A New Outer Inverse of a Matrix and Its Characterizations

Divya Shenoy Purushothama, Member, IAENG

Abstract—**This paper introduces a new outer inverse known as the Drazin-secondary generalized inverse (D-sg inverse), which combines the properties of the Drazin inverse and the secondary generalized inverse. We provide a representation of the D-sg inverse emphasizing its specific column space and row space characteristics. Several critical characterizations of the D-sg inverse are derived, demonstrating its significance. Additionally, we explore an application of the D-sg inverse in solving systems of linear equations, illustrating its practical utility in this context.**

Index Terms—**Drazin inverse, Secondary generalized inverse, Secondary transpose, Column space**

I. Introduction

THE Moore Penrose inverse $A^\dagger$ of a matrix A is a generalization of the classic inverse of a matrix. However, the Moore Penrose inverse can be obtained for any matrix, even if the given matrix is rectangular.

Definition 1. *[1] Let A be any square or rectangular matrix. Then the Moore-Penrose inverse $A^\dagger$ of A, is the unique matrix satisfying the following conditions:*

$$(1)\ AA^\dagger A = A \qquad (2)\ A^\dagger AA^\dagger = A$$
$$(3)\ (AA^\dagger)^* = AA^\dagger \qquad (4)\ (A^\dagger A)^* = (A^\dagger A)$$

Another matrix inverse that is of greater interest is the Drazin inverse which is defined for a square matrix of index k. The index k of a matrix A is the least nonnegative integer such that $rank(A^{k+1}) = rank(A)$.

Definition 2. *[1] The Drazin inverse of a matrix A is the unique matrix A^D satisfying the conditions:*

$$A^D AA^D = A^D, \quad AA^D = A^D A, \quad A^{k+1}A^D = A^k$$

Different types of generalized inverses are characterized by differing sets of defining conditions. Core inverse [2], Core EP inverse [3], [4], secondary generalized inverse [5]. A recent trend is to combine these inverses and define new classes of generalized inverses, which will have several applications in solving the system of equations, extending the given class of inverses, etc. Mallik and Thome [6] introduced the DMP inverse for a square matrix of arbitrary index by combining the Drazin inverse and the Moore Penrose inverse of the matrix. The DMP inverse of A is $A^{d,\dagger} = A^D AA^\dagger$. CMP inverse, i.e., $A^{c,\dagger} = A^{\dagger s}AA^D AA^\dagger$ is introduced by Mehdipour and Salemi [7]. The Drazin star matrices [8] and Drazin theta matrices [9] are particular types of matrices that act as the outer inverse of $(A^\dagger)^*$ and $(A^{\dagger s})^S$, respectively. There are many more generalized inverses, such as Outer theta inverse [10], 1D inverse and D1 inverse [11], $m-DMP$ inverse [12], MPCEP inverse [13], MPWG inverse [14] etc.

Here, we define a new generalized inverse by combining the Drazin inverse and the secondary generalized inverse and name it the Drazin secondary generalized inverse (D-sg inverse). Here are a few characterizations, an analytic approach, and a geometrical representation of the new inverse. Also, the D-sg inverse is represented as an outer inverse with specific column space and null space.

II. Preliminaries

In this article, the representation $\mathbb{C}^{n\times n}$ denotes the set of complex matrices of order $n \times n$. The range space, the null space, the index and the rank of a matrix A are denoted by $\mathcal{C}(A), \mathcal{N}(A), ind(A)$ and $rank(A)$ respectively. $P_{L,M}$ is the projector onto L along M, where L and M are complementary subspaces.

Additionally, A^S represents the secondary transpose of a matrix. The concept of the secondary transpose of a matrix is introduced by Anna Lee [15].

Definition 3. *[15] Let $A \in \mathbb{C}^{n\times n}$. Then the conjugate secondary transpose of A is denoted by A^S and is defined as $A^S = (c_{ij})$ where $c_{ij} = a_{n-j+1,n-i+1}$.*

Based on the idea of secondary transpose, Savitha et al. [5] defined secondary generalized inverse $A^{\dagger s}$, which is analogous to Moore-Penrose inverse.

Definition 4. *Let $A \in \mathbb{C}^{m\times n}$. The unique matrix X satisfying the conditions*

$$(1)\ AXA = A \qquad (2)\ XAX = X$$
$$(3)\ (AX)^S = AX \qquad (4)\ (XA)^S = XA.$$

is called the secondary generalized inverse of A and is denoted as $A^{\dagger s}$.

A necessary and sufficient condition for the existence of secondary generalized inverse $A^{\dagger s}$ is $rank(AA^S) = rank(A^S A) = rank(A)$.

The following lemma can be verified directly using the properties of column space and null space of the matrix A.

Lemma 1. *Let $A \in \mathbb{C}^{m\times n}$ be such that $A^{\dagger s}$ exists. Then,*
1) $\mathcal{C}(A^{\dagger s}) = \mathcal{C}(A^S)$ *and* $\mathcal{N}(A^{\dagger s}) = \mathcal{N}(A^S)$;
2) $AA^{\dagger s} = P_{\mathcal{C}(A),\mathcal{N}(A^S)}$;
3) $A^{\dagger s}A = P_{\mathcal{C}(A^S),\mathcal{N}(A)}$.

The representation of the Drazin inverse in terms of its column space and row space is given in the following lemma. This particular lemma helps to prove many results of this article.

Manuscript received July 15, 2024; revised November 30, 2024.

Divya Shenoy is an Assistant Professor at the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India - 576104. e-mail: divya.shenoy@manipal.edu

Lemma 2. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. Then,*
1) $\mathcal{C}(A^D) = \mathcal{C}(A^k)$ *and* $\mathcal{N}(A^D) = \mathcal{N}(A^k)$;
2) $AA^D = A^D A = P_{\mathcal{C}(A^k),\mathcal{N}(A^k)}$

Lemma 3. *[1] Let $A, B, C \in \mathbb{C}^{n\times n}$. Then the matrix equation $AXB = C$ is consistent if and only if for some $A^{(1)} \in A\{1\}, B^{(1)} \in B\{1\}$,*

$$AA^{(1)}CB^{(1)}B = C,$$

in which the general solution is

$$X = A^{(1)}CB^{(1)} + Z - A^{(1)}AZBB^{(1)}$$

for arbitrary $Z \in \mathbb{C}^{n\times n}$.

Note that $A^{(1)}$ is a generalized inverse of A, whereas $A\{1\}$ is the set of all generalized inverses of A.

III. Results

Theorem 1. *Consider $A \in \mathbb{C}^{n\times n}$ with index k satisfying the condition $rank(AA^S) = rank(A^S A) = rank(A)$. Then, the system of equations*

$$GAG = G \quad GA = A^D A \quad A^k G = A^k A^{\dagger s} \tag{1}$$

has a unique solution whenever the solution exists.

Proof: Assume that G_1 and G_2 satisfy (1).
i.e., $G_1AG_1 = G_1, G_1A = A^D A, A^k G_1 = A^k A^{\dagger s}$
and
$G_2AG_2 = G_2, G_2A = A^D A$ *and* $A^k G_2 = A^k A^{\dagger s}$.
Since, $A^D A$ is a projector and $AA^D = A^D A$ we get,

$$\begin{aligned} G_1 &= G_1AG_1 = A^D AG_1 = (A^D A)^k G_1 \\ &= (A^D)^k A^k G_1 = (A^D)^k A^k A^{\dagger s} \\ &= (A^D)^k A^k G_2 = (A^D A)^k G_2 \\ &= A^D AG_2 = G_2AG_2 = G_2. \end{aligned}$$

Hence, the uniqueness. ■

Theorem 2. *The system of equations (1) is consistent and has a unique solution $G = A^D AA^{\dagger s}$.*

Proof: It is easy to see that $A^D AA^{\dagger s}$ satisfies the three equations in the system (1). Now, theorem (1) gives the uniqueness. ■

Thus, for a given matrix A, the matrix $A^D AA^{\dagger s}$ is the unique matrix satisfying system of equations (1).

Definition 5. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k (not necessarily $k \leq 1$). The D-sg inverse of A, denoted by $A^{D,\dagger s}$, is defined to be the matrix*

$$A^{D,\dagger s} = A^D AA^{\dagger s}.$$

An example for D-sg inverse of a matrix A is given below:

Example 1. *Consider a matrix* $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.
Here, $rank(A) = 1$. The secondary generalized inverse of the matrix is $A^{\dagger s} = \begin{pmatrix} 2/8 & 1/8 \\ 2/8 & 1/8 \end{pmatrix}$ *and*
Drazin inverse is $A^D = \begin{pmatrix} 1/45 & 2/45 \\ 2/45 & 4/45 \end{pmatrix}$

The Drazin secondary generalized inverse is given by

$$A^{D,\dagger s} = \begin{pmatrix} 1/6 & 1/12 \\ 1/3 & 1/6 \end{pmatrix}$$

Also, it can be observed that the Drazin secondary generalized inverse differs from the Drazin Moore-Penrose inverse (DMP inverse) since the DMP inverse of A is

$$A^{D,\dagger} = \begin{pmatrix} 1/15 & 2/15 \\ 2/15 & 4/15 \end{pmatrix}$$

Remark 1. *Similar to the D-sg inverse, one can define secondary generalized Drazin inverse (sg-D inverse) for a square matrix with index k.*
Consider $A \in \mathbb{C}^{n\times n}$ with index k such that $A^{\dagger s}$ exists. Then the matrix $G = A^{\dagger s} AA^D$ is a unique solution for the system of equations

$$GAG = G \quad AG = GA^D \quad GA^k = A^{\dagger s} A^k$$

This particular G can be named as secondary generalized Drazin inverse.

Even though the secondary generalized Drazin inverse is an outer inverse of A, it differs from the Drazin secondary generalized inverse. This is clear from the example given below.

Example 2. *Consider the same matrix in example 1.*
i.e., $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.
The secondary generalized Drazin inverse is

$$\begin{aligned} A^{\dagger s} AA^D &= \begin{pmatrix} 2/8 & 1/8 \\ 2/8 & 1/8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1/45 & 2/45 \\ 2/45 & 4/45 \end{pmatrix} \\ &= \begin{pmatrix} 1/30 & 1/15 \\ 1/30 & 1/15 \end{pmatrix} \end{aligned}$$

In the following theorem, the D-sg inverse is represented as an outer inverse with prescribed column space and null space.

Theorem 3. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with index k such that $rank(A^S AA^S) = rank(A)$. Then*
1) $rank(A^{D,\dagger s}) = rank(A^k)$;
2) $\mathcal{C}(A^{D,\dagger s}) = \mathcal{C}(A^k)$ *and* $\mathcal{N}(A^{D,\dagger s}) = \mathcal{N}(A^k A^{\dagger s})$;
3) $A^{D,\dagger s} = A^{(2)}_{\mathcal{C}(A^k),\mathcal{N}(A^k A^{\dagger s})}$;
4) $AA^{D,\dagger s} = P_{\mathcal{C}(A^k),\mathcal{N}(A^k A^{\dagger s})}$;
5) $A^{D,\dagger s} A = A^D A = P_{\mathcal{C}(A^k),\mathcal{N}(A^k)}$.

Proof: (1) By the lemma (2), we have that

$$\begin{aligned} rank(A) &= rank(A^D A) = rank(A^D AA^{\dagger s} A) \\ &\leq rank(A^D AA^{\dagger s}) \leq rank(A^D) \end{aligned}$$

which together with the definition (5) shows that $rank(A^{D,\dagger s}) = rank(A^k)$.
(2) Using the definition (5) and the item (1), directly we get $\mathcal{C}(A^{D,\dagger s}) = \mathcal{C}(A^k)$.
Since

$$rank(A^k) = rank(A^k A^{\dagger s} A) \leq rank(A^k A^{\dagger s}) \leq rank(A^k)$$

again by the item (1), we get

$$rank(A^k A^{\dagger s}) = rank(A^k) = rank(A^{D,\dagger s}).$$

Moreover,

$$\begin{aligned} \mathcal{N}(A^k A^{\dagger s}) &\subseteq \mathcal{N}((A^D A)^k A^{\dagger s}) \\ &= \mathcal{N}(A^D AA^{\dagger s}) = \mathcal{N}(A^{D,\dagger s}) \end{aligned}$$

(3) It is evident by theorem 1 and the item (2).
(4) In terms of $ind(A) = k$ and the item (2), we infer

$$\mathcal{C}(AA^{D,\dagger s}) = A\mathcal{C}(A^k) = \mathcal{C}(A^{k+1}) = \mathcal{C}(A^k)$$

Evidently, $rank(AA^{D,\dagger s}) = rank(A^k)$, which together with the items (1) and (2), shows that

$$\mathcal{N}(AA^{D,\dagger s}) = \mathcal{N}(A^{D,\dagger s}) = \mathcal{N}(A^k A^{\dagger s})$$

Since $A^{D,\dagger s}$ is an outer inverse of A, we have $AA^{D,\dagger s} = P_{\mathcal{C}(A^k),\mathcal{N}(A^k A^{\dagger s})}$.
(5) It is easily obtained by definition 5 and (2) of lemma 2. ■

A geometrical approach to characterize the Drazin secondary generalized inverse is given below:

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$ with index k and $rank(A^S AA^S) = rank(A)$. Then $A^{D,\dagger s}$ is the unique matrix $G \in \mathbb{C}^{n\times n}$ such that*

$$AG = P_{\mathcal{C}(A^k),\mathcal{N}(A^k A^{\dagger s})}, \quad \mathcal{C}(G) \subseteq \mathcal{C}(A^k) \qquad (2)$$

Proof: Clearly, from $(2)^{nd}$ and $(4)^{th}$ conditions of theorem 3, $A^{D,\dagger s}$ is a solution to (2).
To prove the uniqueness, assume that G_1 and G_2 are solutions of (2). Now, since $AG_1 = AG_2 = P_{\mathcal{C}(A^k),\mathcal{N}(A^k A^{\dagger s})}$, we get $\mathcal{C}(G_1 - G_2) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(A^k)$. Moreover, from $\mathcal{C}(G_1) \subseteq \mathcal{C}(A^k)$ and $\mathcal{C}(G_2) \subseteq \mathcal{C}(A^k)$ we get $\mathcal{C}(G_1 - G_2) \subseteq \mathcal{C}(A^k)$. Now, $\mathcal{C}(G_1 - G_2) \subseteq \mathcal{C}(A^k) \cap \mathcal{N}(A^k) = \{0\}$ is direct which means $G_1 = G_2$.
Hence, the uniqueness. ■

Theorem 5. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. Let $rank(AA^S) = rank(A^S A) = rank(A)$. Then $A^{D,\dagger s} = A_{\mathcal{C}(A^k),\mathcal{N}(A^k A^{\dagger s})}$*

Proof: D-sg inverse is an outerinverse by definition. So it is enough to show that $\mathcal{C}(A^{D,\dagger s}) = \mathcal{C}(A^k)$ and $\mathcal{N}(A^{D,\dagger s}) = \mathcal{N}(A^k A^{\dagger s})$. Evidently

$$\begin{aligned}\mathcal{C}(A^{D,\dagger s}) &= \mathcal{C}(A^D AA^{D,\dagger s}) \subseteq \mathcal{C}(A^k)\\ &= \mathcal{C}(A^D A^{k+1}) = \mathcal{C}(A^D AA^{\dagger_s} A^{k+1})\\ &\subseteq \mathcal{C}(A^{D,\dagger s})\end{aligned}$$

Therefore, $\mathcal{C}(A^{D,\dagger s}) = \mathcal{C}(A^k)$. Also

$$\begin{aligned}\mathcal{N}(A^{D,\dagger s}) &\subseteq \mathcal{N}(A^k A^{D,\dagger s})\\ &= \mathcal{N}(A^k A^{\dagger s}) \subseteq \mathcal{N}((A^D)^k A^k A^{\dagger s})\\ &= \mathcal{N}(A^D AA^{\dagger s}) = \mathcal{N}(A^{D,\dagger s})\end{aligned}$$

so, $\mathcal{N}(A^{D,\dagger s}) = \mathcal{N}(A^k A^{\dagger s})$. ■

Different characterizations of D-sg inverse in terms of column space of $A^{D,\dagger s}$ are obtained here.

Theorem 6. *Let $A, G \in \mathbb{C}^{n\times n}$ and $ind(A) = k$. Assume $rank(AA^S) = rank(A^S A) = rank(A)$. Then the following conditions are equivalent:*

1) $G = A^{D,\dagger s}$;
2) $A^k G = A^k A^{\dagger s}, \mathcal{C}(G) = \mathcal{C}(A^k)$;
3) $\mathcal{C}(G) = \mathcal{C}(A^k), \mathcal{N}(G) = \mathcal{N}(A^k A^{\dagger s}), GA \in \mathbb{C}_n^P$;
4) $\mathcal{C}(G) = \mathcal{C}(A^k), \mathcal{N}(G) = \mathcal{N}(A^k A^{\dagger s}), AG \in \mathbb{C}_n^P$;
5) $\mathcal{C}(G) = \mathcal{C}(A^k), \mathcal{N}(G) = \mathcal{N}(A^k A^{\dagger s}), GA^D = (A^D)^2$;
6) $GA^{k+1} = A^k, \mathcal{N}(G) = \mathcal{N}(A^k A^{\dagger s})$;
7) $GAA^D = A^D, rank(G) = rank(A^k), A^k G = A^k A^{\dagger s}$;
8) $GAA^D = A^D, \mathcal{N}(G) = \mathcal{N}(A^k A^{\dagger s})$.

Proof: (1) $\Longrightarrow$ (2) − (7) can be verified directly using theorem (5) and the definition of D-sg inverse.
(2) $\Longrightarrow$ (1). Since $A^D A = P_{\mathcal{C}(A^k),\mathcal{N}(A^k)}$ from $\mathcal{C}(G) = \mathcal{C}(A^k)$, we have $A^D AG = G$. Since $A^D A$ is an idempotent, it follows that

$$\begin{aligned}G &= A^D AG = (A^D A)^k G = (A^D)^k A^k A^{\dagger s}\\ &= A^D AA^{\dagger s} = A^{D,\dagger s}\end{aligned}$$

(3) $\Longrightarrow$ (1). Since GA is an idempotent, $\mathcal{C}(A - AGA) \subseteq \mathcal{N}(G) = \mathcal{N}(A^k A^{\dagger s})$, so $A^k A^{\dagger s} A = A^k A^{\dagger s} AGA$, i.e., $A^k = A^k GA$. Multiplying the last equality by $A^{\dagger s}$ from the right, $A^k A^{\dagger s} AGA$, i.e., $A^k = A^k GA$. Multiplying the last equality by $A^{\dagger s}$ from the right, we get $A^k A^{\dagger s} = A^k GAA^{\dagger s}$. Finally, since $\mathcal{C}(I - AA^{\dagger s}) \subseteq \mathcal{N}(A^k A^{\dagger s}) = \mathcal{N}(G)$ we have $G = GAA^{\dagger s}$. Hence, $A^k A^{\dagger s} = A^k G$.
The proof of (4) $\Longrightarrow$ (1) follows in the similar line as in the part (3) $\Longrightarrow$ (1).
(5) $\Longrightarrow$ (2). By $\mathcal{C}(G) = \mathcal{C}(A^k)$ we have $A^D AG = G$. Also, $GA^D = (A^D)^2$ implies $GA^D A = A^D$. These two conditions together give $G = GA^D A^2 G$. Hence $\mathcal{C}(I - A^D A^2 G) \subseteq \mathcal{N}(G) = \mathcal{N}(A^k A^{\dagger s})$, so $A^k A^{\dagger s} = A^{k+1} A^D G$, i.e., $A^k A^{\dagger s} = A^k G$.
(6) $\Longrightarrow$ (5). Multiplying $GA^{k+1} = A^k$ from the right side by $(A^D)^{k+2}$ we get $GA^D = (A^D)^2$. Now using $\mathcal{C}(A^k A^{\dagger s}) = \mathcal{C}(A^k)$, $\mathcal{N}(A^k) = \mathcal{N}(A^k A^{\dagger s})$ and $dim\mathcal{C}(A^k) = dim\mathcal{C}(G)$ along with $\mathcal{C}(A^k) \subseteq \mathcal{C}(G)$ yields $\mathcal{C}(A^k) = \mathcal{C}(G)$.
(7) $\Longrightarrow$ (2). Notice that $GAA^D = A^D$ is equivalent to $GA^{k+1} = A^k$ which gives $\mathcal{C}(A^k) \subseteq \mathcal{C}(G)$. Now by $rank(G) = rank(A^k)$ we get $\mathcal{C}(A^k) = \mathcal{C}(G)$.
(8) $\Longleftrightarrow$ (6). This equivalent condition is followed by the equivalence of $GAA^D = A^D$ and $GA^{k+1} = A^k$. ■

Theorem 7. *Let $A, G \in \mathbb{C}^{n\times n}$ and $ind(A) = k$. Assume $rank(AA^S) = rank(A^S A) = rank(A)$. Then the following conditions are equivalent:*

1) $G = A^{D,\dagger s}$;
2) $AG = A^2 A^D A^{\dagger s}, \mathcal{C}(G) \subseteq \mathcal{C}(A^k)$;
3) $AGA = AA^D A, \mathcal{C}(G) \subseteq \mathcal{C}(A^k), \mathcal{N}(A^k A^{\dagger s}) \subseteq \mathcal{N}(G)$;
4) $GA = AA^D, \mathcal{N}(A^k A^{\dagger s}) \subseteq \mathcal{N}(G)$;
5) $AG^2 = G, A^k G = A^k A^{\dagger s}$;
6) $AG^2 = G, AG = P_{\mathcal{C}(A^k),\mathcal{N}(A^k A^{\dagger s})}$;
7) $AG^2 = G, AG = A^2 A^D A^{\dagger s}$

Proof: (1) implies (2-7) can be proved directly by definition (5). We have to prove other implications.
(2) $\Longrightarrow$ (1). Since $AG = A^{D,\dagger s} C_A A^{\dagger s} = A^2 A^D A^{\dagger s}$, we have $A^k G = A^k A^{\dagger s}$. Also by $\mathcal{C}(G) \subseteq \mathcal{C}(A^k)$ and $A^k G = A^k A^{\dagger s}$, we get $\mathcal{C}(G) = \mathcal{C}(A^k)$. Hence, by theorem (6), it follows that $G = A^{D,\dagger s}$.
(3) $\Longrightarrow$ (1). Since $\mathcal{C}(G) \subseteq \mathcal{C}(A^k)$, it follows that $A^D AG = G$. Multiplying $AGA = AA^D A$ from the left side by A^D we get $GA = A^D A$. Since $\mathcal{C}(I - AA^{\dagger s}) \subseteq \mathcal{N}(A^k A^{\dagger s}) \subseteq \mathcal{N}(G)$ we have that $G = GAA^{\dagger s}$. Now,

$$A^k G = A^k GAA^{\dagger s} = A^k A^D AA^{\dagger s} = A^k A^{\dagger s}$$

Proof of (4) $\Longrightarrow$ (1) follows in the similar lines as that of (3) $\Longrightarrow$ (1).
(5) $\Longrightarrow$ (1). Since $AG^2 = G$ implies $G = A^k G^{k+1}$, $\mathcal{C}(G) \subseteq \mathcal{C}(A^k)$. Furthermore, by $A^k G = A^k A^{\dagger s}$, it is easy to check $\mathcal{C}(G) = \mathcal{C}(A^k)$. Hence, $G = A^{D,\dagger s}$ by (2) $\Longrightarrow$ (1)

of theorem 6.
(6) $\implies$ (1). As in the part (5) $\implies$ (1), and from $AG^2 = G$ we get $A^DAG = G$. Since $\mathcal{C}(I - AG) = \mathcal{N}(A^kA^{\dagger s})$, it follows that

$$A^kA^{\dagger s} = A^kA^{\dagger s}AG = A^kG.$$

(7) $\implies$ (5). Since $AG = A^2A^DA^{\dagger s}$, we have $A^kG = A^kA^{\dagger s}$. ■

Theorem 8. *Consider a matrix $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ Assume $rank(A^SAA^S) = rank(A)$. Then,*

1) $A^{D,\dagger s} = A^DP_{\mathcal{R}(A),\mathcal{N}(A^S)}$;
2)

$$(A^{D,\dagger s})^l = \begin{cases} (A^DA^{\dagger s})^{\frac{l}{2}} & if \;\; l \;\; is\ even, \\ A(A^DA^{\dagger s})^{\frac{l+1}{2}} & if \;\; l \;\; is\ odd \end{cases}$$

3) $(A^{D,\dagger s}) = (A^2A^{\dagger s})^D$;
4) $((A^{D,\dagger s})^D)^D = A^{D,\dagger s}$;
5) $AA^{D,\dagger s} = A^{D,\dagger s}A$ *if and only if* $A^{\dagger s} = A^D$ *if and only if* $\mathcal{N}(A^S) \subseteq \mathcal{N}(A^k)$;
6) $A^{D,\dagger s} = 0$ *if and only if A is nilpotent.*

Proof: (1) It is clear by definition (5) and $(2)^{nd}$ part of lemma (1).
(2) From definition (5) we have

$$\begin{aligned}(A^{D,\dagger s})^2 &= A^DAA^{\dagger s}A^DAA^{\dagger s} \\ &= A^DAA^{\dagger s}AA^DA^{\dagger s} = A^DA^{\dagger s}\end{aligned}$$

Then, for an even number l, it follows that

$$(A^{D,\dagger s})^l = ((A^{D,\dagger s})^2)^{\frac{l}{2}} = (A^DA^{\dagger s})^{\frac{l}{2}} \tag{3}$$

Moreover, if l is odd, then from Equation (3) and definition (5), we get that

$$\begin{aligned}(A^{D,\dagger s})^l &= A^{D,\dagger s}(A^{D\dagger s})^{l-1} \\ &= A^{D,\dagger s}(A^DA^{\dagger s})^{\frac{l-1}{2}} \\ &= AA^DA^{\dagger s}(A^DA^{\dagger s})^{\frac{l-1}{2}} \\ &= A(A^DA^{\dagger s})^{\frac{l+1}{2}}\end{aligned}$$

(3) Using Cline's formula, [16], $(XY)^D = X((YX)^D)^2Y$ for $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{n\times m}$, from definition 5,

$$\begin{aligned}(A^2A^{\dagger s})^D &= (A(AA^{\dagger s}))^D = A((AA^{\dagger s}A)^D)^2AA^{\dagger s} \\ &= A(A^D)^2AA^{\dagger s} = A^DAA^{\dagger s} = A^{D,\dagger s}\end{aligned}$$

(4). Again, using Cline's formula, from definition (5) and $(2)^{nd}$ part of lemma (1) and $(1)^{st}$ part of lemma (2) we have

$$\begin{aligned}(A^{D,\dagger s})^D &= (A^D(AA^{\dagger s}))^D = A^D((AA^{\dagger s}A^D)^D)^2AA^{\dagger s} \\ &= A^D((A^D)^D)^2AA^{\dagger s} = (A^D)^DAA^{\dagger s} \\ &= (A^D)^{\#}AA^{\dagger s}\end{aligned}$$

Again by Cline's formula,

$$\begin{aligned}((A^{D,\dagger s})^D)^D &= ((A^D)^{\#}AA^{\dagger s})^D \\ &= (A^D)^{\#}((AA^{\dagger s}(A^D)^{\#})^D)^2AA^{\dagger s} \\ &= (A^D)^{\#}(((A^D)^{\#})^{\#})^2AA^{\dagger s} = A^DAA^{\dagger s}\end{aligned}$$

(5) According to definition 5 and (2) of lemma (1) and (2) of lemma (2) we see that

$$\begin{aligned}AA^{D,\dagger s} &= A^{D,\dagger s}A \\ &\iff AA^D(AA^{\dagger s} - I_n) = 0 \\ &\iff \mathcal{N}(A^S) \subseteq \mathcal{N}(A^k) \\ &\iff A^D(AA^{\dagger s} - I_n) = 0 \\ &\iff A^{D,\dagger s} = A^D.\end{aligned}$$

■

Zuo et al. in [17] gave an interesting result of the DMP inverse, that is, for $A \in \mathbb{C}_k^{n\times n}$,

$$A^{D,\dagger} = AA^{\dagger}(I_n - \overline{A}AA^{\dagger})^D = (I_n - \overline{A}AA^{\dagger})^DAA^{\dagger}$$

where $\overline{A} = I_n - A$.
The following theorem turns out analogous expressions of the D-sg inverse.

Theorem 9. *Let $A \in \mathbb{C}_k^{n\times n}$ with $rank(A^SAA^S) = rank(A)$, and let $\overline{A} = I_n - A$. Then,*

$$\begin{aligned}A^{D,\dagger s} &= AA^{\dagger s}(I_n - \overline{A}AA^{\dagger s})^D && (4) \\ &= (I_n - \overline{A}AA^{\dagger s})^DAA^{\dagger s} && (5)\end{aligned}$$

Proof: Using corollary 1 of [18], i.e., $(X + Y)^D = X^D + Y^D$, where $X, Y \in \mathbb{C}^{n\times n}$ satisfies $XY = YX = 0$ and a clear fact

$$(I_n - AA^{\dagger s})A^2A^{\dagger s} = A^2A^{\dagger s}(I_n - AA^{\dagger s}) = 0,$$

we can directly have

$$\begin{aligned}(I_n - AA^{\dagger s} + A^2A^{\dagger s})^D &= (I_n - AA^{\dagger s})^D + (A^2A^{\dagger s})^D \\ &= I_n - AA^{\dagger s} + (A^2A^{\dagger s})^D\end{aligned}$$

Hence, it follows from theorem 8, theorem 3 and lemma 1 that

$$\begin{aligned}&AA^{\dagger s}(I_n - \overline{A}AA^{\dagger s})^D \\ &= AA^{\dagger s}(I_n - AA^{\dagger s} + A^2A^{\dagger s})^D \\ &= AA^{\dagger s}(I_n - AA^{\dagger s}) + AA^{\dagger s}(A^2A^{\dagger s})^D \\ &= AA^{\dagger s}A^{D,\dagger s} = A^{D,\dagger s}\end{aligned}$$

and

$$\begin{aligned}&(I_n - \overline{A}AA^{\dagger s})^DAA^{\dagger s} \\ &= (I_n - AA^{\dagger s} + A^2A^{\dagger s})^DAA^{\dagger s} \\ &= (I_n - AA^{\dagger s})AA^{\dagger s} + (A^2A^{\dagger s})^DAA^{\dagger s} \\ &= A^{D,\dagger s}AA^{\dagger s} = A^{D,\dagger s}\end{aligned}$$

which shows that equation (4) and equation (5) are true.

Theorem 10. *Let $A \in \mathbb{C}^{n\times n}$ such that $ind(A) = k$. Assume $rank(AA^S) = rank(A^SA) = rank(A) = r$. Let U be a matrix such that $AU = UA$ and $A^{k+1}X = A^k$. Let $V \in A\{1\}$. Then the following conditions are equivalent:*

1) $A^{D,\dagger s} = UAV$

2) $UA = A^D A, A^k V = A^k A^{\dagger s}$.
3) $U = A^D + X(I_n - P_A), V = A^{\dagger s} + (I_n - Q_{A^k})Y$ *for arbitrary* $X, Y \in \mathbb{C}^{n\times n}$.

Proof: (1) $\implies$ (2) Since $A^{D,\dagger s} = UAV$,

$$A^{D,\dagger s}A = UAVA = UA.$$

Now, $UA = A^{D,\dagger s}A = A^D AA^{\dagger s}A = A^D A$.
On the other hand, since $A^{D,\dagger s} = UAV$,

$$A^k A^{D,\dagger s} = A^k UAV = A^{k+1}UV = A^k V.$$

By the definition of Drazin inverse, we have
$A^k V = A^k A^{D,\dagger s} = A^k A^D AA^{\dagger s} = A^k A^{\dagger s}$.
(2) $\implies$ (3) Clearly, A^D satisfies the equation $A^D A = UA$. By applying 3 the general solution of $A^D A = UA$ is given by $U = A^D + X(I_n - P_A)$, for arbitrary $X \in \mathbb{C}^{n\times n}$. By repeated application of lemma 3, we obtain the general solution of $A^k A^{\dagger s} = A^k V$ is given by
$V = A^{\dagger s} + (I_n - Q_{A^k})Y$ for arbitrary $Y \in \mathbb{C}^{n\times n}$.
(3) $\implies$ (1) Assume that $U = A^D + X(I_n - P_A)$ and $V = A^{\dagger s} + (I_n - Q_{A^k})Y$ for arbitrary $X, Y \in \mathbb{C}^{n\times n}$.
Now,

$$\begin{aligned} UAV &= A^D + X(I_n - P_A)AA^{\dagger s} + (I_n - Q_{A^k})Y \\ &= A^D A(A^{\dagger s} + (I_n - Q_{A^k})Y) \\ &= A^{D,\dagger s} \end{aligned}$$

■

Remark 2. *These characterizations obtained for Drazin secondary generalized inverse can also be explored for secondary generalized Drazin inverses.*

■

A. An application of D-sg inverse in solving the system of linear equations

Theorem 11. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $ind(A) = k$ *and* $rank(A^S AA^S) = rank(A)$. *Let* $y \in \mathbb{C}^n$ *and let the system of linear equations be*

$$A^k x = A^k A^{\dagger s} y \tag{6}$$

Then the general solution of the system (6) is

$$x = A^{D,\dagger s}y + (I_n - A^{D,\dagger s}A)z, \tag{7}$$

where $z \in \mathbb{C}^n$ *is arbitrary. Moreover,*

$$x = A^{D,\dagger s}y$$

is the unique solution to the system (6) on $\mathcal{C}(A^k)$.

Proof: By definition 5, clearly $A^{D,\dagger s}$ is a solution to equation (6). Hence, by using theorem 3, we have the set of solutions of (6) given by

$$\begin{aligned} &\{A^{D,\dagger s}y + b \mid b \in \mathcal{N}(A^k)\} \\ &= \{A^{D,\dagger s} + b \mid b \in \mathcal{C}(I_n - A^{D,\dagger s}A)\} \end{aligned}$$

which shows that the general solution of (6) is (7). Moreover, since $\mathcal{C}(A^k) \oplus \mathcal{N}(A^k) = \mathcal{C}^n$, by using (2) of theorem (3) we can see that $A^{D,\dagger s} \in \mathcal{C}(A^k)$ is a unique solution to (6) on $\mathcal{C}(A^k)$. ■

IV. Conclusion

In conclusion, this article has introduced and characterized a new outer inverse, the Drazin secondary generalized inverse. We have explored the application of the Drazin secondary generalized inverse (D-sg inverse) in solving systems of linear equations, demonstrating its utility in dealing with singular matrices. Furthermore, we propose that the D-sg inverse can be extended to rectangular matrices, thus broadening its applicability. Additionally, the results concerning the D-sg inverse can be further generalized to Hilbert and Banach spaces, opening new avenues for research and application in functional analysis.

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley, New York, 1974.
[2] O. M. Baksalary and G. Trenkler, ”Core inverse of matrices,“ *Linear and Multilinear Algebra*, vol. 58, no. 6, pp. 681–697, 2009.
[3] K. M. Prasad, and K. S. Mohana, ”Core–EP inverse,“ *Linear and Multilinear Algebra*, vol. 62, no. 6, pp. 792–802, 2013.
[4] K. M. Prasad, N. Nandini, D. P. Shenoy, “Rank and dimension functions,”*Electronic Journal of Linear Algebra*, vol. 29, no. 1, Article No. 11, pp. 144-155, 2015.
[5] V. Savitha, D. P. Shenoy, K. Umashankar and R. B. Bapat, ”Secondary transpose of a matrix and generalized inverses,“ *Journal of Algebra and its applications*, vol. 23, no. 3, Article No. 2450052, 2024.
[6] S. B. Malik, N. Thome, ”On a new generalized inverse for matrices of an arbitrary index,“ *Appl. Math. Comput.*, vol. 226, pp. 575–580, 2014.
[7] M. Mehdipour, A. Salemi, ”On a new generalized inverse of matrices,“ *Linear and Multilinear algebra*, vol. 66, no. 5, pp. 1046–1053, 2018.
[8] D. Mosić, ”Drazin-Star and Star-Drazin matrices,“ *Results Math.*, vol. 75, pp. 1-21, 2020.
[9] D. P. Shenoy, ”Drazin theta and theta Drazin matrices,“ *Numerical Algebra, Control and Optimization*, vol. 14, no. 2, pp. 273-283, 2024.
[10] D. P. Shenoy, ”Outer theta and theta outer inverses,“ *IAENG International Journal of Applied Mathematics*, 52(4), pp. 1020-1024, 2022.
[11] J. K. Sahoo, G. Maharana, B. Sitha, and N. Thome, ”1D inverse and D1 inverse of square matrices“ vol. 25, no. 1, pp. 445–463, 2024.
[12] J. Gao, Q. Wang and K. Zuo, ”The m-DMP inverse in Minkowski space and its applications,“ https://arxiv.org/abs/2303.13944v1, https://doi.org/10.48550/arXiv.2303.13944.
[13] J. L. Chen, D. Mosić, S. Z. Xu, ”On a new generalized inverse for Hilbert space operators,“ *Quaest. Math.*, vol.43,pp. 1331–1348, 2020.
[14] C. Wang, X. Liu, H. Jin, ”The MP weak group inverse and its application,“ *Filomat*, vol. 36, no. 18, pp. 6085–6102, 2022.
[15] A. Lee, ”Secondary symmetric, skew symmetric and orthogonal matrices,“ *Period. Math. Hung.*, vol. 7, pp. 63-70, 1976.
[16] R. E. Cline, ”An application of representation for the generalized inverse of a matrix,“ *MRC Technical Report*, 592, 1965.
[17] K. Zuo, D. Cvetkovi ́c-Ili ́c, Y. Cheng, ”Different characterizations of DMP-inverse of matrices,“ *Linear and Multilinear Algebra*, vol. 70, no. 3, pp. 411–418, 2022.
[18] M. P. Drazin, ”Pseudo-inverses in associative rings and semigroups,“ *Amer. Math. Monthly*, vol. 65, no. 7, pp. 506–514, 1958.

Divya Shenoy Purushothama is an Assistant Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India - 576104. She obtained Ph.D from Manipal Academy of Higher Education in the year 2017. Her research fields are generalized inverse of matrices, Drazin inverse, matrix partial orders and secondary generalized inverse.