Observer-Based Sliding Mode Control for Memristive Chua's Circuit Systems

Xinyu Hu, Zhilian Yan

Abstract—**This paper deals with the issue of observer-based sliding mode control (SMC) for memristive Chua's circuit (MCC) systems. The objective is to design an observer-based SMC controller to stabilize MCC systems with external disturbances and uncertainties. With the aid of a Lyapunov function, a sufficient condition is proposed to ensure the closed-loop MCC system is asymptotically stable with a prescribed disturbance attenuation level. Based on this, a SMC controller is designed to guarantee that the state variables of the considered MCC system are driven to the sliding mode surface (SMS) in finite time and move towards the origin along the SMS in the subsequent time. Finally, an example is used to demonstrate the effectiveness of the proposed controller design.**

Index Terms—**Memristive Chua's circuit, sliding mode control, uncertainty, disturbance attenuation**

I. Introduction

Chua's circuit has received widespread attention because it possesses rich nonlinear dynamic phenomena and is thought to provide proof for the existence of chaos in the circuit [1–3]. One step further, Chua's circuit has conveniently been utilized to construct chaotic sensors and reveal the possibility of controlling and synchronizing chaotic systems [4–6]. Actually, Chua's circuit is a simple nonlinear electronic circuit mainly consisting of a self-excited circuit and Chua's diode (CD). Among them, the self-excited circuit is composed of a resistor, a capacitor, and an inductor. It can control the oscillation frequency to achieve frequency amplification and stable output of the signal. Whereas, CD is a nonlinear resistor and can produce rich dynamic behaviors such as single attractor, double attractor, and periodic state by simply changing the parameters of the components [7, 8]. Therefore, Chua's circuit is known for exhibit prominent chaotic behavior [9]. However, CD in Chua's circuit is easily affected by external factors such as temperature and voltage, resulting in instability and low efficiency of the circuit [10]. In contrast, memristors, which are nonlinear passive components with two terminals, provide several advantages, including nanoscale dimensions, large data storage capacity, and low power consumption [11]. These desirable features have motivated the replacement of CD with memristors, leading to the development of a modified structure known as the memristive Chua's circuit (MCC) [12, 13]. And then, MCC has potential applications in a variety of fields, including secure communication, image encryption, and image decryption [14–16].

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On the other hand, MCC have complex transient dynamics since their stabilities are dependent on the inner initial conditions of the memristors rather than the circuit parameters [17]. Therefore, numerous control strategies have been introduced for MCC systems in recent years. For example, in [18], Wen et al. studied the synchronization problem of MCC using an adaptive control method. In [19], Zhang et al. proposed an adaptive sliding mode control (SMC) approach based on radial basis function neural networks. By refining the SMC strategy, this approach effectively addresses the control challenges of radial basis function neural networks in the presence of disturbances. In [20], Su et al. proposed a dynamic periodic event-triggered control to address the stabilization problem of MCC. Nonetheless, under disturbances and uncertainties, these control schemes often fall short of short of meeting high-precision control requirements. SMC is especially effective for systems with external disturbances and uncertainties [21]. Because SMC is insensitive to parameter changes and external interference, it exhibits high robustness and anti-interference performance [22]. In [23], Wen et al. studied the problem of uncertain MCC systems using event-triggered SMC. Xue et al. investigated the uncertain MCC utilizing adaptive SMC and achieved state feedback stabilization of the uncertain system [24].

Often in control design of dynamic systems, the real-time state knowledge is essential [25]. However, it is almost impossible to get full-state information for most engineering applications because of financial or technological constraints. Hence, the state observer is introduced to estimate unmeasurable variables in control design [26]. The combination of SMC and state observer can collect the current system state more accurately, allowing the designed controller to accurately ensure the tracking system's stability and provide the system state to the sliding mode surface (SMS) in less time [27–29]. The controller adjusts according to the state estimate provided by the state observer, thereby achieving more accurate state tracking performance and improving system control accuracy [30]. In [31], Estelle et al. studied the synchronization problem of MCC by using observer-based SMC. However, the stabilization problem of MCC with external disturbances via observer-based SMC has not been investigated yet, which is the motivation for this paper.

Following the analysis presented above, this paper is dedicated to observer-based SMC for MCC systems. The challenge at hand is to stabilize the MCC systems under external disturbances and uncertainties using SMC. The remaining sections of this writing are as follows: In Section II, the structure and description of the MCC systems are given, and a SMS based on the observer is constructed. Finally, the problem of this paper is proposed, and the SMC controller to be designed is described. In Section III, the Lyapunov functions of the system state and SMS are constructed, respectively. The linear matrix inequality technology is used to analyse the stability of the MCC system in the sliding section and the arrival of the MCC system to the SMS in a finite time in the arrival section. In Section IV, an example of an MCC is offered to demonstrate the usefulness of the design described in this study. In conclusion, Section V offers a summary of the work discussed in this paper.

II. PRELIMINARIES

Throughout this paper, the notations adhere to the definitions given in [32–34] except where explicitly clarified otherwise.

A. MCC System

The terminal voltage and current of a memristor exhibit a nonlinear relationship, expressed as

$$i = W(\varphi)v, v = M(q)i.$$

Here, $W(\varphi)$ and $M(q)$ represent the memductance and memristance, respectively. These quantities are defined as

$$M(q) = \frac{d\varphi}{dq}, W(\varphi) = \frac{dq}{d\varphi},$$

where the charge q and φ are mutually dependent, such that $q = q(\varphi)$ and $\varphi = \varphi(q)$. The memristor, therefore, describes the relationship between the charge and flux.

Consider the MCC shown in Fig.1 [35]. The circuit contains two capacitors, C_1 and C_2, with voltages V_1 and V_2 across them. L denotes the inductance, while i represents the current flowing through it. R and r are resistors in the circuit. A conductance $-G$ is connected in parallel with the memductance $W(\varphi)$ to facilitate the creation of an active memductance [36].

Using the relationship between circuit elements, (1) can be obtained through circuit principles.

$$\begin{cases} \dot{V}_2 &= \frac{V_1-V_2}{RC_2} - \frac{1}{C_2}W(\varphi(t))V_2 + \frac{G}{C_2}V_2 + \bar{u}(\varphi(t)), \\ \dot{V}_1 &= \frac{1}{C_1}i - \frac{V_1-V_2}{RC_1}, \\ \dot{i} &= \frac{V_1}{L} - \frac{r}{L}i, \\ \dot{\varphi} &= V_2, \end{cases} \tag{1}$$

where

$$\begin{cases} W(\varphi) &= \frac{dq}{d\varphi}, \\ q(\varphi) &= a_2\varphi + 0.5(a_1 - a_2)(\mid \varphi + 1 \mid - \mid \varphi - 1 \mid). \end{cases}$$

It can know that $\bar{u}(t)$ is dependent on the flux $\varphi(t)$. Let $x_1 = V_2$, $x_2 = V_1$, $x_3 = -i$, and $x_4 = \varphi$. Thus

$$\dot{x}(t) = A_i x(t) + B\bar{u}(x_4(t)), \tag{2}$$

where $x(t)$=col$\{x_1(t), x_2(t), x_3(t), x_4(t)\}$, B=col$\{1, 0, 0, 0\}$,

$$A_i = \begin{bmatrix} \frac{-W(x_4(t))R+GR-1}{C_2R} & \frac{1}{C_2R} & 0 & 0 \\ \frac{1}{C_1R} & -\frac{1}{C_1R} & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{L} & -\frac{r}{L} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

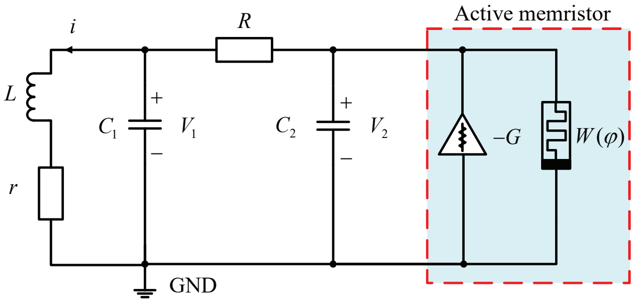

Fig. 1. MCC

The function $W(x_4(t))$ is defined as follows [35]:

$$W(x_4(t)) = \begin{cases} a_1, & |x_4(t)| \leqslant 1, \\ a_2, & |x_4(t)| > 1, \end{cases}$$

where a_1 and a_2 are positive constants (a_1, $a_2 > 0$). The control input $\bar{u}(t)$ is expressed as:

$$\bar{u}(t) = \begin{cases} \bar{u}_1(t), & |x_4(t)| \leqslant 1, \\ \bar{u}_2(t), & |x_4(t)| > 1. \end{cases}$$

Consequently, system (2) takes the following equivalent form:

$$\dot{x}(t) = \begin{cases} A_1x(t) + B\bar{u}_1(t), & |x_4(t)| \leqslant 1, \\ A_2x(t) + B\bar{u}_2(t), & |x_4(t)| > 1, \end{cases} \tag{3}$$

where the matrices A_1 and A_2 are defined as:

$$A_1 = \begin{bmatrix} \frac{-a_1R+GR-1}{C_2R} & \frac{1}{C_2R} & 0 & 0 \\ \frac{1}{C_1R} & -\frac{1}{C_1R} & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{L} & -\frac{r}{L} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{-a_2R+GR-1}{C_2R} & \frac{1}{C_2R} & 0 & 0 \\ \frac{1}{C_1R} & -\frac{1}{C_1R} & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{L} & -\frac{r}{L} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Define

$$\lambda_1 = \begin{cases} 1, & |x_4(t)| \leqslant 1, \\ 0, & |x_4(t)| > 1, \end{cases}$$

$$\lambda_2 = \begin{cases} 0, & |x_4(t)| \leqslant 1, \\ 1, & |x_4(t)| > 1. \end{cases}$$

Then, it follows that

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4}$$

where $A = \sum_{i=1}^{2} \lambda_i A_i$ and $u(t) = \sum_{i=1}^{2} \lambda_i \bar{u}_i(t)$.

According to system (4), MCC subject to external disturbance can be described as

$$\begin{cases} \dot{x}(t) = (A + \triangle\mathcal{A}(t))x(t) + B(u(t) + w(t)), \\ y(t) = Cx(t), \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ denote the state, control input, disturbance and measure output vetor, respectively. $w(t)$ satisfies the constrain $\|w(t)\| \leqslant w$, where w is a known positive constant. The system uncertainty, $\triangle\mathcal{A}(t)$ is described by the following equation:

$$\triangle\mathcal{A}(t) = \mathcal{D}\mathcal{U}(t)\mathcal{F},$$

where $\mathcal{D} = \sum_{i=1}^{2}\lambda_i\mathcal{D}_i$ and $\mathcal{F} = \sum_{i=1}^{2}\lambda_i\mathcal{F}_i$ are the constant matrices of appropriate dimensions. $\mathcal{U}(t)$ is an unknown time-varying matrix function satisfying

$$\mathcal{U}^T(t)\mathcal{U}(t) \leqslant I.$$

B. Observer-Based SMC

A state observer, which can estimate the unmeasured state in (5), is formulated as

$$\begin{cases}\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)),\\ \hat{y}(t) = C\hat{x}(t),\end{cases} \tag{6}$$

where $\hat{x}(t) \in \mathbb{R}^n$ and $L \in \mathbb{R}^{n\times p}$ denote the estimation of $x(t)$ and the observer gain matrix to be designed later, respectively.

Set $e(t) = x(t) - \hat{x}(t)$. Combining (5) and (6) leads to the following estimation error dynamics.

$$\dot{e}(t) = (A + \triangle\mathcal{A}(t) - LC)e(t) + \triangle\mathcal{A}(t)\hat{x}(t) + Bw(t). \tag{7}$$

Consider the following integral SMS function

$$s(t) = Z\hat{x}(t) - \int_0^t Z(A + BK)\hat{x}(s)ds, \tag{8}$$

where $Z \in \mathbb{R}^{m\times n}$ is a given constant matrix and $K \in \mathbb{R}^{m\times n}$ is a control gain matrix to be designed later.

The observer-based SMC law is designed as

$$u(t) = u_\epsilon(t) + u_{eq}(t). \tag{9}$$

where $u_\epsilon(t)$ is responsible for determining the SMS $\dot{s}(t) = 0$ reaching time of system variables and $u_\epsilon(t)$ is applied to system (6) to ensure the system trajectories stay on the SMS and move to the origin thereafter. we assume that $u_\epsilon(t) = -\epsilon sign(s(t))$, where ϵ denotes the convergence rate to be determined. Moreover, $u_{eq}(t)$, known as the equivalent control [37], is closely related to constraint $\dot{s}(t) = 0$. From (8) it yields that

$$\begin{aligned}\dot{s}(t) &= Z\dot{\hat{x}}(t) - Z(A + BK)\hat{x}(t),\\ &= ZBu(t) + ZL(y(t) - C\hat{x}(t)) - ZBK\hat{x}(t).\end{aligned} \tag{10}$$

The equivalent control law that satisfies $\dot{s}(t) = 0$ can be easily computed as

$$u_{eq}(t) = K\hat{x}(t) - (ZB)^{-1}ZLCe(t). \tag{11}$$

Substituting (11) into (6), we obtain

$$\dot{\hat{x}}(t) = (A + BK)\hat{x}(t) + (I - B(ZB)^{-1}Z)LCe(t). \tag{12}$$

Therefore, the stability of system (12) will be investigated in the following part.

Remark 1. SMC input $u(t)$ is composed of two components, $u_{eq}(t)$, responsible for ensure system stability and $u_\epsilon(t)$, which governs the reachability of SMS. The rate at which the system state converges to the SMS is determined by the choice of the reaching law ϵ, which is directly related to the system parameters. Consequently, the selection of ϵ is unaffected by external disturbances, thereby preventing instability in the convergence time.

C. Necessary Definitions and Lemma

The following are the definitions and lemmas required for this paper.

Definition 1. [38] Define the reference output as $y(t) = Cx(t)$; for system (12), the system is H_∞ asymptotically stable, in the sense that

1) It is asymptotically stable when $w(t) = 0$;

2) It has a specified H_∞ disturbance-rejection performance level γ, meaning that under the zero initial condition,

$$\int_0^t y^T(\iota)y(\iota)d\iota \leqslant \gamma^2\int_0^t w^T(\iota)w(\iota)d\iota.$$

Definition 2. [39] For the closed loop system (12), there exists a function $\mathcal{V}(x) \in \mathbb{R}^n \rightarrow R$, if the following inequations

$$(i)\ \forall x \neq 0, \mathcal{V}(x) > 0,$$
$$(ii)\ \forall x \neq 0, \dot{\mathcal{V}}(x) \leqslant 0,$$

hold. Then the closed-loop system (12) is asymptotically stable.

Lemma 1. [40] Given a symmetric matrix

$$\zeta = \begin{bmatrix}\zeta_{11} & \zeta_{12}\\ * & \zeta_{22}\end{bmatrix},$$

where $\zeta_{11} \in \mathbb{R}^{n\times n}$, the following three conditions are equivalent:

$$(i)\ \zeta < 0;$$
$$(ii)\ \zeta_{11} < 0, \zeta_{22} - \zeta_{12}^T\zeta_{11}^{-1}\zeta_{12} < 0;$$
$$(iii)\ \zeta_{22} < 0, \zeta_{11} - \zeta_{12}^T\zeta_{22}^{-1}\zeta_{12} < 0.$$

Lemma 2. [41] When there are matrices X and Y of suitable dimensions, the following relationship is satisfied:

$$2X^TY \leqslant \sigma^{-1}X^TX + \sigma Y^TY,$$

where σ is a positive scalar.

Thus, the main problem of this paper is to design a SMC controller $u(t)$ to achieve the goal of MCC reaching the SMS in finite time and being asymptotically stable under the H_∞ performance on the SMS.

III. MAIN RESULTS

This section gives a sufficient condition for ensuring the asymptotic stability of the MCC closed-loop systems under H_∞ level. Moreover, the controller gain, observer gain, and the convergence rate parameter in the controller are designed jointly.

Theorem 1. For given a scalar $\gamma > 0$, and given matrices K and L, suppose that there exist matrices $P_1 > 0$ and $P_2 > 0$ such that

$$\Xi = \begin{bmatrix}\Xi_{11} & \Xi_{12} & 0\\ * & \Xi_{21} & P_2B\\ * & * & -\gamma^2 I\end{bmatrix} < 0 \tag{13}$$

holds, where

$$\Xi_{11} = He(P_1A + P_1BK) + C^TC + \beta\mathcal{F}^T\mathcal{F},$$
$$\Xi_{12} = P_1HLC + C^TC,$$

$$\Xi_{21} = He(P_2A - P_2LC) + C^TC + \Upsilon,$$
$$\Upsilon = \alpha^{-1}P_2\mathcal{D}\mathcal{D}^TP_2 + \alpha\mathcal{F}^T\mathcal{F} + \beta^{-1}P_2\mathcal{D}\mathcal{D}^TP_2,$$
$$H = I - B(ZB)^{-1}Z.$$

Thus, the system (12) achieves asymptotic stability while maintaining H_∞ performance.

Proof: Consider the Lyapunov function as follows:

$$V(t) = \hat{x}^T(t)P_1\hat{x}(t) + e^T(t)P_2e(t). \tag{14}$$

Set $w(t) = 0$. In this case, combining (7) and (12), it follows from (14) that

$$\begin{aligned}\dot{V}(t) &= 2\hat{x}^T(t)P_1(A+BK)\hat{x}(t) + 2\hat{x}^T(t)P_1(HLCe(t)\\ &\quad + 2e^T(t)P_2(A\triangle\mathcal{A}(t) - LC)e(t)\\ &\quad + 2e^T(t)P_2\triangle\mathcal{A}(t)\hat{x}(t).\end{aligned}$$

According to Lemma 2, $2e^T(t)P_2A\triangle\mathcal{A}(t)e(t)$ and $2e^T(t)P_2\triangle\mathcal{A}(t)\hat{x}(t)$ can be processed as follows:

$$\begin{aligned}2e^T(t)P_2A\triangle\mathcal{A}(t)e(t) &\leqslant \alpha^{-1}e^T(t)P_2A\mathcal{D}\mathcal{D}^TA^TP_2e(t)\\ &\quad + \alpha e^T(t)\mathcal{F}^T\mathcal{F}e(t),\\ 2e^T(t)P_2\triangle\mathcal{A}(t)\hat{x}(t) &\leqslant \beta^{-1}e^T(t)P_2\mathcal{D}\mathcal{D}^TP_2e(t)\\ &\quad + \beta\hat{x}^T(t)\mathcal{F}^T\mathcal{F}\hat{x}(t).\end{aligned}$$

Therefore, $\dot{V}(t)$ can be converted to

$$\begin{aligned}\dot{V}(t) &\leqslant 2\hat{x}^T(t)P_1(A+BK)\hat{x}(t) + 2\hat{x}^T(t)P_1HLCe(t)\\ &\quad + 2e^T(t)P_2Ae(t) - 2e^T(t)P_2LCe(t) + \alpha^{-1}e^T(t)\\ &\quad \times P_2A\mathcal{D}\mathcal{D}^TA^TP_2e(t) + \alpha e^T(t)\mathcal{F}^T\mathcal{F}e(t) + \beta^{-1}\\ &\quad \times e^T(t)P_2\mathcal{D}\mathcal{D}^TP_2e(t) + \beta\hat{x}^T(t)\mathcal{F}^T\mathcal{F}\hat{x}(t),\\ &= \eta^T(t)\Theta\eta(t),\end{aligned} \tag{15}$$

where

$$\eta^T(t) = \begin{bmatrix}\hat{x}^T(t) & e^T(t)\end{bmatrix}^T,$$
$$\Theta_1 = He(P_1A + P_1BK) + \beta\mathcal{F}^T\mathcal{F},$$
$$\Theta_2 = He(P_2A - P_2LC) + \Upsilon,$$
$$\Upsilon = \alpha^{-1}P_2A\mathcal{D}\mathcal{D}^TA^TP_2 + \alpha\mathcal{F}^T\mathcal{F} + \beta^{-1}P_2\mathcal{D}\mathcal{D}^TP_2,$$

and

$$\Theta = \begin{bmatrix}\Theta_1 & P_1HLC\\ * & \Theta_2\end{bmatrix}.$$

From (13), we can conclude that $\Theta < 0$. Consequently, when $w(t) = 0$, it follows that $\dot{V}(t) < 0$. Based on definition 2, we ascertain that system (12), under the condition $w(t) = 0$, is asymptotically stable.

Next, the Lyapunov function is

$$\begin{aligned}\dot{V}(t) &= 2\hat{x}^T(t)P_1(A+BK)\hat{x}(t) + 2\hat{x}^T(t)P_1HLCe(t)\\ &\quad + 2e^T(t)P_2(A + \triangle\mathcal{A}(t) - LC)e(t)\\ &\quad + 2e^T(t)P_2\triangle\mathcal{A}(t)\hat{x}(t) + 2e^T(t)P_2Bw(t),\end{aligned} \tag{16}$$

we show the system (12) is asymptotically stable with performance level γ.

Define

$$\varrho(t) = y^T(t)y(t) - \gamma^2w^T(t)w(t).$$

In fact, under zero initial condition, we can deduce that

$$\begin{aligned}\dot{V}(t) + \varrho(t) &= 2\hat{x}^T(t)P_1(A+BK)\hat{x}(t) + 2\hat{x}^T(t)P_1HL\\ &\quad \times Ce(t) + 2e^T(t)P_2(A + \triangle\mathcal{A}(t) - LC)\\ &\quad \times e(t) + 2e^T(t)P_2\triangle\mathcal{A}(t)\hat{x}(t) + 2e^T(t)\\ &\quad \times P_2Bw(t) + y^T(t)y(t) - \gamma^2w^T(t)w(t),\\ &\leqslant 2\hat{x}^T(t)P_1(A+BK)\hat{x}(t) + 2\hat{x}^T(t)P_1HL\\ &\quad \times Ce(t) + 2e^T(t)P_2Ae(t) - 2e^T(t)P_2LC\\ &\quad \times e(t) + \alpha^{-1}e^T(t)P_2A\mathcal{D}\mathcal{D}^TA^TP_2e(t)\\ &\quad + \alpha e^T(t)\mathcal{F}^T\mathcal{F}e(t) + \beta^{-1}e^T(t)P_2\mathcal{D}\mathcal{D}^TP_2\\ &\quad \times e(t)\beta\hat{x}^T(t)\mathcal{F}^T\mathcal{F}\hat{x}(t) + e^T(t)C^TCe(t)\\ &\quad + e^T(t)C^TC\hat{x}(t) + \hat{x}^T(t)C^TCe(t)\\ &\quad + \hat{x}^T(t)C^TC\hat{x}(t) - \gamma^2w^T(t)w(t)\\ &\quad + 2e^T(t)P_2Bw(t),\\ &= \xi^T(t)\Xi\xi(t),\end{aligned} \tag{17}$$

where $\xi^T = \begin{bmatrix}\hat{x}^T(t) & e^T(t) & w^T(t)\end{bmatrix}^T$. Then from (13), it is easy to show that $\Xi < 0$, because $V(t) > 0$. It holds that

$$\dot{V}(t) + \varrho(t) \leqslant 0. \tag{18}$$

Integrating both sides of (18) from 0 to t allows us to obtain

$$\int_0^t(\dot{V}(\tau) + \varrho(\tau))d\tau = V(t) - V(0) + \int_0^t\varrho(\tau)d\tau \leqslant 0. \tag{19}$$

Noticing that $V(t) \geqslant 0$ and $V(0) = 0$, from (19) we obtain

$$\int_0^t\varrho(\tau)d\tau \leqslant 0. \tag{20}$$

As $t \to \infty$, from the above inequalities (18)–(20), we can write

$$\int_0^\infty y^T(\iota)y(\iota)d\iota \leqslant \gamma^2\int_0^\infty w^T(\iota)w(\iota)d\iota, \tag{21}$$

which indicates that the system derived from (12) is H_∞ asymptotically stable by means of Definition 2 and Definition 1. The proof is finished. ■

Theorem 2. Given scalars $\gamma > 0$, $\alpha > 0$, and $\beta > 0$, suppose that there exist matrices $P_2 > 0$ and $R > 0$, and any matrices M, N, and L such that

$$\begin{bmatrix}\Psi_{11} & \Psi_{12} & \Psi_{13}\\ * & \Psi_2 & 0\\ * & * & \Psi_3\end{bmatrix} < 0 \tag{22}$$

holds, where

$$H = I - B(ZB)^{-1}Z,$$
$$\Lambda_1 = He(AR + BM),$$
$$\Lambda_2 = He(P_2A - NC) + \alpha\mathcal{F}^T\mathcal{F},$$
$$\Psi_2 = diag\{-\gamma^2I, -I\},$$
$$\Psi_3 = diag\{-\beta^{-1}I, -(\alpha^{-1})^{-1}I, -(\beta^{-1})^{-1}I\},$$
$$\Psi_{11} = \begin{bmatrix}\Lambda_1 & HLC\\ * & \Lambda_2\end{bmatrix},$$
$$\Psi_{12} = \begin{bmatrix}0 & R^TC^T\\ P_2B & C^T\end{bmatrix},$$
$$\Psi_{13} = \begin{bmatrix}R^T\mathcal{F}^T & 0 & 0\\ 0 & P_2A\mathcal{D} & P_2\mathcal{D}\end{bmatrix}.$$

Then the closed-loop systems composed of (12) is asymptotically stable with H_∞ performance.

Moreover, for the MCC (5), the SMC controller is designed as

$$u(t) = K\hat{x}(t) - (ZB)^{-1}(ZLCe(t) + \epsilon \text{sign}(s(t))), \quad (23)$$

with the state observer gain $L = P_2^{-1}N$, control gain $K = MR^{-1}$.

Proof: Utilizing Lemma 1 to (22) yields that

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 & 0 \\ * & \Pi_3 & P_2B \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (24)$$

where

$$\Pi_1 = He(AR + BM) + R^TC^TCR + \beta R\mathcal{F}^T\mathcal{F}R,$$
$$\Pi_2 = HLC + R^TC^TC,$$
$$\Pi_3 = He(P_2A - NC) + C^TC + \Upsilon.$$

Then pre- and post-multiplying (24) by $diag\{P_1, I, I\}$. (13) can be obtained with $R = P_1^{-1}$, $M = KR$, and $N = P_2L$.

Additionally, consider the following Lyapunov function:

$$\tilde{V}(t) = \frac{1}{2}s^T(t)s(t).$$

Next is the derivative of $\tilde{V}$:

$$\begin{aligned} \dot{\tilde{V}}(t) &= s^T(t)\dot{s}(t), \\ &= s^T(t)(ZBu(t) + ZLCe(t) - ZBK\hat{x}(t)), \\ &= s^T(t)(ZB(K\hat{x}(t) - (ZB)^{-1}(ZLCe(t) \\ &\quad + \epsilon\text{sign}(s(t)))) + ZLCe(t) - ZBK\hat{x}(t)), \\ &= -s^T(t)\epsilon\text{sign}(s(t)), \\ &\leqslant -\epsilon\|s(t)\|, \\ &= -\sqrt{2}\epsilon\tilde{V}^{\frac{1}{2}}(t), \\ &= -\epsilon_0\tilde{V}^{\frac{1}{2}}(t). \end{aligned} \quad (25)$$

Based on the above discussion, it can be seen from (25) that when $t^* = \frac{2\sqrt{\tilde{V}(0)}}{\epsilon_0}$, $\tilde{V}(t) = 0$, then when $t > t^*$, $s(t) = 0$. ϵ is a positive constant, where $\epsilon = \frac{1}{\sqrt{2}}\epsilon_0$. Therefore, under the SMC controller (23), the system's trajectories can reach the designated SMS (8) within a finite time. Consequently, achieving the asymptotic stability of the MCC concludes the proof. ■

Remark 2. This paper consists of the derivation of two theorems. Theorem 1 performs an H_∞ stability analysis of the estimation error system (7) and the closed-loop MCC system (12) using a Lyapunov function, deriving the conditions under which both the estimation error system and the closed-loop MCC system achieve asymptotic stability with an H_∞ performance index γ. Theorem 2, based on the first, designs an SMC controller to obtain the control gains.

IV. Slimulation Results

In this section, an example is provided to demonstrate the effectiveness of the SMC controller derived from Theorem 2.

Consider MCC system (1) with the following parameter values and initial condition [35],

$$\begin{cases} C_1 = 1, C_2 = 0.1, L = \frac{1}{13}, G = 1.5, R = 1, a_1 = 0.3, \\ \frac{r}{L} = 0.35, a_2 = 0.8, x(0) = [1, 0, 0, 0]^T. \end{cases}$$

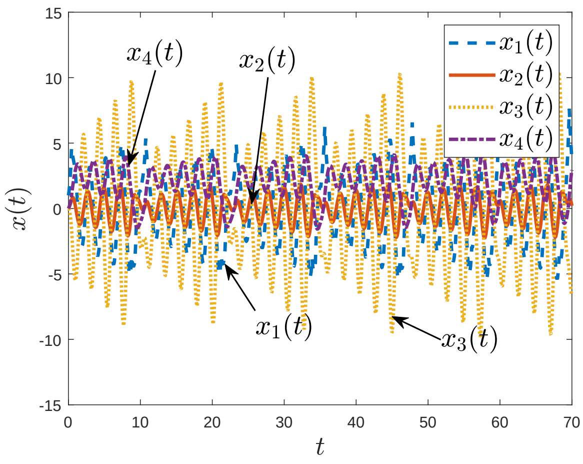

Fig. 2. State trajectories without control input

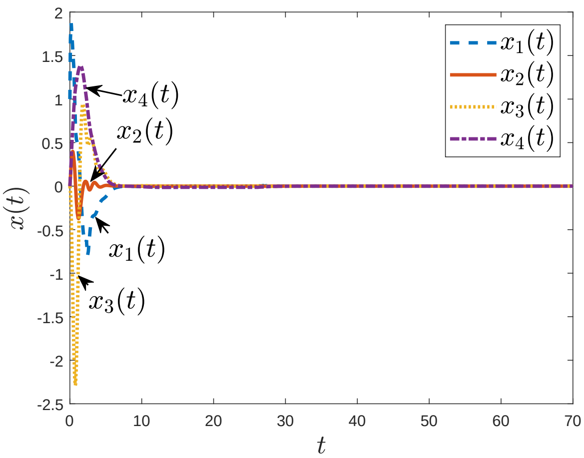

Fig. 3. The trajectories of system state based on Theorem 2

Then, the MCC system (5) has the following matrices

$$A_1 = \begin{bmatrix} 2 & 10 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -13 & -0.35 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -3 & 10 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -13 & -0.35 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\mathcal{D}_1 = \mathcal{D}_2 = \begin{bmatrix} 0.1 \\ 0.05 \\ -0.01 \\ 0.1 \end{bmatrix}, \mathcal{F}_1 = \mathcal{F}_2 = \begin{bmatrix} 0 \\ -0.01 \\ 0.2 \\ 0 \end{bmatrix}^T.$$

Choose the equation $\mathcal{U}(t)$ in the uncertainty term:

$$\mathcal{U}(t) = \begin{cases} sin(t), & |x_4(t)| \leqslant 1, \\ cos(t), & |x_4(t)| > 1. \end{cases}$$

Selecting the disturbance $w(t) = e^{-0.4t}\cos(-0.8t)$, the MCC system exhibits unstable and chaotic behavior in the absence of control input, as illustrated in Fig. 2.

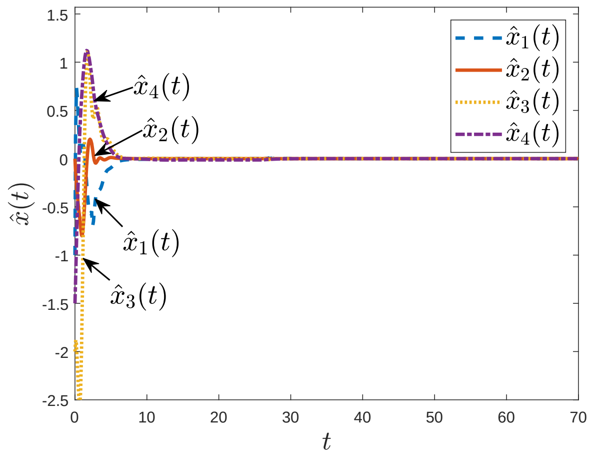

Fig. 4. The trajectories of estimated state based on Theorem 2

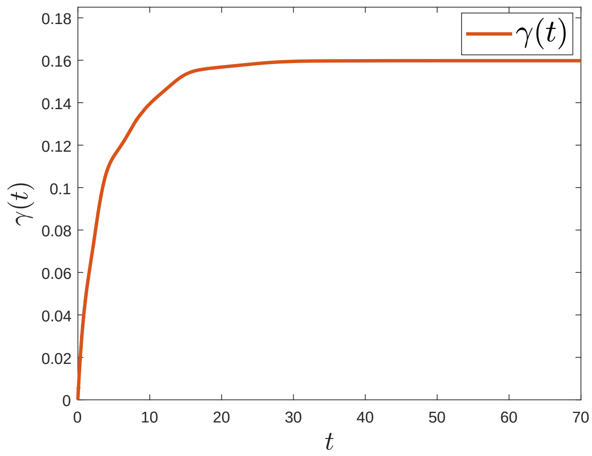

Fig. 6. The trajectory of $\gamma(t)$

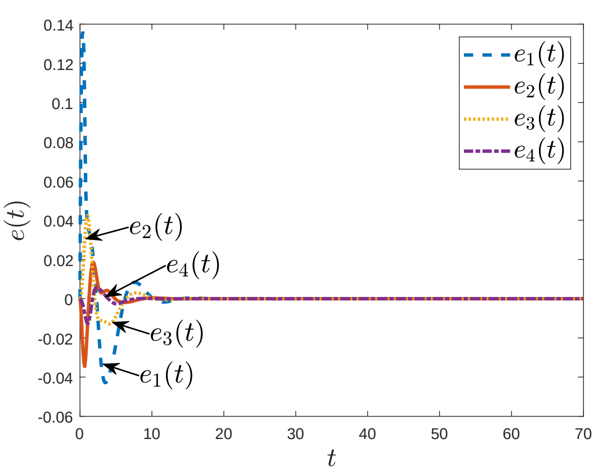

Fig. 5. The trajectories of error state

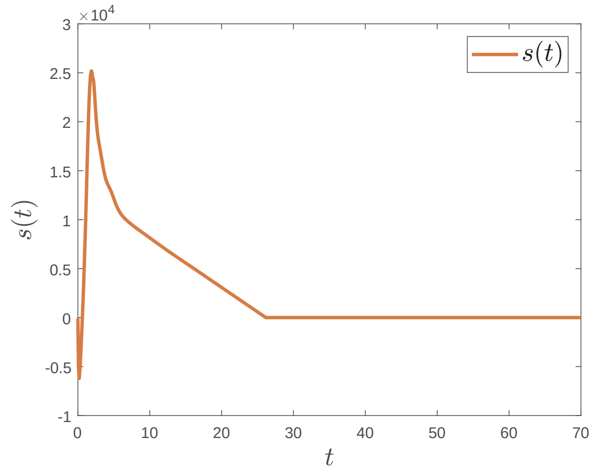

Fig. 7. The trajectory of SMS

In the following content, the focus is on verifying the asymptotic stability of the MCC system under the SMC controller. The correlation matrix and parameter selection are as follows: $Z = [5.2 \; -4.3 \; -4 \; -6.5]$, with constants $\alpha = \beta = \epsilon_0 = 0.5$ and $\gamma = 1.5$. The initial states of the system are given as $\hat{x}(0) = [-1, 0, -2, -1.5]^T$. It is proved in Theorem 2 that by utilizing the MOSEK toolboxes to solve the linear matrix inequality (23), the observer gain and control gain can be calculated as follows:

$$K = \begin{bmatrix} -10.5448 & -24.7889 & 1.333 & -7.1758 \end{bmatrix},$$

$$L = \begin{bmatrix} 7.0144 & 4.6492 & -0.3013 & -0.0049 \\ 1.7879 & -0.1311 & -1.2990 & 0.0468 \\ -0.1033 & -10.7455 & 0.5890 & 0.0018 \\ 1.0257 & 0.0844 & -0.1322 & 1.0384 \end{bmatrix}.$$

As illustrated in Figs. 4 and 5, the trajectories of the state estimation and the estimation error dynamics demonstrate that, under the influence of the SMC controller, the state estimation gradually converges to zero.

Next, it is the time to verify the H_∞ performance under zero initial condition. As [42], we define

$$\gamma(t) = \sqrt{\frac{\int_0^t y^T(\iota) y(\iota) d\iota}{\int_0^t w^T(\iota) w(\iota) d\iota}}. \tag{26}$$

The trajectory of $\gamma(t)$, reflecting the disturbance effect on the control output, is illustrated in Fig. 6, indicating that $\gamma(\infty) = 0.1598$ $(< \gamma_{min} = 0.3502)$. From the simulation results, it is observed that $\gamma(t)$ converges to the constant 0.1598, which is below the specified H_∞ performance bound.

Figs. 3 and 7 depict the closed-loop state with the MCC incorporated into the controller and the trajectory of the SMS, respectively. It is evident from these figures that the system state reaches the SMS within a finite time and achieves stability on this surface. The results corresponding to the designed SMC controller are presented in Fig. 8. In conclusion, numerical simulations confirm the effectiveness of the observer-based SMC for MCC systems.

V. Conclusion

This paper addressed the problem of observer-based SMC for MCC systems. An observer-based SMC controller was designed to stabilize the MCC system in the presence of external disturbances and uncertainties. To begin with, a sufficient condition was established in Theorem 1 to guarantee asymptotic stability of the closed-loop MCC system under a prescribed disturbance attenuation level. Building upon this condition, the joint design of controller gain, observer gain, and convergence law parameter was proposed in Theorem 2. Subsequently, the design has been proven to ensure that the system state reached the SMS in a finite time. An example was provided to validate the effectiveness of the research findings. SMC is often susceptible to chattering phenomena in practical applications, which can significantly

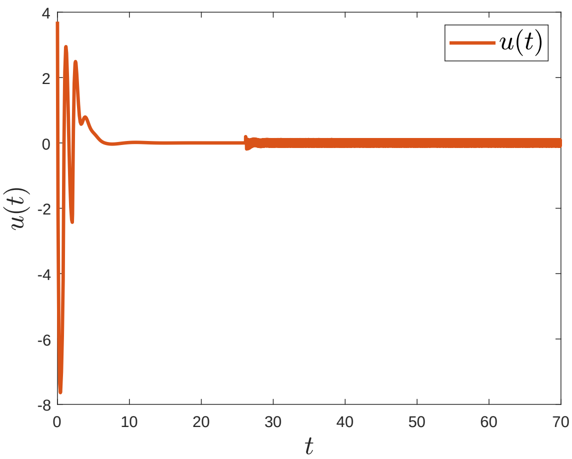

Fig. 8. The trajectory of controller

affect system performance. Future research will focus on exploring higher-order SMC methods to effectively mitigate the occurrence of chattering and improve control precision and robustness.

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