Two Accurate Semi-analytical Techniques for Solving (2+1)-D and (3+1)-D Schrodinger Equations

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***Abstract* - In this paper, the solutions of the (2+1)-D and (3+1)-D Schrodinger equations are investigated mathematically using two efficient semi-analytical techniques. One proposed technique is based on the combination of the formable transform and the homotopy perturbation method (FTHPM), whereas another technique is the classical variational iteration method (VIM). A comparison study between the formable transform-based homotopy perturbation method (FTHPM) and the variational iteration method (VIM) for solving these equations is discussed. Some theorems are presented to illustrate the convergence of both semi-analytical techniques. To verify the accuracy and efficiency of the proposed schemes, two test examples are discussed.**

***Index Terms* - Formable transform, Homotopy perturbation method (HPM), Variation iteration method (VIM), (2+1)-D and (3+1)-D Schrödinger equations, Test examples.**

I. INTRODUCTION

THE higher-dimensional partial differential equations are very important in fields like physics, engineering, biology, and other sciences. The (2+1)-D and (3+1)-D Schrödinger equations are especially useful in these areas, and finding their solutions is an active area of research. These equations are key to exploring wave functions and energy states of particles in quantum mechanics. Such equations continue to attract researchers because of their importance and complexity. Advances in analytical and semi-analytical methods, have made it easier to study and understand complex multidimensional systems. In this study, we present a new technique that combines the formable transform and the homotopy perturbation method to solve the (2+1)-D and (3+1)-D Schrödinger equations. Examples validate the effectiveness, simplicity, and accuracy of the suggested method.

We also solve the (2+1)-D and (3+1)-D Schrödinger equations using the variational iteration method and compare the results from both methods. Through graphical representations, the examples offer a detailed view of the behavior of both the examples offer a detailed view of the behavior of both the real and imaginary parts of the solution within a fixed domain. These findings highlight the potential of the proposed methods, making them useful tools for tackling similar problems in scientific and engineering fields.

The homotopy perturbation method (HPM) and the variational iteration method (VIM) are semi-analytical methods used to handle a wide range of linear and nonlinear differential equations. HPM and VIM methods were pioneered by J.H.He [11, 12]. A new approach that integrates homotopy and perturbation techniques has been developed to tackle nonlinear problems by J.H. He [13]. The homotopy perturbation method has been employed to address the solution of non-linear wave equation by J.H.He [14]. Convergence analysis of this method, which applies to both systems of PDEs and PDEs, is discussed by J. Biazar et al. in [9, 10]. In [1], A. Ghorbani presented an alternative method to the Adomian method, utilizing He's polynomial, which was derived from the HPM, instead of Adomian polynomials. In [22], a new transform, the formable integral transform, was developed by Rania Saadeh and Bayan Ghazal, while its various properties and applications were explored by Basit et al. [17] and Saadeh et al. [21]. Eljaily and Elzaki employed ETHPM to tackle Schrödinger equations in [19], while LTVIM has been employed to solve the Schrodinger equation by G. Singh and I. Singh [8]. Ghanbari [4] used HAM for solving (2+1)-D Schrödinger equations. Wazwaz [2, 3] used VIM for tackling some linear and nonlinear PDEs. E. Rama [6] solved various problems of differential equations via VIM. J. H. He [15] gave some results and new interpretation of VIM. He and Wu [16] presented some applications and developments of VIM. Abdou and Soliman [18] studied new applications of VIM. Tomar et al. [23] presented the VIM as a new approach to obtain the Lagrange multiplier. In [24], the convergence analysis of VIM is discussed by Odibat. In [5], both VIM and HPM have been employed on different evolution equations by D. D. Ganji. In [7], G. Singh et al. employed the Laplace variational iterative method to solve higher-order Burgers' equations, incorporating a modified VIM and Laplace transform. In [20], O. E. Ige et al. tackled the sine-Gordon equations by Adomian polynomial and Elzaki transform method.

In this research, we discuss an innovative technique by combining the formable transform and the HPM (FTHPM) to address both the (2+1)-D and (3+1)-D Schrodinger equations. Additionally, we employ VIM to solve these equations, conduct a comparative analysis of their solutions, and evaluate absolute errors.

(a) The (2+1)-D Schrödinger Equation

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$$-i\frac{\partial\eta}{\partial t}=\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+w(x,y)\eta$$

With initial conditions

$$\eta(x,\breve{y},0)=g_1(x,\breve{y}),\qquad \eta_t(x,\breve{y},0)=g_2(x,\breve{y}).$$

(b) The (3+1)-D Schrödinger Equation

$$-i\frac{\partial\eta}{\partial t}=\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}+w(x,y,z)\eta$$

in continuous domain with initial conditions

$\eta(x,\breve{y},z,0)=g_3(x,\breve{y},z)$,

$\eta_t(x,\breve{y},z,0)=g_4(x,\breve{y},z)$.

The function w is an arbitrary potential function.

This research paper is organized**:** An Introduction of the formable integral transform and its properties is discussed in Section II. Section III contains a complete discussion of the homotopy perturbation method (HPM). The formable transform-based homotopy perturbation method (FTHPM) is discussed in Section IV. In Section V, convergence analysis of HPM is explored. Section VI introduces the variational iteration method, while Section VII presents the convergence analysis of the variational iteration method. To check the accuracy and investigate the solutions of the (2+1)-D and (3+1)-D Schrödinger equations, some numerical illustrations are discussed in section VIII. In section IX, the results and discussion about the figures and tables can be found. In section X, the conclusion of this work is presented.

II. FORMABLE TRANSFORM

The formable transform of the function $g_1(t)$ is formally defined as (see [23]):

$$\mathcal{R}\{g_1(t)\}=\mathcal{B}(s,u)=s\int_0^\infty e^{-st}g_1(ut)dt,$$

$s>0, u>0, t\in[0,\infty)$.

Formable transform of derivative is given below:

$$\mathcal{R}\left[g^{(n)}(t)\right]=\frac{s^n}{u^n}\mathcal{B}(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k}g^{(k)}(0),$$

$$n=0,1,2\ldots$$

Formable transform of some functions:

$$\mathcal{R}(1)=1,\qquad \mathcal{R}(t)=\frac{u}{s},\qquad \mathcal{R}\left(\frac{t^n}{n!}\right)=\frac{u^n}{s^n},$$

$$\mathcal{R}(e^{at})=\frac{s}{s-au},\quad \mathcal{R}(\sin(at))=\frac{asu}{s^2+a^2u^2},$$

$\mathcal{R}[\cos(at)]=\frac{s^2}{s^2+a^2u^2}$.

III. INRODUCTION OF HE'S HPM [11]

The homotopy perturbation method combines classical perturbation and homotopy techniques to overcome traditional limitations. To demonstrate its application in solving nonlinear differential equations, consider a differential equation (see [11])

$$\hat{A}(\eta)-f(r)=o,\qquad r\in\Omega \tag{1}$$

Let the boundary condition is,

$$B\left(\eta,\frac{\partial\eta}{\partial n}\right)=0,\qquad r\in\Gamma$$

Here $\hat{A}$ is differential operator, B is boundary operator,$f(r)$ is known analytic function, Ω represents the domain with boundaryΓ. Now $\hat{A}$ is divide into $\mathcal{L}$, which is linear and $\mathcal{N}$, which is non-linear. Now (1), is expressed as follow:

$$\mathcal{L}(\eta)+\mathcal{N}(\eta)-f(r)=0,$$

develop a homotopy$\hat{w}(r,p):\Omega\times[0,1]\to\mathcal{R}$, and satisfies

$$\mathcal{H}(w,p)=(1-p)[\mathcal{L}(w)-\mathcal{L}(\eta_0)]+p\left[\hat{A}(w)-f(r)\right]=0,$$

$$p\in[0,1], r\in\Omega \tag{2}$$

Here, $p\in[0,1]$is the embedding parameter, and $\eta_0$is initial approximation of equation which satisfies boundary conditions. The solution of equation (2), can be written as:

$$w=w_0+pw_1+p^2w_2+\cdots$$

Letting $p=1$, the resulting approximation for equation (1), is:

$$\eta=\lim_{p\to1}w=w_0+w_1+w_2+\cdots$$

IV. THE FORMABLE TRANSFORM-BASED HOMOTOPY PERTURBATION METHOD (FTHPM)

Consider a non-homogeneous and non-linear partial differential equation

$$D\{\eta(x,\breve{y},t)\}+R\{\eta(x,\breve{y},t)\}+N\{\eta(x,\breve{y},t)\}=g(x,\breve{y},t), \tag{3}$$

With the initial condition

$$\eta(x,\breve{y},0)=f_1(x,\breve{y}),\qquad \eta_t(x,\breve{y},0)=f_2(x,\breve{y},t),$$

Here,D represents a linear differential-operator of the second-order, $\acute{R}$ is linear differential-operator of order less than order of D. Additionally, consider N as a non-linear differential-operator and introduce a source term $g(x,\breve{y},t)$. By using formable transform to equation (3), we acquire the result

$$\mathcal{R}[D\{\eta(x,\breve{y},t)\}]+\mathcal{R}[R\{\eta(x,\breve{y},t)\}]+\mathcal{R}[N\{\eta(x,\breve{y},t)\}]=\mathcal{R}[g(x,\breve{y},t)], \tag{4}$$

Utilizing the differential property inherent in the formable transform and subsequently employing the inverse formable transform, we get

$$\eta(x,\breve{y},t)=\mathcal{S}(x,\breve{y},t)-\mathcal{R}^{-1}\left[\frac{u}{s}\{\mathcal{R}\{\eta(x,\breve{y},t)\}+N\{\eta(x,\breve{y},t)\}\right], \tag{5}$$

Here $\mathcal{S}(x,\breve{y},t)$ represent the terms that arise from the IC and source terms.

According to homotopy perturbation method

$$\eta(x,\breve{y},t)=\sum_{n=0}^\infty p^n\eta_n(x,\breve{y},t), \tag{6}$$

The decomposition of non-linear term may be as follow

$$N[\eta(x,\breve{y},t)]=\sum_{n=0}^\infty p^nH_n(\eta), \tag{7}$$

Here $H_n(\eta)$ is the He's polynomial [6] and is given as:

$$H_n(\eta_0,\eta_1,\eta_2,\ldots,\eta_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^n p^i\eta_i\right)\right]_{p=0},$$

$$n=0,1,2,3, \tag{8}$$

Substitute (6) and (7)in equation (5), then we obtain

$$\sum_{n=0}^\infty p^n\eta_n(x,\breve{y},t)=\mathcal{S}(x,\breve{y},t)-p\mathcal{R}^{-1}\left[\frac{u}{s}\{\mathcal{R}\{\sum_{n=0}^\infty p^n\eta_n(x,\breve{y},t)\}+\sum_{n=0}^\infty p^nH_n(\eta)\}\right], \tag{9}$$

which is combined form of formable transform and homotopy perturbation method. Compare the coefficients associated with corresponding indices of p,

$$p^0:\eta_0(x,\breve{y},t)=\mathcal{S}(x,\breve{y},t),$$

$$p^1:\eta_1(x,\breve{y},t)=-\mathcal{R}^{-1}\left\{\frac{u}{s}\mathcal{R}[\eta_0(x,\breve{y},t)+H_0(\eta)]\right\},$$

$$p^2:\eta_2(x,\breve{y},t)=\mathcal{R}^{-1}\left\{\frac{u}{s}\mathcal{R}[\eta_1(x,\breve{y},t)+H_1(\eta)]\right\},$$

$$p^3: \eta_3(\mathfrak{x}, \breve{y}, t) = \mathcal{R}^{-1}\left\{\frac{u}{s}\mathcal{R}[\eta_2(\mathfrak{x}, \breve{y}, t) + H_2(\eta)]\right\},$$

Continue the process, the solution is:

$$\eta(\mathfrak{x}, \breve{y}, t) = \lim_{p\to 1} \eta_n(\mathfrak{x}, \breve{y}, t),$$

This implies

$$\eta(\mathfrak{x}, \breve{y}, t) = \eta_0(\mathfrak{x}, \breve{y}, t) + \eta_1(\mathfrak{x}, \breve{y}, t) + \eta_2(\mathfrak{x}, \breve{y}, t) + \cdots \quad (10)$$

V. CONVERGENCE ANALYSIS OF HOMOTOPY PERTURBATION METHOD [9,10]

In this section, we have explored the theorems that illustrate the convergence of HPM (see [9],[10]).

Theorem: Let Ґand Ҡ be Banach spaces, consider a mapping Φ: Ґ → Ҡ that is contractive and non-linear, and ∀ v, $\tilde{v}$ ∈ Ґ,

$$\|\Phi(v) - \Phi(\tilde{v})\| \le \gamma\|v - \tilde{v}\|, \quad 0 < \gamma < 1$$

As per the "Banach fixed point theorem" the mapping Φ possesses a unique fixed point u, that is

$$\Phi(u) = u,$$

In the context of homotopy perturbation method

$$V_n = \Phi(V_{n-1}),$$

$$V_{n-1} = \sum_{i=0}^{n-1} u_i, \quad n = 1,2,3 \ldots$$

assume, $V_0 = v_0 = u_0 \in B_r(u)$

where$B_r(u) = \{u^* \in X : \|u^* - u\| < r\}$ then

i. $\|V_n - u\| \le \gamma^n\|v_0 - u\|$

ii. $V_n \in B_r(u)$

iii. $\lim_{n\to\infty} V_n = u$

Proof:

i. By utilizing an inductive approach with the base case when $n = 1$

$$\|V_1 - u\| = \|\Phi(V_o) - \Phi(u)\| \le \gamma\|v_0 - u\|,$$

Assume that

$$\|V_{n-1} - u\| \le \gamma^{n-1}\|v_0 - u\|,$$

So as

$$\|V_n - u\| = \|\Phi(V_{n-1}) - \Phi(u)\| = \gamma\|V_{n-1} - u\| = \gamma^n\|v_0 - u\|,$$

Using (i)

$$\|V_n - u\| \le \gamma^n\|v_0 - \|u \le \gamma^n r < r \Rightarrow V_n \in B_r(u).$$

ii. Because of $\|V_n - u\| \le \gamma^n\|v_0 - u\|$ and $\lim_{n\to\infty}\gamma^n = 0$, $\lim_{n\to\infty}\|V_n - u\| = 0$,
that is $\lim_{n\to\infty} V_n = u$.

VI. VARIATIONAL ITERATION METHOD (VIM) [12]

To elucidate the fundamental concept of the variational iteration method (VIM) (see [12]), let's examine the following differential equation

$$Lu + Nu = g(x),$$

Here a linear-operator L, a non-linear operator N, and an inhomogeneous term $g(x)$ are involved in the differential equation. From "variation iteration method" (VIM), the construction of a correction function takes the following form:

$$u_{n+1}(\mathfrak{x}, \breve{y}, t) = u_n(\mathfrak{x}, \breve{y}, t) + \int_0^t \lambda\{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\}\,d\xi,$$

The general Lagrange's multiplier denoted as λ, and utilizing variational theory to identify its optimal value, the variational iteration method (VIM) involves generating successive approximations as $u_{n+1}(\mathfrak{x}, \breve{y}, t)$, $n \ge 0$. The initial conditions $u(\mathfrak{x}, \breve{y}, 0)$and $u_t(\mathfrak{x}, \breve{y}, 0)$ are used for selective zeroth approximation u_0. The solution is given by

$$u = \lim_{n\to\infty} u_n.$$

VII. CONVERGENCE ANALYSIS OF VARIATIONAL ITERATION METHOD [24]

In this section, we have explored the theorems that illustrate convergence of variational iteration method as outlined in [24].

Theorem. Let A be the operator from Hilbert space $\mathbb{H}$ to $\mathbb{H}$. The series solution

$u(t) = \sum_{k=o}^{\infty} v_k(t)$ converges if $\exists\, 0 < \gamma < 1$, then

$\|A[v_0 + v_1 + \cdots + v_{k+1}]\| \le \gamma\|A[v_0 + v_1 + \ldots + v_{k+1}]\|$,

that is $(\|v_{k+1}\| \le \gamma\|v_k\|)$. $\quad \forall k \in N \cup \{0\}$.

Proof: Define a sequence$\{\mathcal{S}_n\}_{n=0}^{\infty}$as

$\mathcal{S}_0 = v_0$, and

$$\mathcal{S}_n = v_0 + v_1 + \cdots + v_n,$$

We will show that, the sequence$\{\mathcal{S}_n\}_{n=0}^{\infty}$is a Cauchy sequence in Hilbert space $\mathbb{H}$. To establish this, let us assume that

$$\|\mathcal{S}_{n+1} - \mathcal{S}_n\| = \|v_{n+1}\| \le \gamma\|v_n\| \le \gamma^2\|v_{n-1}\| \le \cdots \le \gamma^{n+1}\|v_0\|,$$

For every $n, j \in N$, $n \ge j$. So we have

$$\|\mathcal{S}_n - \mathcal{S}_i\| = \|(\mathcal{S}_n - \mathcal{S}_{n-1}) + (\mathcal{S}_{n-1} - \mathcal{S}_{n-2}) + \cdots + (\mathcal{S}_{i-1} - \mathcal{S}_i)\|,$$

$$\le \|\mathcal{S}_n - \mathcal{S}_{n-1}\| + \|\mathcal{S}_{n-1} - \mathcal{S}_{n-2}\| + \cdots + \|\mathcal{S}_{i-1} - \mathcal{S}_i\|,$$

$$\le \gamma^n\|v_0\| + \gamma^{n-1}\|v_0\| + \cdots + \gamma^{j+1}\|v_0\|,$$

$$= \frac{1 - \gamma^{n-i}}{1 - \gamma}\gamma^{i+1}\|v_0\|,$$

Since $0 < \gamma < 1$, we get

$$\lim_{n,j\to\infty}\|\mathcal{S}_n - \mathcal{S}_i\| = 0,$$

Therefore, $\{\mathcal{S}_n\}_{n=0}^{\infty}$forms a Cauchy sequence within a Hilbert space Ḥ, indicating the convergence of the series solution $u(t) = \sum_{k=o}^{\infty} v_k(t)$.

VIII. NUMERICAL EXPERIMENTS

Let us discuss some examples; these examples will help us to exhibit the working of the hybrid method (FTHPM) and VIM.

Example 1: Examine the (2+1)-D Schrödinger equation

$$-i\eta_t = \eta_{\mathfrak{x}\mathfrak{x}} + \eta_{\breve{y}\breve{y}} + w(\mathfrak{x}, \breve{y}), \quad (11)$$

Where $w(\mathfrak{x}, \breve{y}) = 3 - 2\tan h^2(\mathfrak{x}) - 2\tan h^2(\breve{y})$. The initial conditions is

$$\eta(\mathfrak{x}, \breve{y}, 0) = \frac{i}{\cosh \mathfrak{x}.\cosh \breve{y}},$$

The exact solution is

$$\eta(\mathfrak{x}, \breve{y}, t) = \frac{ie^{it}}{\cosh \mathfrak{x}.\cos \breve{y}},$$

Solution: Method 1.

<u>Formable Transform Homotopy Perturbation Method (FTHPM)</u>

By applying the formable transform to (11)

$$\mathcal{R}(-i\eta_t) = \mathcal{R}\left[\eta_{\mathfrak{x}\mathfrak{x}} + \eta_{\breve{y}\breve{y}} + \left(3 - 2\tan h^2(\mathfrak{x}) - 2\tan h^2(\breve{y})\right)\eta\right],$$

This implies

$$\left\{\frac{s}{u}\mathcal{R}\eta(\mathfrak{x},\breve{y},\mathfrak{t})-\frac{s}{u}.\eta(\mathfrak{x},\breve{y},0)\right\} = i.\mathcal{R}\{[\eta_{\mathfrak{x}\mathfrak{x}}+\eta_{\breve{y}\breve{y}} + \left((3-2\tan h^2(\mathfrak{x})-2\tan h^2(\breve{y})\right)\eta]\},$$

Applying the inverse formable transform, we obtain

$$\eta(\mathfrak{x},\breve{y},\mathfrak{t}) = \mathcal{R}^{-1}\left\{\frac{i}{\cosh\mathfrak{x}.\cos\breve{y}}\right\} + i.\mathcal{R}^{-1}\{[\eta_{\mathfrak{x}\mathfrak{x}}+\eta_{\breve{y}\breve{y}} + \left((3-2\tan h^2(\mathfrak{x})--2\tan h^2(\breve{y})\right)\eta]\},$$

Applying the HPM, we obtain

$$\sum_{n=0}^{\infty}p^n\,\eta_n(\mathfrak{x},\breve{y},\mathfrak{t}) = \frac{i}{\cosh\mathfrak{x}.\cosh\breve{y}} + i.p.\left\{\mathcal{R}^{-1}\left[\frac{u}{s}.\mathcal{R}\left(\sum_{n=0}^{\infty}p^n\,H_n(\eta)\right)\right]\right\}, \quad (12)$$

where $H_n(\eta)$ is He's polynomial

$$H_n(\eta_0,\eta_1,\eta_2,\dots) = \frac{1}{n!}\left[\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n}p^i\eta_i\right)\right)\right]_{p=0}, \quad (13)$$

From (12), we obtain

$$\eta_0+p\,\eta_1+p^2\eta_2+\cdots = \frac{i}{\cosh\mathfrak{x}\cosh\breve{y}} + i.p.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\{H_0(\eta)+pH_1(\eta) + p^2H_2(\eta)+\cdots\}\right\},$$

Comparing the equal powers of p, so we obtain

$$p^0:\quad \eta_0 = \frac{i}{\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$p^1:\eta_1 = i.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\left(H_0(\eta)\right)\right\} = \frac{i^2\mathfrak{t}}{\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$H_0(\eta) = \frac{i}{\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$p^2:\eta_2 = i.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\left(H_1(\eta)\right)\right\} = \frac{i^3\mathfrak{t}^2}{(2!)\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$H_1(\eta) = \frac{i^2\mathfrak{t}}{\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$p^3:\eta_3 = i.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\left(H_2(\eta)\right)\right\} = \frac{i^4\mathfrak{t}^3}{(3!)\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$H_2(\eta) = \frac{i^3\mathfrak{t}^2}{\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$p^4:\eta_4 = i.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\left(H_3(\eta)\right)\right\} = \frac{i^5\mathfrak{t}^4}{(4!)\cosh\mathfrak{x}.\cosh\breve{y}},$$

$$H_3(\eta) = \frac{i^4\mathfrak{t}^3}{\cosh x.\cosh y},$$

and so on.

Therefore, the solution is:

$$\eta(\mathfrak{x},\breve{y},\mathfrak{t}) = \eta_0+\eta_1+\eta_2+\cdots,$$

$$= \frac{i}{\cosh\mathfrak{x}.\cosh\breve{y}} + \frac{1}{\cosh\mathfrak{x}.\cosh\breve{y}}.\frac{i^2\mathfrak{t}}{1!} + \frac{1}{\cosh\mathfrak{x}.\cosh\breve{y}}.\frac{i^3\mathfrak{t}^2}{2!} + \frac{1}{\cosh\mathfrak{x}.\cosh\breve{y}}.\frac{i^4\mathfrak{t}^3}{2!}+\cdots,$$

$$= \frac{i}{\cosh\mathfrak{x}.\cos x\breve{y}}\left(1+(i\mathfrak{t})+\frac{(i\mathfrak{t})^2}{2!}+\frac{(i\mathfrak{t})^3}{3!}+\cdots\right),$$

$$= \frac{ie^{i\mathfrak{t}}}{\cosh\mathfrak{x}\cosh\breve{y}}.$$

Method 2: <u>Variational Iteration Method (VIM)</u>

In variational iteration method (VIM), it was found that the Lagrange's multiplier $\lambda=-1$, and the correction functional is given by

$$u_{n+1}(\mathfrak{x},\breve{y},\mathfrak{t}) = u_n(\mathfrak{x},\breve{y},\mathfrak{t}) + \int_0^t \lambda\left[\frac{\partial u_n}{\partial\xi} - i\left(\frac{\partial^2 u_n}{\partial\mathfrak{x}^2}+\frac{\partial^2 u_n}{\partial\breve{y}^2}\right) - i\left(3-2\tan h^2(\mathfrak{x}) - 2\tan h^2(\breve{y})\right)u_n\right]d\xi,$$

$$u_1(\mathfrak{x},\breve{y},\mathfrak{t}) = u_0(\mathfrak{x},\breve{y},\mathfrak{t}) - \int_0^t\left[\frac{\partial u_0}{\partial\xi} - i\left(\frac{\partial^2 u_0}{\partial\mathfrak{x}^2}+\frac{\partial^2 u_0}{\partial\breve{y}^2}\right) - i\left(3-2\tan h^2(\mathfrak{x}) - 2\tan h^2(\breve{y})\right)u_0\right]d\xi, u_1(\mathfrak{x},\breve{y},\mathfrak{t}) = \frac{(i-\mathfrak{t})}{\cos h\mathfrak{x}.\cos h\,\breve{y}},$$

$$u_2(\mathfrak{x},\breve{y},\mathfrak{t}) = u_1(\mathfrak{x},\breve{y},\mathfrak{t}) - \int_0^t\left[\frac{\partial u_1}{\partial\xi} - i\left(\frac{\partial^2 u_1}{\partial\mathfrak{x}^2}+\frac{\partial^2 u_1}{\partial\breve{y}^2}\right) - i\left(3-2\tan h^2(\mathfrak{x}) - 2\tan h^2(\breve{y})\right)u_1\right]d\xi,$$

$$u_2(\mathfrak{x},\breve{y},\mathfrak{t}) = \frac{(i-\mathfrak{t}-\frac{i\mathfrak{t}^2}{2})}{\cos h\mathfrak{x}.\cos h\,\breve{y}},$$

$$u_3(\mathfrak{x},\breve{y},\mathfrak{t}) = \frac{(i-\mathfrak{t}-\frac{i\mathfrak{t}^2}{2}+\frac{\mathfrak{t}^3}{6})}{\cos h\mathfrak{x}.\cos h\,\breve{y}},$$

$$u_4(\mathfrak{x},\breve{y},\mathfrak{t}) = \frac{(i-\mathfrak{t}-\frac{i\mathfrak{t}^2}{2}+\frac{\mathfrak{t}^3}{6}-\frac{\mathfrak{t}^4}{24})}{\cos h\mathfrak{x}.\cos h\,\breve{y}},$$

and so on. The solution is

$$u(\mathfrak{x},\breve{y},\mathfrak{t}) = \lim_{n\to\infty}u_n(\mathfrak{x},\breve{y},\mathfrak{t}),$$

$$= \lim_{n\to\infty}\frac{i}{\cos h\,\mathfrak{x}.\cos h\,\breve{y}}\left(1+i\mathfrak{t}-\frac{\mathfrak{t}^2}{2!}-\frac{i\mathfrak{t}^3}{3!}+\frac{\mathfrak{t}^4}{4!}+\cdots\right),$$

$$= \frac{ie^{i\mathfrak{t}}}{\cos h\mathfrak{x}.\cos h\breve{y}}.$$

which is exact solution.

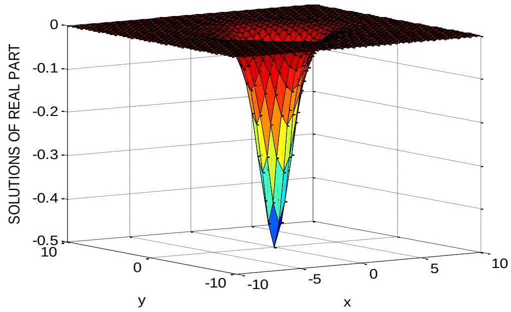

Fig 1. Dynamic behavior real part of the solution for $\mathfrak{t}=\frac{\pi}{6}$.

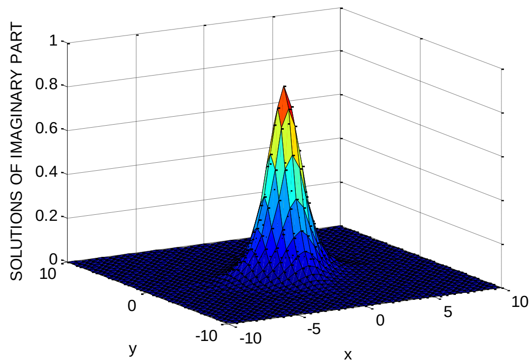

Fig 2.Dynamic behavior of imaginary part of the solution for $\text{ŧ} = \frac{\pi}{6}$.

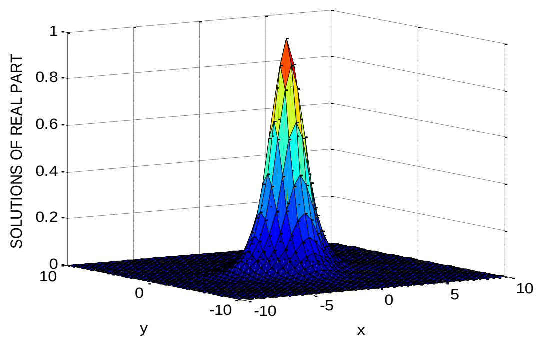

Fig 3. Dynamic behavior real part of the solution for $\text{ŧ} = 3.5\pi$.

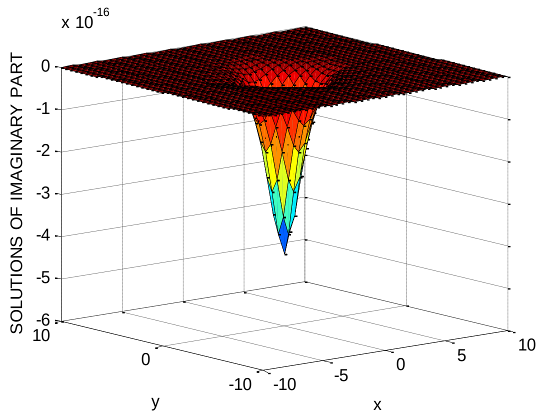

Fig 4. Dynamic behavior of imaginary part of the solution for $\text{ŧ} = 3.5\pi$.

Fig 1 and Fig 2 represent the dynamic behavior of the solutions of the real and imaginary parts of example 1 at different ranges of x, y, and $t = \frac{\pi}{6}$. Whereas Fig 3 and Fig 4 represent the dynamic behavior of solutions of the real part and imaginary part of the example 1 at different ranges of x, y, and $t = 3.5\,\pi$. Table 1 presents the comparision of absolute errors of solution obtained by FTHPM and VIM for example 1.

Example 2: Consider the (3+1)-D Schrödinger equation of the form:

$$-i\eta_{\text{ŧ}} = \nabla^2\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) + w(\mathrm{x},\check{y},\mathrm{z},\text{ŧ})\eta, \qquad (14)$$

Where $\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = 1 - \frac{2}{\mathrm{x}^2} - \frac{2}{\check{y}^2} - \frac{2}{\mathrm{z}^2}$.

The initial conditions are: $\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = \mathrm{x}^2\check{y}^2\mathrm{z}^2$ and $\eta_t(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = i\mathrm{x}^2\check{y}^2\mathrm{z}^2$.

The exact solution is: $\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = \mathrm{x}^2\check{y}^2\mathrm{z}^2 e^{i\text{ŧ}}$.

Solution: Method 1.

<u>Formable Transform Homotopy Perturbation Method (FTHPM)</u>

Use formable transform to (14), we acquire the result

$$\mathcal{R}(-i\eta_{\text{ŧ}}) = \mathcal{R}\left[\nabla^2\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) + \left(1 - \frac{2}{\mathrm{x}^2} - \frac{2}{\check{y}^2} - \frac{2}{\mathrm{z}^2}\right)\eta\right],$$

This implies

$$-i\left\{\frac{s}{u}\mathcal{R}[\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ})] - \frac{s}{u}.\eta(\mathrm{x},\check{y},\mathrm{z},0)\right\} = \mathcal{R}\left[\nabla^2\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) + \left(1 - \frac{2}{\mathrm{x}^2} - \frac{2}{\check{y}^2} - \frac{2}{\mathrm{z}^2}\right)\eta\right].$$

$$\mathcal{R}[\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ})] = \eta(\mathrm{x},\check{y},\mathrm{z},0) + i\frac{u}{s}\mathcal{R}\left[\nabla^2\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) + \left(1 - \frac{2}{\mathrm{x}^2} - \frac{2}{\check{y}^2} - \frac{2}{\mathrm{z}^2}\right)\eta\right].$$

Taking the inverse formable transform, we obtain

$$\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = \mathcal{R}^{-1}[\mathrm{x}^2\check{y}^2\mathrm{z}^2] + i.\mathcal{R}^{-1}\left[\frac{u}{s}.\mathcal{R}\left(\nabla^2\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) + \left(1 - \frac{2}{\mathrm{x}^2} - \frac{2}{\check{y}^2} - \frac{2}{\mathrm{z}^2}\right)\eta\right)\right],$$

Using the HPM, we obtain

$$\sum_{n=0}^{\infty} p^n\,\eta_n(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = \mathrm{x}^2\check{y}^2\mathrm{z}^2 + i.p\left\{\mathcal{R}^{-1}\left[\frac{u}{s}\mathcal{R}\left(\sum_{n=0}^{\infty} p^n\,H_n(\eta)\right)\right]\right\}, \qquad (15)$$

where $H_n(\eta)$ is He's polynomial

$$H_n(\eta_0,\eta_1,\eta_2,\dots) = \frac{1}{n!}\left[\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i\eta_i\right)\right)\right]_{p=0}, \qquad (16)$$

From "(15)", we obtain

$$\eta_0 + p\eta_1 + p^2\eta_2 + \cdots = \mathrm{x}^2\check{y}^2\mathrm{z}^2 + i.p.\mathcal{R}^{-1}\left\{\frac{u}{s}\mathcal{R}\{H_0(\eta) + pH_1(\eta) + p^2H_2(\eta) + \cdots\}\right\},$$

Comparing the equal powers of p, we obtain

$$p^0:\quad \eta_0 = \mathrm{x}^2\check{y}^2\mathrm{z}^2,$$

$$p^1: \eta_1 = i.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\big(H_0(\eta)\big)\right\} = \mathrm{i}.\mathrm{x}^2\check{y}^2\mathrm{z}^2\text{ŧ},$$

$$H_0(\eta) = \mathrm{x}^2\check{y}^2\mathrm{z}^2,$$

$$p^2: \eta_2 = i.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\big(H_1(\eta)\big)\right\} = i^2\mathrm{x}^2\check{y}^2\mathrm{z}^2\frac{\text{ŧ}^2}{2!},$$

$$H_1(\eta_0,\eta_1) = i\text{ŧ}\mathrm{x}^2\check{y}^2\mathrm{z}^2,$$

$$p^3: \eta_3 = i.\mathcal{R}^{-1}\left\{\frac{u}{s}.\mathcal{R}\big(H_1(\eta)\big)\right\} = i^3\mathrm{x}^2\check{y}^2\mathrm{z}^2\frac{\text{ŧ}^3}{3!},$$

$$H_2(\eta_0,\eta_1,\eta_2) = \frac{i^2\text{ŧ}^2}{2}\mathrm{x}^2\check{y}^2\mathrm{z}^2,$$

and so on. Therefore, the solution is:

$$\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = \eta_0 + \eta_1 + \eta_2 + \cdots,$$

$$\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = \mathrm{x}^2\check{y}^2\mathrm{z}^2 + \mathrm{i}.\mathrm{x}^2\check{y}^2\mathrm{z}^2\text{ŧ} + i^2\mathrm{x}^2\check{y}^2\mathrm{z}^2\frac{\text{ŧ}^2}{2!} + \cdots$$

This implies

$$\eta(\mathrm{x},\check{y},\mathrm{z},\text{ŧ}) = \mathrm{x}^2\check{y}^2\mathrm{z}^2\left(1 + (i\text{ŧ}) + \frac{(i\text{ŧ})^2}{2!}\dots\right), = \mathrm{x}^2\check{y}^2\mathrm{z}^2 e^{i\text{ŧ}}.$$

Method 2. **<u>Variational Iteration Method (VIM)</u>**

By VIM, it was found that the Lagrange multiplier $\lambda = -1$, and the correction functional is given by

$$\eta_{n+1}(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \eta_n(x,y,t) + \int_0^t \lambda \left[\frac{\partial \eta_n}{\partial \xi} - i\left(\frac{\partial^2 \eta_n}{\partial x^2} + \frac{\partial^2 \eta_n}{\partial y^2} + \frac{\partial^2 \eta_n}{\partial z^2}\right) - i\left(1 - \frac{2}{x^2} - \frac{2}{y^2} - \frac{2}{z^2}\right)\eta_n\right] d\xi,$$

$$\eta_1(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \eta_0(x,y,t) - \int_0^t \left[\frac{\partial \eta_0}{\partial \xi} - i\left(\frac{\partial^2 \eta_0}{\partial \text{ӽ}^2} + \frac{\partial^2 \eta_0}{\partial \breve{y}^2} + \frac{\partial^2 \eta_0}{\partial \mathrm{z}^2}\right) - i\left(1 - \frac{2}{\text{ӽ}^2} - \frac{2}{\breve{y}^2} - \frac{2}{\mathrm{z}^2}\right)\eta_0\right] d\xi,$$

$$\eta_1(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \text{ӽ}^2\breve{y}^2\mathrm{z}^2(1 + i\text{ŧ}),$$

$$\eta_2(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \eta_1(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) - \int_0^t \left[\frac{\partial \eta_1}{\partial \xi} - i\left(\frac{\partial^2 \eta_1}{\partial \text{ӽ}^2} + \frac{\partial^2 \eta_1}{\partial \breve{y}^2} + \frac{\partial^2 \eta_1}{\partial \mathrm{z}^2}\right) - i\left(1 - \frac{2}{\text{ӽ}^2} - \frac{2}{\breve{y}^2} - \frac{2}{\mathrm{z}^2}\right)\eta_1\right] d\xi,$$

$$\eta_2(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \text{ӽ}^2\breve{y}^2\mathrm{z}^2\left(1 + i\text{ŧ} - \frac{\text{ŧ}^2}{2}\right),$$

$$\eta_3(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \text{ӽ}^2\breve{y}^2\mathrm{z}^2\left(1 + i\text{ŧ} - \frac{\text{ŧ}^2}{2} - i\frac{\text{ŧ}^3}{6}\right),$$

$$\eta_4(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \text{ӽ}^2\breve{y}^2\mathrm{z}^2\left(1 + i\text{ŧ} - \frac{\text{ŧ}^2}{2} - i\frac{\text{ŧ}^3}{6} + \frac{\text{ŧ}^4}{24} \ldots\right),$$

and so on. Therefore, the solution
$\eta = \lim_{n\to\infty} \eta_n(\text{ӽ},\breve{y},\mathrm{z},\text{ŧ}) = \text{ӽ}^2\breve{y}^2\mathrm{z}^2 e^{i\text{ŧ}}$.
Which is exact solution.

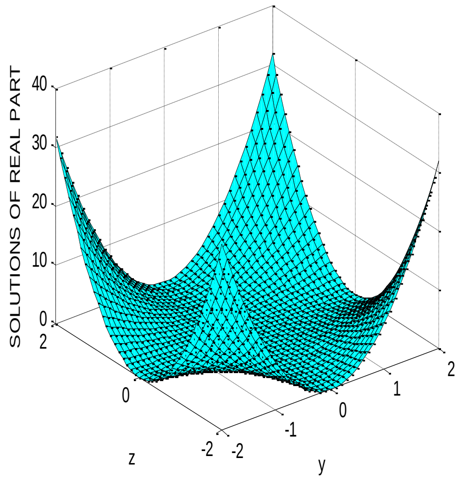

Fig 5. Dynamic behavior of real part of the solutions for t = $\frac{\pi}{3}$ and $x = 2$.

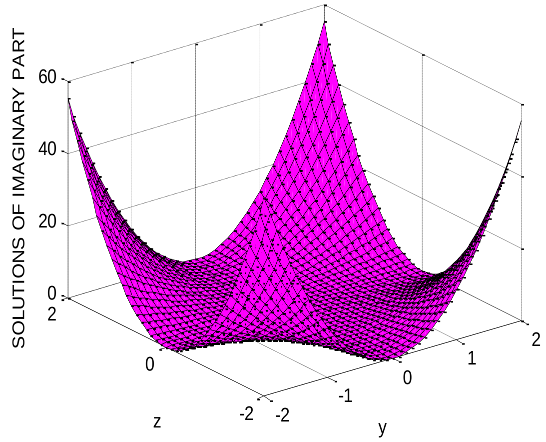

Fig 6. Dynamic behavior of imaginary part of the solutions for t = $\frac{\pi}{3}$ and $x = 2$.

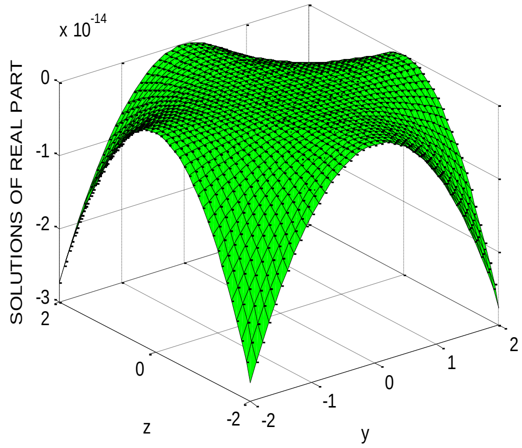

Fig 7. Dynamic behavior of real part of the solutions for t = 3.5 π and $x = 2$.

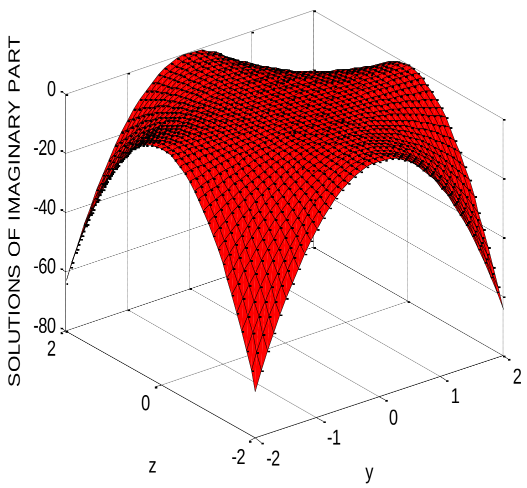

Fig 8. Dynamic behavior of imaginary part of the solutions for t = 3.5 π and $x = 2$.

Table I: Comparison of exact solutions by FTHPM and VIM for example 1.

(x, y)	Exact Solutions		FTHPM (First Four Terms)		VIM (First Four Terms)	
	Real	Imaginary	Real	Imaginary	Real	Imaginary
(0.1,0.1)	-4.7466e-001	8.6886e-001	-4.7441e-001	8.6631e-001	-4.7441e-001	8.6631e-001
(0.2,0.2)	-4.6075e-001	8.4339e-001	-4.6050e-001	8.4091e-001	-4.6050e-001	8.4091e-001
(0.3,0.3)	-4.3874e-001	8.0311e-001	-4.3850e-001	8.0074e-001	-4.3850e-001	8.0074e-001
(0.4,0.4)	-4.1022e-001	7.5089e-001	-4.0999e-001	7.4868e-001	-4.0999e-001	7.4868e-001
(0.5,0.5)	-3.7704e-001	6.9017e-001	-3.7684e-001	6.8814e-001	-3.7684e-001	6.8814e-001
(0.6,0.6)	-3.4115e-001	6.2447e-001	-3.4096e-001	6.2263e-001	-3.4096e-001	6.2263e-001
(0.7,0.7)	-3.0431e-001	5.5704e-001	-3.0415e-001	5.5540e-001	-3.0415e-001	5.5540e-001
(0.8,0.8)	-2.6803e-001	4.9062e-001	-2.6788e-001	4.8917e-001	-2.6788e-001	4.8917e-001
(0.9,0.9)	-2.3344e-001	4.2731e-001	-2.3331e-001	4.2605e-001	-2.3331e-001	4.2605e-001

Table II: Comparison of absolute errors in solutions by FTHPM and VIM for example 2.

(x, y)	FTHPM (First Seven Terms)		VIM (First Seven Terms)	
	Errors (Real) $\lvert\eta_{exact}-\eta_{FTHPM}\rvert$	Errors (Imaginary) $\lvert\eta_{exact}-\eta_{FTHPM}\rvert$	Errors (Real) $\lvert\eta_{exact}-\eta_{VIM}\rvert$	Errors (Imaginary) $\lvert\eta_{exact}-\eta_{VIM}\rvert$
(0.1,0.1)	9.6613e-012	1.5447e-010	9.6613e-012	1.5447e-010
(0.2,0.2)	1.5458e-010	2.4716e-009	1.5458e-010	2.4716e-009
(0.3,0.3)	7.8256e-010	1.2512e-008	7.8256e-010	1.2512e-008
(0.4,0.4)	2.4733e-009	3.9545e-008	2.4733e-009	3.9545e-008
(0.5,0.5)	6.0383e-009	9.6546e-008	6.0383e-009	9.6546e-008
(0.6,0.6)	1.2521e-008	2.0020e-007	1.2521e-008	2.0020e-007
(0.7,0.7)	2.3197e-008	3.7089e-007	2.3197e-008	3.7089e-007
(0.8,0.8)	3.9573e-008	6.3272e-007	3.9573e-008	6.3272e-007
(0.9,0.9)	6.3388e-008	1.0135e-006	6.3388e-008	1.0135e-006

IX. RESULTS AND DISCUSSIONS

Fig 1 and Fig 2 represent the dynamic behavior of the solutions of the real and imaginary parts of example 1 at different ranges of x, y, and $t = \frac{\pi}{6}$. Whereas Fig 3 and Fig 4 represent the dynamic behavior of solutions of the real part and imaginary part of the example 1 at different ranges of x, y, and $t = 3.5\,\pi$. Table 1 presents the comparison of absolute errors of solutions obtained by FTHPM and VIM for example 1.

Fig 5 and Fig 6 represent the dynamic behavior of the solutions of the real and imaginary parts of example 2 at $t = \frac{\pi}{3}$ and $x = 2$. Whereas Fig 7 and Fig 8 represent the dynamic behavior of solutions of the real part and imaginary part of the example 2 at $t = 3.5\,\pi$ and $x = 2$. Table 2 presents the comparison of absolute errors of solutions obtained by FTHPM and VIM for example 2.

X. CONCLUSION

In this paper, we compare two techniques, namely the variational iteration method (VIM) and the formable transform homotopy perturbation method (FTHPM), by applying them to the (2+1)-D and (3+1)-D Schrödinger equations. The computational results reveal significant insights, showing that both methods provide solutions as infinite convergence series with easily computable components. From the error estimation tables, it is clear that the hybrid scheme is as effective as the classical variational iteration method. However, the computational size is comparatively smaller in the hybrid scheme. We used MATHEMATICA and MATLAB software for graphical representation and error analysis. We concluded that both schemes are accurate and efficient for solving the (2+1) D and (3+1) D Schrödinger equations.

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