

Harvesting Policy in a Stage-Structured Model of Single-Species Fish Populations with Time-Dependent Mortality

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Abstract—In this study, we introduce a model for a single-species fish population characterized by two distinct stages: mature and immature. Traditionally, assuming a constant natural mortality rate for any fish species is impractical. Therefore, we propose a time-dependent function to represent natural mortality, acknowledging its exponential decrease over time. However, an indefinite decrease would inaccurately imply zero mortality over infinite time, which contradicts real-world conditions. To address this, we explore a mortality step function that declines exponentially between successive birth events within a generation. Our analysis focuses on examining the local and global stability of equilibrium points. Additionally, we develop a sustainable harvesting strategy using a straightforward quadratic Lyapunov function.

Index Terms—Stage Structure, Time dependent mortality, Global Stability, Lyapunov Function, Harvesting.

I. INTRODUCTION

A Major proportion of the earth's surface is encompassed by water surfaces, most particularly by the oceans, and so they have a significant influence on the terrestrial environment. Oceanography encompasses various scientific disciplines, including biology, physics, chemistry, geology, and meteorology, all of which are involved in studying the oceans and their related phenomena. The study of marine animals and how they interact with their environment and one another is known as marine biology. Fish is a crucial source of the protein that people eat worldwide. The world's natural reserve fisheries are currently being overfished and are not being harvested sustainably to the tune of roughly 70 percent. Fisheries managers must understand biological processes and factors to accurately define what constitutes overfishing. Since collecting biological data to assess fish stocks is costly and fish populations are not easily observed, fisheries managers often estimate biological characteristics based on catch data and fishing inputs. Schaefer (1954) introduced the approach that has become the accepted method for using fisheries data in stock assessment: first, one must specify an equation that links output (catch) to fishing inputs (effort) and latent fish abundance (stock), and then one must specify a state equation for biological dynamics in terms of the latent stock variable. Thus, catch and effort data are the sole sources

for reconstructing population parameters econometrically. Both fields are concerned with determining the sustainable extraction levels of resources and recognize the link between biological dynamics and economic production functions. As a result, inferring fish stocks from fishery data has historically drawn the interest of both economists and fishery biologists (Comitini and Huang, 1967; Pella and Tomlinson, 1969; Wilen, 1976). However, more recently, investigations into the biological dynamics and connections between economic output have grown apart.

The maritime ecology is abundant in resources. Exploiting the biological community helps countries' economies grow. Fishing is one of the activities included in marine biology. In a marine ecosystem zone, a fishery is one that studies the interactions between the marine population and the harvesting system. We've seen in recent years that unlawful exploitation has a severe influence on fish populations in the oceans, contributing to the destruction of the environment.

Interesting studies have been conducted on the dynamics of fishery populations. In [1], Fulgence et al. explicitly examined the effects of fractional-order derivatives on the dynamics of the fishery model, and they used the Jacobian matrix and the Lyapunov direct technique to look into the local and global asymptotic stability of the equilibrium points. In [2], a Holling II functional response prey-predator model with harvesting for fisheries resources in a two-patch environment—a free fishing zone and a reserve zone where fishing is severely prohibited—was proposed and studied by Yunfei et al. By including a temporal delay in the harvesting period, the author took into account a prey-predator fisheries model and discussed the selective harvesting of fish over a specific age or size in [3]. In [4], the bioeconomic harvesting of a prey-predator fishery when both species are affected by toxins generated by other species was discussed by T. Das et al. Flugence Mansal et al. presented a mathematical bioeconomic model of a fishery with a variable price in [5]. M. Javidi et al. introduced a fractional-order prey-predator model and dealt with the mathematical behaviour of the model in [6]. In [7], a delayed stage-structured predator-prey model with non-monotone functional responses is proposed by the author, Youngi Xia et al. T. K. Kar et al. proposed a multispecies harvesting model with interference [10]. In [11], the issue of non-selective harvesting in a prey-predator system when both prey and predator species follow the law of logistic growth is examined in the study by K. S. Chaudhuri et al. Jia Wo et al. used a multispecies size-spectrum model (MSSM) deployed in the coastal ecosystem of the North Yellow Sea, China, to analyze the dynamics of multispecies fisheries under data-limited conditions

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[12].Kinfe Hailemariam Hntsa et al. studied fish resource harvest using a predator at the highest sustainable yield according to mathematical modelling [13]. In [14-16], the authors give a lot of attention to stage-structured models. In [20], S. Shalini Priya et al. studied a multicompartmental mathematical model for HIV/AIDS transmission in Ethiopia. In [21], Susila Bahri et al. analyzed the spread of the Omicron virus using the Next Generation Matrices method, considering various vaccination rates of 10. In [8] Tang, Chen discussed a single species of fish population with two stage structures, viz., an immature and mature population, where they suggested equal death rates for both population segments. But in [9] Gao, Chen rectified this naturally unreasonable concept by changing the equal death rates to two different constant rates of mortality. It is a fact that natural mortality decreases exponentially with time (i.e., age of the fish), but this decrement cannot go infinitely because otherwise, at infinite time, there will be no mortality at all, which is totally unrealistic. This idea of natural mortality allows us to consider a mortality step function that decreases exponentially within a new birth and the next birth process, i.e., within a generation. These functional values repeat in every new generation.

II. METHODS

This paper is organized as follows: Section 1 introduces the formulation of the two-stage structured fish population model incorporating step-function mortality. Section A examines the existence and uniqueness of solutions to validate the model. Section B focuses on the analysis of local stability at equilibrium points. Finally, Section C addresses the global stability of the model using a straightforward quadratic Lyapunov function.

1) *Model Description:* In [1], the authors explored a single-species fish population model with two distinct stage structures, assuming a constant mortality rate for both stages. However, this assumption is not realistic. To address this limitation, we propose a model where the mortality rate depends on both population size and time. It is well-established that the risk of natural mortality is inversely proportional to age (i.e., time). Therefore, if the mortality function decreases over time within each interval between birth events, the mortality function for an immature population can be expressed as follows:

$$f(t) = \begin{cases} e^{-(t-\tau)} & 0 \leq t \leq \tau \\ e^{-(t-2\tau)} & \tau \leq t \leq 2\tau \\ e^{-(t-3\tau)} & 2\tau \leq t \leq 3\tau \\ \vdots & \vdots \\ \vdots & \vdots \\ e^{-(t-n\tau)} & (n-1)\tau \leq t \leq n\tau \end{cases}$$

Here t represents instantaneous time and τ represents birth pulse.

Thus our two stage structured fish population model takes the form

$$\frac{dx}{dt} = be^{-\frac{y(t)}{y_0}} y(t) - cx(t) - d_1 f(t)x(t), \tag{1}$$

$$\frac{dy}{dt} = cx(t) - d_2 y(t) - hy(t), \tag{2}$$

In this model, $x(t)$ and $y(t)$ denote the population densities of immature and mature fish at time t, respectively. The parameter b represents the reproduction rate, while y_0 indicates the maximum recruitment capacity of an adult fish within the ecosystem, which we refer to as the maximum recruitment parameter. The maturity rate is denoted by c, d_1 and d_2 represent the mortality rates for immature and mature fish, respectively, and h is the harvesting rate applied to the mature population.

A. Existence and uniqueness of a solution of the model

In this section, we construct a two-stage structured fishery model with time-dependent mortality represented in Equations (1) and (2). The main aim of this section is to find the condition of existence and uniqueness of the fishery model. Suppose we consider the following functions:

$$u_1(t, x) = be^{-\frac{y(t)}{y_0}} y(t) - cx(t) - d_1 f(t)x(t), \tag{3}$$

$$u_2(t, y) = cx(t) - d_2 y(t) - hy(t). \tag{4}$$

We define the following sets: $C([0, n\tau], \mathbf{R})$ denote the Banach space of all continuous and differentiable functions from $[0, n\tau]$ to \mathbf{R} and

$$C[0, n\tau] = \{x : [0, n\tau] \rightarrow \mathbf{R} \text{ such that } \frac{dx}{dt} \in c[0, n\tau]\}, \tag{5}$$

n being a positive integer.

For simplification we consider the maximum norm which is defined by the following equation $\|u_1\| = \sup_{t \in [0, n\tau]} |u_1(t, x)|$. Here we consider the partial derivative equation defined by

$$\frac{dx}{dt} = u_1(t, x), \frac{dy}{dt} = u_2(t, x).$$

Now we proceed to validate our proposed model in the following way:

Theorem 2.1: We assume that the functions x and y admit upper bounds. Then the function u_1 and u_2 are Lipschitz continuous with Lipschitz constant α and β .

Proof: Using the defined norm, we obtain the following relationship:

$$\begin{aligned} & \|u_1(t, x_1) - u_1(t, x_2)\| \\ &= \|be^{-\frac{y}{y_0}} y - cx_1 - d_1 e^{-(t-n\tau)} x_1 - \{be^{-\frac{y}{y_0}} y - cx_2 - d_1 e^{-(t-n\tau)} x_2\}\|, \quad \text{for } (n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+, \\ &= \|c(x_1 - x_2) + d_1 e^{-(t-n\tau)}(x_1 - x_2)\| \\ &\leq c\|x_1 - x_2\| + d_1 e^{-(t-n\tau)}\|x_1 - x_2\|, \\ &= (c + d_1 e^{-(t-n\tau)})\|x_1 - x_2\| \\ &< (c + d_1 e^\tau)\|x_1 - x_2\|, \\ &= \alpha\|x_1 - x_2\|, \end{aligned}$$

where $\alpha = c + d_1 e^\tau$. This shows that u_1 is Lipschitz continuous and α is the Lipschitz Constant.

Next to show that u_2 is Lipschitz continuous. we take the following steps:

$$\begin{aligned} & \|u_2(t, y_1) - u_2(t, y_2)\| \\ &= \|\{cx - d_2 y - hy_1\} - \{cx - d_2 y_2 - hy_2\}\|, \\ &= \|d_2(y_1 - y_2) + h(y_1 - y_2)\|, \\ &\leq d_2\|y_1 - y_2\| + h\|y_1 - y_2\|, \\ &= (d_2 + h)\|y_1 - y_2\|, \end{aligned}$$

where $\beta = d_2 + h$. Therefore u_2 is Lipschitz continuous and β is the Lipschitz constant. ■

Theorem 2.2: The Picard operators defined by the following relations are bounded

$$Tx = x(0) + \int_{(n-1)\tau}^{n\tau} u_1(t, x) dt, \tag{6}$$

$$My = y(0) + \int_{(n-1)\tau}^{n\tau} u_2(t, y) dt, \tag{7}$$

$$(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+.$$

Proof: Applying the defined norm and Using triangular inequality, we derive the relationship

$$\begin{aligned} \|Tx - x(0)\| &= \left\| \int_{(n-1)\tau}^{n\tau} u_1(t, x) dt \right\|, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+, \\ &\leq \int_{(n-1)\tau}^{n\tau} \|u_1(t, x)\| dt. \end{aligned} \tag{8}$$

From theorem 2.1, the function u_1 is Lipschitz continuous, which implies that the function u_1 is continuous and bounded as well. That is $\|u_1\| \leq \epsilon$, where ϵ is positive constant. Equation (8) can be now written in the following form

$$\begin{aligned} \|Tx - x(0)\| &\leq \int_{(n-1)\tau}^{n\tau} \epsilon dt, \\ &= \epsilon\tau. \end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned} \|My - y(0)\| &= \left\| \int_{(n-1)\tau}^{n\tau} u_2(t, x) dt \right\|, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+, \\ &\leq \int_{(n-1)\tau}^{n\tau} \|u_2(t, x)\| dt. \end{aligned} \tag{10}$$

Again from theorem 2.1, the function u_2 is Lipschitz continuous, i.e the function u_1 is continuous and also bounded. That is $\|u_2\| \leq \delta$, where δ is a positive constant. Equation can now be written as

$$\begin{aligned} \|My - y(0)\| &\leq \int_0^\tau \delta dt, \\ &= \delta\tau. \end{aligned} \tag{11}$$

Theorem 2.3: The Picard's operators T and M of theorem 2.2 define contractions if

$$\tau \leq \frac{1}{\alpha}, \tag{12}$$

$$\tau \leq \frac{1}{\beta}. \tag{13}$$

Proof: We have

$$\begin{aligned} \|Tx_1 - Tx_2\| &= \left\| \int_{(n-1)\tau}^{n\tau} [u_1(t, x_1) - u_1(t, x_2)] dt \right\|, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+, \\ &\leq \int_{(n-1)\tau}^{n\tau} \| [u_1(t, x_1) - u_1(t, x_2)] \| dt, \\ &= \int_{(n-1)\tau}^{n\tau} \| [c(x_1 - x_2) + d_1 e^{-(t-\tau)} \\ &(x_1 - x_2)] \| dt, \\ &= \int_{(n-1)\tau}^{n\tau} (c + d_1) \|x_1 - x_2\| dt, \\ &= (c + d_1) \int_{(n-1)\tau}^{n\tau} \|x_1 - x_2\| dt, \\ &= (c + d_1 e^\tau)\tau \|x_1 - x_2\|. \end{aligned} \tag{14}$$

From equation (14) the operator T is a contraction if the following condition holds:

$$\tau \leq \frac{1}{\alpha}. \tag{15}$$

Again,

$$\begin{aligned} \|Mx_1 - Mx_2\| &= \left\| \int_{(n-1)\tau}^{n\tau} [u_2(t, y_1) - u_2(t, y_2)] dt \right\|, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+, \\ &= \int_{(n-1)\tau}^{n\tau} \| [u_2(t, y_1) - u_2(t, y_2)] \| dt, \\ &= \int_{(n-1)\tau}^{n\tau} \| [d_2(y_1 - y_2) + h(y_1 - y_2)] \| dt, \\ &\leq \int_{(n-1)\tau}^{n\tau} (d_2 + h) \|y_1 - y_2\| dt, \\ &= (d_2 + h)\tau \|y_1 - y_2\|. \end{aligned} \tag{16}$$

From equation (16), the operator T is a contraction if the following conditions hold:

$$\tau \leq \frac{1}{\beta}, \quad \beta = d_2 + h. \tag{17}$$

Theorem 2.4: We assume the functions x_1 and x_2 be the solutions of Equation (1) and the functions y_1 and y_2 be the solutions of Equation (2). Then we have the following condition

$$x_1 = x_2 \text{ and } y_1 = y_2. \tag{18}$$

Proof: Let us consider the function x_1 be the solution of Equation (1) then it satisfies the property expressed in the following Equation

$$\begin{aligned} x_1 &= x_1(0) + \int_{(n-1)\tau}^{n\tau} u_1(t, x_1) dt, \quad (n-1)\tau < t \leq n\tau, \\ &n \in \mathbf{Z}^+. \end{aligned} \tag{19}$$

Again, if the function x_2 is the solution of Equation (1), then it satisfies the property expressed in the following Equation

$$\begin{aligned} x_2 &= x_2(0) + \int_{(n-1)\tau}^{n\tau} u_1(t, x_2) dt, \quad (n-1)\tau < t \leq n\tau, \\ &n \in \mathbf{Z}^+. \end{aligned} \tag{20}$$

Then,

$$\begin{aligned} \|x_1 - x_2\| &= \left\| \int_{(n-1)\tau}^{n\tau} [u_1(t, x_1) - u_1(t, x_2)] dt \right\|, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+, \\ &\leq \int_{(n-1)\tau}^{n\tau} \|u_1(t, x_1) - u_1(t, x_2)\| dt, \\ &\leq \int_{(n-1)\tau}^{n\tau} (c + d_1) \|x_1 - x_2\| dt, \\ &= (c + d_1) \|x_1 - x_2\| \tau. \end{aligned} \tag{21}$$

From Equation (21) we have the following condition :

$$[1 - (c + d_1)\tau] \|x_1 - x_2\| \leq 0. \tag{22}$$

Since M is a contraction mapping if $[1 - (c + d_1)\tau] \geq 0$, which implies that $\|x_1 - x_2\| \leq 0$ if T is a contraction mapping. Using the definition of our norm we have,

$$\|x_1 - x_2\| = 0 \text{ which implies that } x_1 = x_2. \tag{23}$$

Next, let function y_1 be the solution of Equation (2), then it satisfies the following property:

$$\begin{aligned} y_1 &= y_1(0) + \int_{(n-1)\tau}^{n\tau} u_2(t, y_1) dt, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+. \end{aligned} \tag{24}$$

Furthermore, if we assume that the function y_2 is the solution of Equation (2), then it satisfies the following property of the form

$$\begin{aligned} y_2 &= y_2(0) + \int_{(n-1)\tau}^{n\tau} u_2(t, y_2) dt, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+. \end{aligned} \tag{25}$$

Then using the difference between equations (24) and (25) we obtain the following relationship:

$$\begin{aligned} \|y_1 - y_2\| &= \left\| \int_{(n-1)\tau}^{n\tau} [u_2(t, y_1) - u_2(t, y_2)] dt \right\|, \\ &(n-1)\tau < t \leq n\tau, n \in \mathbf{Z}^+, \\ &\leq \int_{(n-1)\tau}^{n\tau} \|u_2(t, y_1) - u_2(t, y_2)\| dt, \\ &\leq \int_{(n-1)\tau}^{n\tau} (d_2 + h) \|y_1 - y_2\| dt, \\ &= (d_2 + h)\tau \|y_1 - y_2\|. \end{aligned} \tag{26}$$

From Equation (26), we have the following condition:

$$[1 - (d_2 + h)\tau] \|y_1 - y_2\| \leq 0. \tag{27}$$

Since M is contraction mapping if $[1 - (d_2 + h)\tau] \geq 0$ which implies that $\|y_1 - y_2\| \leq 0$ if M is a contraction mapping. Then the definition of our norm gives

$$\|y_1 - y_2\| = 0 \text{ which implies that } y_1 = y_2. \tag{28}$$

The above theorems prove that the solution of the proposed model is unique. ■

B. Local stability analysis of the model's equilibrium points

In this section, we analyze the local stability of our fishery model using differential calculus. We employ the Routh-Hurwitz criteria to assess stability in the context of differential equations. The differential equations are defined as follows:

$$\frac{dx}{dt} = u_1(x, y), \tag{29}$$

$$\frac{dy}{dt} = u_2(x, y). \tag{30}$$

The equilibrium points satisfy the equations (29) and (30) by the following relationship:

$$u_1(x_{eq}, y_{eq}) = 0 \text{ and } u_2(x_{eq}, y_{eq}) = 0. \tag{31}$$

Here (x_{eq}, y_{eq}) indicates the equilibrium point of the fish population model. After solving the equations, we get the two equilibrium points. The first one is trivial equilibrium points, i.e., $(x_{eq}, y_{eq}) = (0, 0)$. The trivial equilibrium point represents the extinction of both populations. Stabilisation of a zero population is an undesirable situation in any fishery model. Thus, we aim to unstabilize this equilibrium point and find the harvesting parameter range. We find the Jacobian at the trivial equilibrium point, i.e., $(x_{eq}, y_{eq}) = (0, 0)$ as

$$J_{(0,0)} = \begin{bmatrix} -c - d_1 f(t) & b \\ c & -d_2 - h \end{bmatrix}. \tag{32}$$

Assuming that λ is the eigenvalue of the Jacobian matrix. The characteristic equation of the Jacobian matrix becomes

$$a_1 \lambda^2 + a_2 \lambda + a_3 = 0, \tag{33}$$

Where the parameter values are given by the following

$$a_1 = 1, \tag{34}$$

$$a_2 = c + d_1 f(t) + d_2 + h, \tag{35}$$

$$a_3 = (c + d_1 f(t))(d_2 + h) - cb. \tag{36}$$

Using Rowth-Hurwitz Criteria for stability we get

$$s_1 : a_1 \quad a_3,$$

$$s_2 : a_2 \quad 0,$$

$$s_3 : \frac{a_2 a_3}{a_2} = a_3.$$

The equilibrium point $(0, 0)$ is stable if

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0.$$

Therefore, we obtain the stability condition for the trivial equilibrium point as

$$h > \frac{cb}{c + d_1 f(t)} - d_2. \tag{37}$$

since the stability of the trivial equilibrium point is undesirable. To unstabilise the equilibrium point $(0, 0)$ we must have

$$h \leq \frac{cb}{c + d_1 f(t)} - d_2. \tag{38}$$

Next we assume that (x_{eq}, y_{eq}) be the coexisting equilibrium point that satisfy the Equation (29) and (30).

Then,

$$\begin{aligned} be^{-\frac{y_{eq}}{y_0}} - cx_{eq} - d_1x_{eq} &= 0, \\ \implies be^{-\frac{y_{eq}}{y_0}} - (c + d_1f(t))x_{eq} &= 0, \\ \implies x_{eq} &= \frac{be^{-\frac{y_{eq}}{y_0}}}{(c + d_1f(t))}. \end{aligned} \tag{39}$$

Now Putting value of x_{eq} in the Equation (30) we have

$$\begin{aligned} cx_{eq} - d_2y_{eq} - hy_{eq} &= 0, \\ \implies c \frac{be^{-\frac{y_{eq}}{y_0}}}{(c + d_1f(t))} - d_2y_{eq} - hy_{eq} &= 0, \\ \implies cbe^{-\frac{y_{eq}}{y_0}} - (c + d_1f(t))d_2 - (c + d_1f(t))h &= 0, \\ \implies e^{-\frac{y_{eq}}{y_0}} &= \frac{(c + d_1f(t))(d_2 + h)}{cb}, \\ \implies \frac{-y_{eq}}{y_0} &= \ln \frac{(c + d_1f(t))(d_2 + h)}{cb}, \\ \implies y_{eq} &= y_0 \ln \frac{cb}{(c + d_1f(t))(d_2 + h)}. \end{aligned} \tag{40}$$

Putting the value of y_{eq} in the equation (39) we have

$$x_{eq} = \frac{y_0(d_2 + h)}{cb} \ln \frac{cb}{(c + d_1f(t))(d_2 + h)}.$$

Therefore our time-dependent coexisting equilibrium point is (x_{eq}, y_{eq}) , Where

$$x_{eq} = \frac{y_0(d_2 + h)}{cb} \ln \frac{cb}{(c + d_1f(t))(d_2 + h)}, \tag{41}$$

$$y_{eq} = y_0 \ln \frac{cb}{(c + d_1f(t))(d_2 + h)}. \tag{42}$$

We begin our study of local stability by focusing on the coexisting equilibrium point i.e., $(x_{eq}, y_{eq}) = (\frac{y_0(d_2+h)}{cb} \ln \frac{cb}{(c+d_1f(t))(d_2+h)}, y_0 \ln \frac{cb}{(c+d_1f(t))(d_2+h)})$. The Jacobian matrix at the coexisting equilibrium point is

$$J_{(x_{eq}, y_{eq})} = \begin{bmatrix} c - d_1f(t) & be^{-\frac{y_{eq}}{y_0}} - \frac{b}{y_0}e^{-\frac{y_{eq}}{y_0}}y_{eq} \\ c & -d_2 - h \end{bmatrix}. \tag{43}$$

If λ is an eigenvalue of the Jacobian matrix defined by equation (43), then the characteristic equation of that matrix is the following:

$$b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{44}$$

Where the parameter values are given by the following

$$b_1 = 1, \tag{45}$$

$$b_2 = (c + d_1f(t))(d_2 + h), \tag{46}$$

$$b_3 = (c + d_1f(t))(d_2 + h) \ln \frac{cb}{(c + d_1f(t))(d_2 + h)}. \tag{47}$$

Using Rowth-Hurwitz Criteria for stability we get

$$s_1 : b_1 \quad b_3,$$

$$s_2 : b_2 \quad 0,$$

$$s_3 : \frac{b_2b_3}{b_2} = b_3.$$

The coexisting equilibrium point (x_{eq}, y_{eq}) is stable if

$$b_1 > 0, \quad b_2 > 0, \quad b_3 > 0.$$

Which gives

$$(c + d_1f(t))(d_2 + h) > 0 \quad (\text{obvious})$$

and

$$\begin{aligned} \ln \frac{cb}{(c + d_1f(t))(d_2 + h)} &> 0, \\ \implies h &< \frac{cb}{c + d_1f(t)} - d_2. \end{aligned}$$

Therefore, to locally stabilise the coexisting equilibrium point, we must have

$$h < \frac{cb}{c + d_1f(t)} - d_2. \tag{48}$$

C. Analysing the harvesting strategy by constructing a Lyapunov Function

In this part, we examine the global asymptotic stability of the two-stage structure fish population model's non-trivial equilibrium point by defining a suitable Lyapunov function. We examine the stability of the non-trivial equilibrium point, which is defined by (x^*, y^*) . Where

$$x^* = \frac{y_0(d_2 + h)}{cb} \ln \frac{cb}{(c + d_1f(t))(d_2 + h)}, \tag{49}$$

$$y^* = y_0 \ln \frac{cb}{(c + d_1f(t))(d_2 + h)}. \tag{50}$$

We define the Lyapunov function as

$$V = \left(\frac{x - x^*}{x^*}\right)^2 + \left(\frac{y - y^*}{y^*}\right)^2. \tag{51}$$

Differentiating V with respect to t we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{2(x - x^*)}{x^{*2}} \frac{dx}{dt} + \frac{2(y - y^*)}{y^{*2}} \frac{dy}{dt}, \\ &= \frac{2(x - x^*)}{x^{*2}} [b(e^{-\frac{y}{y_0}}y - e^{-\frac{y^*}{y_0}}y^*) - (c + d_1f(t))(x - x^*)] \\ &\quad + \frac{2(y - y^*)}{y^{*2}} [c(x - x^*) - (d_2 + h)(y - y^*)], \\ &\leq \frac{2(x - x^*)}{x^{*2}} [b(y - y^*) - (c + d_1f(t))(x - x^*)] \\ &\quad + \frac{2(y - y^*)}{y^{*2}} [c(x - x^*) - (d_2 + h)(y - y^*)], \\ &= -\frac{2(x - x^*)^2}{x^{*2}}(c + d_1f(t)) + \frac{2(x - x^*)(y - y^*)}{x^{*2}}b \\ &\quad + \frac{2(x - x^*)(y - y^*)}{x^{*2}}c - \frac{2(y - y^*)^2}{y^{*2}}(d_2 + h), \tag{52} \\ &= -X^TAX. \tag{53} \end{aligned}$$

[obviously for $y \geq y^*$, $e^{-\frac{y}{y_0}}y - be^{-\frac{y^*}{y_0}} \leq y - y^*$. Also for $y < y^*$, $e^{-\frac{y}{y_0}}y - be^{-\frac{y^*}{y_0}} < y - y^*$ provided maximum recruitment parameter y_0 is less or equal to the equilibrium position for mature fish population represented by y^* , which is very much logical from the practical point of view.]

The above equation (52) is in quadratic Form. where

$$X^T = \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} \text{ column matrix}$$

and

$$A = \begin{bmatrix} \frac{2(c+d_1f(t))}{x^{*2}} & -\left(\frac{b}{x^{*2}} + \frac{c}{y^{*2}}\right) \\ -\left(\frac{b}{x^{*2}} + \frac{c}{y^{*2}}\right) & \frac{2(d_2+h)}{y^{*2}} \end{bmatrix}. \tag{54}$$

If the matrix A defined on (54) is positive-definite, then the coexisting equilibrium point is stable, and we can find the range for the harvesting parameter. The following steps are taken to show A is positive-definite.

$$a_{11} = \frac{2(c + d_1 f(t))}{x^{*2}} > 0 \text{ always}$$

and

$$\begin{aligned} \text{if } & a_{11}a_{22} - a_{12}a_{21} > 0, \\ \implies & \frac{4(c + d_1 f(t))(d_2 + h)}{x^{*2}y^{*2}} - \left(\frac{b}{x^{*2}} + \frac{c}{y^{*2}}\right) > 0, \\ \implies & d_2 + h > \frac{(b + c)^2}{4x^{*2}y^{*2}(c + d_1 f(t))} - d_2, \\ \implies & h > \frac{(b + c)^2}{4x^{*2}y^{*2}(c + d_1 f(t))}. \end{aligned}$$

Thus, to globally stabilise the coexisting equilibrium point (x^*, y^*) we have the range

$$\frac{(b + c)^2}{4x^{*2}y^{*2}(c + d_1 f(t))} < h < \frac{cb}{c + d_1 f(t)} - d_2. \quad (55)$$

III. RESULTS AND DISCUSSION

This paper presents and analyzes a fish population model with a two-stage structure and time-varying mortality rates. The model differentiates between immature and mature fish stages. We first establish the existence of equilibrium points and then investigate their stability from both local and global perspectives. To illustrate the model's behavior, we perform numerical simulations using the Mathematica programming language, demonstrating the system's dynamics and outcomes.

Fig. 1 presents the phase diagram of the system for the set of parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 10$. This clearly shows that all solutions ultimately stabilizes to the coexisting equilibrium point, which we intend to.

Fig. 2 presents the time series plot of the immature fish population for the parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 2$. The graph shows oscillations within a small range of values.

Fig. 3 presents the time series plot of the mature fish population for the parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 2$. The graph shows a very little oscillation and almost stabilizing nature.

Fig. 4 presents the time series plot of the immature fish population for the parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 10$. The graph Shows increasing first then decreasing nature and then oscillations.

Fig. 5 presents the time series plot of the mature fish population for the parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 10$. The graph shows no oscillations at all and stabilizes to a fixed value.

Fig. 6 presents the time series plot of the immature fish population for the parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 100$. The graph shows increasing then decreasing and again increasing nature and then oscillations but the length of oscillations interval is less now.

Fig. 7 presents the time series plot of the mature fish population for the parameter values $d_1 = 0.1, d_2 = 0.01, c =$

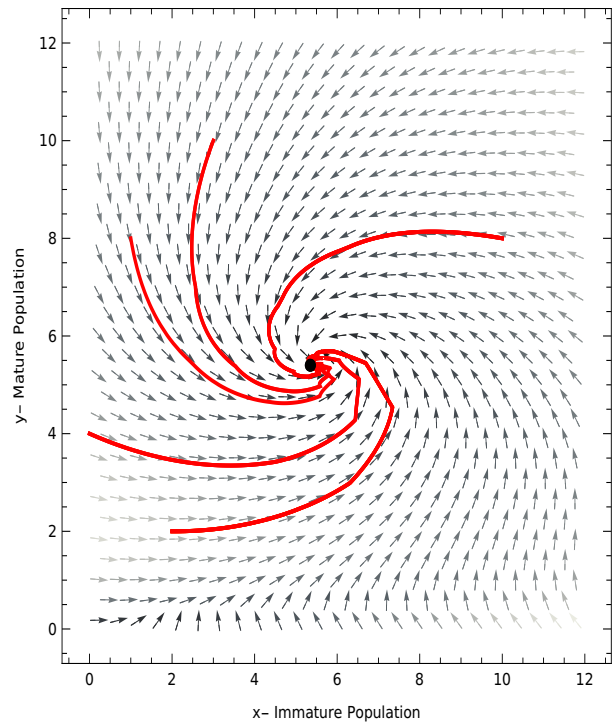


Fig. 1. Phase portrait of the fish population model for parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 10$

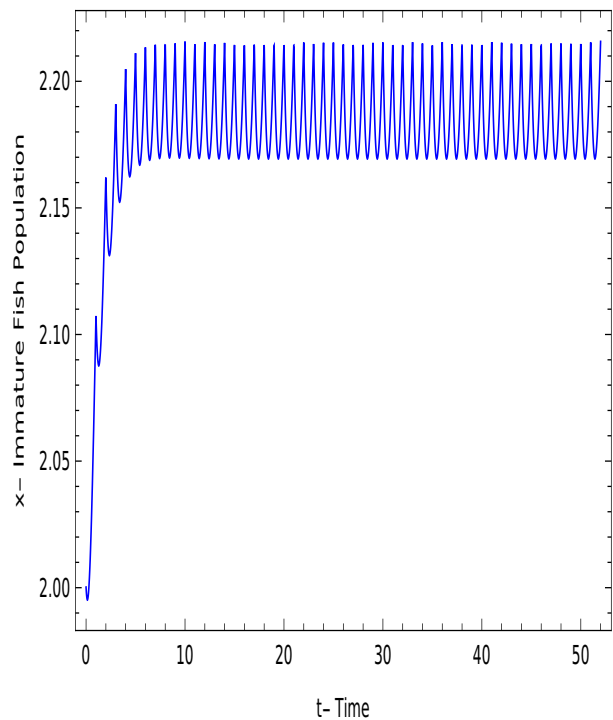


Fig. 2. Time series plot of immature fish population x with respect to time t for parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 2$

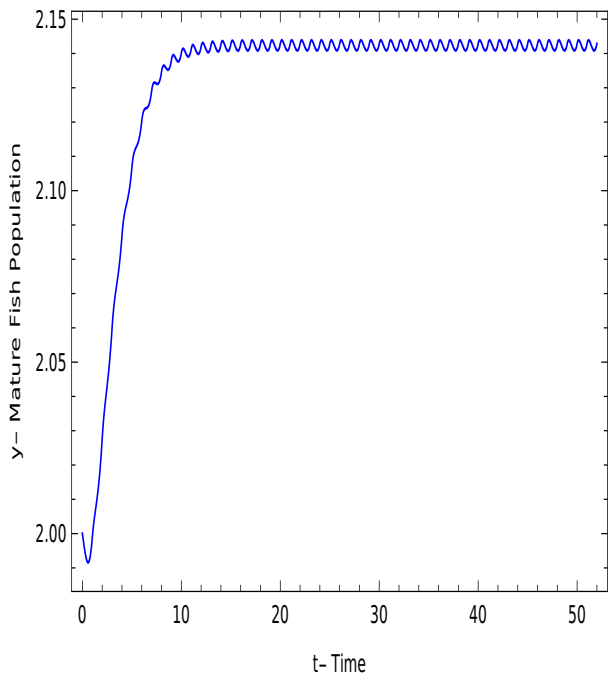


Fig. 3. Time series plot of mature fish population y with respect to time t for parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 2$

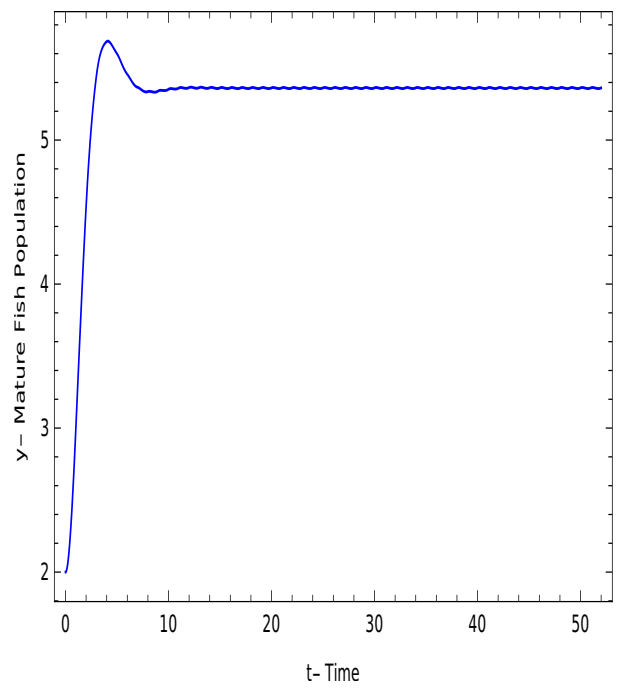


Fig. 5. Time series plot of mature fish population y with respect to time t for parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 10$

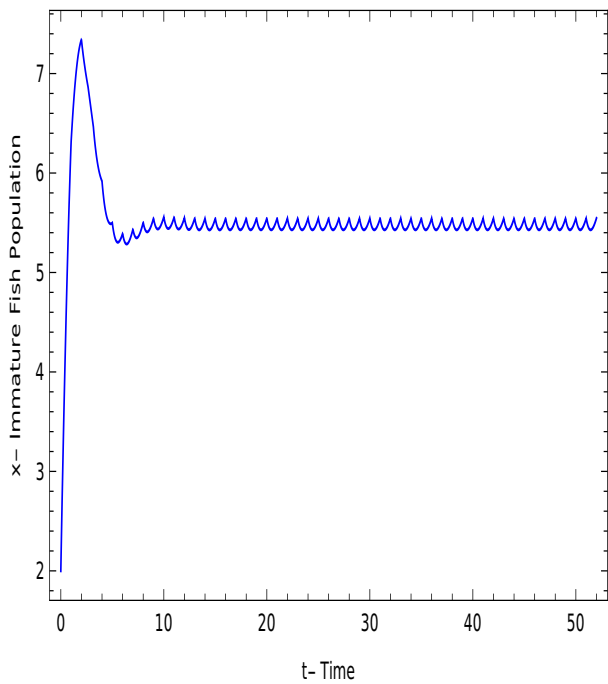


Fig. 4. Time series plot of immature fish population x with respect to time t for parameter values $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 10$

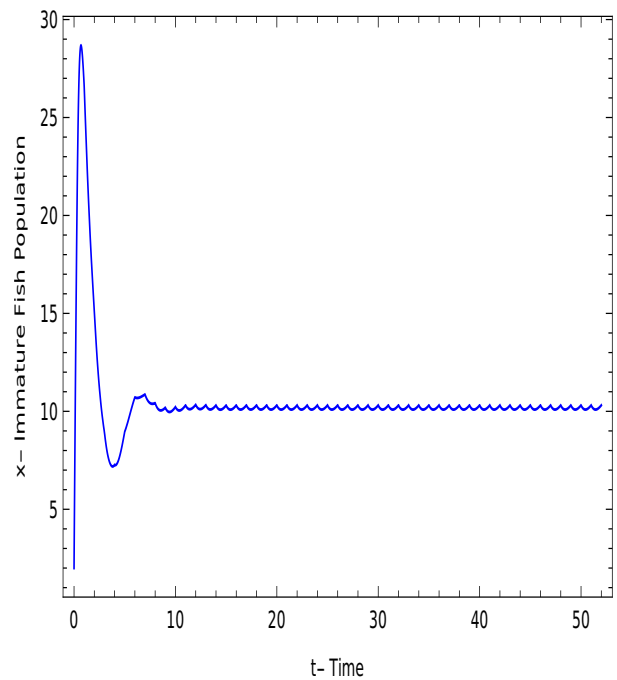


Fig. 6. Time series for immature fish population at $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 100$

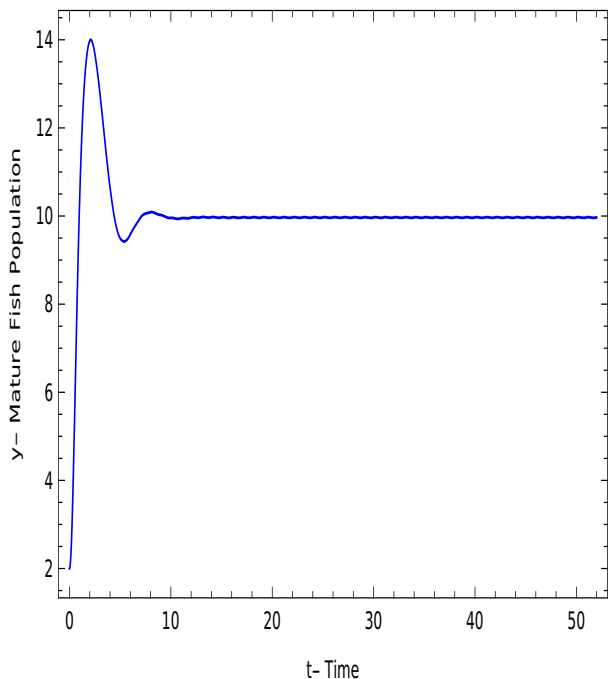


Fig. 7. Time series for mature fish population at $d_1 = 0.1, d_2 = 0.01, c = 0.5, y_0 = 5, \tau = 1, h = 0.5, b = 100$

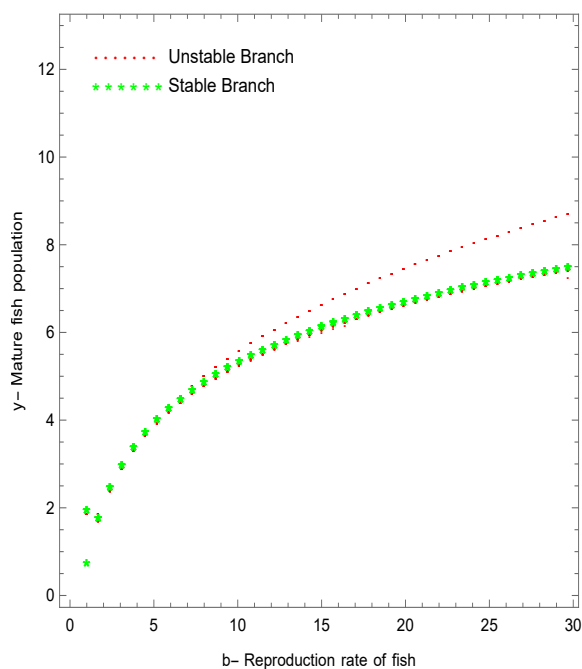


Fig. 9. Bifurcation diagram for mature fish population with respect to the control parameter as the reproduction rate "b"

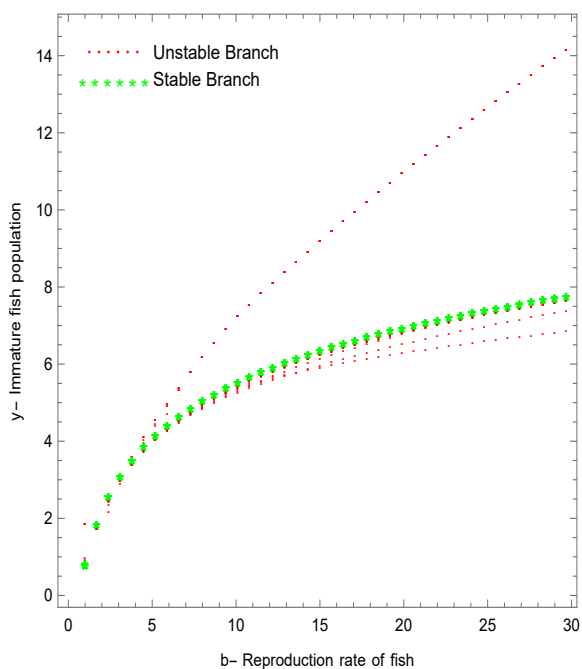


Fig. 8. Bifurcation diagram for immature fish population with respect to the control parameter as the reproduction rate "b"

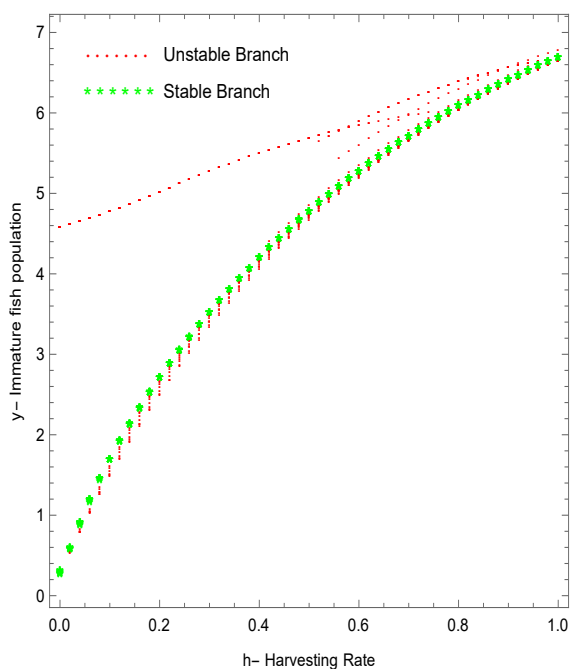


Fig. 10. Bifurcation diagram for immature fish population with respect to the control parameter as the harvesting rate "h"

0.5, $y_0 = 5, \tau = 1, h = 0.5, b = 100$. The graph shows stable nature.

It is noticed from the time series plots that the reproduction rate "b" plays an important role in the dynamics of the system. This forces us to investigate the bifurcation diagram taking the reproduction rate 'b' as the controlling parameter.

Fig. 8 presents the bifurcation diagram for the immature fish population corresponding to the control parameter "b". The unstable branches are represented by red colour and green colour represents the stable branch. It is seen from

the diagram that there are many unstable branches and only one stable branch, which confirms that in the long run the system stabilizes to a single point whatever time in the year is considered for harvesting; provided the permissible harvesting parameter range is ensured.

Also from the time series plots it is seen that the system stabilizes earlier for higher values of "b".

Fig. 9 presents the bifurcation diagram for the mature fish population with respect to the control parameter "b".

Fig. 10 and Fig. 11 presents the bifurcation diagram for the

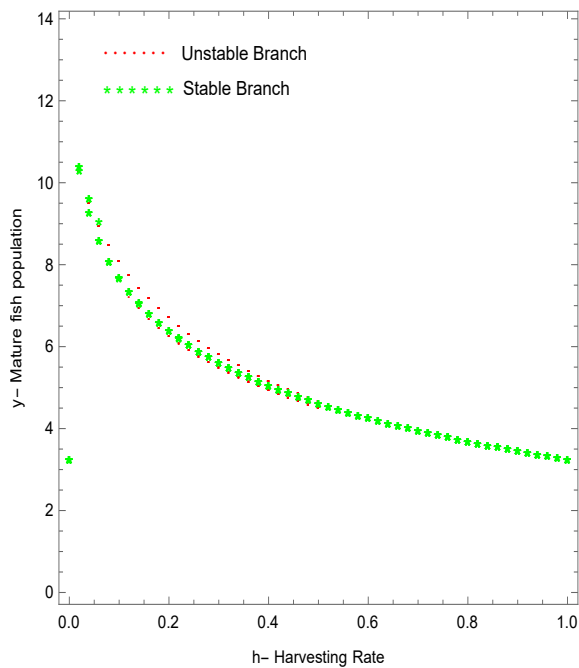


Fig. 11. Bifurcation diagram for mature fish population with respect to the control parameter as the harvesting rate "h"

immature and mature fish population taking the harvesting parameter as the control parameter fixing the reproduction rate "b=7". This figures again confirms the fact that in long run the system stabilizes to a single value.

A. Limitations

This model does not account for mortality arising from sudden environmental changes or predation. To enhance its realism, future work could incorporate environmental variables, account for predation by other species, or include additional stages in the species' life cycle.

IV. CONCLUSIONS

In this paper, we propose a two-stage structured fish population model that incorporates a time-dependent step-function for mortality. Our analysis reveals that the range of harvesting parameters required to maintain a sustainable population—one that avoids extinction at any stage—greatly depends on the timing of the harvesting activities. We observe that minimal harvesting is optimal when the mortality rate is at its peak, which aligns with real-world observations. Our findings are corroborated through numerical simulations and illustrated with graphical representations.

REFERENCES

[1] Fulgence Mansal, Ndolane Sene. Analysis of fractional fishery model with reserve area in the context of time-fractional order derivative. *Chaos, Solitons and Fractal* 2020;140: 0960-0779.
 [2] Lv Y , Yuan R , Pei Y . A prey-predator model with harvesting for fishery resource with reserve area. *Appl Math Model* 2013;37:3048–62.
 [3] T. Kar, Selective harvesting in a prey–predator fishery with time delay, *Math. Comput. Model.* 38 (2003) 449–458.
 [4] T. Das, R. Mukherjee, K. Chaudhuri, Harvesting of a prey–predator fishery in the presence of toxicity, *Appl. Math. Model.* (2008) 2282–2292.
 [5] Mansal F , Auger TNHP , Balde M . A mathematical model of a fishery with variable market price: sustainable fishery/over-exploitation. *Acta Biotheor* 2014;62:305–23.

[6] Javidi M, Nyamoradi N . Dynamic analysis of a fractional order prey-predator interaction with harvesting. *Appl Math Model* 2013;37:8946–56.
 [7] Y. Xia, J. Cao, S.S. Cheng, Multiple periodic solutions of a delayed stage-structured predator–prey model with non-monotone functional responses, *Appl. Math. Model.* 31 (2007) 1947–1959.
 [8] Tang S, Chen L. Density-dependent birth rate, birth pulses and their population dynamic consequences. *J Math Biol* 2002;64:169–84.
 [9] Gao S, Chen L. Dynamic complexities in a single-species discrete population model with stage structure and birth pulses. *Chaos, Solitons Fractals* 2005;23:519–27.
 [10] T. K. Kar and K. S. Chaudhuri, Harvesting in a two-prey onepredator fishery: a bioeconomic model. *The ANZIAM Journal*, vol. 45, no. 3, pp. 443–456, 2004.
 [11] T. K. Kar and K. S. Chaudhuri. On nonselective harvesting of a multi-species fishery. *International Journal of Mathematical Education in Science and Technology*, vol. 33, no. 4, pp. 543–556, 2002.
 [12] Wo, J., Zhang, C., Pan, X., Xu, B., Xue, Y., Ren, Y. (2020). Modeling the Dynamics of Multispecies Fisheries: A Case Study in the Coastal Water of North Yellow Sea, China. *Frontiers in Marine Science*, 7.
 [13] KinfeHailemariamHntsa, ZebebeTekaMengesha (PhD). Mathematical modeling of Fish Resources Harvesting with Predator at Maximum Sustainable Yield. *Mathematical Theory and Modeling*. Vol.6, No.6, 2016.
 [14] Aiello WG, Freedman HI. A time-delay model of single-species growth with stage structure. *Math Biosci* 1990;101:139–53.
 [15] Aiello WG, Freedman HI, Wu J. Analysis of a model representing stage-structured population growth with state-dependent time delay. *SIAM J Appl Math* 1992;3:855–69.
 [16] Cai Y, Fan J, Gard TC. The effects of a stage-structured population growth model. *Nonlin Anal Th Mech Appl* 1992;16:95–105.
 [17] Gurney WSC, Nisbet RM, Lawton JH. The systematic formulation of tractable single species population models incorporating age structure. *J Animal Ecol* 1983;52:479–85.
 [18] Hastings A. Delay in recruitment at different trophic levels: effects on stability. *J Math Biol* 1984;21:35–44.
 [19] Freedman HI, Wu J. Persistence and global asymptotic stability of single species dispersal models with stage structure. *Quart Appl Math* 1991;2:351–71.
 [20] S. Shalini Priya, and K. Ganesan, "An SEIAT Endemic Model for the Control of HIV/AIDS in Ethiopia," *IAENG International Journal of Applied Mathematics*, vol. 53, no.4, pp1418-1426, 2023.
 [21] Susila Bahri, Ainil Mardiyah, Ahmad Iqbal Baqi, and Abqorry Zakkiyyah, "Local Stability Analysis and Simulation of Omicron Virus Spread Using the Omicron SSvIR Model," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 5, pp797-803, 2024.