[2,8]$ coloring π. If $|C_2(u_2)| = 8$, then both t_2 and t_3 have neighbors of color 1_b. We erase the color of vertex $u_{2,1}$. If $1_a \notin \pi_1'(t_2)$, then recolor t_2 with 1_a; otherwise, keep t_2 unchanged. t_3 is done in the similar way. Thus, $|C_2(u_{2,1})| \leq 2 + 2 + 2 = 6$ and $|A_2(u_{2,1})| \geq 2$. Recolor $u_{2,1}$ with an available 2-color, set $\pi(u_2) = \pi'(u_{2,1}) = 1_a$ and $\pi(u) = 1_b$, G has a packing $[2,8]$ coloring π, a contradiction. Therefore, at least one of x_1 and x_2 is colored 1-color. If $\{1_a, 1_b\} \neq \{\pi'(x_1), \pi'(x_2)\}$, then recolor x with the rest 1-color; otherwise, keep x unchanged. Now, $|C_2(u_2)| \leq 2 + 2 + 1 + 2 = 7$, then $|A_2(u_2)| \geq 1$. Color u_2 with an available 2-color and u with 1_a or 1_b, G has a packing $[2,8]$ coloring π, a contradiction.

Suppose $\pi'(x)$ is 1-color. Let $\pi'(x) = 1_b$, then at least one of the vertices x_1 and x_2 is colored 1_a. Suppose $\pi'(x_1) = 1_a$, then x_1 has a neighbor different from x of color 1_b. Then $|C_2(u_2)| \leq 2 + 3 + 1 + 1 + 1 = 8$. If $|C_2(u_2)| \leq 7$, then $|A_2(u_2)| \geq 1$. Color u_2 with an available 2-color and u with 1_a or 1_b, G has a packing $[2,8]$ coloring π. If $|C_2(u_2)| = 8$, then x_2 has a neighbor of color 1_a. Besides, the neighbors of

$u_{2,2}$, except for the vertex u_2, are all 2-colors. We erase the color of vertex x. If $1_b \notin \pi_1^{'}(x_2)$, then recolor x_2 with 1_b; otherwise, keep x_2 unchanged. Thus, $|C_2(x)| \leq 2+2+2+1=7$ and $|A_2(x)| \geq 1$. Color x with an available 2-color, recolor $u_{2,2}$ with 1_a, set $\pi(u_2) = \pi^{'}(x) = 1_b$ and $\pi(u) = 1_a$, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2: $\pi^{'}(u_1)$ and $\pi^{'}(u_2)$ are not the same 2-color.

We only need to color the vertex u. If $\{1_a, 1_b\} \nsubseteq \pi_1^{'}(u)$, then color u with 1_a or 1_b, G has a packing $[2,8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1^{'}(u)$.

Case 2.1: $\pi_1^{'}(u) = \{1_a, 1_b\}$.

Without loss of generality, suppose $\pi^{'}(u_1) = 1_a$, $\pi^{'}(u_3) = 1_b$. Then u_1 has a neighbor of color 1_b and u_3 has a neighbor of color 1_a. Hence, there are at least two vertices with 1-color in $N_2(u)$. Since $|N_2(u)| \leq 8$, $|C_2(u)| \leq 8-2=6$ and $|A_2(u)| \geq 2$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2.2: $\pi_1^{'}(u) = \{1_a, 1_b, 2_a\}$.

Similar to Case 2.1, there are at least two vertices with 1-color in $N_2(u)$. Since $|N_2(u)| \leq 8$, $|C_2(u)| \leq 8+1-2=7$ and $|A_2(u)| \geq 1$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction. ☐

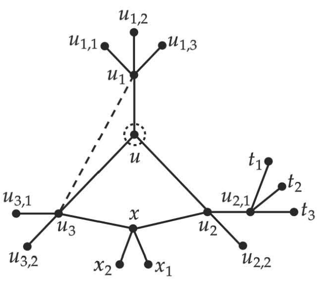

Fig. 2: Illustrations of Lemma 2.5

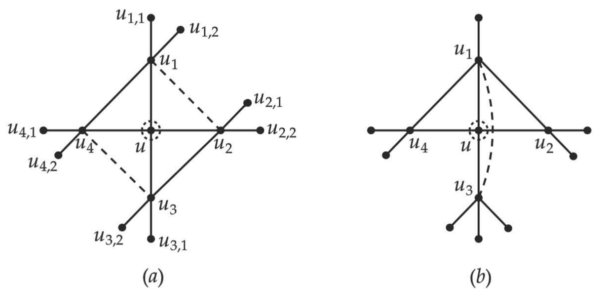

Fig. 3: Illustrations of Lemma 2.6

By Lemma 2.3-2.5, every vertex of G has degree 3 or 4, and all faces of G have the following properties.

Every 3-face is (4,4,4)-face;

Every 4-face is (4,4,4,4)-face;

Every 5-face is (3,4,3,4,4)-face, (3,4,4,4,4)-face or (4,4,4,4,4)-face.

Next, we focus on the question of how many 3-, 4- or 5-faces that the 4-vertex is incident with.

Lemma 2.6 In graph G, every 4-vertex is incident with at most one 3-face.

Proof: Let u be a 4-vertex with $N(u) = \{u_1, u_2, u_3, u_4\}$. On the contrary, assume u is incident with two 3-faces.

Case 1: Suppose these two 3-faces are $[uu_1u_4]$ and $[uu_2u_3]$. Let $N(u_1) = \{u, u_4, u_{1,1}, u_{1,2}\}$, $N(u_2) = \{u, u_3, u_{2,1}, u_{2,2}\}$, $N(u_3) = \{u, u_2, u_{3,1}, u_{3,2}\}$, $N(u_4) = \{u, u_1, u_{4,1}, u_{4,2}\}$. Let $G^{'} = G - u + \{u_1u_2, u_3u_4\}$ (see Fig. 3(a)). By the minimality of G, $G^{'}$ has a packing $[2,8]$ coloring $\pi^{'}$. Then we extend to a packing $[2,8]$ coloring π of G. Since $d_{G'}(u_i, u_j) \leq 2$, where $i, j \in \{1,2,3,4\}$, we only need to color the vertex u.

If $\{1_a, 1_b\} \nsubseteq \pi_1^{'}(u)$, then color u with 1_a or 1_b, G has a packing $[2,8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1^{'}(u)$.

Case 1.1: $\pi_1^{'}(u) = \{1_a, 1_b\}$.

Without loss of generality, suppose $\pi^{'}(u_1) = \pi^{'}(u_3) = 1_a$, $\pi^{'}(u_2) = \pi^{'}(u_4) = 1_b$. Since $|N_2(u)| \leq 8$, then the vertices in $N_2(u)$ are colored different 2-colors; otherwise, $|C_2(u)| \leq 7$ and $|A_2(u)| \geq 1$, color u with an available 2-color, G has a packing $[2,8]$ coloring π.

If $1_b \notin \pi_1^{'}(u_{1,1})$, $1_b \notin \pi_1^{'}(u_{1,2})$, $1_b \notin \pi_1^{'}(u_{3,1})$ or $1_b \notin \pi_1^{'}(u_{3,2})$, then recolor $u_{1,1}$, $u_{1,2}$, $u_{3,1}$ or $u_{3,2}$ with 1_b. Then $|C_2(u)| \leq 8-1=7$ and $|A_2(u)| \geq 1$, color u with an available 2-color, G has a packing $[2,8]$ coloring π. Therefore, $1_b \in \pi_1^{'}(u_{1,1})$, $1_b \in \pi_1^{'}(u_{1,2})$, $1_b \in \pi_1^{'}(u_{3,1})$ and $1_b \in \pi_1^{'}(u_{3,2})$.

We first erase the color of vertices u_1 and u_3. If $1_a \notin \pi_1^{'}(u_{1,1})$, then we recolor $u_{1,1}$ with 1_a; otherwise, keep $u_{1,1}$ unchanged. $u_{1,2}$, $u_{3,1}$ and $u_{3,2}$ are done in the similar way. Thus, $|C_2(u_1)| \leq 2+2+2=6$ and $|A_2(u_1)| \geq 2$, $|C_2(u_3)| \leq 2+2+2=6$ and $|A_2(u_3)| \geq 2$. Color u_1 and u_3 with an available 2-color and set $\pi(u) = \pi^{'}(u_1) = \pi^{'}(u_3) = 1_a$, G has a packing $[2,8]$ coloring π, a contradiction.

Case 1.2: $\pi_1^{'}(u) = \{1_a, 1_b, 2_a\}$.

By symmetry, suppose $\pi^{'}(u_1) = 2_a$, $\pi^{'}(u_2) = \pi^{'}(u_4) = 1_b$, $\pi^{'}(u_3) = 1_a$. If $1_a \notin \pi_1^{'}(u_1)$, then recolor u_1 with 1_a. By Case 1.1, G has a packing $[2,8]$ coloring π, a contradiction. Therefore, $1_a \in \pi_1^{'}(u_1)$. Similar to the discussion in Case 1.1, $1_b \in \pi_1^{'}(u_{3,1})$ and $1_b \in \pi_1^{'}(u_{3,2})$. We erase the color of vertex u_3. If $1_a \notin \pi_1^{'}(u_{3,1})$, then we recolor $u_{3,1}$ with 1_a; otherwise, keep $u_{3,1}$ unchanged. $u_{3,2}$ is done in the similar way. Thus, $|C_2(u_3)| \leq 2+2+2+1=7$ and $|A_2(u_3)| \geq 1$. Color u_3 with an available 2-color and set $\pi(u) = \pi^{'}(u_3) = 1_a$, G has a packing $[2,8]$ coloring π, a contradiction.

Case 1.3: $\pi_1^{'}(u) = \{1_a, 1_b, 2_a, 2_b\}$.

Suppose u_1 and u_2 are colored 2-colors (or u_1 and u_3 are colored 2-colors). Let $\pi^{'}(u_1) = 2_a$, $\pi^{'}(u_2) = 2_b$, $\pi^{'}(u_3) = 1_a$, $\pi^{'}(u_4) = 1_b$. Then u_3 has a neighbor of color 1_b and u_4 has a neighbor of color 1_a; otherwise, recolor u_3 by 1_b and color u with 1_a (or recolor u_4 by 1_a and color u with 1_b), G has a packing $[2,8]$ coloring π, a contradiction. Now, $|C_2(u)| \leq 8+2-2=8$. Then the rest vertices in $N_2(u)$ are colored different 2-colors and are different from 2_a and 2_b; otherwise, $|C_2(u)| \leq 7$, color u with an available 2-color, G has a packing $[2,8]$ coloring π. We can recolor u_2 with 1_b and set $\pi(u) = \pi^{'}(u_2) = 2_b$, G has a packing $[2,8]$ coloring π, a contradiction.

Suppose u_1 and u_4 are colored 2-colors. Let $\pi^{'}(u_1) = 2_a$, $\pi^{'}(u_4) = 2_b$, $\pi^{'}(u_2) = 1_b$, $\pi^{'}(u_3) = 1_a$. If $1_a \notin \pi_1^{'}(u_1)$ or $1_b \notin \pi_1^{'}(u_4)$, then recolor u_1 by 1_a (or recolor u_4 by 1_b). By Case 1.2, G has a packing $[2,8]$ coloring π. Therefore, $1_a \in \pi_1^{'}(u_1)$ and $1_b \in \pi_1^{'}(u_4)$. Now, $|C_2(u)| \leq 8+2-2=8$. Then the rest vertices in $N_2(u)$ are colored different 2-colors and are different from 2_a and 2_b. We can recolor u_4 with 1_a and set $\pi(u) = \pi^{'}(u_4) = 2_b$, G has a packing $[2,8]$ coloring

π, a contradiction.

Case 2: Suppose these two 3-faces are $[uu_1u_4]$ and $[uu_1u_2]$. Let $G^{'} = G - u + \{u_1u_3\}$ (see Fig. 3(b)). By the minimality of G, $G^{'}$ has a packing $[2,8]$ coloring $\pi^{'}$. Then we extend to a packing $[2,8]$ coloring π of G. Since $d_{G'}(u_i,u_j) \le 2$, where $i,j \in \{1,2,3,4\}$, we only need to color the vertex u. Similar to the proof of Case 1, G has a packing $[2,8]$ coloring π, a contradiction. □

Lemma 2.7 If a 4-vertex u is incident with one 3-face, then u is incident with at most one 4-face.

Proof: Let u be a 4-vertex with $N(u) = \{u_1,u_2,u_3,u_4\}$. On the contrary, suppose the 4-vertex u is incident with a 3-face and two 4-faces. Without loss of generality, suppose this 3-face is $[uu_1u_4]$.

Case 1: Suppose these two 4-faces are $[uu_1x_1u_2]$ and $[uu_2x_2u_3]$. Let $G^{'} = G - u + \{u_1u_2, u_3u_4\}$ (see Fig. 4(a)). By the minimality of G, $G^{'}$ has a packing $[2,8]$ coloring $\pi^{'}$. Then we extend to a packing $[2,8]$ coloring π of G. Since $d_{G'}(u_i,u_j) \le 2$, where $i,j \in \{1,2,3,4\}$, we only need to color the vertex u. Similar to the discussion of Lemma 2.6, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2: Suppose these two 4-faces are $[uu_1x_1u_2]$ and $[uu_3x_2u_4]$. Let $N(u_1) = \{u,u_4,x_1,u_{1,1}\}$, $N(u_2) = \{u,x_1,u_{2,1},u_{2,2}\}$, $N(u_3) = \{u,x_2,u_{3,1},u_{3,2}\}$, $N(u_4) = \{u,u_1,x_2,u_{4,1}\}$.

Let $G^{'} = G - u + \{u_2u_3\}$ (see Fig. 4(b)). By the minimality of G, $G^{'}$ has a packing $[2,8]$ coloring $\pi^{'}$. Then we extend to a packing $[2,8]$ coloring π of G. Since $d_{G'}(u_1,u_3) = 3$ and $d_{G'}(u_2,u_4) = 3$, u_1, u_3 may have the same 2-color in $G^{'}$ and u_2, u_4 may have the same 2-color in $G^{'}$.

Case 2.1: $\pi^{'}(u_1)$, $\pi^{'}(u_3)$ are not the same 2-color and $\pi^{'}(u_2)$, $\pi^{'}(u_4)$ are not the same 2-color.

We only need to color the vertex u. If $\{1_a,1_b\} \nsubseteq \pi_1^{'}(u)$, then color u with 1_a or 1_b, G has a packing $[2,8]$ coloring π. Therefore, $\{1_a,1_b\} \subset \pi_1^{'}(u)$.

Case 2.1.1: $\pi_1^{'}(u) = \{1_a,1_b\}$.

Without loss of generality, suppose $\pi^{'}(u_1) = 1_a$, $\pi^{'}(u_4) = 1_b$. Since $|N_2(u)| \le 8$, $|C_2(u)| \le 8$. If $|C_2(u)| \le 7$, then color u with an available 2-color, G has a packing $[2,8]$ coloring π. If $|C_2(u)| = 8$, then the vertices in $N_2(u)$ are colored different 2-colors. If $1_b \notin \pi_1^{'}(u_{1,1})$ or $1_b \notin \pi_1^{'}(x_1)$, then recolor $u_{1,1}$ or x_1 with 1_b and then $|C_2(u)| \le 7$, a contradiction. Therefore, $1_b \in \pi_1^{'}(u_{1,1})$ and $1_b \in \pi_1^{'}(x_1)$. We erase the color of vertex u_1. If $1_a \notin \pi_1^{'}(u_{1,1})$, then recolor $u_{1,1}$ with 1_a; otherwise, keep $u_{1,1}$ unchanged. x_1 is done in the similar way. Then $|C_2(u_1)| \le 2+2+2 = 6$ and $|A_2(u)| \ge 2$. Color u_1 with an available 2-color, recolor u_2 and u_3 with 1_b and color u with 1_a, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2.1.2: $\pi_1^{'}(u) = \{1_a,1_b,2_a\}$.

If $\pi^{'}(u_2)$ is 2-color, then $\{1_a,1_b\} \subset \{\pi^{'}(x_1), \pi^{'}(u_{2,1}), \pi^{'}(u_{2,2})\}$; otherwise, recolor u_2 with the rest 1-color and then we can get a contradiction by Case 2.1.1. Thus, $|C_2(u)| \le 8+1-2 = 7$ and $|A_2(u)| \ge 1$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction. The proof that $\pi^{'}(u_3)$ is 2-color is similar.

If $\pi^{'}(u_1)$ or $\pi^{'}(u_4)$ is 2-color, then there are at least two vertices with 1-color in $N_2(u)$. Thus, $|C_2(u)| \le 8+1-2 = 7$ and $|A_2(u)| \ge 1$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2.1.3: $\pi_1^{'}(u) = \{1_a,1_b,2_a,2_b\}$.

If $\pi^{'}(u_2)$ and $\pi^{'}(u_3)$ are 2-color, then $\{1_a,1_b\} \subset \pi_1^{'}(u_2)$ and $\{1_a,1_b\} \subset \pi_1^{'}(u_3)$; otherwise, recolor u_2 or u_3 with the rest 1-color, and then we can get a contradiction by Case 2.1.2. Thus, there are at least four vertices with 1-color in $N_2(u)$, and then $|C_2(u)| \le 8+2-4 = 6$ and $|A_2(u)| \ge 2$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction. Therefore, at least one of the vertices u_1 and u_4 is colored 2-color. Then there are at least two vertices with 1-color in $N_2(u)$. Hence, $|C_2(u)| \le 8+2-2 = 8$. If $|C_2(u)| \le 7$, then color u with an available 2-color, G has a packing $[2,8]$ coloring π. If $|C_2(u)| = 8$, then the rest vertices in $N_2(u)$ are colored different 2-colors and are different from 2_a and 2_b. We can recolor u_1 or u_4 with the rest 1-color and set $\pi(u) = \pi^{'}(u_1)$ or $\pi(u) = \pi^{'}(u_4)$, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2.2: $\pi^{'}(u_1)$ and $\pi^{'}(u_3)$ are the same 2-color (symmetry, $\pi^{'}(u_2)$ and $\pi^{'}(u_4)$ are the same 2-color).

Without loss of generality, suppose $\pi^{'}(u_1) = \pi^{'}(u_3) = 2_a$. Then $\{1_a,1_b\} \subset \pi_1^{'}(u_1)$ and $\{1_a,1_b\} \subset \pi_1^{'}(u_3)$; otherwise, recolor u_1 or u_3 with the rest 1-color, and then $\pi^{'}(u_1)$ and $\pi^{'}(u_3)$ are not the same 2-color, we can get a contradiction by Case 2.1.

Case 2.2.1: u_2 and u_4 are colored 1-color.

Since $|C_2(u_1)| \le 2+2+1+1 = 6$ and $|A_2(u_1)| \ge 2$, recolor u_1 with an available 2-color. Since $|C_2(u)| \le 8+2-3 = 7$ and $|A_2(u)| \ge 1$, color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2.2.2: One of u_2 and u_4 is colored 2-color.

Suppose $\pi^{'}(u_2)$ is 2-color (symmetry, $\pi^{'}(u_4)$ is 2-color). Let $\pi^{'}(u_2) = 2_b$. If $\{1_a,1_b\} \nsubseteq \pi_1^{'}(u_2)$, then recolor u_2 with the rest 1-color, we can get a contradiction by Case 2.2.1. Therefore, $\{1_a,1_b\} \subset \pi_1^{'}(u_2)$. Since $\{1_a,1_b\} \subset \pi_1^{'}(u_1)$ and $\{1_a,1_b\} \subset \pi_1^{'}(u_3)$, $|C_2(u_1)| \le 3+1+1+2 = 7$ and $|A_2(u_1)| \ge 1$. Recolor u_1 with an available 2-color and color u with the rest 1-color, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2.2.3: u_2 and u_4 are colored 2-color.

Suppose $\pi^{'}(u_2) \ne \pi^{'}(u_4)$. Let $\pi^{'}(u_2) = 2_b$, $\pi^{'}(u_4) = 2_c$. $\{1_a,1_b\} \subset \pi_1^{'}(u_2)$ and $\{1_a,1_b\} \subset \pi_1^{'}(u_4)$; otherwise, recolor u_2 or u_4 with the rest 1-color, we can get a contradiction by Case 2.2.2. Since $\{1_a,1_b\} \subset \pi_1^{'}(u_1)$ and $\{1_a,1_b\} \subset \pi_1^{'}(u_3)$, $|C_2(u_1)| \le 2+1+3 = 6$ and $|A_2(u_1)| \ge 2$. Recolor u_1 with an available 2-color and color u with 1-color, G has a packing $[2,8]$ coloring π, a contradiction.

Suppose $\pi^{'}(u_2) = \pi^{'}(u_4)$. Let $\pi^{'}(u_2) = \pi^{'}(u_4) = 2_b$. Similarly, $\{1_a,1_b\} \subset \pi_1^{'}(u_2)$ and $\{1_a,1_b\} \subset \pi_1^{'}(u_4)$. First we erase the color of vertices u_1 and u_4. Since $|C_2(u_i)| \le 2+1+2 = 5$ and $|A_2(u_i)| \ge 3$, $i \in \{1,4\}$, color u_1 and u_4 with an available 2-color in order and color u with 1-color, G has a packing $[2,8]$ coloring π, a contradiction. □

Lemma 2.8 If a 4-vertex u is incident with a 3-face and a 4-face, then u has no 3-neighbors.

Proof: Let u be a 4-vertex with $N(u) = \{u_1,u_2,u_3,u_4\}$.

Case 1: Suppose the 3-face is $[uu_1u_4]$ and the 4-face is $[uu_2xu_3]$ (see Fig. 5(a)). By Lemma 2.4 and Lemma 2.5, no 3-vertex is incident with any 4^{-}-faces. Then u_1, u_2, u_3 and u_4 are 4-vertices. Thus, the 4-vertex u has no 3-neighbors.

Case 2: Suppose the 3-face is $[uu_1u_4]$ and the 4-face is $[uu_1xu_2]$ (see Fig. 5(b)). By Lemma 2.4 and Lemma 2.5, no 3-vertex is incident with any 4^{-}-faces. Then u_1, u_2 and u_4

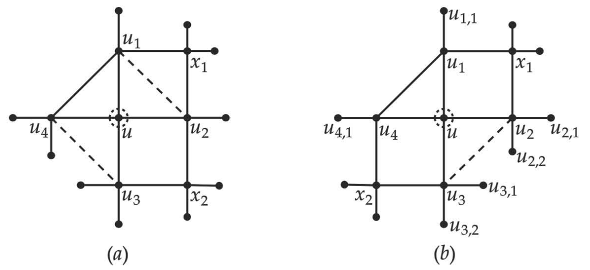

Fig. 4: Illustrations of Lemma 2.7

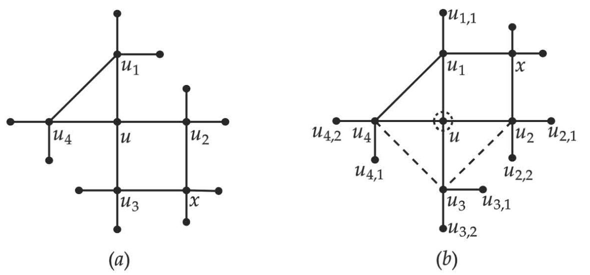

Fig. 5: Illustrations of Lemma 2.8

are 4-vertices. On the contrary, suppose u has a 3-neighbor. Then this 3-neighbor is u_3. Let $N(u_1) = \{u, u_4, x, u_{1,1}\}$, $N(u_2) = \{u, x, u_{2,1}, u_{2,2}\}$, $N(u_3) = \{u, u_{3,1}, u_{3,2}\}$, $N(u_4) = \{u, u_1, u_{4,1}, u_{4,2}\}$.

Let $G^{'} = G - u + \{u_2u_3, u_3u_4\}$. By the minimality of G, $G^{'}$ has a packing $[2,8]$ coloring $\pi^{'}$. Then we extend to a packing $[2,8]$ coloring π of G. Since $d_{G'}(u_i, u_j) \leq 2$, where $i, j \in \{1,2,3,4\}$, we only need to color the vertex u.

If $\{1_a, 1_b\} \nsubseteq \pi_1^{'}(u)$, then color u with 1_a or 1_b, G has a packing $[2,8]$ coloring π. Therefore, $\{1_a, 1_b\} \subset \pi_1^{'}(u)$.

Case 2.1: $\pi_1^{'}(u) = \{1_a, 1_b\}$.

Without loss of generality, suppose $\pi^{'}(u_1) = \pi^{'}(u_3) = 1_a$, $\pi^{'}(u_2) = \pi^{'}(u_4) = 1_b$. Since $|N_2(u)| \leq 8$, then the vertices in $N_2(u)$ are colored different 2-colors; otherwise, $|C_2(u)| \leq 7$ and $|A_2(u)| \geq 1$, color u with an available 2-color, G has a packing $[2,8]$ coloring π.

If $1_b \notin \pi_1^{'}(u_{1,1})$, then recolor $u_{1,1}$ with 1_b and set $\pi(u) = \pi^{'}(u_{1,1})$, G has a packing $[2,8]$ coloring π. Therefore, $1_b \in \pi_1^{'}(u_{1,1})$. We erase the color of vertex u_1. If $1_a \notin \pi_1^{'}(u_{1,1})$, then we recolor $u_{1,1}$ by 1_a; otherwise, keep $u_{1,1}$ unchanged. x is done in the similar way. Thus, $|C_2(u_1)| \leq 2+2+2 = 6$ and $|A_2(u_1)| \geq 2$. Color u_1 with an available 2-color, recolor u_3 by 1_b and set $\pi(u) = \pi^{'}(u_1) = \pi^{'}(u_3) = 1_a$, G has a packing $[2,8]$ coloring π, a contradiction.

Case 2.2: $\pi_1^{'}(u) = \{1_a, 1_b, 2_a\}$.

Suppose $\pi^{'}(u_1)$ is 2-color. Let $\pi^{'}(u_1) = 2_a$, $\pi^{'}(u_2) = \pi^{'}(u_4) = 1_b$, $\pi^{'}(u_3) = 1_a$. Then u_3 has a neighbor of color 1_b and u_2 or u_4 has a neighbor of color 1_a. Thus, $|C_2(u)| \leq 8+1-2 = 7$ and $|A_2(u)| \geq 1$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction. The proof that $\pi^{'}(u_4)$ is 2-color is similar.

Suppose $\pi^{'}(u_2)$ is 2-color. Let $\pi^{'}(u_2) = 2_a$, $\pi^{'}(u_1) = \pi^{'}(u_3) = 1_a$, $\pi^{'}(u_4) = 1_b$. If $1_b \notin \pi_1^{'}(u_2)$, then recolor u_2 by 1_b. By Case 2.1, G has a packing $[2,8]$ coloring π. Therefore, $1_b \in \pi_1^{'}(u_2)$. Now, $|C_2(u)| \leq 8+1-1 = 8$. If $|C_2(u)| \leq 7$, then color u with an available 2-color, G has a packing $[2,8]$ coloring π. If $|C_2(u)| = 8$, then the rest vertices in $N_2(u)$ are colored different 2-colors and are different from 2_a. We can recolor u_2 with 1_a and set $\pi(u) = \pi^{'}(u_2) = 2_a$, G has a packing $[2,8]$ coloring π, a contradiction. The proof that $\pi^{'}(u_3)$ is 2-color is similar.

Case 2.3: $\pi_1^{'}(u) = \{1_a, 1_b, 2_a, 2_b\}$.

If $\pi^{'}(u_2)$ and $\pi^{'}(u_3)$ are 2-colors, then $\{1_a, 1_b\} \subset \pi_1^{'}(u_2)$ and $\{1_a, 1_b\} \subset \pi_1^{'}(u_3)$; otherwise, recolor u_2 or u_3 by the rest 1-color, and then we can get a contradiction by Case 2.2. Thus, there are at least four vertices with 1-color in $N_2(u)$ and then $|C_2(u)| \leq 8+2-4 = 6$ and $|A_2(u)| \geq 2$. Color u with an available 2-color, G has a packing $[2,8]$ coloring π, a contradiction. Therefore, at least one of the vertices u_1 and u_4 is colored 2-color. Suppose $\pi^{'}(u_1)$ is 2-color. Then there are at least two vertices with 1-color in $N_2(u)$. Hence, $|C_2(u)| \leq 8+2-2 = 8$. If $|C_2(u)| \leq 7$, then color u with an available 2-color, G has a packing $[2,8]$ coloring π. If $|C_2(u)| = 8$, then the rest vertices in $N_2(u)$ are colored different 2-colors and are different from 2_a and 2_b. We can recolor u_1 by the rest 1-color and set $\pi(u) = \pi^{'}(u_1)$, G has a packing $[2,8]$ coloring π, a contradiction. □

III. Discharging

In this section, we apply discharge rules to complete the proof of Theorem 1.2. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$\sum_{u \in V(G)} (d(u) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.$$

Note that the total charge is fixed in the process of discharging. We assign an initial charge of $d(u) - 6$ to each vertex and an initial charge of $2d(f) - 6$ to each face of G, then only 3- and 4-vertices have negative initial charges. Next, we design appropriate discharge rules and redistribute the charge among vertices and faces to get the final contradiction.

Now we design the following discharge rules:

$R1$ Every 4-face sends $\frac{1}{2}$ to each vertex on its boundary.

$R2$ Every 5-face sends 1 to each 3-vertex on its boundary.

$R3$ Every (3,4,3,4,4)-face sends $\frac{2}{3}$ to each 4-vertex on its boundary.

$R4$ Every (3,4,4,4,4)-face sends $\frac{3}{4}$ to each 4-vertex on its boundary.

$R5$ Every (4,4,4,4,4)-face sends $\frac{4}{5}$ to each 4-vertex on its boundary.

$R6$ Every 6^+-face sends 1 to each vertex on its boundary.

Next, we will prove that after discharging, each face and vertex has a non-negative new charge, leading to the final contradiction. Obviously, the final charge of 3-face is non-negative. We only check the final charge of 3-vertex, 4-vertex and 4^+-face.

First, we prove that the final charge of each face is non-negative.

4-face: By Lemma 2.3-2.5, every 4-face is (4,4,4,4)-face. The initial charge of (4,4,4,4)-face is $2d(f) - 6 = 2\times4 - 6 = 2$. The finial charge is $2 - 4 \times \frac{1}{2} = 0$ by $R1$.

5-face: By Lemma 2.3, every 5-face has at most two 3-vertices on its boundary. The initial charge of 5-face is $2d(f) - 6 = 2\times5 - 6 = 4$. We have the following three cases about 5-face:

1) (3,4,3,4,4)- face: The finial charge is $4 - 2\times1 - 3\times\frac{2}{3} = 0$ by $R2$ and $R3$.

2) (3,4,4,4,4)-face: The finial charge is $4-1\times1-4\times\frac{3}{4}=0$ by $R2$ and $R4$.

3) (4,4,4,4,4)-face: The finial charge of is $4-5\times\frac{4}{5}=0$ by $R5$.

6^+-face: The initial charge of 6^+-face is $2d(f)-6$. Hence, the finial charge is $2d(f)-6-d(f)\times1=d(f)-6\geq0$ by $R6$.

Next, we prove that the final charge of each vertex is non-negative.

3-vertex: By Lemma 2.4 and Lemma 2.5, no 3-vertex is incident with any 4^--faces. And the initial charge of 3-vertex is $d(u)-6=3-6=-3$. By $R2$ and $R6$, the finial charge is $-3+d(u)\times1=-3+3=0$.

4-vertex: The initial charge of 4-vertex is $d(u)-6=4-6=-2$. By Lemma 2.6, every 4-vertex u is incident with at most one 3-face.

1) If a 4-vertex u is incident with a 3-face, then u is incident with at most one 4-face by Lemma 2.7.

1.1) If a 4-vertex u is incident with a 3-face and a 4-face, then u has no 3-neighbors by Lemma 2.8. Then u is incident with two 5^+-faces, where the 5-faces are (3,4,4,4,4)- or (4,4,4,4,4)-faces. By $R1$ and $R4$-6, the finial charge of u is at least $-2+1\times\frac{1}{2}+2\times\frac{3}{4}=0$.

1.2) If a 4-vertex u is incident with a 3-face but not with a 4-face, then u is incident with three 5^+-face. By $R1$ and $R3$-6, the finial charge of u is at least $-2+3\times\frac{2}{3}=0$.

2) If a 4-vertex u is not incident with 3-face, then the finial charge of u is at least $-2+4\times\frac{1}{2}=0$ by $R1$ and $R3$-6.

Now the final charge of all vertices and faces are non-negative, which contradicts the initial charge -12. Thus, there is no counterexample G existing and we complete the proof of Theorem 1.2.

IV. Conclusion

In fact, the smaller the maximum degree, the more difficult it is to use the discharge method. A further research is to reduce the number of 2-colors.

Packing coloring has a broader application background. Different distance requirements correspond to different real-world contexts. Here we define packing $[2,8]$ coloring to represent the packing S coloring for $S=(1,1,2,2,2,2,2,2,2,2)$. Similar definitions can be established; for instance, packing $[2,2,4]$ coloring refers to packing S coloring for $S=(1,1,2,2,3,3,3,3)$. M. Mortada and O. Togni [14] in their paper used exponents to denote repetitions of integers in a sequence, for example, $(1^2,2^2,3^4)=(1,1,2,2,3,3,3,3)$.

References

[1] W. Goddard, S. Hedetniemi, S. Hedetniemi, J. Harris, and D. Rall, "Braodcast chromatic numbers of graphs." *Ars Comb.*, vol. 86, 01 2008.

[2] W. Goddard and H. Xu, "The s-packing chromatic number of a graph," *Discussiones Mathematicae Graph Theory*, vol. 32, no. 4, pp. 795–806, 2012.

[3] N. Gastineau and O. Togni, "S-packing colorings of cubic graphs," *Discrete Mathematics*, vol. 339, no. 10, pp. 2461–2470, 2016.

[4] W. Yang and B. Wu, "On packing s-colorings of subcubic graphs," *Discrete Applied Mathematics*, vol. 334, pp. 1–14, 2023.

[5] M. Mortada and O. Togni, "About s-packing coloring of subcubic graphs," *Discrete Mathematics*, vol. 347, no. 5, p. 113917, 2024.

[6] M. Chen, L. Miao, and S. Zhou, "2-distance coloring of planar graphs with maximum degree 5," *Discrete Mathematics*, vol. 345, no. 4, p. 112766, 2022.

[7] J. Hou, Y. Jin, L. Miao, and Q. Zhao, "Coloring squares of planar graphs with maximum degree at most five," *Graphs and Combinatorics*, vol. 39, no. 2, p. 20, 2023.

[8] K. Aoki, "Improved 2-distance coloring of planar graphs with maximum degree 5," *arXiv preprint arXiv:2307.16394*, 2023.

[9] J. Zou, M. Han, and H.-J. Lai, "Square coloring of planar graphs with maximum degree at most five," *Discrete Applied Mathematics*, vol. 353, pp. 4–11, 2024.

[10] Z. Deniz, "On 2-distance 16-coloring of planar graphs with maximum degree at most five," *arXiv preprint arXiv:2310.15899*, 2023.

[11] G. Wegner, "Graphs with given diameter and a coloring problem," 1977.

[12] J. Balogh, A. Kostochka, and X. Liu, "Packing chromatic number of subdivisions of cubic graphs," *Graphs and Combinatorics*, vol. 35, pp. 513–537, 2019.

[13] X. Liu, X. Zhang, and Y. Zhang, "Every subcubic graph is packing $(1,1,2,2,3)$-colorable," *arXiv preprint arXiv:2404.09337*, 2024.

[14] M. Mortada and O. Togni, "Further results and questions on s-packing coloring of subcubic graphs," *arXiv preprint arXiv:2407.07424*, 2024.

[15] C. Thomassen, "The square of a planar cubic graph is 7-colorable," *Journal of Combinatorial Theory, Series B*, vol. 128, pp. 192–218, 2018.

[16] N. Bousquet, Q. Deschamps, L. De Meyer, and T. Pierron, "Square coloring planar graphs with automatic discharging," *SIAM Journal on Discrete Mathematics*, vol. 38, no. 1, pp. 504–528, 2024.