Tripolar Fuzzy Ideals which Coincide in Semigroups

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***Abstract*—Rao introduced the concept of tripolar fuzzy ideals in 2018. The concept is a generalization of fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets. In this paper, we study and define tripolar fuzzy ideals. We can find necessary and sufficient conditions for types of tripolar fuzzy ideals in semigroups. Finally, we prove the relationship between types ideals and type tripolar fuzzy ideals in semigroups.**

***Index Terms*—Regular, Intra-regular, semisimple, Tripolar fuzzy sets, Tripolar fuzzy ideals.**

I. INTRODUCTION

THE FUNDAMETAL concept of fuzzy sets theory was discussed and introduced by L. A. Zadeh [1] in 1965 as the most appropriate theory for dealing with uncertainty. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory and topology, computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine. The fuzzification of semigroup was introduced by Kuroki in 1979, [2]. In 1986, K. T. Attsnsov [3] investigated an intuitionistic fuzzy set as an exten-sion of a fuzzy set to deal with uncertainties more efficiently in the actualsituation. Later in 1994, w . Zhang [4] studied concepts of bipolar fuzzy sets which is a generalization of fuzzy sets. In 2000, K. M. Lee [5] studied bipolar valued fuzzy sets and applied it to algebraic structure. The studies of types bipolar fuzzy ideals, such as M. K. Kang [6], studied bipolar fuzzy subsemigroups in semigroups, V. Chinnadurau and K. Arulmozhi [7] discussed the bipolar fuzzy ideal in ordered Γ-semigroups, P. Khamrot and M. Siripitukdet [8] explained generalized bipolar fuzzy subsemigroups in semigroups. T. Gaketem and P. Khamrot [9] studied bipolar weakly interior ideals in semigorups. In 2018, M. M. K. Rao [10] was introduced the concepts tripolar fuzzy set, which is a generalization of fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets. In 2019, M. M. K. Rao and B. Venkateswarlu [11] studied tripolar fuzzy ideals Γ-semirings. In 2020, M. M. K. Rao and B. Venkateswarlu [12] studied tripolar fuzzy soft interior ideals Γ-semirings. In 2022, N. Wattansiripong et al. [13] present properties of tripolar fuzzy pure ideals in ordered semigroups. In the same year N. Wattansiripong et al.[14] studied tripolar fuzzy interior ideals in ordered semigroups and characterized semisimple ordered semigroups in terms of tripolar fuzzy interior ideals. Recently, T. Promai et al. [15] studied tripolar fuzzy ideals and proved some basic properties of tripolar fuzzy ideals in semigroups.

In this paper, we give the definition of tripolar fuzzy ideals in semigroups. We discuss necessary and sufficient conditions of types of tripolar fuzzy ideals in types of semigroups. Moreover, we prove relationship between types ideals and type tripolar fuzzy ideals in semigroups.

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II. PRELIMINARIES

In this section, we recall some basic definitions and results that are used in the study of this paper.

Definition 2.1. *A non-empty subset $\mathcal{B}$ of a semigroup $\mathcal{S}$ is called*

(1) *a subsemigroup of $\mathcal{S}$ if $\mathcal{B}^2 \subseteq \mathcal{B}$,*
(2) *a left (right) ideal of $\mathcal{S}$ if $\mathcal{S}\mathcal{B} \subseteq \mathcal{B}$ $(\mathcal{B}\mathcal{S} \subseteq \mathcal{B})$. By an* ideal *$\mathcal{B}$ of a semigroup $\mathcal{S}$ we mean a left ideal and a right ideal of $\mathcal{S}$,*
(3) *an interior ideal of $\mathcal{S}$ if $\mathcal{B}$ is a subsemigroup of $\mathcal{S}$ and $\mathcal{S}\mathcal{B}\mathcal{S} \subseteq \mathcal{B}$,*
(4) *a generalized bi-ideal of $\mathcal{S}$ if $\mathcal{B}\mathcal{S}\mathcal{B} \subseteq \mathcal{B}$,*
(5) *a bi-ideal of $\mathcal{S}$ if $\mathcal{B}$ is a subsemigroup of $\mathcal{S}$ and $\mathcal{B}\mathcal{S}\mathcal{B} \subseteq \mathcal{B}$,*
(6) *a $(1,2)$-ideal of $\mathcal{S}$ if $\mathcal{B}$ is a subsemigroup of $\mathcal{S}$ and $\mathcal{B}\mathcal{S}\mathcal{B}^2 \subseteq \mathcal{B}$,*
(7) *a quasi-ideal of $\mathcal{S}$ if $\mathcal{B}\mathcal{S} \cap \mathcal{S}\mathcal{B} \subseteq \mathcal{B}$.*

A semigroup $\mathcal{S}$ is said to be *regular* if for each element $h \in \mathcal{S}$, there exists an element $r \in \mathcal{S}$ such that $h = hrh$. A semigroup $\mathcal{S}$ is called *intra-regular* if for every $h \in \mathcal{S}$ there exist $r, t \in \mathcal{S}$ such that $h = rh^2t$. A semigroup $\mathcal{S}$ is said to be *left* (right) regular if for each element $h \in \mathfrak{S}$, there exists an element $r \in \mathcal{S}$ such that $h = rh^2(h = h^2r)$. A semigroup $\mathcal{S}$ is called *semisimple* if for every $h \in \mathcal{S}$, there exist $r, t, u \in \mathcal{S}$ such that $h = rhthu$. A semigroup $\mathcal{S}$ is called *weakly regular* if for every $h \in \mathcal{S}$ there exist $r, t \in \mathcal{S}$ such that $h = hrht$. A semigroup $\mathcal{S}$ is a *left (right) quasi-regular* if for every $h \in \mathcal{S}$, there exist $r, t \in \mathcal{S}$ such that $h = rhth$ $(h = hthyt)$.

For any $h_i \in [0,1]$, $i \in \mathcal{F}$, define

$$\underset{i\in\mathcal{F}}{\vee} h_i := \sup_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \underset{i\in\mathcal{F}}{\wedge} h_i := \inf_{i\in\mathcal{F}}\{h_i\}.$$

We see that for any $h, r \in [0,1]$, we have

$$h \vee r = \max\{h, r\} \quad \text{and} \quad h \wedge r = \min\{h, r\}.$$

A *fuzzy set (fuzzy subset)* of a non-empty set $\mathcal{E}$ is a function $\rho : \mathcal{E} \to [0,1]$.

For any two fuzzy sets ρ and ν of a non-empty set $\mathcal{E}$, define the symbol as follows:

(1) $\rho \subseteq \nu \Leftrightarrow \rho(h) \leq \nu(h)$ for all $h \in \mathcal{E}$,
(2) $\rho = \nu \Leftrightarrow \rho \subseteq \nu$ and $\nu \subseteq \rho$,
(3) $(\rho \wedge \nu)(h) = \rho(h) \wedge \nu(h)$ and $(\rho \vee \nu)(h) = \rho(h) \vee \nu(h)$ for all $h \in \mathcal{E}$, For the symbol $\rho \supseteq \nu$, we mean $\nu \subseteq \rho$.

Let $k \in \mathcal{S}$. Then $\mathcal{F}_k := \{(m,n) \in \mathcal{S} \times \mathcal{S} \mid k = mn\}$.

For any two fuzzy sets ρ and ν of a semigroup $\mathcal{S}$. The product of fuzzy subsets ρ and ν of $\mathcal{S}$ is defined as follow, for all $h \in \mathcal{S}$

$$(\rho \circ \nu)(h) = \begin{cases} \bigvee_{(m,n)\in\mathcal{F}_k} \{\rho(k) \wedge \nu(r)\} & \text{if} \quad h = mn, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic function of a subset $\mathcal{B}$ of a nonempty set $\mathcal{S}$ is a fuzzy set of $\mathcal{S}$

$$\lambda_{\mathcal{B}}(h) = \begin{cases} 1 & \text{if} \quad h \in \mathcal{B}, \\ 0 & \text{if} \quad h \notin \mathcal{B}. \end{cases}$$

for all $h \in \mathcal{S}$.

Definition 2.2. *[2] A fuzzy set ρ of a semigroup $\mathcal{S}$ is said to be*

(1) *a fuzzy subsemigroup of $\mathcal{S}$ if $\rho(h) \wedge \rho(r) \leq \rho(hr)$, for all $h, r \in \mathcal{S}$,*
(2) *a fuzzy left (right) ideal of $\mathcal{S}$ if $\rho(r) \leq \rho(hr)$ ($\rho(h) \leq \rho(hr)$), for all $h, r \in \mathcal{S}$.*
(3) *afuzzy ideal of $\mathcal{S}$ if it is both a fuzzy left ideal and a fuzzy right ideal of $\mathcal{S}$,*
(4) *a fuzzy interior ideal of $\mathcal{S}$ if ρ is a fuzzy subsemigroup of $\mathcal{S}$ and $\rho(r) \leq \rho(hrt)$, for all $h, r, t \in \mathcal{S}$,*
(5) *a fuzzy generalized bi-ideal of $\mathcal{S}$ if $\rho(h) \wedge \rho(t) \leq \rho(hrt)$, for all $h, r, t \in \mathcal{S}$,*
(6) *a fuzzy bi-ideal of $\mathcal{S}$ if ρ is a fuzzy subsemigroup of $\mathcal{S}$ and $\rho(h) \wedge \rho(t) \leq \rho(hrt)$, for all $h, r, t \in \mathcal{S}$,*
(7) *a fuzzy $(1,2)$-ideal of $\mathcal{S}$ if ρ is a fuzzy subsemigroup of $\mathcal{S}$ and $\rho(hk(tr)) \geq \rho(h) \wedge \rho(t) \wedge \rho(r)$, for all $h, k, r, t \in \mathcal{S}$,*
(8) *a fuzzy quasi-ideal of $\mathcal{S}$ if $(\lambda_{\mathcal{S}} \circ \rho)(h) \wedge (\rho \circ \lambda_{\mathcal{S}})(h) \leq \rho(h)$, for all $h \in \mathcal{S}$.*

Definition 2.3. *[10] The* tripolar fuzzy set $\mathcal{TF}$ *on a non-empty set $\mathcal{E}$ if*

$$\mathcal{TF} := \{(h, \rho(h), \nu(h), \delta(h)) \mid h \in \mathcal{E}\},$$

where $\rho(h) : \mathcal{E} \to [0,1]$, $\nu(h) : \mathcal{E} \to [0,1]$ and $\delta(h) : \mathcal{E} \to [-1,0]$, such that $0 \leq \rho(h) + \nu(h) \leq 1$ for all $h \in \mathcal{E}$. The membership degree $\rho(h)$ characterizes the extent that the element $\mathcal{E}$ satisfies the property corresponding to tripolar fuzzy set $\mathcal{TF}$, $\nu(h)$ characterizes the extent that the element $\mathcal{E}$ satisfies the not property (irrelevant) corresponding to tripolar fuzzy set ρ, and $\delta(h)$ characterizes the extent that the element $\mathcal{E}$ satisfies the implicit counter property corresponding to tripolar fuzzy set $\mathcal{TF}$. For simplicity $\mathcal{TF} := (\rho, \nu, \delta)$ has been used for $\mathcal{TF} := \{(h, \rho(h), \nu(h), \delta(h)) \mid h \in \mathcal{E}\}$ such that $0 \leq \rho(h) + \nu(h) \leq 1$.

The *characteristic tripolar fuzzy set* $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ of a non-empty subset $\mathcal{B}$ of set $\mathcal{S}$ is defined as follows:

$$\rho_{\mathcal{B}}(k) = \begin{cases} 1 & \text{if } k \in \mathcal{B}, \\ 0 & \text{if } k \notin \mathcal{B}, \end{cases}$$

$$\nu_{\mathcal{B}}(k) = \begin{cases} 0 & \text{if } k \in \mathcal{B}, \\ 1 & \text{if } k \notin \mathcal{B}, \end{cases}$$

$$\delta_{\mathcal{B}}(k) = \begin{cases} \text{-1} & \text{if } k \in \mathcal{B}, \\ 0 & \text{if } k \notin \mathcal{B} \end{cases}$$

for all $k \in \mathcal{S}$. In this case of $\mathcal{B} = \mathcal{S}$ defined $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{S}}, \nu_{\mathcal{S}}, \delta_{\mathcal{S}}) = (1, 0, -1)$.

For $\mathcal{TF}_1 = (\rho, \nu, \delta)$ and $\mathcal{TF}_2 = (\lambda, \mu, \omega)$ be a tripolar fuzzy sets. Defined the product $\mathcal{TF}_1 \circ \mathcal{TF}_2$ of a semigroup $\mathcal{S}$ as follows:

$$(\rho \circ \lambda)(k) = \begin{cases} \bigvee_{(m,n)\in\mathcal{F}_k} \{\rho(m) \wedge \lambda(n)\} & \text{if} \quad \mathcal{F}_k \neq \emptyset, \\ 0 & \text{if} \quad \mathcal{F}_k = \emptyset, \end{cases}$$

$$(\nu \circ \mu)(k) = \begin{cases} \bigwedge_{(m,n)\in\mathcal{F}_k} \{\nu(m) \vee \mu(n)\} & \text{if} \quad \mathcal{F}_k \neq \emptyset, \\ 1 & \text{if} \quad \mathcal{F}_k = \emptyset, \end{cases}$$

$$(\delta \circ \omega)(k) = \begin{cases} \bigwedge_{(m,n)\in\mathcal{F}_k} \{\delta(m) \vee \omega(n)\} & \text{if} \quad \mathcal{F}_k \neq \emptyset, \\ 0 & \text{if} \quad \mathcal{F}_k = \emptyset, \end{cases}$$

for all $k \in \mathcal{E}$. It is easy to verify that the structure $(\mathcal{TF}_1, \circ)$ is a semigroup.

III. Main Resluts

In this section, we define the notions of tripolar fuzzy ideals in semigroups and some properties of them are investigated.

Definition 3.1. *A tripolar fuzzy set is called a $\mathcal{TF} = (\rho, \nu, \delta)$ of a semigroup $\mathcal{S}$ is called*

(1) *a tripolar fuzzy subsemigroup (TFS) of $\mathcal{S}$ if $\rho(hk) \geq \rho(h) \wedge \rho(k)$, $\nu(hk) \leq \nu(h) \vee \nu(k)$ and $\delta(hk) \leq \delta(h) \vee \delta(k)$ for all $h, k \in \mathcal{S}$.*
(2) *a tripolar fuzzy left ideal (TFLI) of $\mathcal{S}$ if $\rho(hk) \geq \rho(k)$, $\nu(hk) \leq \nu(k)$ and $\delta(hk) \leq \delta(k)$ for all $h, k \in \mathcal{S}$.*
(3) *a tripolar fuzzy right ideal (TFRI) of $\mathcal{S}$ if $\rho(hk) \geq \rho(h)$, $\nu(hk) \leq \nu(h)$ and $\delta(hk) \leq \delta(h)$ for all $h, k \in \mathcal{S}$.*
(4) *a tripolar fuzzy ideal (TFI) of $\mathcal{S}$ if it is both TFLI and TFRI of $\mathcal{S}$.*
(5) *a tripolar fuzzy generalized bi-ideal (TFGBI) of $\mathcal{S}$ if $\rho(hkt) \geq \rho(h) \wedge \rho(t), \nu(hkt) \leq \nu(h) \vee \nu(t)$ and $\delta(hkt) \leq \delta(h) \vee \delta(t)$ for all $h, k, t \in \mathcal{S}$.*
(6) *a tripolar fuzzy bi-ideal (TFBI) of $\mathcal{S}$ if $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS and $\rho(hkt) \geq \rho(h) \wedge \rho(t), \nu(hkt) \leq \nu(h) \vee \nu(t)$ $\delta(hkt) \leq \delta(h) \vee \delta(t)$ for all $h, k, t \in \mathcal{S}$.*
(7) *a tripolar fuzzy interior ideal (TFII) of $\mathcal{S}$ if $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS and $\rho(hkt) \geq \rho(k), \nu(hkt) \leq \nu(k)$ $\delta(hkt) \leq \delta(k)$ for all $h, k, t \in \mathcal{S}$.*
(8) *a tripolar fuzzy $(1,2)$-ideal (TF $(1,2)$-I) of $\mathcal{S}$ if $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS and $\rho(hk(tr)) \geq \rho(h) \wedge \rho(t) \wedge \rho(r), \nu(hk(tr)) \leq \nu(h) \vee \nu(t) \vee \nu(r)$ $\delta(hk(tr)) \leq \delta(h) \vee \delta(t) \vee \delta(r)$ for all $h, k, t, r \in \mathcal{S}$.*
(9) *a tripolar fuzzy quasi-ideal (TFQI) of $\mathcal{S}$ if $\rho(h) \geq (\rho_{\mathcal{S}} \circ \rho)(h) \wedge (\rho \circ \rho_{\mathcal{S}})(h)$, $\nu(h) \leq (\nu_{\mathcal{S}} \circ \nu)(h) \vee (\nu \circ \nu_{\mathcal{S}})(h)$ and $\delta(hkt) \leq (\delta_{\mathcal{S}} \circ \delta)(h) \vee (\delta \circ \delta_{\mathcal{S}})(h)$ for all $h \in \mathcal{S}$.*

In Definition 3.1 we have $(4) \Rightarrow (1)$, $(6) \Rightarrow (5)$, $(4) \Rightarrow (6)$, $(4) \Rightarrow (7)$ and $(4) \Rightarrow (8)$.

Example 3.2. *Let $\mathcal{S} = \{w, x, y, z\}$ be semigroup with the following Cayley table:*

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	w	w	w
y	w	w	x	w
z	w	w	x	x

Define $\mathcal{TF} = (\rho, \nu, \delta)$ *by* $\rho(w) = 0.4$, $\rho(x) = 0.7$, $\rho(y) = 0.8$, $\rho(z) = 0.3$*;* $\nu(w) = 0.5$, $\rho(x) = 0.2$, $\rho(y) = 0.1$, $\rho(z) = 0.4$ *and* $\delta(w) = -0.7$, $\delta(x) = -0.5$, $\delta(y) = -0.3$, $\delta(z) = -0.3$*. Then* $\mathcal{TF} = (\rho, \nu, \delta)$ *is a TFII of* $\mathcal{S}$*.*

Example 3.3. *Let* $\mathcal{S} = \{v, w, x, y, z\}$ *be semigroup with the following Cayley table:*

$\cdot$	v	w	x	y	z
v	v	v	v	v	v
w	v	v	v	v	v
x	v	v	x	x	z
y	v	v	x	y	z
z	v	v	x	x	e

Define $\mathcal{TF} = (\rho, \nu, \delta)$ *by* $\rho(v) = 0.6$, $\rho(w) = 0.5$, $\rho(x) = 0.4$, $\rho(y) = 0.4$*,* $\rho(z) = 0.3$*,* $\nu(v) = 0.3$, $\nu(w) = 0.3$, $\nu(x) = 0.4$, $\nu(y) = 0.5$*,* $\nu(z) = 0.6$ *and* $\delta(v) = -0.7$, $\delta(w) = -0.5$, $\delta(x) = -0.3$, $\delta(y) = -0.3$ $\delta(z) = -0.4$*. Then* $\mathcal{TF} = (\rho, \nu, \delta)$ *is a TFBI of* $\mathcal{S}$*.*

The following lemma shows that every TFI is a TFBI of a semigroups.

Lemma 3.4. *Every TFI of a semigroup* $\mathcal{S}$ *is a TFBI of* $\mathcal{S}$*.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$, we have that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Thus,

$$\rho(hr) \geq \rho(h), \quad \nu(hr) \leq \nu(h) \quad \text{and} \quad \delta(hr) \leq \delta(h)$$

and so $\rho(hr) \geq \rho(h) \geq \rho(h) \wedge \rho(r)$, $\nu(hr) \leq \nu(h) \leq \nu(h) \vee \nu(r)$ and $\delta(hr) \leq \delta(h) \leq \delta(h) \vee \delta(r)$. Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\mathcal{S}$. Let $h, r, t \in \mathcal{S}$. Since $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$, we have that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\rho(hrt) = \rho((hr)t) \geq \rho(t)$, $\nu(hrt) = \nu((hr)t) \leq \nu(t)$ and $\delta(hrt) = \delta((hr)t) \leq \delta(t)$ and so $\rho(hrt) \geq \rho(t) \geq \rho(h) \wedge \rho(t)$, $\nu(hrt) \leq \nu(t) \leq \nu(h) \vee \nu(t)$ and $\delta(hrt) \leq \delta(t) \leq \delta(h) \vee \delta(t)$. Hence $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFBI of $\mathcal{S}$. ■

In order to consider the converse of Lemma 3.4, we need to strengthen the condition of a semigroup $\mathcal{S}$.

Theorem 3.5. *In a regular semigroup* $\mathcal{S}$*, the TFBIs and the TFIs coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFBI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is regular, we have $hr \in (h\mathcal{S}h)\mathcal{S} \subseteq h\mathcal{S}$ which implies that $hr = hkh$ for some $k \in \mathcal{S}$. Thus, $\rho(hr) = \rho(hkh) \geq \rho(h) \wedge \rho(h) = \rho(h)$, $\nu(hr) = \nu(hkh) \leq \nu(h) \vee \nu(h) = \nu(h)$ and $\delta(hr) = \delta(hkh) \leq \delta(h) \vee \delta(h) = \delta(h)$. Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Similarly, we can show that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$. ■

In the following results, we show that CFGIBs and CF-BIDs on types of SGs.

Lemma 3.6. *In a regular semigroup of of* $\mathcal{S}$*, the TFGBI and the TFBI coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFGBI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is regular, there exists $k \in \mathcal{S}$ such that $r = rkr$. Thus,

$$\rho(hr) = \rho((h(rkr)) = \rho(h(rk)r) \geq \rho(h) \wedge \rho(r)$$

$$\nu(hr) = \nu(h(rkr)) = \nu(h(rk)r) \leq \nu(h) \vee \nu(r)$$

and

$$\delta(hr) = \delta(h(rkr)) = \delta(h(rk)r) \leq \delta(h) \vee \delta(r)$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\mathcal{S}$. By Definition 3.1(6) we have $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFBI of $\mathcal{S}$. ■

Lemma 3.7. *In left (right) regular semigroup of* $\mathcal{S}$*, the TFGBI and the TFBI coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFGBI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is left regular, there exists $k \in \mathcal{S}$ such that $r = kr^2 = krr$. Thus,

$$\rho(hr) = \rho((h(krr)) = \rho(h(kr)r) \geq \rho(h) \wedge \rho(r)$$

$$\nu(hr) = \nu((h(krr)) = \nu(h(kr)r) \leq \nu(h) \vee \nu(r)$$

and

$$\delta(hr) = \delta((h(krr)) = \delta(h(kr)r) \leq \delta(h) \vee \delta(r)$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\mathcal{S}$. By Definition 3.1(6) we have $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFBI of $\mathcal{S}$. ■

Lemma 3.8. *In left (right) quasi-regular semigroup of* $\mathcal{S}$*, the TFGBI and the TFBI coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFGBI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is left quasi-regular, there exists $k, w \in \mathcal{S}$ such that $r = krwr$. Thus,

$$\rho(hr) = \rho((h(krwr)) = \rho(h(krw)r) \geq \rho(h) \wedge \rho(r)$$

$$\nu(hr) = \nu((h(krwr)) = \nu(h(krw)r) \leq \nu(h) \vee \nu(r)$$

and

$$\delta(hr) = \delta(h(krwr)) = \delta(h(krw)r) \leq \delta(h) \vee \delta(r)$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\mathcal{S}$. By Definition 3.1(6) we have $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFBI of $\mathcal{S}$. ■

Lemma 3.9. *In weakly regular semigroup of* $\mathcal{S}$*, the TFGBI and the TFBI coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFGBI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is weakly regular, there exists $k, w \in \mathcal{S}$ such that $h = hkhw$. Thus,

$$\rho(hr) = \rho((hkhw)r) = \rho(h(khw)r) \geq \rho(h) \wedge \rho(r)$$

$$\nu(hr) = \nu((hkhw)r) = \nu(h(khw)r) \leq \nu(h) \vee \nu(r)$$

and

$$\delta(hr) = \delta((hkhw)r) = \delta(h(khw)r) \leq \delta(h) \vee \delta(r)$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\widetilde{\mathfrak{F}}$. By Definition 3.1(6) we have $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFBI of $\mathcal{S}$. ■

By Lemma 3.6, 3.7, 3.8 and 3.9, we have Theorem 3.10.

Theorem 3.10. *Let* $\mathcal{S}$ *be a semigorup. If* $\mathcal{S}$ *is regular, left (right) regular, left (right) quasi-regular or weakly regular, then the TFGBI and the TFBI coincide.*

The following lemma shows that every TFI is a TFII on a semigroup.

Lemma 3.11. *Every TFI of a semigroup $\mathcal{S}$ is a TFII of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$, we have $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Thus,

$$\rho(hr) \geq \rho(h), \quad \nu(hr) \leq \nu(h) \quad \text{and} \quad \delta(hr) \leq \delta(h)$$

and so $\rho(hr) \geq \rho(h) \geq \rho(h) \wedge \rho(r)$, $\nu(hr) \leq \nu(h) \leq \nu(h) \vee \nu(r)$ and $\delta(hr) \leq \delta(h) \leq \delta(h) \vee \delta(r)$. Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\mathcal{S}$. Let $h, r, t \in \mathcal{S}$. Then, $\rho(hrt) = \rho(h(rt)) \geq \rho(rt) \geq \rho(r)$, $\nu(hrt) = \nu(h(rt)) \leq \nu(rt) \leq \nu(r)$ and $\delta(hrt) = \delta(h(rt)) \leq \delta(rt) \leq \delta(r)$ Thus, $\rho(hrt) \geq \rho(r)$, $\nu(hrt) \leq \nu(r)$ and $\delta(hrt) \leq \delta(r)$. Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFII of $\mathcal{S}$. ■

In order to consider the converse of Lemma 3.11, we need to strengthen the condition of a semigroup $\mathcal{S}$.

Lemma 3.12. *In a regular semigroup $\mathcal{S}$, the TFIIs and the TFIs coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFII of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is regular, there exists $k \in \mathcal{S}$ such that $h = hkh$. Thus,

$$\rho(hr) = \rho((hkh)r) = \rho((hk)hr) \geq \rho(h),$$

$$\nu(hr) = \nu((hkh)r) = \nu((hk)hr) \leq \nu(h),$$

and

$$\delta(hr) = \delta((hkh)r) = \delta((hk)hr) \leq \delta(h).$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Similarly, we can show that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$. ■

Lemma 3.13. *In a left (right) regular semigroup $\mathcal{S}$, the TFIIs and the TFIs coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFII of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is left regular, there exists $k \in \mathcal{S}$ such that $h = kh^2$. Thus,

$$\rho(hr) = \rho((kh^2)r) = \rho(khhr) = \rho((kh)hr) \geq \rho(h),$$

$$\nu(hr) = \nu((kh^2)r) = \nu(khhr) = \nu((kh)hr) \leq \rho(h)$$

and

$$\delta(hr) = \delta((kh^2)r) = \delta(khhr) = \delta((kh)hr) \leq \delta(h).$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Similarly, we can show that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$. ■

Lemma 3.14. *In an intra-regular semigroup $\mathcal{S}$, the TFIIs and the TFIs coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFII of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is intra-regular, there exist $k, t \in \mathcal{S}$ such that $h = kh^2t$. Thus,

$$\rho(hr) = \rho((kh^2t)r) = \rho((khht)r) = \rho((kh)hr) \geq \rho(h),$$

$$\nu(hr) = \nu((kh^2t)r) = \nu((khht)r) = \nu((kh)hr) \leq \nu(h)$$

and

$$\delta(hr) = \delta((kh^2t)r) = \delta((khht)r) = \delta((kh)hr) \leq \delta(h).$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Similarly, we can show that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$. ■

Lemma 3.15. *In a semisiple semigroup $\mathcal{S}$, the TFIIs and the TFIs coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFII of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is semisimple, there exist $k, t, p \in \mathcal{S}$ such that $h = khthp$. Thus,

$$\rho(hr) = \rho((khthp)r) = \rho((kht)h(pr)) \geq \rho(h),$$

$$\nu(hr) = \nu((khthp)r) = \nu((kht)h(pr)) \leq \nu(h)$$

and

$$\delta(hr) = \delta((khthp)r) = \delta((kht)h(pr)) \leq \delta(h).$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Similarly, we can show that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$. ■

Lemma 3.16. *In a let (right) quasi-regular semigroup $\mathcal{S}$, the TFIIs and the TFIs coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFII of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is left quasi-regular, there exist $k, t \in \mathcal{S}$ such that $r = krtr$. Thus,

$$\rho(hr) = \rho(h(krtr)) = \rho((hk)r(tr)) \geq \rho(r),$$

$$\nu(hr) = \nu(h(krtr)) = \nu((hk)r(tr)) \leq \nu(r)$$

and

$$\delta(hr) = \delta(h(krtr)) = \delta((hk)r(tr)) \leq \delta(r).$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Similarly, we can show that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$. ■

Lemma 3.17. *In a weakly regular semigroup $\mathcal{S}$, the TFIIs and the TFIs coincide.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFII of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{S}$ is weakly regular, there exist $k, t \in \mathcal{S}$ such that $h = hkht$. Thus,

$$\rho(hr) = \rho((hkht)r) = \rho((hk)h(tr)) \geq \rho(h),$$

$$\nu(hr) = \nu((hkht)r) = \nu((hk)h(tr)) \leq \nu(h),$$

and

$$\delta(hr) = \delta((hkht)r) = \delta((hk)h(tr)) \leq \delta(h),$$

Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$. Similarly, we can show that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$. ■

By Lemma 3.12, 3.13, 3.14, 3.15, 3.16 and 3.17, we have Theorem 3.18.

Theorem 3.18. *Let $\mathcal{S}$ be a semigroup. If $\mathcal{S}$ is regular, left (right) regular, intra-regular, semisimple, left (right) quasi-regular or weakly regular, then TFIIs and TFIs coincide.*

The following theorem shows that every TFI is a TFQI of a semigroup.

Theorem 3.19. *Every TFLI (TFRL) of a semigroup $\mathcal{S}$ is a TFQI of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$ and let $h \in \mathcal{S}$.

If $\mathcal{F}_h = \emptyset$, then it is easy to verify that $\rho(h) \geq (\rho_{\mathcal{S}} \circ \rho)(h) \wedge (\rho \circ \rho_{\mathcal{S}})(h)$, $\nu(h) \leq (\nu_{\mathcal{S}} \circ \nu)(h) \vee (\rho \circ \nu_{\mathcal{S}})(h)$ and $\delta(h) \leq (\delta_{\mathcal{S}} \circ \delta)(h) \vee (\delta \circ \delta_{\mathcal{S}})(h)$.

If $\mathcal{F}_h \neq \emptyset$, then

$$\begin{aligned}(\rho_{\mathcal{S}} \circ \rho)(h) &= \bigvee_{(i,j)\in\mathcal{F}_h} \{\rho_{\mathcal{S}}(i) \wedge \rho(j)\} \\ &= \bigvee_{(i,j)\in\mathcal{F}_h} \{1 \wedge \rho(j)\} \\ &\leq \bigvee_{(i,j)\in\mathcal{F}_h} \{\rho(j)\} \\ &\leq \bigvee_{(i,j)\in\mathcal{F}_h} \{\rho(ij)\} \\ &\leq \rho(h),\end{aligned}$$

$$\begin{aligned}(\nu_{\mathcal{S}} \circ \nu)(h) &= \bigwedge_{(i,j)\in\mathcal{F}_h} \{\nu_{\mathcal{S}}(i) \vee \nu(j)\} \\ &= \bigwedge_{(i,j)\in\mathcal{F}_h} \{0 \vee \nu(j)\} \\ &\geq \bigwedge_{(i,j)\in\mathcal{F}_h} \{\nu(j)\} \\ &\geq \bigwedge_{(i,j)\in\mathcal{F}_h} \{\nu(ij)\} \\ &\geq \nu(h),\end{aligned}$$

and

$$\begin{aligned}(\delta_{\mathcal{S}} \circ \delta)(h) &= \bigwedge_{(i,j)\in\mathcal{F}_h} \{\delta_{\mathcal{S}}(i) \vee \delta(j)\} \\ &= \bigwedge_{(i,j)\in\mathcal{F}_h} \{-1 \vee \delta(j)\} \\ &\geq \bigwedge_{(i,j)\in\mathcal{F}_h} \{\delta(j)\} \\ &\geq \bigwedge_{(i,j)\in\mathcal{F}_h} \{\delta(ij)\} \\ &\geq \delta(h).\end{aligned}$$

Thus, $\rho(h) \geq (\rho_{\mathcal{S}} \circ \rho)(h)$, $\nu(h) \leq (\nu_{\mathcal{S}} \circ \nu)(h)$ and $\delta(h) \leq (\delta_{\mathcal{S}} \circ \delta)(h)$ and so $\rho(h) \geq (\rho_{\mathcal{S}} \circ \rho)(h) \wedge (\rho \circ \rho_{\mathcal{S}})(h)$, $\nu(h) \leq (\nu_{\mathcal{S}} \circ \nu)(h) \vee (\rho \circ \nu_{\mathcal{S}})(h)$ and $\delta(h) \leq (\delta_{\mathcal{S}} \circ \delta)(h) \vee (\delta \circ \delta_{\mathcal{S}})(h)$. Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFQI of $\mathcal{S}$. Similarly, if $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI, then $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFQI of $\mathcal{S}$. ■

The following theorem shows that every TFQI is a TFS on a semigroup.

Theorem 3.20. *Every TFQI of a semigroup $\mathcal{S}$ is a TFS of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFQI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$.

$$\begin{aligned}\rho(hr) &\geq (\rho \circ \rho_{\mathcal{S}})(hr) \wedge (\rho_{\mathcal{S}} \circ \rho)(hr) \\ &= \bigvee_{(p,q)\in\mathcal{F}_{hr}} \{\rho(p) \wedge \rho_{\mathcal{S}}(q)\} \wedge \\ &\quad \bigvee_{(a,b)\in\mathcal{F}_{hr}} \{\rho_{\mathcal{S}}(a) \wedge \rho(b)\} \\ &\geq (\rho(h) \wedge \rho_{\mathcal{S}}(r)) \wedge (\rho_{\mathcal{S}}(h)) \wedge \rho(r) \\ &= (\rho(h) \wedge 1) \wedge (1 \wedge \rho(r)) \\ &\geq \rho(h) \wedge \rho(r),\end{aligned}$$

$$\begin{aligned}\nu(hr) &\leq (\nu \circ \nu_{\mathcal{S}})(hr) \vee (\nu_{\mathcal{S}} \circ \nu)(hr) \\ &= \bigwedge_{(p,q)\in\mathcal{F}_{hr}} \{\nu(p) \vee \nu_{\mathcal{S}}(q)\} \vee \\ &\quad \bigwedge_{(a,b)\in\mathcal{F}_{hr}} \{\nu_{\mathcal{S}}(a) \vee \nu(b)\} \\ &\leq (\nu(h) \vee \nu_{\mathcal{S}}(r)) \vee (\nu_{\mathcal{S}}(h)) \vee \nu(r) \\ &= (\nu(h) \vee 0) \vee (0 \vee \nu(r)) \\ &\leq \nu(h) \vee \nu(r),\end{aligned}$$

and

$$\begin{aligned}\delta(hr) &\leq (\delta \circ \delta_{\mathcal{S}})(hr) \vee (\delta_{\mathcal{S}} \circ \delta)(hr) \\ &= \bigwedge_{(p,q)\in\mathcal{F}_{hr}} \{\delta(p) \vee \delta_{\mathcal{S}}(q)\} \vee \\ &\quad \bigwedge_{(a,b)\in\mathcal{F}_{hr}} \{\delta_{\mathcal{S}}(a) \vee \delta(b)\} \\ &\leq (\delta(h) \vee \delta_{\mathcal{S}}(r)) \vee (\delta_{\mathcal{S}}(h)) \vee \delta(r) \\ &= (\delta(h) \vee -1) \vee (-1 \vee \delta(r)) \\ &\leq \delta(h) \vee \delta(r)\end{aligned}$$

Thus, $\rho(hr) \geq \rho(h) \wedge \rho(r)$, $\nu(hr) \leq \nu(h) \vee \nu(r)$ and $\delta(hr) \leq \delta(h) \vee \delta(r)$. Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\mathcal{S}$. ■

The following theorem shows that every TFQI is a TFBI on a semigroup.

Theorem 3.21. *Every TFQI of a semigroup $\mathcal{S}$ is a TFBI of $\mathcal{S}$.*

Proof: Assume that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFQI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Then by Theorem 3.20, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFS of $\mathcal{S}$.
Let $h, r, t \in \mathcal{S}$. Then

$$\begin{aligned}\rho(hrt) &\geq (\rho \circ \rho_{\mathcal{S}})(hrt) \wedge (\rho_{\mathcal{S}} \circ \rho)(hrt) \\ &= \bigvee_{(p,q)\in\mathcal{F}_{hrt}} \{\rho(p) \wedge \rho_{\mathcal{S}}(q)\} \wedge \\ &\quad \bigvee_{(a,b)\in\mathcal{F}_{hrt}} \{\rho_{\mathcal{S}}(a) \wedge \rho(b)\} \\ &\geq (\rho(h) \wedge \rho_{\mathcal{S}}(rt)) \wedge (\rho_{\mathcal{S}}(hr)) \wedge \rho(t) \\ &= (\rho(h) \wedge 1) \wedge (1 \wedge \rho(t)) \\ &\geq \rho(h) \wedge \rho(t),\end{aligned}$$

$$\begin{aligned}\nu(hrt) &\leq (\nu \circ \nu_{\mathcal{S}})(hrt) \vee (\nu_{\mathcal{S}} \circ \nu)(hrt) \\ &= \bigwedge_{(p,q)\in\mathcal{F}_{hrt}} \{\nu(p) \vee \nu_{\mathcal{S}}(q)\} \vee \\ &\quad \bigwedge_{(a,b)\in\mathcal{F}_{hrt}} \{\nu_{\mathcal{S}}(a) \vee \nu(b)\} \\ &\leq (\nu(h) \vee \nu_{\mathcal{S}}(rt)) \vee (\nu_{\mathcal{S}}(hr)) \vee \nu(t) \\ &= (\nu(h) \vee 0) \vee (0 \vee \nu(t)) \\ &\leq \nu(h) \vee \nu(t),\end{aligned}$$

and

$$\begin{aligned}\delta(hrt) &\leq (\delta \circ \delta_{\mathcal{S}})(hrt) \vee (\delta_{\mathcal{S}} \circ \delta)(hrt) \\ &= \bigwedge_{(p,q)\in\mathcal{F}_{hrt}} \{\delta(p) \vee \delta_{\mathcal{S}}(q)\} \vee \\ &\quad \bigwedge_{(a,b)\in\mathcal{F}_{hrt}} \{\delta_{\mathcal{S}}(a) \vee \delta(b)\} \\ &\leq (\delta(h) \vee \delta_{\mathcal{S}}(rt)) \vee (\delta_{\mathcal{S}}(hr)) \vee \delta(t) \\ &= (\delta(h) \vee -1) \vee (-1 \vee \delta(t)) \\ &\leq \delta(h) \vee \delta(t)\end{aligned}$$

Thus, $\rho(hrt) \geq \rho(h) \wedge \rho(t)$, $\nu(hrt) \leq \nu(h) \vee \nu(t)$ and $\delta(hrt) \leq \delta(h) \vee \delta(t)$. Hence, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFB of $\mathcal{S}$. ■

The following theorem shows that every TFI is a TF $(1, 2)$-I on a semigroup

Theorem 3.22. *Every TFI of a semigroup of $\mathcal{S}$ is a TF $(1, 2)$-I of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFI of $\mathcal{S}$ and let $h, r \in \mathcal{S}$. Since $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFI of $\mathcal{S}$, we have that $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFRI of $\mathcal{S}$. Thus, $\rho(hr) \geq \rho(h)$ $\nu(hr) \leq \nu(h)$ and $\delta(hr) \leq \delta(h)$ and so, $\rho(hr) \geq \rho(h) \geq \rho(h) \wedge \rho(r)$ $\nu(hr) \leq \nu(h) \leq \nu(h) \vee \nu(r)$ and $\delta(hr) \leq \delta(h) \leq \delta(h) \vee \delta(r)$. Hence, $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFS of $\mathcal{S}$. Let $h, r, k, w \in \mathcal{S}$. Since $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFI of $\mathcal{S}$, we have that $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFLI of $\mathcal{S}$. Thus, $\rho(hrkw) = \rho((hrk)w) \geq \rho(w)$ $\nu(hrkw) = \nu((hrk)w) \leq \nu(w)$ and $\delta(hrkw) = \delta((hrk)w) \leq \delta(w)$. and so,

$$\rho(hr(kw)) \geq \rho(w) \geq \rho(h) \wedge \rho(k) \wedge \rho(w),$$

$$\nu(hr(kw)) \leq \nu(w) \leq \nu(h) \vee \nu(k) \vee \nu(w)$$

and

$$\delta(hr(kw)) \leq \delta(w) \leq \delta(h) \vee \delta(k) \vee \delta(w)$$

Hence, $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TF $(1, 2)$-I of $\mathcal{S}$. ∎

IV. Relation between Ideal and Tripolar ideal in semigroups

Theorem 4.1. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then $\mathcal{B}$ is a subsemigroup of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFS of $\mathcal{S}$.*

Suppose that $\mathcal{B}$ is a semigruop of $\mathcal{S}$ and let $h, k \in \mathcal{S}$. Then $\mathcal{B}^2 \subseteq \mathcal{B}$.

If $h, k \in \mathcal{B}$, then $hk \in \mathcal{B}$. Thus, $\rho_{\mathcal{B}}(h) = \rho_{\mathcal{B}}(k) = \rho_{\mathcal{B}}(hk) = 1$, $\nu_{\mathcal{B}}(h) = \nu_{\mathcal{B}}(k) = \nu_{\mathcal{B}}(hk) = 0$ and $\delta_{\mathcal{B}}(h) = \delta_{\mathcal{B}}(k) = \delta_{\mathcal{B}}(hk) = -1$. Hence, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(k)$.

If $h \notin \mathcal{B}$ or $k \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(k)$.

Therefore, $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFSG of $\mathcal{S}$.

For the converse, assume that $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFSG of $\mathcal{S}$, let $h, k \in \mathcal{S}$ with $h, k \in \mathcal{B}$. Then $\rho_{\mathcal{B}}(h) = \rho_{\mathcal{B}}(k) = 1$, $\nu_{\mathcal{B}}(h) = \nu_{\mathcal{B}}(k) = 0$ and $\delta_{\mathcal{B}}(h) = \delta_{\mathcal{B}}(k) = -1$. By assumption, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(k)$. Thus, $hk \in \mathcal{B}$. Therefore $\mathcal{B}$ is a SG of $\mathcal{S}$.

Theorem 4.2. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then $\mathcal{B}$ is a left (right) ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFLI (TFRI) of $\mathcal{S}$.*

Proof: Assume that $\mathcal{B}$ is a left ideal of $\mathcal{S}$ and $h, k \in \mathcal{S}$. If $k \in \mathcal{B}$, then $hk \in \mathcal{B}$ Thus, $\rho_{\mathcal{B}}(k) = 1$, $\nu_{\mathcal{B}}(k) = 0$, $\delta_{\mathcal{B}}(k) = -1$ and $\rho_{\mathcal{B}}(hk) = 1$, $\nu_{\mathcal{B}}(hk) = 0$, $\delta_{\mathcal{B}}(hk) = -1$. Hence, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(k)$.

If $k \notin \mathcal{B}$, then $hk \in \mathcal{B}$ Thus, $\rho_{\mathcal{B}}(k) = 0$, $\nu_{\mathcal{B}}(k) = 1$, $\delta_{\mathcal{B}}(k) = 0$ and $\rho_{\mathcal{B}}(hk) = 1$, $\nu_{\mathcal{B}}(hk) = 0$, $\delta_{\mathcal{B}}(hk) = -1$. Hence, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(k)$.

Therefore $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFLI of $\mathcal{S}$.

Conversely, assume that $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFLI of $\mathcal{S}$ and $h, k \in \mathcal{S}$ such that $k \in \mathcal{B}$. Then then $\rho_{\mathcal{B}}(k) = 1$, $\nu_{\mathcal{B}}(k) = 0$, $\delta_{\mathcal{B}}(k) = -1$. If $hk \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hk) = 0$, $\nu_{\mathcal{B}}(hk) = 1$, $\delta_{\mathcal{B}}(hk) = 0$. Thus, $0 = \rho_{\mathcal{B}}(hk) < 1 = \rho_{\mathcal{B}}(k)$, $1 = \nu_{\mathcal{B}}(hk) > 0 = \nu_{\mathcal{B}}(k)$ and $0 = \delta_{\mathcal{B}}(hk) > -1\delta_{\mathcal{B}}(k)$. By assumption, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(k)$. It is a contradiction so, $hk \in \mathcal{B}$. Hence, $\mathcal{B}$ is a left ideal of $\mathcal{S}$. ∎

Theorem 4.3. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then $\mathcal{B}$ is a generalized bi-ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFGBI of $\mathcal{S}$.*

Proof: Assume that $\mathcal{B}$ is a generalized bi-ideal of $\mathcal{S}$ and $h, k, t \in \mathcal{S}$. If $h, t \in \mathcal{B}$, then $hkt \in \mathcal{B}$ Thus, $\rho_{\mathcal{B}}(h) = 1 = \rho_{\mathcal{B}}(t)$, $\nu_{\mathcal{B}}(h) = 0 = \nu_{\mathcal{B}}(t)$, $\delta_{\mathcal{B}}(h) = -1 = \delta_{\mathcal{B}}(t)$ and $\rho_{\mathcal{B}}(hkt) = 1$, $\nu_{\mathcal{B}}(hkt) = 0$, $\delta_{\mathcal{B}}(hkt) = -1$. Hence, $\rho_{\mathcal{B}}(hkt) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(t)$, $\nu(hkt) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(t)$ and $\delta_{\mathcal{B}}(hkt) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t)$.

If $h \notin \mathcal{B}$ or $t \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hkt) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(t)$, $\nu_{\mathcal{B}}(hkt) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(t)$ and $\delta_{\mathcal{B}}(hkt) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t)$.

Therefore $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFGBI of $\mathcal{S}$.

Conversely, assume that $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFGBI of $\mathcal{S}$ and $h, k, t \in \mathcal{S}$ such that $h, t \in \mathcal{B}$. Then then $\rho_{\mathcal{B}}(h) = 1 = \rho_{\mathcal{B}}(t)$, $\nu_{\mathcal{B}}(h) = 0 = \rho_{\mathcal{B}}(t)$, $\delta_{\mathcal{B}}(h) = -1 = \rho_{\mathcal{B}}(t)$. If $hkt \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hkt) = 0$, $\nu_{\mathcal{B}}(hkt) = 1$, $\delta_{\mathcal{B}}(hkt) = 0$. Thus, $0 = \rho(hkt) < 1\rho(h) \wedge \rho_{\mathcal{B}}(t)$, $1 = \nu(hkt) > 0 = \vee\nu_{\mathcal{B}}(t)$ and $0 = \delta(hkt) > -1\delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t)$. By assumption, $\rho(hkt) \geq \rho(h) \wedge \rho_{\mathcal{B}}(t)$, $\nu_{\mathcal{B}}(hkt) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(t)$ and $\delta_{\mathcal{B}}(hkt) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t)$. It is a contradiction so, $hkt \in \mathcal{B}$. Hence, $\mathcal{B}$ is a generalized bi-ideal of $\mathcal{S}$. ∎

Theorem 4.4. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then $\mathcal{B}$ is a bi-ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFBI of $\mathcal{S}$.*

Proof: By Theorem 4.1 and Theorem 4.3. ∎

Theorem 4.5. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then $\mathcal{B}$ is an interior ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFII of $\mathcal{S}$.*

Proof: Assume that $\mathcal{B}$ is an interior ideal of $\mathcal{S}$. Then $\mathcal{B}$ is a subsemigroup of $\mathcal{S}$. Thus, by Theorem 4.1, $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFS of $\mathcal{S}$. Let $h, k, t \in \mathcal{S}$. If $k \in \mathcal{B}$, then $hkt \in \mathcal{B}$ Thus, $\rho_{\mathcal{B}}(k) = 1$, $\nu_{\mathcal{B}}(k) = 0$, $\delta_{\mathcal{B}}(k) = -1$ and $\rho_{\mathcal{B}}(hkt) = 1$, $\nu_{\mathcal{B}}(hkt) = 0$, $\delta_{\mathcal{B}}(hkt) = -1$. Hence, $\rho_{\mathcal{B}}(hkt) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hkt) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hkt) \leq \delta_{\mathcal{B}}(k)$.

If $k \notin \mathcal{B}$, then $hkt \in \mathcal{B}$ Thus, $\rho_{\mathcal{B}}(k) = 0$, $\nu_{\mathcal{B}}(k) = 1$, $\delta_{\mathcal{B}}(k) = 0$ and $\rho_{\mathcal{B}}(hkt) = 1$, $\nu_{\mathcal{B}}(hkt) = 0$, $\delta_{\mathcal{B}}(hkt) = -1$. Hence, $\rho_{\mathcal{B}}(hkt) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hkt) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hkt) \leq \delta_{\mathcal{B}}(k)$.

Therefore, $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFLI of $\mathcal{S}$.

Conversely, assume that $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFII of $\mathcal{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFS of $\mathcal{S}$. Thus, by Theorem 4.1, $\mathcal{B}$ is a subsemigorup of $\mathcal{S}$. Let $h, k, t \in \mathcal{S}$ such that $k \in \mathcal{B}$. Then then $\rho_{\mathcal{B}}(k) = 1$, $\nu_{\mathcal{B}}(k) = 0$, $\delta_{\mathcal{B}}(k) = -1$. If $hkt \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hkt) = 0$, $\nu_{\mathcal{B}}(hkt) = 1$, $\delta_{\mathcal{B}}(hkt) = 0$. Thus, $0 = \rho_{\mathcal{B}}(hkt) < 1 = \rho_{\mathcal{B}}(k)$, $1 = \nu_{\mathcal{B}}(hkt) > 0 = \nu_{\mathcal{B}}(k)$ and $0 = \delta_{\mathcal{B}}(hkt) > -1\delta_{\mathcal{B}}(k)$. By assumption, $\rho_{\mathcal{B}}(hkt) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hkt) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hkt) \leq \delta_{\mathcal{B}}(k)$. It is a contradiction so, $hkt \in \mathcal{B}$. Hence, $\mathcal{B}$ is an interior ideal of $\mathcal{S}$. ∎

Theorem 4.6. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then $\mathcal{B}$ is a $(1, 2)$-ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TF $(1, 2)$-I of $\mathcal{S}$.*

Proof: Assume that $\mathcal{B}$ is a $(1, 2)$-ideal of $\mathcal{S}$ Then $\mathcal{B}$ is a subsemigroup of $\mathcal{S}$. Thus, by Theorem 4.1, $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}},\, \nu_{\mathcal{B}},\, \delta_{\mathcal{B}})$ is a TFS of $\mathcal{S}$. Let $h, k, t, r \in \mathcal{S}$. If $h, t, r \in \mathcal{B}$, then $hktr \in \mathcal{B}$ Thus, $\rho_{\mathcal{B}}(h) = 1 = \rho_{\mathcal{B}}(t) = \rho_{\mathcal{B}}(r)$, $\nu_{\mathcal{B}}(h) = 0 = \nu_{\mathcal{B}}(t) = \nu_{\mathcal{B}}(r)$ $\delta_{\mathcal{B}}(h) = -1 = \delta_{\mathcal{B}}(t) = \delta_{\mathcal{B}}(r)$

and $\rho_{\mathcal{B}}(hk(tr)) = 1$, $\nu_{\mathcal{B}}(hk(tr)) = 0$, $\delta_{\mathcal{B}}(hk(tr)) = -1$. Hence, $\rho_{\mathcal{B}}(hk(tr)) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(t) \wedge \rho_{\mathcal{B}}(r)$, $\nu(hk(tr)) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(t) \vee \nu_{\mathcal{B}}(r)$ and $\delta_{\mathcal{B}}(hk(tr)) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t) \vee \delta_{\mathcal{B}}$.

If $h \notin \mathcal{B}$ or $t \notin \mathcal{B}$ or $r \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hk(tr)) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(t) \wedge \rho_{\mathcal{B}}(r)$, $\nu(hk(tr)) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(t) \vee \nu_{\mathcal{B}}(r)$ and $\delta_{\mathcal{B}}(hk(tr)) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t) \vee \delta_{\mathcal{B}}$.

Therefore, $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a TF $(1, 2)$-I of $\mathcal{S}$.

Conversely, assume that $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a TF $(1, 2)$-I of $\mathcal{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a TFS. Thus, by Theorem 4.1, $\mathcal{B}$ is a subsemigroup of $\mathcal{S}$. Let $h, k, t, r \in \mathcal{S}$ such that $h, t, r \in \mathcal{B}$. Then $\rho_{\mathcal{B}}(h) = 1 = \rho_{\mathcal{B}}(t) = \rho_{\mathcal{B}}(r)$, $\nu_{\mathcal{B}}(h) = 0 = \rho_{\mathcal{B}}(t) = \nu_{\mathcal{B}}(r)$, $\delta_{\mathcal{B}}(h) = -1 = \rho_{\mathcal{B}}(t) = \delta_{\mathcal{B}}(r)$. If $hktr \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hk(tr)) = 0$, $\nu_{\mathcal{B}}(hk(tr)) = 1$, $\delta_{\mathcal{B}}(hk(tr)) = 0$. Thus, $0 = \rho(hk(tr)) < 1\rho(h) \wedge \rho_{\mathcal{B}}(t)\rho_{\mathcal{B}}$, $1 = \nu(hk(tr)) > 0 = \vee\nu_{\mathcal{B}}(t) \vee \nu_{\mathcal{B}}$ and $0 = \delta(hk(tr)) > -1\delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t)\ vee\delta_{\mathcal{B}}$. By assumption, $\rho_{\mathcal{B}}(hk(tr)) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(t) \wedge \rho_{\mathcal{B}}(r)$, $\nu(hk(tr)) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(t) \vee \nu_{\mathcal{B}}(r)$ and $\delta_{\mathcal{B}}(hk(tr)) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(t) \vee \delta_{\mathcal{B}}$. It is a contradiction so, $hktr \in \mathcal{B}$. Hence, $\mathcal{B}$ is a $(1, 2)$-ideal of $\mathcal{S}$. ∎

V. Minimal and Maximal Tripolar Fuzzy types Ideals

Definition 5.1. *An interior ideal $\mathcal{B}$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every interior ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{B}$, we have $\mathcal{J} = \mathcal{B}$,*

(2) *a* maximal *if for every interior ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{J}$, we have $\mathcal{B} = \mathcal{I}$,*

Definition 5.2. *A TRFI $\mathcal{TF} = (\rho, \nu, \delta)$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every TRFII of $\mathcal{TF}_1 = (\lambda, \mu, \omega)$ of $\mathcal{S}$ such that $\mathcal{TF}_1 \subseteq \mathcal{TF}$, we have* $\text{supp}(\mathcal{TF}_1) = \text{supp}(\mathcal{TF})$,

(2) *a* maximal *if for every TRFII of $\mathcal{TF}_1 = (\lambda, \mu, \omega)$ of $\mathcal{S}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_1$, we have* $\text{supp}(\mathcal{TF}_1) = \text{supp}(\mathcal{TF})$.

Theorem 5.3. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds.*

(1) *$\mathcal{B}$ is a minimal interior ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a minimal IRFI of $\mathcal{S}$,*

(2) *$\mathcal{B}$ is a maximal interior ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathfrak{B}}$ is a maximal TRFII of $\mathcal{S}$.*

Proof:

(1) Suppose that $\mathcal{B}$ is a minimal interior ideal of $\mathcal{S}$. Then $\mathcal{B}$ is an interior ideal of $\mathcal{S}$. By Theorem 4.5, $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}, \omega_{\mathcal{B}})$ is a TRFII of $\mathcal{S}$. Let $\mathcal{TF} = (\lambda, \mu, \omega)$ be a TRFII of $\mathcal{S}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_{\mathcal{B}}$. Then $\text{supp}(\mathcal{TF}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Thus, $\text{supp}(\mathcal{TF}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathcal{B}$. Hence, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B}$. Since $\mathcal{TF} = (\lambda, \mu, \omega)$ is a TRFII of $\mathcal{S}$ we have $\text{supp}(\mathcal{TF})$ is an IID of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Hence, $\mathcal{TF}_{\mathcal{B}}$ is a minimal TRFII of $\mathfrak{S}$.

Conversely, $\mathcal{TF}_{\mathcal{B}}$ is a minimal TRFII of $\mathfrak{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}, \omega_{\mathcal{B}})$ is a TRFII of $\mathcal{S}$. By Theorem 4.5, $\mathcal{B}$ is an interior ideal of $\mathcal{S}$. Let $\mathcal{J}$ be an interior ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{B}$. Then by Theorem 4.5, $\mathcal{TF}_{\mathcal{J}} = (\lambda_{\mathcal{J}}, \mu_{\mathcal{J}}, \omega_{\mathcal{J}})$ is a TRFII of $\mathfrak{S}$ such that $\mathcal{TF}_{\mathcal{J}} \subseteq \mathcal{TF}_{\mathcal{B}}$. Hence, $\mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathcal{B}$. By assumption, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. So, $\mathcal{B} = \chi_{\mathcal{I}}$. Hence, $\mathcal{B}$ is a minimal interior ideal of $\mathfrak{S}$.

(2) Suppose that $\mathcal{B}$ is a maximal interior ideal of $\mathcal{S}$. Then $\mathcal{B}$ is an interior ideal of $\mathcal{S}$. By Theorem 4.5, $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a TRFII of $\mathcal{S}$. Let $\mathcal{TF} = (\lambda, \mu, \omega)$ be a TRFII of $\mathcal{S}$ such that $\mathcal{TF}_{\mathcal{B}} \subseteq \mathcal{TF}$. Then $\text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF})$. Thus, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF})$. Hence, $\mathcal{B} \subseteq \text{supp}(\mathcal{TF})$. Since $\mathcal{TF} = (\lambda, \mu, \omega)$ is a TRFII of $\mathcal{S}$ we have $\text{supp}(\mathcal{TF})$ is an IID of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Hence, $\mathcal{TF}_{\mathcal{B}}$ is a maximal TRFII of $\mathfrak{S}$.

Conversely, $\mathcal{TF}_{\mathcal{B}}$ is a maximal TRFII of $\mathfrak{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a TRFII of $\mathcal{S}$. By Theorem 4.5, $\mathcal{B}$ is an interior ideal of $\mathcal{S}$. Let $\mathcal{J}$ be an interior ideal of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{J}$. Then by Theorem 4.5, $\mathcal{TF}_{\mathfrak{J}} = (\lambda_{\mathcal{J}}, \mu_{\mathcal{J}}\, \omega_{\mathcal{J}})$ is a TRFII of $\mathfrak{S}$ such that $\mathcal{TF}_{\mathcal{J}} \subseteq \mathcal{TF}_{\mathcal{B}}$. Hence, $\mathfrak{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. By assumption, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. So, $\mathcal{B} = \chi_{\mathcal{I}}$. Hence, $\mathcal{B}$ is a minimal interior ideal of $\mathfrak{S}$. ∎

Definition 5.4. *An $(1, 2)$-ideal $\mathcal{B}$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every $(1, 2)$-ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{B}$, we have $\mathcal{J} = \mathcal{B}$,*

(2) *a* maximal *if for every $(1, 2)$-ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{J}$, we have $\mathcal{B} = \mathcal{I}$,*

Definition 5.5. *A TRFI $\mathcal{TF} = (\rho, \nu, \delta)$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every TF $(1, 2)$-I of $\mathcal{TF}_1 = (\lambda, \mu, \omega)$ of $\mathcal{S}$ such that $\mathcal{TF}_1 \subseteq \mathcal{TF}$, we have* $\text{supp}(\mathcal{TF}_1) = \text{supp}(\mathcal{TF})$,

(2) *a* maximal *if for every TF $(1, 2)$-I of $\mathcal{TF}_1 = (\lambda, \mu, \omega)$ of $\mathcal{S}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_1$, we have* $\text{supp}(\mathcal{TF}_1) = \text{supp}(\mathcal{TF})$.

Theorem 5.6. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds.*

(1) *$\mathcal{B}$ is a minimal $(1, 2)$-ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a minimal TF $(1, 2)$-I of $\mathcal{S}$,*

(2) *$\mathcal{B}$ is a maximal $(1, 2)$-ideal of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathfrak{B}}$ is a maximal TF $(1, 2)$-I of $\mathcal{S}$.*

Proof:

(1) Suppose that $\mathcal{B}$ is a minimal $(1, 2)$-ideal of $\mathcal{S}$. Then $\mathcal{B}$ is a $(1, 2)$-ideal of $\mathcal{S}$. By Theorem 4.6, $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}, \omega_{\mathcal{B}})$ is a TF $(1, 2)$-I of $\mathcal{S}$. Let $\mathcal{TF} = (\lambda, \mu, \omega)$ be a TF $(1, 2)$-I of $\mathcal{S}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_{\mathcal{B}}$. Then $\text{supp}(\mathcal{TF}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Thus, $\text{supp}(\mathcal{TF}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathcal{B}$. Hence, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B}$. Since $\mathcal{TF} = (\lambda, \mu, \omega)$ is a $(1, 2)$-I of $\mathcal{S}$ we have $\text{supp}(\mathcal{TF})$ is a $(1, 2)$-ideal of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Hence, $\mathcal{TF}_{\mathcal{B}}$ is a minimal TF $(1, 2)$-I of $\mathfrak{S}$.

Conversely, $\mathcal{TF}_{\mathcal{B}}$ is a minimal TF $(1, 2)$-I of $\mathfrak{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}, \omega_{\mathcal{B}})$ is a TRFII of $\mathcal{S}$. By Theorem 4.5, $\mathcal{B}$ is a $(1, 2)$-ideal of $\mathcal{S}$. Let $\mathcal{J}$ be a $(1, 2)$-ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{B}$. Then by Theorem 4.6, $\mathcal{TF}_{\mathcal{J}} = (\lambda_{\mathcal{J}}, \mu_{\mathcal{J}}, \omega_{\mathcal{J}})$ is a TRF $(1, 2)$-I of $\mathfrak{S}$ such that $\mathcal{TF}_{\mathcal{J}} \subseteq \mathcal{TF}_{\mathcal{B}}$. Hence, $\mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathcal{B}$. By assumption, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. So, $\mathcal{B} = \chi_{\mathcal{I}}$. Hence, $\mathcal{B}$ is a minimal $(1, 2)$-ideal of $\mathfrak{S}$.

(2) Suppose that $\mathcal{B}$ is a maximal $(1, 2)$-ideal of $\mathcal{S}$. Then $\mathcal{B}$ is a $(1, 2)$-ideal of $\mathcal{S}$. By Theorem 4.6, $\mathcal{TF}_{\mathcal{B}} =$

$(\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}, \omega_{\mathcal{B}})$ is a TF $(1,2)$-I of $\mathcal{S}$. Let $\mathcal{TF} = (\lambda, \mu, \omega)$ be a TF $(1,2)$-I of $\mathcal{S}$ such that $\mathcal{TF}_{\mathcal{B}} \subseteq \mathcal{TF}$. Then $\text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF})$. Thus, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF})$. Hence, $\mathcal{B} \subseteq \text{supp}(\mathcal{TF})$. Since $\mathcal{TF} = (\lambda, \mu, \omega)$ is a TRF $(1,2)$-I of $\mathcal{S}$ we have $\text{supp}(\mathcal{TF})$ is a $(1,2)$-ideal of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Hence, $\mathcal{TF}_{\mathcal{B}}$ is a maximal TRF $(1,2)$-I of $\mathfrak{S}$.

Conversely, $\mathcal{TF}_{\mathcal{B}}$ is a maximal TF $(1,2)$-I of $\mathfrak{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}, \omega_{\mathcal{B}})$ is a TRF $(1,2)$-I of $\mathcal{S}$. By Theorem 4.5, $\mathcal{B}$ is a $(1,2)$-ideal of $\mathcal{S}$. Let $\mathcal{J}$ be a $(1,2)$-ideal of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{J}$. Then by Theorem 4.5, $\mathcal{TF}_{\mathcal{J}} = (\lambda_{\mathcal{J}}, \mu_{\mathcal{J}}, \omega_{\mathcal{J}})$ is a TRF $(1,2)$-I of $\mathfrak{S}$ such that $\mathcal{TF}_{\mathcal{J}} \subseteq \mathcal{TF}_{\mathcal{B}}$. Hence, $\mathfrak{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. By assumption, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. So, $\mathcal{B} = \chi_{\mathcal{I}}$. Hence, $\mathcal{B}$ is a minimal $(1,2)$-ideal of $\mathfrak{S}$.

■

VI. Conclusion

In paper, we study define tripolar fuzzy ideals in semigroup which is a generalization of fuzzy set, bipolar fuzzy sets, and intuitionistic fuzzy sets. In the important results, we found necessary and sufficient conditions of coincidences of types of tripolar fuzzy ideals in types of semigroups which we explained in Theorem 3.5 and 3.18, respectively. Finally, we study connection between of types ideals and types tripolar fuzzy ideals in semigroups.

In future work, we can study the characterization of some properties of the semigroup in terms of tripolar fuzzy subsemigroups.

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