An Optimal Green Inventory Model for a Deteriorating Product with Non-Linear Quadratic Demand and Permissible Payment Systems

M. BALASUBRAMANIAN and R. KAMALI

Abstract – **This paper is an optimal approach to the Inventory Model. Here, we estimate efficient inventory control for the supply chain process to balance customer demand, minimize costs, and promote sustainable practices. It is imperative to integrate a non-linear, time-dependent quadratic demand function and a judiciously designed delayed payment policy to formulate an efficient inventory model for a diminishing commodity. We aim to maximize inventory choices while considering the effects of variations in demand, deteriorating products, and the flexibility of delayed payment plans. And also, the study begins by formulating a mathematical model that captures the non-linear relationship between time-dependent quadratic demand and inventory management. The model considers the effect of product deterioration over time and introduces a delay payment policy to accommodate customers' financial preferences. The objective is to optimize the total inventory cost concerning different decision variables associated with delayed payments.**

Index Term – **green inventory, deteriorating product, time-dependent demand, delay payment, optimization, simulation-based methods, total cost, and customer service.**

I. INTRODUCTION

Effective inventory management significantly contributes to the success of businesses as it directly influences customer satisfaction, operational expenses, and environmental sustainability. There is a growing emphasis on implementing green practices in various aspects of business operations, including inventory management. Green inventory management optimizes inventory decisions while considering the environmental impact and sustainable objectives. This paper focuses on developing an optimal green inventory, such as perishable goods, pharmaceuticals, and electronics. Businesses need to consider the impact of product deterioration in their inventory management strategies to minimize loss and ensure product quality. Additionally, customer demand for products often exhibits non-linear, time-dependent patterns due to certain factors such as seasonality, promotional activities, and market trends. Incorporating these non-linear demand patterns into inventory models can lead to more accurate forecasting and improved inventory control decisions. Furthermore, allowing customers to delay payment for their purchases can be an effective strategy for businesses to attract and retain customers.

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Providing a permissible delay payment option gives customers the flexibility to manage their cash flow while ensuring timely order fulfilment. However, businesses must carefully balance the financial implications of delayed payment with the costs associated with inventory holding, ordering, and potential stock-outs. This study aims to create an optimum green inventory model that tackles the issues of a deteriorating product, non-linear quadratic demand, and delayed payment. The model aims to optimize the total inventory cost concerning different decision variables such as holding costs, ordering costs, shortage costs, and penalty costs associated with delayed payments.

The proposed model will reduce waste, minimize environmental impact, and promote sustainability in supply chain operations by integrating green practices into the inventory management process. To achieve the research objective, a mathematical model has been formulated that captures the non-linear relationship between time-dependent quadratic demand, product deterioration, and delay payment policies.

The model will be optimized using numerical optimization techniques and simulation-based methods. The proposed model will be validated through a case study in the relevant industry, providing practical insights and demonstrating the model's effectiveness in real-world scenarios. Y. Huang et al. [1] investigated optimizing pricing strategies and inventory replenishment decisions to maximize the supply chain's overall profit, considering consumer demand and potential lost sales. J. Zhang et al. [2] created appropriate pricing and lot-sizing rules that maximize the total profit of the supply chain while achieving the specified service level. S. Liu et al. [3] researched pricing and inventory control systems that maximize the overall profit, taking into account the deterioration rate of the item and the likelihood of backlogging. B. C. Giri et al. [4] considering the non-instantaneous degradation rate of the products, it was explored to find the best values for these decision factors to maximize the overall profit. M. Ghahremanloo et al. [5] examined a novel method for creating a fuzzy inventory model for non-instantaneously deteriorating products with partial shortage backlogs. The goal is to create a cost-effective inventory control plan that considers changes in demand and the potential for a partial backlog. S. D. Wu et al. [6] created to calculate the ideal price and lot-sizing options that maximize the overall profit, including the influence of stock-dependent demand and the potential of partial backlogging. S. S. Sana et al. [7] analyzed the ideal pricing and replenishment techniques to maximize overall profit, considering the dynamic nature of stock-dependent demand and the potential for partial backlog. I. Konstantaras et al. [8] developed pricing and inventory control rules that maximize the total profit, considering the deterioration rate of the products and the likelihood of partial backlogging. K. Arshinder et al. [9] examined the impact of stock-dependent demand and the risk of partial backlogging to identify optimal pricing and inventory replenishment methods that enhance overall profit. The research specifically delved into price and inventory control for deteriorating items. S. Yang and J. Ma [10] analyzed the best inventory policies for non-instantaneous degrading products to discover the ideal replenishment and price options that maximize the overall profit, given the stock-dependent demand pattern and the risk of partial backlogging.

A. A. Taleizadeh et al. [11] developed the best replenishment plans for items that don't deteriorate instantly to find the best inventory control strategies that optimize total profit, taking into account the impact of demand that depends on stock levels and the possibility of partial back-ordering. Y. Niu and S. Wang [12] devised a production-inventory model tailored for deteriorating products, incorporating factors such as trade credit terms, the time value of money, inflation, and the perishable nature of goods. This model aims to identify optimal production and inventory replenishment strategies that maximize overall profit. X. Wang et al. [13] explored the determination of optimal price and inventory replenishment techniques that optimize total profit, considering stock-dependent demand patterns and the potential for partial backlogging. Their research emphasized optimal pricing and inventory control for non-immediately degrading items. In a study by R. N. Mishra et al. [14], optimal replenishment alternatives for non-instantaneous deteriorating items were established. Their approach leveraged stock-dependent demand to derive optimal inventory control strategies, considering the dynamic nature of stock-dependent demand and the time-varying deterioration rate. K. Eshghi et al. [15] created inventory models for deteriorating products to build optimum inventory control techniques that maximize the overall profit, incorporating the dynamic nature of stock-dependent demand, the potential of partial backlogging, and the effect of advance payments. G. Zhang and J. Zhang [16] examined pricing and replenishment decisions for items experiencing non-instantaneous deterioration. They focused on determining the most effective pricing and replenishment policies that optimize overall profit, considering the dynamic nature of stock-dependent demand and the likelihood of partial backlog. L. A. San José et al. [17] formulated pricing and replenishment policies for non-instantaneous deteriorating items to identify optimal strategies that maximize total profit. Their analysis considered the dynamic nature of stock-dependent demand, the potential for partial backlog, and various operational constraints. M. Y. Jaber and M. Bonney [18] addressed the escalating inventory replenishment model, devising optimal pricing and inventory control strategies that maximize overall profit while incorporating stock-dependent demand and the potential for partial backlog. H. Zhang and G. Zhang [19] the study delved into an optimal pricing and inventory model, addressing the optimal pricing and inventory choices for items experiencing non-instantaneous deterioration. This model considered factors such as stock-dependent demand and partial backlog, considering the influence of user evaluations. The study aims to identify the best price and inventory control tactics for maximizing overall profit while considering the dynamic nature of stock-dependent demand, the likelihood of partial backlog, and the influence of user evaluations. X. Yu and X. Wang [20] devised the pricing inventory to identify the ideal price and inventory control strategies that maximize the overall profit, incorporating the dynamic nature of stock-dependent demand, the potential of partial backlogging, and the effect of carbon emissions legislation. Y. Huang and G. Q. Huang [21] delved into a numerical study to understand the influence of different parameters on the decisions and profits of the supply chain and its constituent members. Y. Huang and G. Q. Huang [22] devised that when product cost is more significant than a certain echelon, the chain members' profits will increase as the market becomes more sensitive to the retail price.

The outcomes of this study will add to the current literature on green inventory management. They will assist firms in enhancing their inventory decisions while considering environmental sustainability and financial performance. By adopting a practical, optimal green inventory model, businesses can achieve cost savings, improve customer service levels, and reduce their carbon footprint, contributing to a greener supply chain. Overall, a comprehensive and practical approach is presented for managing deteriorating product inventory with non-linear time-dependent quadratic demand and allowable delay payment, emphasizing the importance of green practices in inventory management and highlighting the benefits of sustainable supply chain operations.

II. DECISION PARAMETERS AND CONSIDERATION

The following notations are used: the ordering cost (r), the level of green inventory I(t) at t, the deterioration rate θ, the inventory purchase cost P, h denotes the holding cost, s denotes the shortage cost for stock out period, α denote opportunity cost, due to end sale, Ie denote the interest earned, and Ir denote the interest charges with $Ir \geq Ie$.

Decision Variables

M, T, T_1, TC (T_1, T), TC_1 (T_1, T), for $T_1 \geq M$ and TC_2 (T_1, T) for $T_1 < M$

Research Focus

The proposed approach is validated using the automotive industry, where non-linear demand patterns and shortage situations are every day. The results demonstrate the quadratic demand non-linear approach over traditional linear models regarding inventory performance metrics, such as fill rate, customer satisfaction, and total inventory costs.

$$D(t) = \begin{cases} a + bt + ct^2; & 0 \leq t < T_1 \\ a & ; T_1 \leq t < T \end{cases}$$

Where a > 0, 0 < b < 1, 0 < c < 1

III. MODEL FORMULATION

The optimal inventory control decisions are developed; numerical optimization techniques and simulation-based methods are utilized. Also, the optimization process seeks to find the optimal replenishment time, order quantity, and permissible delay period that minimizes the total cost while satisfying customer demand requirements are established. The Simulation-based method is employed to assess the robustness of the proposed model under various demand scenarios and evaluate its performance in real-world applications.

An optimal green inventory system is developed for a deteriorating product with quadratic demand considering time-dependent and allows customers to delay payment within a specified period. The Mathematical model that captures the non-linear relationship between time-dependent quadratic demand and inventory management is developed in the interval [0, T1).

Hence, in the interval [0, T1), the non-linear model is formulated as follows:

$$\frac{dI(t)}{dt} = \begin{cases} -a - bt - ct^2 - \theta I(t)\,;\ 0 \le t \le T_1 \\ -\frac{a}{1+\delta(T-t)} \quad ;\ T_1 \le t \le T \end{cases} \tag{1}$$

The inventory level is obtained as,

$$I(t)e^{\theta t} = \int (-a - bt - ct^2)e^{\theta t}\,dt + c_1$$

Use the boundary condition $I(T_1) = 0$ we get

$$I(t) = \frac{a\theta^2 - b\theta + 2c}{\theta^3}\left[e^{\theta(T_1-t)} - 1\right] - \frac{2c}{\theta^2}\left[T_1 e^{\theta(T_1-t)} - t\right] + \frac{1}{\theta}\left[T_1(b + cT_1)e^{\theta(T_1-t)} - (bt + ct^2)\right]$$

For the interval $[T_1, T)$.we obtain

$$I(t) = \frac{a}{\delta}\log\log[1 + \delta(T-t)] + c_2,$$

$$I(t) = \frac{a}{\delta}\begin{bmatrix} \log\left(1+\delta(T-t)\right) \\ -\log(1+\delta(T-T_1)) \end{bmatrix}$$

Hence, the change in non-linear demand we get,

$$I(T) = \frac{a\theta^2 - b\theta + 2c}{\theta^3}\left[e^{\theta(T_1-t)} - 1\right] - \frac{2c}{\theta^2}\left[T_1 e^{\theta(T_1-t)} - t\right] + \frac{1}{\theta}\left[T_1(b + cT_1)e^{\theta(T_1-t)} - (bt + ct^2)\right];\ 0 \le t < T_1$$

$$\frac{a}{\delta}\begin{bmatrix} \log\left(1+\delta(T-t)\right) - \\ \log\left(1+\delta(T-T_1)\right) \end{bmatrix};\ T_1 \le t < T \tag{2}$$

Various Inventory Costs by Using Non-Linear Demand Pattern

Green Inventory Holding Cost Estimation

$$HC = h\int_0^{T_1} I(t)dt$$
$$= h\left\{(a\theta^2 - b\theta + 2c)\left(\frac{e^{\theta T_1} - 1}{\theta^4}\right) - \frac{2cT_1}{\theta^3}e^{\theta T_1} + \frac{T_1 e^{\theta T_1}}{\theta^2}(b + cT_1) - \frac{T_1}{6\theta}(6a + 3bT_1 + 2cT_1^2)\right\} \tag{3}$$

Estimation of Deterioration Cost

$$DC = P\theta\int_0^{T_1} I(t)dt$$
$$= P\theta\left\{(a\theta^2 - b\theta + 2c)\left(\frac{e^{\theta T_1} - 1}{\theta^4}\right) - \frac{2cT_1}{\theta^3}e^{\theta T_1} + \frac{T_1 e^{\theta T_1}}{\theta^2}(b + cT_1) - \frac{T_1}{6\theta}(6a + 3bT_1 + 2cT_1^2)\right\} \tag{4}$$

Stock-Out Period Estimation of Shortage Cost

$$SC = s\int_{T_1}^{T} I(t)dt$$
$$= s\left\{\frac{a}{\delta^2}\left[\delta(T - T_1) - \log(1 + \delta(T - T_1))\right]\right\} \tag{5}$$

Estimation of Opportunity Cost

$$OC = \alpha\int_{T_1}^{T}\left(a - \frac{a}{1+\delta(T-t)}\right)dt$$
$$= \frac{a\alpha}{\delta}\{\delta(T - T_1) - \log(1 + \delta(T - T_1))\} \tag{6}$$

Deterioration of products is a common phenomenon in industries such as perishable goods, pharmaceuticals, and electronics. Businesses need to consider the impact of product deterioration in their inventory management strategies to minimize losses and ensure product quality. Additionally, customer demand for products often exhibits non-linear, time-dependent patterns due to seasonality, promotional activities, and market trends. Incorporating these non-linear demand patterns into inventory models can lead to more accurate forecasting and improved inventory control decisions.

Furthermore, allowing customers to delay payment for their purchases can be an effective strategy for businesses to attract and retain customers. Providing a permissible delay in payment option gives the customer the flexibility to manage their cash flow while ensuring timely order fulfilment. However, businesses must carefully balance the financial implications of delayed payment with the costs associated with inventory holding, ordering, and potential stock-outs. Therefore, the credit period M is estimated as the permissible delay in settling the accounts in two cases.

Case1: $M \le T_1$

Estimation of Interest Earned

$$IE_1 = PI_e\int_0^{T_1}(T_1 - t)(a + bt + ct^2)\,dt$$
$$= \frac{PI_e T_1^2}{12}\left[6a + 2bT_1 + cT_1^2\right] \tag{7}$$

Estimation of Interest Payable

$$I_p = PI_r\int_M^{T_1} I(t)\,dt$$
$$= PI_r\left\{\frac{a\theta^2 - b\theta + 2c}{\theta^4}\left[e^{\theta(T_1-M)} + M\theta - (1 + \theta T_1)\right] - \frac{2c}{\theta^3}\left[T_1\left(e^{\theta(T_1-M)} - 1\right) + \frac{\theta}{2}(M^2 - T_1^2)\right] + \frac{1}{\theta^2}\begin{bmatrix} T_1(b + cT_1)\left(e^{\theta(T_1-M)} - 1\right) + \frac{b\theta}{2}(M^2 - T_1^2) \\ + \frac{c\theta}{3}(M^3 - T_1^3) \end{bmatrix}\right\} \tag{8}$$

A combination of numerical optimization techniques and simulation-based methods is utilized to obtain optimal inventory control decisions. The optimization aims to establish the best replenishment time, order amount, and allowable wait duration to minimize overall cost while meeting client demand criteria. Simulation-based methods are employed to assess the robustness of the proposed model under various demand scenarios and evaluate its performance in real-world applications. The proposed optimal green inventory model is validated through the manufacturing industry's case study. The case study considers a deteriorating product with time-dependent quadratic demand and allows buyers to postpone payment for a set amount of time. The results demonstrate that the proposed model leads to significant cost savings, improved customer service levels, and reduced environmental impact by promoting sustainable inventory management practices. The total system cost is as follows:

$$TC_1 = \frac{r + HC + DC + SC + OC + I_p - IE_1}{T} \tag{9}$$

$$= \frac{1}{T}\Big\{r + h\left\{(a\theta^2 - b\theta + 2c)\left(\frac{e^{\theta T_1} - 1}{\theta^4}\right) - \frac{2cT_1}{\theta^3}e^{\theta T_1} + \frac{T_{1}e^{\theta T_1}}{\theta^2}(b + cT_1) - \frac{T_1}{6\theta}(6a + 3bT_1 + 2cT_1^2)\right\} + P\theta\left\{(a\theta^2 - b\theta + 2c)\left(\frac{e^{\theta T_1} - 1}{\theta^4}\right) - \frac{2cT_1}{\theta^3}e^{\theta T_1} + \frac{T_{1}e^{\theta T_1}}{\theta^2}(b + cT_1) - \frac{T_1}{6\theta}(6a + 3bT_1 + 2cT_1^2)\right\} + s\left\{\frac{a}{\delta^2}\left[\delta(T - T_1) - \log\left(1 + \delta(T - T_1)\right)\right] + \frac{b}{2}\left[\frac{T^3 + 2T_1^3}{3} - TT_1^2\right]\right\} + \frac{a\alpha}{\delta}\left\{\delta(T - T_1) - \log\left(1 + \delta(T - T_1)\right)\right\}$$
$$PI_r\left\{\frac{a\theta^2 - b\theta + 2c}{\theta^4}\left[e^{\theta(T_1-M)} + M\theta - (1 + \theta T_1)\right] - \frac{2c}{\theta^3}\left[T_1\left(e^{\theta(T_1-M)} - 1\right) + \frac{\theta}{2}(M^2 - T_1^2)\right] + \frac{1}{\theta^2}\left[T_1(b + cT_1)\left(e^{\theta(T_1-M)} - 1\right) + \frac{b\theta}{2}(M^2 - T_1^2) + \frac{c\theta}{3}(M^3 - T_1^3)\right]\right\} - \frac{PI_e T_1^2}{12}\left[6a + 2bT_1 + cT_1^2\right]\Big\}$$

$$= \frac{1}{T}\Big\{r + (h + P\theta)\left\{(a\theta^2 - b\theta + 2c)\left(\frac{e^{\theta T_1} - 1}{\theta^4}\right) + \frac{T_{1}e^{\theta T_1}}{\theta^3}\left[\theta(b + cT_1) - 2c\right] - \frac{T_1}{6\theta}(6a + 3bT_1 + 2cT_1^2)\right\} +$$

$\frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1)-log\left(1+\delta(T-T_1)\right)\right\}+\frac{sb}{2}\left[\frac{T^3+2T_1^3}{3}-TT_1^2\right]+PI_r\{\frac{a\theta^2-b\theta+2c}{\theta^4}\left[e^{\theta(T_1-M)}+M\theta-(1+\theta T_1)\right]-\frac{2c}{\theta^3}\left[T_1\left(e^{\theta(T_1-M)}-1\right)+\frac{\theta}{2}(M^2-T_1^2)\right]+\frac{1}{\theta^2}\left[T_1(b+cT_1)\left(e^{\theta(T_1-M)}-1\right)+\frac{b\theta}{2}(M^2-T_1^2)+\frac{c\theta}{3}(M^3-T_1^3)\right]\}$ - $\frac{PI_eT_1^2}{12}[6a+2bT_1+cT_1^2]\}$ (10)

To solve the below equations, we obtain $T^*T_1^*$

$$\frac{\partial TC_1(T_1,T)}{\partial T_1}=0 \quad and \quad \frac{\partial TC_1(T_1,T)}{\partial T}=0 \qquad (11)$$

Provided they satisfy the sufficient conditions

$$\left[\frac{\partial^2TC_1(T_1,T)}{\partial T_1^2}\right]_{at\ (T_1^*,T^*)}>0\ \left[\frac{\partial^2TC_1(T_1,T)}{\partial T^2}\right]_{at\ (T_1^*,T^*)}>0$$

$$and\ \left[\left(\frac{\partial^2TC_1(T_1,T)}{\partial T_1^2}\right)\left(\frac{\partial^2TC_1(T_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2TC_1(T_1,T)}{\partial T_1\partial T}\right)^2\right]>0$$

$$\frac{\partial TC_1(T_1,T)}{\partial T_1}=0$$

$$\frac{1}{T}\{(h+P\theta)\{(a\theta^2-b\theta+2c)\left(\frac{e^{\theta T_1}}{\theta^3}\right)+\frac{e^{\theta T_1}}{\theta^3}[\theta(b+2cT_1)-2c]+\frac{e^{\theta T_1}}{\theta^2}[\theta(bT_1+cT_1^2)-2cT_1]-\frac{T_1}{6\theta}(3b+4cT_1)-\frac{1}{6\theta}(6a+3bT_1+2cT_1^2)\}-\frac{a(s+\delta\alpha)(T-T_1)}{1+\delta(T-T_1)}+PI_r\left\{\frac{a\theta^2-b\theta+2c}{\theta^5}\left[e^{\theta(T_1-M)}-\theta^2\right)\right]-\frac{2c}{\theta^3}\left[\frac{e^{\theta(T_1-M)}}{\theta}(T_1+\theta)-\theta T_1-1\right]+\frac{1}{\theta^2}\left[(bT_1+cT_1^2)\frac{e^{\theta(T_1-M)}}{\theta}+\left(e^{\theta(T_1-M)}-1\right)(b+2cT_1)-\theta T_1(b+cT_1)\right]\Big\}-\frac{PI_eT_1}{6}[6a+3bT_1+2cT_1^2]\}=0 \qquad (12)$$

$$\frac{\partial TC_1(T_1,T)}{\partial T}=0 \quad \Rightarrow\frac{1}{T}\left\{\frac{a(s+\delta\alpha)(T-T_1)}{1+\delta(T-T_1)}\right\}-\frac{1}{T^2}\{r+(h+P\theta)\left\{(a\theta^2-b\theta+2c)\left(\frac{e^{\theta T_1}-1}{\theta^4}\right)+\frac{T_1e^{\theta T_1}}{\theta^3}[\theta(b+cT_1)-2c]-\frac{T_1}{6\theta}(6a+3bT_1+2cT_1^2)\right\}+\frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1)-log\left(1+\delta(T-T_1)\right)\right\}+PI_r\{\frac{a\theta^2-b\theta+2c}{\theta^4}\left[e^{\theta(T_1-M)}+M\theta-(1+\theta T_1)\right]-\frac{2c}{\theta^3}\left[T_1\left(e^{\theta(T_1-M)}-1\right)+\frac{\theta}{2}(M^2-T_1^2)\right]+\frac{1}{\theta^2}\left[T_1(b+cT_1)\left(e^{\theta(T_1-M)}-1\right)+\frac{b\theta}{2}(M^2-T_1^2)+\frac{c\theta}{3}(M^3-T_1^3)\right]\}-\frac{PI_eT_1^2}{12}[6a+2bT_1+cT_1^2]\}=0 \qquad (13)$$

IV. SOLUTION PROCEDURE ALGORITHM

To find the optimal time period T_1^*,T^* and minimize $C_1(T_1,T)$

Step 1 Find i) to iv)

i) Put $T_{1,(1)}=M$

ii) use $T_{1\,(1)}$ into 12), find $T_{(1)}$

iii) Using $T_{(1)}$ we obtain $T_{1,(2)}$ from (13)

iv) Continue (ii) and (iii) until optimum of T_1 and T

Step 2 to find T_1, M.

i) If $M\le T_1$, T_1 is optimum, go to Step3

ii) If $M>T_1$, T_1 is not optimum. Set $T_1=M$ to find T from (13) and go to step 3.

Step 3 Calculate $TC_1(T_1^*,T^*)$

Case2: $T_1<M$

Estimation of Interest Earned

$$IE_2=PI_e\left\{\begin{matrix}\int_0^{T_1}(T_1-t)(a+bt+ct^2)\,dt+\\(M-T_1)\int_0^{T_1}(a+bt+ct^2)\,dt\end{matrix}\right\}$$

$$=\frac{-PI_eT_1^2}{12}[6a+4bT_1+3cT_1^2]+\frac{pI_eT_1M}{6}[6a+3bT_1+2cT_1^2] \qquad (14)$$

Therefore, the total average cost

$$TC_2=\frac{r+HC+DC+SC+OC-IE_2}{T} \qquad (15)$$

$$=\frac{1}{T}\{r+h\left\{(a\theta^2-b\theta+2c)\left(\frac{e^{\theta T_1}-1}{\theta^4}\right)-\frac{2cT_1}{\theta^3}e^{\theta T_1}+\frac{T_{1e^{\theta T_1}}}{\theta^2}(b+cT_1)-\frac{T_1}{6\theta}(6a+3bT_1+2cT_1^2)\right\}+p\theta\left\{(a\theta^2-b\theta+2c)\left(\frac{e^{\theta T_1}-1}{\theta^4}\right)-\frac{2cT_1}{\theta^3}e^{\theta T_1}+\frac{T_{1e^{\theta T_1}}}{\theta^2}(b+cT_1)-\frac{T_1}{6\theta}(6a+3bT_1+2cT_1^2)\right\}+s\left\{\frac{a}{\delta^2}\left[\delta(T-T_1)-\log\left(1+\delta(T-T_1)\right)\right]\right\}+\frac{a\alpha}{\delta}\left\{\delta(T-T_1)-\log\left(1+\delta(T-T_1)\right)\right\}-\frac{pI_eT_1^2}{12}[6a+4bT_1+3cT_1^2]+\frac{pI_eT_1M}{6}[6a+3bT_1+2cT_1^2]\}$$

$$=\frac{1}{T}\{r+(h+P\theta)\left\{(a\theta^2-b\theta+2c)\left(\frac{e^{\theta T_1}-1}{\theta^4}\right)+\frac{T_{1e^{\theta T_1}}}{\theta^3}[\theta(b+cT_1)-2c]-\frac{T_1}{6\theta}(6a+3bT_1+2cT_1^2)\right\}+\frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1)-log\left(1+\delta(T-T_1)\right)\right\}-\frac{PI_eT_1^2}{12}[6a+4bT_1+3cT_1^2]+\frac{PI_eT_1M}{6}[6a+3bT_1+2cT_1^2]\} \qquad (16)$$

To find the optimal T*, T1* to get the optimum total inventory cost.

$$\frac{\partial TC_2(T_1,T)}{\partial T_1}=0 \quad and \quad \frac{\partial TC_2(T_1,T)}{\partial T}=0 \qquad (17)$$

Provided the sufficient conditions

$$\left[\frac{\partial^2TC_2(T_1,T)}{\partial T_1^2}\right]_{at\ (T_1^*,T^*)}>0\ ,\left[\frac{\partial^2TC_2(T_1,T)}{\partial T^2}\right]_{at\ (T_1^*,T^*)}>0\ and\ \left[\left(\frac{\partial^2TC_2(T_1,T)}{\partial T_1^2}\right)\left(\frac{\partial^2TC_2(T_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2TC_2(T_1,T)}{\partial T_1\partial T}\right)^2\right]>0$$

Now, we shall solve

$$\frac{\partial TC_2(T_1,T)}{\partial T_1}=0\ and\ \frac{\partial TC_2(T_1,T)}{\partial T}=0:\frac{\partial TC_2(T_1,T)}{\partial T_1}0$$

$$\Rightarrow\frac{1}{T}\left\{(h+P\theta)\left\{(a\theta^2-b\theta+2c)\left(\frac{e^{\theta T_1}}{\theta^3}\right)+\frac{e^{\theta T_1}}{\theta^3}[\theta(b+2cT_1)-2c]+\frac{e^{\theta T_1}}{\theta^2}[\theta(bT_1+cT_1^2)-2cT_1]-\frac{T_1}{6\theta}(3b+4cT_1)-\frac{1}{6\theta}(6a+3bT_1+2cT_1^2)\right\}-\frac{a(s+\delta\alpha)(T-T_1)}{1+\delta(T-T_1)}\ PI_eT_1[a+bT_1+T_1^2]\ pI_eM\,[a+bT_1+cT_1^2]\right\}=0 \qquad (18)$$

$$\frac{\partial TC_2(T_1,T)}{\partial T}=0\Rightarrow\frac{1}{T}\left\{\frac{a(s+\delta\alpha)(T-T_1)}{1+\delta(T-T_1)}\right\}-\frac{1}{T^2}\{r+(h\,P\theta)\left\{(a\theta^2-b\theta+2c)\left(\frac{e^{\theta T_1}-1}{\theta^4}\right)+\frac{T_{1e^{\theta T_1}}}{\theta^3}[\theta(b+cT_1)-2c]-\frac{T_1}{6\theta}(6a+3bT_1+2cT_1^2)\right\}+\frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1)-log\left(1+\delta(T-T_1)\right)\right\}-\frac{PI_eT_1^2}{12}[6a+4bT_1+3cT_1^2]+\frac{pI_eT_1M}{6}[6a+3bT_1+2cT_1^2]\}=0 \qquad (19)$$

Estimation of T^* and T_1^* and Minimize $TC_2(T_1,T)$ By using the following algorithm

Analysis of T_1 and T

Discuss (i) - (iv)

i) Set $T_{1,(1)} = M$
ii) Put $T_{1,(1)}$ into (18), find $T_{(1)}$
iii) Using $T_{(1)}$ to find $T_{1,(2)}$, from (19)
iv) Continue again (ii) and (iii) until optimum values of T_1 and T

Analysis of T_1 and M
i) If $T_1 < M$, T_1 is optimum, go to Step 3.
ii) If $T_1 \geq M$, T_1 is not optimum, set $T_1 = M$, to find T From (19) and then goes to Step 3.
iii) To find $TC_2(T_1^*, T^*)$ we obtain,
$TC(T_1, T) = \min(TC_1(T_1^*, T^*), TC_2(T_1^*, T^*))$

V. SOLUTION PROCEDURE FOR VARIOUS PARAMETERS

Table I. ANALYSIS OF A DECISION VARIABLE WITH NON-LINEAR DEMAND PATTERN

	$\boldsymbol{\delta}$	
M	1	5
5 TC T_1^* T^*	2.0338×10^6 40.1351 39.1288	3.7142×10^6 51.1335 50.9332
10 TC T_1^* T^*	1.1791×10^6 30.3806 29.3668	2.2862×10^6 42.6151 42.4149
15 TC T_1^* T^*	3.2121×10^5 11.0863 10.0056	5.7590×10^6 19.4125 19.2094
40 TC T_1^* T^*	1.0078×10^8 102.4787 101.4786	9.4914×10^{35} 935.6173 935.4173
50 TC T_1^* T^*	3.9369×10^9 153.5997 152.5997	5.6070×10^{63} 1.7427×10^3 1.7425×10^3

Table II. ANALYSIS OF VARIOUS PARAMETERS WITH NON-LINEAR DEMAND PATTERN

		M	
$\boldsymbol{\delta}$	5	10	15
1 TC T_1^* T^*	2.0338×10^6 40.1351 39.1288	1.1791×10^6 30.3806 29.3668	1.6868×10^6 36.7828 35.7746
2 TC T_1^* T^*	2.1346×10^6 41.2534 40.7517	1.1869×10^6 30.8397 30.3353	5.4686×10^6 57.6537 57.1532
3 TC T_1^* T^*	2.4012×10^6 43.4248 43.0906	1.3220×10^6 32.8306 32.4951	5.2863×10^7 93.1734 92.8401
4 TC T_1^* T^*	2.8832×10^6 46.6932 46.4427	2.2862×10^6 36.6153 36.3643	2.9507×10^9 149.7342 149.4842
5 TC T_1^* T^*	3.7142×10^6 51.1335 50.9332	5.6070×10^6 42.6154 42.4149	1.7832×10^{12} 235.4709 235.2709

The study results demonstrate the effectiveness of the proposed green inventory model in managing deteriorating products under non-linear, time-dependent demand patterns. Here are the key findings:

COST REDUCTION
The quadratic demand model significantly reduces total inventory costs compared to traditional linear models. This is achieved by more accurately predicting demand variations, which leads to better replenishment strategies and optimized order quantities.

ENVIRONMENTAL IMPACT
The implementation of a green inventory system leads to lower environmental costs by minimizing product wastage due to deterioration. Businesses can maintain sustainability by managing stock levels more efficiently.

CUSTOMER SATISFACTION
By addressing stockouts and improving fill rates, the model enhances customer satisfaction. The quadratic demand model provides more precise stock levels, ensuring products are available when needed, even during periods of fluctuating demand.

FINANCIAL FLEXIBILITY
The introduction of a permissible delay in payment allows businesses to manage cash flow more effectively. This delay reduces financial strain while ensuring that businesses can meet customer demand without overstocking, leading to better financial performance.

REAL-WORLD VALIDATION
The model is validated through a case study in the automotive industry. It demonstrates robustness in handling complex demand patterns, with significant improvements in key metrics like inventory holding cost, deterioration cost, and shortage cost.

OPTIMIZATION RESULTS
The numerical results from the case study show the optimal replenishment time T* and inventory level T_1* lead to minimized total costs (TC). The optimized values for T*and T_1* help businesses balance holding, ordering, and shortage costs, improving overall profitability.

VI. RESULTS AND CONCLUSION

The green inventory with a deteriorating product model that addresses the complex dynamics of non-linear time-dependent quadratic demand and incorporates the concept of permissible delay payment is analyzed. The model provided a practical approach to optimizing inventory control decisions, balancing the trade-off between maintaining sufficient inventory levels and minimizing costs associated with deterioration and holding costs. This enables decision-makers to make environmentally conscious choices while ensuring optimal inventory performance. Integrating the non-linear time-dependent quadratic demand pattern captures the realistic demand variability, allowing for more accurate inventory forecasting and control. The inclusion of permissible delay payment enables businesses to manage cash flow effectively by delaying payment while maintaining an optimal inventory level, leading to improved financial flexibility. It provided a structured framework to guide decision-making related to inventory control, leading to improved efficiency and profitability. This study adds to the body of knowledge in inventory management by presenting a complete method that considers the interaction of demand dynamics, product deterioration, sustainability, and financial considerations. Further research can be developed by incorporating additional factors such as transportation costs, lead time variability, or product substitution.

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TABLE III. A COMPARATIVE ANALYSIS WITH TRADITIONAL MODEL

Aspect	Non-Linear Demand Model (Based on Table I and II)	Traditional Linear/Constant Demand Models	Quadratic Demand Models
Cost Behaviour (TC)	**Highly reactive to δ and M**: As the deterioration rate increases, the total cost rises sharply, particularly at larger M values. Extremely high costs for larger δ values (e.g., from 2.0338×10^{6} to 5.6070×10^{63}).	**More stable costs**: Total costs tend to increase predictably as demand remains constant or grows linearly over time. Less responsive to δ, as deterioration is often simplified or excluded.	**Moderately complex**: Costs increase more gradually compared to non-linear models but are more realistic than linear models due to the quadratic terms accounting for demand fluctuations.
Cycle Time (T_1^*, T^*)	**Shortens with increasing M** (e.g., from 11.0863 to 10.0056 at M=15 for δ=1). Lengthens as δ increases, indicating more frequent replenishment is needed for products that deteriorate more quickly.	**More stable cycle times**: Typically, longer, as these models assume constant or predictable demand, requiring fewer replenishments. Deterioration may have less impact on cycle time if it's not a significant factor in the model.	**Similar trends but less extreme**: Cycle times decrease as M increases, but the sensitivity is lower than in non-linear models.

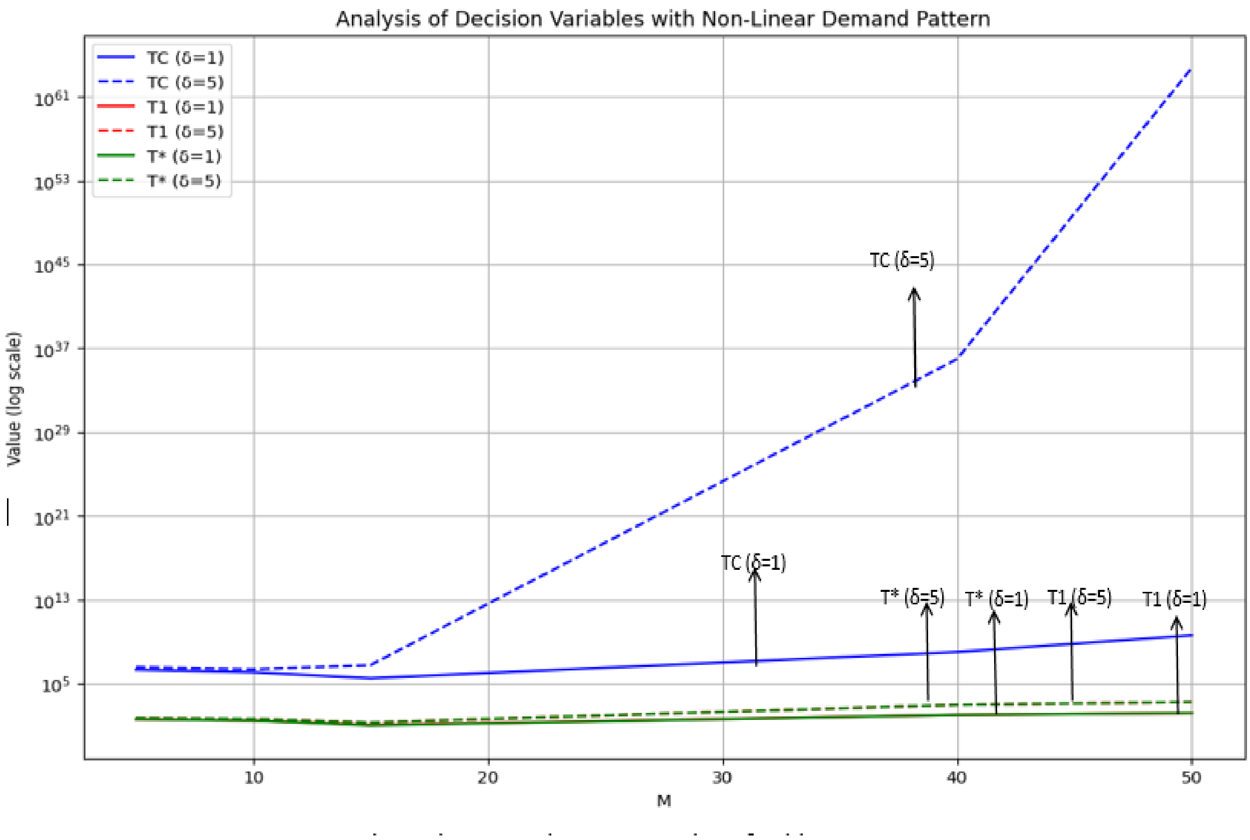

Fig 1. Diagrammatic Representation of Table I.

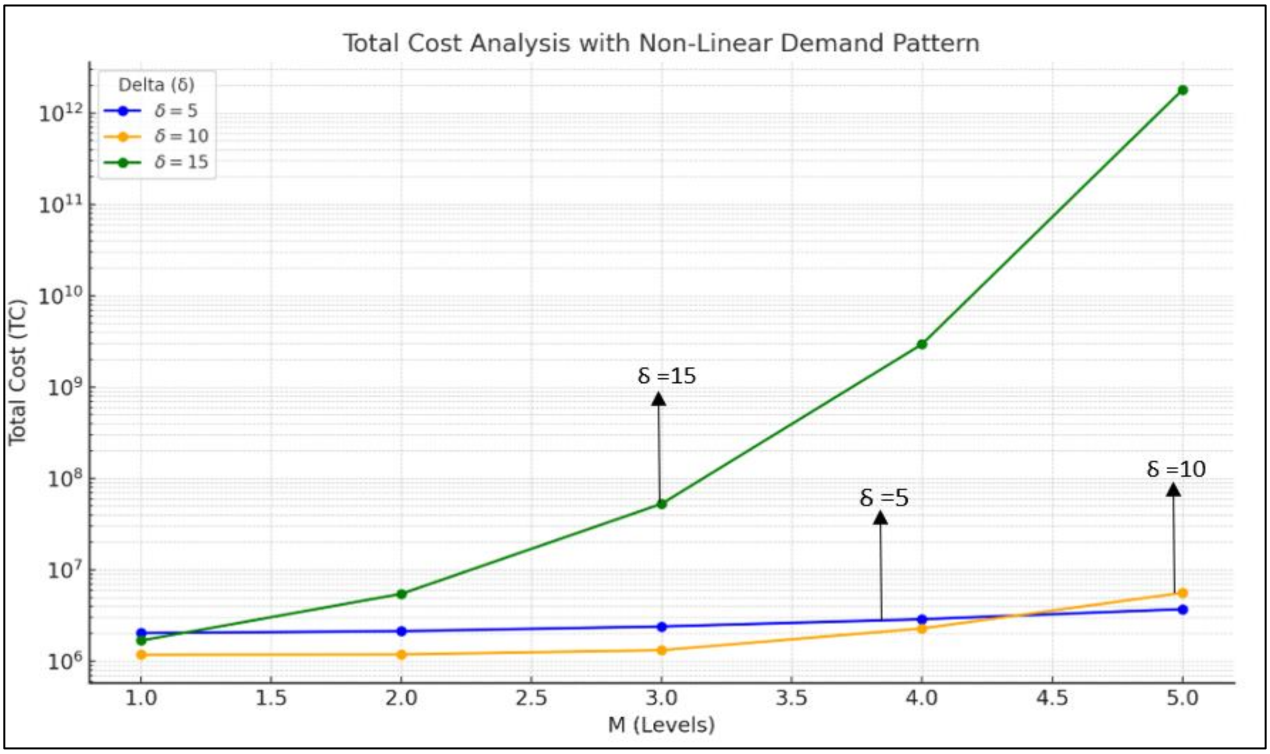

Fig 2. Diagrammatic Representation of Table II.