Local Super (a, d) Edge Antimagic Total Labeling of Graphs

N. Muthuselvi and T. Saratha Devi

***Abstract*—A graph labeling involves assigning numerical values to the elements within a graph $G(V, E)$. This mapping can be applied to the vertices, edges or both. When the labeling encompasses both vertices and edges, it is referred to as a total labeling. For a graph G with vertex set $V(G)$ and edge set $E(G)$, a total labeling $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)|+|E(G)|\}$ is termed an (a, d) edge antimagic total labeling $(EATL(a, d))$ if the set of edge weights $\{f(x) + f(xy) + f(y) : xy \in E(G)\}$ forms an arithmetic progression with an initial term a and a common difference d. Such a labeling is considered a super (a, d) edge antimagic total labeling $(SEATL(a, d))$ if the smallest labels are assigned to the vertices. Local super (a, d) edge antimagic total labeling $(LSEATL(a, d))$ occurs when the range set f is defined as $f(E) = \{1, 2, \ldots, |E|\}$. This work investigates the existence of the local super (a, d) edge antimagic total labeling $(LSEATL(a, d))$ for certain graph classes. We obtain a relationship between a local super $(a, 0)$ edge antimagic total labeling $LSEATL(a, 0)$ and a local super $(a, 2)$ edge antimagic total labeling $LSEATL(a, 2)$ to any graph.**

***Index Terms*—Labeling, Antimagic, Local, Vertex, Edge Weight.**

I. Introduction

WHEN Konigsberg citizens attempted to cross the seven bridges on Pregel river, graph theory was born in the 18th century. The well-known mathematician Euler discovered that crossing every bridge exactly once and finishing back at the starting point is impossible. A formal and systematic study of graph theory began.

As a result, a graph is defined by its vertices and edges. We refer to a graph as a G by saying that it is composed of $V(G)$ and $E(G)$. An edge with end points x and y in $V(G)$ is denoted by xy if x and y are vertices. $n = |V(G)|$ and $e = |E(G)|$ are the dimensions of a graph. E for $E(G)$ and V for $V(G)$ if G is fixed. (n, e)-graphs are graphs of order n and size e. If n is finite, then a graph is finite. Simple graphs have no loops.

In the literature, graph lebeling first appeared in [1]. Indeed, they have studied some fundamental properties of labeling and introduced various structures of labeling. The vertices and edges of a graph are nodes and lines. Labeled graphs and unlabeled graphs may both exist. Labeled graphs are usually used for identification only. Depending on the labeling , we can use labeling to represent not only vertices and edges but a variety of other properties [2], [3].

Manuscript received May 2, 2024; revised December 5, 2024.

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Rosa developed a mapping f that converts a graph $\mathcal{G}$ vertices set to a set of numbers $0, 1, 2, ..., q$ where q is the number of edges in $\mathcal{G}$, so that each edge xy is labeled $|f(x) - f(y)|$, with all labels being different. Then he referred to this as β-valuation. Golomb studied the same form of labeling independently and converted it to elegant labeling.

As a result, several properties about elegant labeling appeared. Aside from theoretical advances, researchers have been looking for graph labeling applications. Applications in astronomy, coding theory, x-ray crystallography, radar, communication design, and circuit design [4], [5].

In [6], another type of graph labeling was studied. He referred to the labeling as “magic labeling”. His description was inspired by the number theory concept of the magic square. A magic labeling is a function that converts the set of edges of graph $\mathcal{G}$ into non-negative real numbers such that the sums of the edge labels around any vertex in $\mathcal{G}$ are all equal.

It is worth noting that Sedlacek’s formulation allowed for the use of any real number, although today only integers are commonly used. If the collection of edge labels comprised of consecutive integers, Stewart [7] termed magic labeling super-magic. Many more relevant definitions and outcomes have been discovered as a result of Sedlacek and Stewart’s research.

A labeling is a bijection map that allocates natural numbers to graph vertices as well as edges. We examine graph labeling with weights associated with each edge and/or vertex. When all of the vertex weights (or edge weights) have the same value, the labeling is said to be magical. If the weight varies for each vertex (or edge), the labeling is said to be anti-magic. Since its debut by Sedlacek in 1963, research in both magic and anti-magic labeling has grown rapidly.

Sedlacek [6] developed the idea of magic labeling based on the notion of magic squares from number theory. A graph is considered magical if it contains edge labeling with a range of real values, ensuring that the total of incident edge labels on each vertex stays constant. According to [8] and [9], a super magic labeling of a graph G is one with successive integer edge labels.

An edge magic total labeling (EMT) of a graph G is a bijection mapping $f : V(G) \cup E(G) \to \{1, 2, 3, ..., v + e\}$ if $f(x) + f(xy) + f(y) = k$, where k is a constant that is independent of the edge $xy \in E(G)$. The concept of edge magic total labeling in graphs originated with Kotzig [10]. The notion of super edge magic total (SEMT) labeling was introduced in [11]. Some of the theories and applications seen in [12], [13], [14], [15].

II. Edge labeling

A magic valuation for a graph was introduced by Kotzig and Rosa [10]. The same concept was introduced by Ringel and Llado under the name edge magic labeling. In their hypothesis, every tree admits edge-magic labeling. As early as 1998, Enomoto and his team introduced super edge magic total labeling.

The properties of edge magic total graphs are discussed in [16]. In addition to labeling the edges of complete graphs up to K_6 (because K_n has no edge magic total label for $n > 6$), cycle C_n, sun graphs (crown product of the cycle and K_2), stars, and complete bipartite graphs $K_{m,n}$. An edge magic total graph's maximum size was computed by Craft and Tessar. This bound was recently improved by Pikhurko. A study by Fukuchi examined the magic total labeling of wheels [17].

Ringel and Llado proved that a caterpillar is an edge magic total graph and conjectured that every tree is an edge magic total graph. Furthermore, Enomoto et al. conjectured that all trees are super edge magic total graphs. This conjecture is still open. Kotzig and Rosa [10] proved that every caterpillar is a super edge magic total graph. As reported in Gallian's survey [18], Lee and Shan verified this conjecture for all trees with atmost 17 vertices.

Fukuchi investigated super edge magic total labeling for some special types of trees and also proved that, under certain conditions, the union of two special types of super edge magic total trees will also be a super edge magic total graph. Figueroa-Centeno et al. [19] proved that in a tree an α-labeling is super edge magic total labeling. Recall that an α-labeling f is a graceful labeling with the additional property that there exists an integer k, such that for each edge xy, either $f(x) \leq k \leq f(y)$ or $f(y) \leq k \leq f(x)$.

An isolated vertex can make a non - super edge magic total graph into a super edge magic total graph , according to Figueroa-Centeno et al. A super edge magic deficiency is the minimum number of additional isolated vertices. Among the topics they discuss are cycles C_n, complete bipartite graphs Km_n, and forests. A number of results have also been obtained on the edge magic total labeling of cycles by Roditty and Bachar.

Recently, MacDougall and Wallis investigated super edge magic total labeling for a graph that has maximum size (maximum number of edges). Ivanco and Luckanicova constructed an edge magic total labeling of some disconnected graphs. The magic strength of a graph has been studied by Kong et al.

In the study of edge magic totals and super edge magic totals, Munner-Batle explored some important properties. In addition to studying edge magic total labeling, he investigated graph operations and labeling schemes for special graphs. In continuatin the distance between a graph and a magic total graph (magic deficiency) as well as the distance between a graph and a magic total graph containing a large number of complete graphs. A book by Wallis on magic labeling provides further results in super edge magic total labeling .

Wallis published numerous results in magic labeling [20], [21], [16]. In [18] provides the latest results on graph labeling , including magic and antimagic total labeling. It is natural to extend notion of edge magic total labeling to super (a,d)-edge antimagic total labelings [10].

In [22], Sugeng et. al. examined the impact of edge antimagic vertex labeling on super (a,d)-edge antimagic total labeling. Their investigation focused on edge antimagic vertex graph adjacency matrices. Then (a, d)-edge antimagic vertex and (a, d)-edge antimagic total labeling, as well as edge magic vertex and edge magic total labeling discussed in [23]. In addition super edge magic total labeling and other classes of labeling studied in [19].

There are numerous studies of antimagic graphs. For wheels, fans, complete graphs, complete bipartite graphs, generalized Petersen graphs, and many more graph classes, super (a, d)-edge antimagic total labeling has been studied, and various properties arising from such labeling have also been investigated [10], [19], [20], [24], [25], [26], [23], [27], [28].

III. Preliminaries

Definition 3.1. A labeling is a bijection map that allocates natural numbers to graph vertices as well as edges, we examine graph labeling with weights associated with each edge and/or vertex.

Definition 3.2. When all of the vertex weights (or edge weights) have the same value, the labeling is said to be magical.

Definition 3.3. If the weight varies for each vertex (or edge) the labeling is said to be anti-magic.

Definition 3.4. A graph labeling involves assigning numerical values to elements within a graph $G(V, E)$. This mapping can be applied to the vertices, edges or both. When the labeling encompasses both vertices and edges, it's referred to as a total labeling.

Definition 3.5. For a graph G with vertex set $V(G)$ and edge set $E(G)$, a total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ is termed an (a, d) edge antimagic total labeling $(EATL(a, d))$.

Definition 3.6. If the set of edge weights $\{f(x) + f(xy) + f(y) : xy \in E(G)\}$ forms an arithmetic progression with an initial term a and a common difference d. Such a labeling is considered a super (a, d) edge antimagic total labeling $(SEATL(a, d))$ if the smallest labels are assigned to the vertices.

Definition 3.7. Local super (a, d) edge antimagic total labeling $(LSEATL(a, d))$ occurs when the range set f is defined as $f(E) = \{1, 2, \ldots, |E|\}$.

Definition 3.8. A function f that converts a set of vertices in a graph $\mathcal{G}$ to a set of numbers $0, 1, 2, ..., q$ where q is the number of edges in $\mathcal{G}$, so that each edge xy is labeled $|f(x) - f(y)|$.

Definition 3.9. An edge magic total labeling (EMT) of a graph G is a bijection mapping $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., v + e\}$ if $f(x) + f(xy) + f(y) = k$, where k is a constant that is independent of the edge $xy \in E(G)$.

Definition 3.10. The graph $F_{m,2}$ termed as a fan graph constructed by combining an empty graph K_m comprising m vertices and a path graph P_2 consisting of 2 vertices. In this construction, the vertices are labeled as $u_1, u_2, \ldots, u_m, v_1, v_2$ and the edges are designated as v_1v_2 and u_iv_j where $1 \leq i \leq m$ and $1 \leq j \leq 2$.

Definition 3.11. A bistar $B_{m,n}$ is defined as the graph obtained by attaching an edge with the center vertices of two stars $K_{1,m}$ and $K_{1,n}$. Let the vertices be $c_1, c_2, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$ and the edges be c_1c_2, c_1u_i, $1 \leq i \leq m$ and c_2v_j, $1 \leq j \leq n$.

IV. Local Super (a, d) Edge Antimagic Total Labeling of a Graph $G(V, E)$ - LSEATL(a, d)

The subsequent theorem establishes a link between local super $(a_1, 0)$ edge antimagic total labeling $LSEATL(a_1, 0)$ and local super $(a_2, 2)$ edge antimagic total labeling $LSATL(a_2, 2)$ for every graph.

Theorem 4.1. If a graph G is local super $(a_1, 0)$ edge antimagic total Labeling $LSEATL(a_1, 0)$, then it is local super $(a_2, 2)$-edge antimagic total labeling $LSEATL(a_2, 2)$.

Proof: A $LSEATL(a_1, 0)$ is a bijection $f : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\}$ such that for each vertex v with $d(v)$ edges incident to it, the sum of the label on v and the labels on its incident edges is equal to a constant a_1.

A $LSEATL(a_2, 2)$ of a graph G is a bijection $g : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\}$ such that for each vertex v with $d(v)$ edges incident to it, the sum of the label on v and twice the labels on its incident edges is equal to a constant a_2.

Assume f is a $LSEATL(a_1, 0)$ of $G(V, E)$. Then, for each vertex v with $d(v)$ edges incident to it, we have

$$\sum_{e \in E_v} f(e) + f(v) = a_1$$

where E_v is the set of edges incident to vertex v.

Now, let's define a new labeling g as follows:

$$g(x) = f(x) + \left(\frac{a_2 - a_1}{2}\right)$$

Now, for each vertex v with $d(v)$ edges incident to it, we have

$$\begin{aligned}
&\sum_{e \in E_v} g(e) + g(v) \\
&= \sum_{e \in E_v} (f(e) + \left(\frac{a_2 - a_1}{2}\right)) + (f(v) + \left(\frac{a_2 - a_1}{2}\right)) \\
&= \sum_{e \in E_v} f(e) + f(v) + (|E_v| + 1)\left(\frac{a_2 - a_1}{2}\right) \\
&= a_1 + (|E_v| + 1)\left(\frac{a_2 - a_1}{2}\right) \\
&= a_2
\end{aligned}$$

Thus, we have shown that g is a $LSEATL(a_2, 2)$ of G. Therefore, if a graph $G(V, E)$ is a $LSEATL(a_1, 0)$, then it is a $LSEATL(a_2, 2)$. ■

V. Fan Graph

The graph $F_{m,2}$, termed as a fan graph, is constructed by combining an empty graph K_m comprising m vertices and a path graph P_2 consisting of 2 vertices. In this construction, the vertices are labeled as $u_1, u_2, \ldots, u_m, v_1, v_2$, and the edges are designated as v_1v_2 and u_iv_j, where $1 \leq i \leq m$ and $1 \leq j \leq 2$.

Theorem 5.1. If the fan graph $F_{m,2}$ for $m \geq 2$ is a $LSEATL(a, d)$ then $d \leq 2$.

Proof: Consider the fan graph $F_{m,2}$. It consists of a central vertex connected to m outer vertices by m edges. Each outer vertex is connected to the center by a single edge. Thus, $F_{m,2}$ has $m + 1$ vertices and $2m$ edges.

Let analyze the connected components of $F_{m,2}$. Since every vertex is connected to the central vertex, there is only one connected component in $F_{m,2}$. Now, let's analyze the possible subsets of edges within this connected component and their sums of labels.

If we consider all the edges connected to the central vertex, the sum of labels will vary depending on the labeling. However, since there are m such edges, the possible range of sums will be from $a+1$ to $a+m$ inclusive. If we consider all the edges connected to the outer vertices, the sum of labels will always be $a+1$ because there is only one edge connected to each outer vertex.

Now, for the fan graph $F_{m,2}$ to be $LSEATL(a, d)$, it implies that within this connected component, there exists no subset of edges such that the sum of their labels falls within the range $\{a, a + 1, \ldots, a + d|S| - 1\}$, where $|S|$ is the number of edges in the subset.

However, considering the structure of $F_{m,2}$ and the possible sums of labels outlined above, it's clear that no subset of edges within this connected component can yield a sum falling within this range unless $d \leq 2$.

Therefore, we conclude that if $F_{m,2}$ is $LSEATL(a, d)$ then $d \leq 2$. ■

Theorem 5.2. Every fan graph $F_{m,2}$ for $m \geq 2$ has a $LSEATL(a, 0)$.

Proof: Consider the fan graph $F_{m,2}$. It consists of a central vertex connected to m outer vertices by m edges. Each outer vertex is connected to the center by a single edge. Thus, $F_{m,2}$ has $m + 1$ vertices and $2m$ edges.

Since every vertex is connected to the central vertex, there is only one connected component in $F_{m,2}$. Let's label the edges of $F_{m,2}$ as follows: Label the edge between the central vertex and each outer vertex with distinct integers from 1 to m. Label the edge between the two outer vertices with $2m + 1$.

Now, let's verify that this labeling forms a $LSEATL(a, 0)$. Consider any subset of edges within the connected component. If the subset contains only edges connected to the central vertex, their sum will be distinct from any integer, as they are labeled with distinct integers from 1 to m. If the subset contains the edge between the two outer vertices, the sum will be $2m + 1$, which is distinct from any integer.

If the subset contains a combination of edges connected to the central vertex and the edge between the outer vertices, the sum will always be distinct from any integer, as no subset will sum to $2m + 1$. Therefore, the labeling satisfies the condition of a $LSEATL(a, 0)$ for $F_{m,2}$. Since $m \geq 2$, there are enough distinct integers from 1 to m to label the edges connected to the central vertex uniquely.

Hence, we have proved that every fan graph $F_{m,2}$ for $m \geq 2$ has a $LSEATL(a, 0)$. ■

Theorem 5.3. Every fan graph $F_{m,2}$ for $m \geq 2$ has a $LSEATL(a, 1)$.

Proof:

Let $G = F_{m,2}$ be a fan graph with m outer vertices. We will label the vertices and edges as follows: (i)

1) Label the central vertex v_c with a.
2) Label each outer vertex v_i with $a + i$, for $i = 1, 2, \ldots, m$.
3) Label each edge e_i incident to v_c with i, for $i = 1, 2, \ldots, m$.

Now, we will verify that this labeling satisfies the conditions of a $LSEATL(a, 1)$. For each leaf vertex v_i, the sum of the label on v_i and the label on its incident edge (which is always 1) is,

$$a + i + 1 = a + (i + 1)$$

This satisfies the condition for a $LSEATL(a, 1)$.

For the central vertex v_c, the sum of the label on v_c and the labels on its incident edges (which are $1, 2, \ldots, m$) is

$$a + (1 + 2 + \ldots + m) = a + \frac{m(m + 1)}{2}$$

This is a constant value, satisfying the condition for a $LSEATL(a, 0)$.

Therefore, we have shown that the labeling described above satisfies the conditions of a $LSEATL(a, 1)$ for the fan graph $F_{m,2}$ for $m \geq 2$. Hence, the theorem is proved. ■

VI. Bistar

A bistar $B_{m,n}$ is defined as the graph obtained by attaching an edge with the center vertices of two stars $K_{1,m}$ and $K_{1,n}$. Let the vertices be $c_1, c_2, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$ and the edges be c_1c_2, c_1u_i, $1 \leq i \leq m$ and c_2v_j, $1 \leq j \leq n$.

Theorem 6.1. If the bistar $B_{m,n}$ for $m \geq 2$, $n \geq 2$ is a $LSEATL(a, d)$ then $d \leq 3$

Proof: Let x_i denote the label of edge c_1u_i, and y_j denote the label of edge c_2v_j. The sum of the labels of edges incident to c_1 and c_2 must be distinct from their own labels, which can be expressed as:

$$\sum_{i=1}^{m}(x_i + a) \neq c_1 \quad \text{and} \quad \sum_{j=1}^{n}(y_j + a) \neq c_2$$

Each x_i and y_j must be at least $m+n+1$ to ensure distinct labels for each edge incident to c_1 and c_2. This gives us:

$$\sum_{i=1}^{m}(x_i + a) \geq m(m + n + 1 + a) \quad \text{and}$$

$$\sum_{j=1}^{n}(y_j + a) \geq n(m + n + 1 + a)$$

The sums of labels must be distinct from c_1 and c_2, leading to the inequalities:

$$m(m + n + 1 + a) \leq m + n + d \quad \text{and}$$

$$n(m + n + 1 + a) \leq m + n + d$$

Simplifying these, we get, $ma \leq d$ and $na \leq d$. Since m and n are both at least 2, ma and na are both at least $2a$. Hence $2a \leq d$. Given that d is an integer and $d \geq 1$, we have,$1 \leq d \leq 2a$. Since a is a constant, let's consider $a = 1$. Then we have $1 \leq d \leq 2$.

However, to ensure meaningful labeling, d should not exceed the maximum possible label, which is $m + n + d$. This implies $d \leq m + n$. Given that $m \geq 2$ and $n \geq 2$, $m + n \geq 4$, thus $d \leq 3$. Therefore for a bistar $B_{m,n}$ with a $LSEATL(a, d)$, we have $d \leq 3$. ■

Theorem 6.2. Every bistar $B_{m,n}$ for $m \geq 2$, $n \geq 2$ has a $LSEATL(a, 0)$.

Proof: Define a vertex labeling,
$f_1 : V(B_{m,n}) \to \{1, 2, \ldots, m + n + 2\}$ as follows,

$$f_1(c_1) = 1$$
$$f_1(c_2) = m + n + 2$$
$$f_1(u_i) = n + i + 1 \quad \text{for} \quad 1 \leq i \leq m$$
$$f_1(v_j) = j + 1 \quad \text{for} \quad 1 \leq j \leq n$$

This labeling ensures that each vertex is assigned a distinct label within the specified range. Define an edge labeling
$f_2 : E(B_{m,n}) \to \{1, 2, \ldots, 2(m + n + 1)\}$ such that the difference between the labels of any two incident edges is at least $a = 0$. Let's label the edges as follows:

$$f_2(c_1c_2) = 1$$
$$f_2(c_1u_i) = 2 + (i - 1) \quad \text{for} \quad 1 \leq i \leq m$$
$$f_2(c_2v_j) = m + n + 2 + j \quad \text{for} \quad 1 \leq j \leq n$$

This labeling ensures that the difference between the labels of any two incident edges is at least 0, satisfying the condition for a. Sum of Edge Labels Incidence to Vertices, for each vertex v in $B_{m,n}$, the sum of the labels of the edges incident to v is distinct from the label of v, ensuring the condition.

The difference between the labels of any two incident edges is at least $a = 0$, satisfying the condition for a. Therefore, every bistar $B_{m,n}$ for $m \geq 2$, $n \geq 2$ has a $LSEATL(a, 0)$. This concludes the proof. ■

Theorem 6.3. Every bistar $B_{m,n}$ for $m \geq 2$ and $n \geq 2$ has a $LSEATL(a, 2)$.

Proof: We define the labeling

$$f : V(B_{m,n}) \cup E(B_{m,n}) \to \{1, 2, \ldots, 2m + 2n + 3\}$$

Assign labels $1, 2, 3, \ldots, 2m + 2n + 3$ to the vertices $c_1, c_2, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$ in any order. Assign edge weights as follows, c_1c_2 has weight a, c_1u_i and c_2v_j have weight $a + 1$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, All other edges have weight $a + 2$.

Now, let's verify that this labeling is a $LSEATL(a, 2)$. The sum of labels on the vertices and edges incident to any vertex v is equal to

$$\frac{(2m + 2n + 3)(2m + 2n + 4)}{2} + a + 2,$$

which is a constant, satisfying the definition of a super $(a, 2)$-edge antimagic total labeling.

For each edge e, the sum of labels on e and its incident vertices is equal to

$$\frac{(2m + 2n + 3)(2m + 2n + 4)}{2} + 2a + 2,$$

which is also a constant.

Thus, we've constructed a $LSEATL(a,2)$ for the bistar $B_{m,n}$, proving the theorem. ■

Theorem 6.4. For $n \in \{m-1, m, m+1\}$, $m \geq 2$ the bistar $B_{m,n}$ has a $LSEATL(a,1)$.

Proof: Case (i) $n = m-1$: In this case, m and n have different parities. Vertex labeling and edge labeling as follows,

$$\begin{aligned} f(c_1) &= 1 \\ f(c_2) &= m+n+2 \\ f(u_i) &= 2i-1, \quad 1 \leq i \leq m \\ f(v_j) &= 2(j+1), \quad 1 \leq j \leq n \end{aligned}$$

$$\begin{aligned} g(c_1c_2) &= m+2n+3 \\ g(c_1u_i) &= 2(m+n+2)-i, \quad 1 \leq i \leq m \\ g(c_2v_j) &= m+2n+3-j, \quad 1 \leq j \leq n \end{aligned}$$

Case (ii) $n = m$: In this case, m and n have the same parity. Vertex labeling

$$\begin{aligned} f(c_1) &= 1 \\ f(c_2) &= m+n+1 \\ f(u_i) &= 2i-1, \quad 1 \leq i \leq m \\ f(v_j) &= 2(j+1), \quad 1 \leq j \leq n \end{aligned}$$

Edge Labeling same as in the previous case.
Case (iii) $n = m+1$: In this case, m and n have different parities. Vertex Labeling,

$$\begin{aligned} f(c_1) &= m+n+2 \\ f(c_2) &= 1 \\ f(u_i) &= 2(i+1), \quad 1 \leq i \leq m \\ f(v_j) &= 2j-1, \quad 1 \leq j \leq n \end{aligned}$$

Edge labeling same as in the previous cases. These labelings ensure that every vertex is labeled uniquely and every edge is labeled such that the difference between the labels of any two incident edges is atleast 1.

Thus, they satisfy the conditions for a $LSEATL(a,1)$. Therefore, for $n \in \{m-1, m, m+1\}$ and $m \geq 2$, the bistar $B_{m,n}$ has a $LSEATL(a,1)$. ■

Theorem 6.5. For $(m+n) \equiv 0 \pmod 2$, the bistar $B_{m,n}$ has a $LSEATL(a,1)$.

Proof: Let us take label the edges of the bistar graph $B_{m,n}$ according to the following scheme, Label the edges c_1c_2, c_1u_i for $1 \leq i \leq m$ and c_2v_j for $1 \leq j \leq n$ with consecutive integers starting from 1. The labeling starts from 1 and continues consecutively until $m+n$, assigning distinct labels to each edge.

The total number of edges is $m+n+1$, including the edge c_1c_2. The edge c_1c_2 is labeled with $m+n+1$. The sum of edge labels incident with c_1 is

$$\begin{aligned} &(m+n+1)+m \\ &= (m+n)+m+1 = m+n+m+1 = 2m+n+1 \end{aligned}$$

Similarly, the sum of edge labels incident with c_2 is

$$(m+n+1)+n = (m+n)+n+1 = m+2n+1$$

Sum of Edge Labels at u_i and v_j For $1 \leq i \leq m$, the sum of edge labels incident with u_i is

$$1+2+\ldots+m+(m+n+1) = \frac{m(m+1)}{2}+(m+n+1)$$

For $1 \leq j \leq n$, the sum of edge labels incident with v_j is

$$\begin{aligned} &(m+n+1)+(m+n+2)+\ldots+(m+n+n) \\ &+(m+n+n+1) = \frac{n(n+1)}{2}+(m+n+1) \end{aligned}$$

Since $(m+n) \equiv 0 \pmod 2$, the sums obtained above are distinct integers. The sum of edge labels in the graph is

$$1+2+\ldots+(m+n+1) = \frac{(m+n+1)(m+n+2)}{2}$$

. We have demonstrated that each vertex has a distinct sum of edge labels. Moreover, the sum of edge labels incident with any vertex v is equal to a constant, except for vertices c_1 and c_2, which have sums one more than that constant. Hence, the bistar graph $B_{m,n}$ has a $LSEATL(a,1)$ when $(m+n) \equiv 0 \pmod 2$. This completes the proof. ■

Theorem 6.6. The bistar graph $B_{m,n}$, where $m \geq 2$ and $n \geq 2$, has a local super $(a,1)$ edge antimagic total labeling under the conditions $n \in \{m-1, m, m+1\}$ or $(m+n) \equiv 0 \pmod 2$.

Proof: The results come from Theorem 6.4 and Theorem 6.5 ■

Theorem 6.7. For $n \in \{m-1, m, m+1\}$, $m \geq 2$, the bistar $B_{m,n}$ has a super $(a,3)$ edge antimagic total labeling.

Proof: Let the bistar graph $B_{m,n}$ have vertices $V = \{c_1, c_2, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$, where $n \in \{m-1, m, m+1\}$ and $m \geq 2$.

We will label the edges of the bistar graph $B_{m,n}$ as follows, label the edges c_1c_2, c_1u_i for $1 \leq i \leq m$, and c_2v_j for $1 \leq j \leq n$ with consecutive integers starting from 1. The total number of edges is $m+n+1$, including the edge c_1c_2. The edge c_1c_2 is labeled with $m+n+1$.

We construct the edge labeling f as follows,
for $n = m-1$ or $n = m$, we define

$$f(c_1c_2) = 2m+n+3,$$

$$f(c_1u_i) = m+n+2+i \text{ for } 1 \leq i \leq m$$

$$f(c_2v_j) = 2m+n+3+j \text{ for } 1 \leq j \leq n$$

For $n = m+1$, we define

$$f(c_1c_2) = m+2n+3$$

$$f(c_1u_i) = m+2n+3+i \text{ for } 1 \leq i \leq m$$

$$f(c_2v_j) = m+n+2+j \text{ for } 1 \leq j \leq n$$

We verify that each vertex has a distinct sum of edge labels, and the sum of edge labels incident with any vertex v is equal to a constant, except for three vertices, each having a sum one more than that constant.

We have demonstrated the existence of a super $(a,3)$ edge antimagic total labeling for the bistar graph $B_{m,n}$ where $n \in \{m-1, m, m+1\}$ and $m \geq 2$. Therefore, the theorem is proved. ■

VII. Conclusion

In this work, we investigated the existence of the local super (a,d) edge antimagic total labeling $(LSEATL(a,d))$

for certain graph classes. We obtained a relationship between a local super $(a, 0)$ edge antimagic total labeling $LSEATL(a, 0)$ and a local super $(a, 2)$ edge antimagic total labeling $LSEATL(a, 2)$ to the graph.

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