Improvement of the Accuracy to Solve Fractional Riccati Differential Equations Via Bernstein Wavelets

Yiping Meng, Rui Jin

Abstract—**Our aim of the article is to report an efficient numerical method to enhance the accuracy in solving fractional Riccati differential equations. The main idea is to change the original differential equation into a simple algebraic one, by employing the operational matrix of Bernstein wavelets merged with the collocation method. The details of our method are summarized in this paper. The numerical examples are devoted to make known that the method can obtain numerical solutions effectively and accurately. Compared with the existing results, the method proposed in the article can reduce errors and improve accuracy.**

Index Terms—**Bernstein wavelets, Riccati differential equations, Collocation method, Absolute error.**

I. Introduction

THE Riccati differential equation (RDE), named after the renowned Italian mathematician Count Jacopo Francesco Riccati [1], plays a pivotal role in numerous applied mathematical fields, including solitary wave theory, dynamic games, stochastic processes, and differential equations, among others(see to, for example, [2], [3] and the references cited herein).

In general, a quadratic RDE has the following form [4]

$$y'(t) = a(t) + r(t)y(t) + k(t)y^2(t), \tag{1}$$

where $k(t) \neq 0$.

Fractional Riccati differential equations (FRDEs) can be derived by substituting differential operators with fraction-order differential operators in equation (1). Owing to that the fractional-order derivatives can describe memory characteristics of various mathematical processes, many mathematical models built by FRDEs seem to be more reasonable, thus much attention of FRDEs has been attracted by many mathematicians [5], [6] in recent years.

In this paper, the following fractional Riccati differential equation (FRDE) was considered [7]:

$$\begin{cases} D^{\beta}y(t) = a(t) + r(t)y(t) + k(t)y^2(t), \quad 0 < \beta \leq 1, \\ y(0) = \lambda, \end{cases} \tag{2}$$

with the functions $a(t)$, $r(t)$, $k(t)$ are defined over $[0, 1]$ and λ is a fixed constant.

Manuscript received August 26, 2024; revised December 8, 2024. The first author is supported by National Natural Science Foundation of China (No.11901249).

Yiping Meng is an associate professor of School of Science, Jiangsu University of Science and Technology, Zhenjiang 212003, P. R. China (corresponding author: phone:+8613852941090, email: ypmeng@just.edu.cn).

Rui Jin is a postgraduate student of mathematics, Jiangsu University, Zhenjiang 212013, P. R. China (email: 2564271391@qq.com).

Recently, there are many works focused on the numerical methods for solving FRDEs. Saker [8] introduced the iterative reproducing kernel Hilbert spaces method (IRKHSM) for solving FRDEs. Taiwo and Osilagun [9] derived numerical solutions for FRDEs using an iterative decomposition algorithm. Jin [7] used the generalized Bell collocation method (GBCM) to obtain more accurate numerical solutions for FRDEs.

The high-precision numerical solutions has consistently been the focus of researchers' efforts. To this end, wavelets are employed as the foundation for the collocation method in numerical computations. As stated in [3], the primary advantage is that wavelets can accurately approximate functions with discontinuities and sharp peaks. There are numerous papers that demonstrate the increment of accuracy for applying wavelets in solving various types of equations. These include the application of wavelet least squares techniques for boundary value problems [10], the deployment of wavelet collocation methods for optimal control problems [11], the fractional-order Boubaker wavelets approach for fractional differential equations [12], etc.

Motivated by the aforementioned considerations, our objective is to employ Bernstein wavelets as the basis for improving the accuracy of solving FRDEs. This entails demonstrating a highly accurate numerical method for solving FRDEs. By following the prescribed steps of this method, (2) can be converted into straightforward nonlinear algebraic equations. Once solutions to these nonlinear algebraic equations have been identified, the numerical solutions can be readily expressed. In comparison with existing results, our method has the potential to reduce errors and improve accuracy.

The other parts of the article are arranged as follows: A variety of necessary preparations are given in Section II. Section III summarizes the specific processes of using wavelets to solve equation (2). In Section IV, some numerical experiments are conducted to verify the conclusions. A brief conclusion is provided in the final section.

II. Preliminary

In this section, we present some preliminaries that will be used further in the introduction to our method. Firstly, we collect some classical definitions of fractional calculus.

Definition 1. ([7]) Define the Riemann-Liouville fractional integral operator by:

$$J^{\beta}f(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1}f(s)ds, & \beta > 0, \\ f(t), & \beta = 0. \end{cases}$$

Definition 2. ([7]) Define the Caputo's fractional derivative by:

$$D^{\beta}f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\beta)}\displaystyle\int_0^t(t-s)^{n-\beta-1}\frac{d^n}{ds^n}f(s)ds,\\ n-1<\beta<n,\\ f^{(n)}(t), \qquad\qquad \beta=n.\end{cases}$$

Next, we give the definitions of Bernstein polynomials and Bernstein wavelets.

Definition 3. ([13]) Bernstein polynomials of degree m are defined on $[0,1]$ as follows

$$B_{m,M}(t)=\binom{M}{m}t^m(1-t)^{M-m}. \tag{3}$$

where $m=0,1,\cdots,M$ and M is a any positive integer.

Moreover, the orthonormal Bernstein polynomials of degree m on $[0,1]$ are

$$\begin{aligned}\bar{B}_{m,M}(t)=&\sqrt{2(M-m)+1}(1-t)^{M-m}\\ &\times\sum_{k=0}^{m}(-1)^k\binom{2M+1-k}{m-k}\binom{m}{k}t^{m-k}.\end{aligned} \tag{4}$$

By Definition 3, the orthonormal Bernstein polynomials are orthonormal in the following sense

$$\int_0^1\bar{B}_{i,M}(t)\bar{B}_{j,M}(t)dt=\delta_{i,j},\ i,j=0,1,\cdots,M, \tag{5}$$

where $\delta_{i,j}$ denotes the Kronecker function.

Then, we introduce the orthonormal Bernstein wavelets defined on $[0,1]$.

Definition 4. ([13]) The orthonormal Bernstein wavelets $\psi_{nm}(t)=\psi(k,n,m,t)$ are defined on $[0,1]$ by

$$\psi_{nm}(t)=\begin{cases}2^{\frac{k}{2}}\bar{B}_{m,M}(2^kt-n), \dfrac{n}{2^k}\le t<\dfrac{n+1}{2^k},\\ 0, \qquad\qquad \text{otherwise},\end{cases}$$

where $n=0,1,\cdots,2^k-1$, $m=0,1,\cdots,M$. Here m denotes the degree of the polynomials.

According to [13], we introduce the unit step function as follows

$$\mu_\eta(t)=\begin{cases}1, t\ge\eta,\\ 0, t<\eta,\end{cases}$$

then we can rephrase the orthonormal Bernstein wavelets as

$$\begin{aligned}\psi_{nm}(t)=&\mu_{\frac{n}{2^k}}(t)2^{\frac{k}{2}}\bar{B}_{m,M}(2^kt-n)\\ &-\mu_{\frac{n+1}{2^k}}(t)2^{\frac{k}{2}}\bar{B}_{m,M}(2^kt-n).\end{aligned} \tag{6}$$

For any function $f(t)$ belonging to $L^2[0,1]$, we can expand it in terms of the basis ψ_{nm} as follows

$$f(t)=\sum_{n=0}^{\infty}\sum_{m\in\mathbf{Z}}g_{nm}\psi_{nm}(t), \tag{7}$$

where

$$g_{nm}=(f,\psi_{nm})=\int_0^1f(t)\psi_{nm}(t)dt.$$

By truncating the infinity series of (7), we can approximate $f(t)$ by

$$f(t)\approx\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}g_{nm}\psi_{nm}(t)=G^T\Psi(t), \tag{8}$$

where G and $\Psi(t)$ denote the $2^k(M+1)$ vectors,

$$\begin{aligned}G&=[G_0,G_1,\cdots,G_{(2^k-1)}]^T,\\ G_i&=[g_{i0},g_{i1},\cdots,g_{iM}],\end{aligned} \tag{9}$$

and

$$\begin{aligned}\Psi(t)&=[\Psi_0(t),\Psi_1(t),\cdots,\Psi_{(2^k-1)}(t)]^T,\\ \Psi_i(t)&=[\psi_{i0}(t),\psi_{i1}(t),\cdots,\psi_{iM}(t)].\end{aligned} \tag{10}$$

We state the operational matrix for Bernstein wavelet which is also used in [13]. Let

$$P(t,\beta)=J^{\beta}(\Psi(t)). \tag{11}$$

It is obvious $P(t,\beta)$ is the $2^k(M+1)$ column vector. We have

$$\begin{aligned}P(t,\beta)=&[J^{\beta}\psi_{00}(t),J^{\beta}\psi_{01}(t),\cdots,J^{\beta}\psi_{0M}(t),J^{\beta}\psi_{10}(t),\cdots,\\ &J^{\beta}\psi_{1M}(t),\cdots,J^{\beta}\psi_{(2^k-1)0}(t),\cdots,J^{\beta}\psi_{(2^k-1)M}(t)]^T,\end{aligned}$$

where

$$J^{\beta}\psi_{nm}(t)=\begin{cases}0, & 0\le t<\frac{n}{2^k},\\ 2^{\frac{k}{2}}\varsigma(m,M)(t-\dfrac{n}{2^k})^{\beta}, & \frac{n}{2^k}\le t<\frac{n+1}{2^k},\\ 2^{\frac{k}{2}}\varsigma(m,M)(t-\dfrac{n}{2^k})^{\beta}-2^{\frac{k}{2}}\bar{\varsigma}(m,M)\\ \times(t-\frac{n+1}{2^k})^{\beta} & \frac{n+1}{2^k}\le t<1,\end{cases}$$

and

$$\begin{aligned}\varsigma(m,M)=&\sqrt{2(M-m)+1}\sum_{i=0}^{m}(-1)^i\frac{\binom{2M+1-i}{m-i}\binom{m}{i}}{\binom{M-i}{m-i}}\\ &\times\sum_{j=m-i}^{M}(-1)^{j-M+i}\binom{M-i}{m-i}\binom{M-m}{j-m+i}\\ &\times 2^{jk}(t-\frac{n}{2^k})^j\frac{\Gamma(j+1)}{\Gamma(\beta+j+1)},\end{aligned}$$

$$\begin{aligned}\bar{\varsigma}(m,M)=&\sqrt{2(M-m)+1}\sum_{i=0}^{m}(-1)^i\frac{\binom{2M+1-i}{m-i}\binom{m}{i}}{\binom{M-i}{m-i}}\\ &\times\sum_{j=M-m}^{M}(-1)^{2j-M+m}\binom{M-i}{M-m}\binom{m-i}{j-M+m}\\ &\times 2^{jk}(t-\frac{n+1}{2^k})^j\frac{\Gamma(j+1)}{\Gamma(\beta+j+1)}.\end{aligned}$$

For example, if we choose $k=1$, $M=3$, $\beta=\frac{1}{2}$ and the collocation points $t_i=\frac{2i-1}{2^{k+1}(M+1)}$, we can first get $2^k(M+1)=8$ and then obtain the following operational matrix

$$\begin{pmatrix}0.8171 & 0.8243 & 0.6217 & 0.4713 & \cdots & 0.2895\\ 0 & 0 & 0 & 0 & \cdots & 0.4713\\ -0.3289 & 0.4359 & 0.8104 & 0.6343 & \cdots & 0.2783\\ 0 & 0 & 0 & 0 & \cdots & 0.6343\\ 0.1195 & -0.3366 & 0.1770 & 0.8417 & \cdots & 0.2419\\ 0 & 0 & 0 & 0 & \cdots & 0.8417\\ -0.0374 & 0.1512 & -0.1951 & 0.0990 & \cdots & 0.1506\\ 0 & 0 & 0 & 0 & \cdots & 0.0990\end{pmatrix}$$

III. Detailed steps of the method for FRDEs

This section outlines the detailed steps of the numerical method utilizing Bernstein wavelets operational matrices to solve Equation (2).

Step 1: $D^{\beta}y(t)$ is supposed to be approximated as

$$D^{\beta}y(t) \simeq \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{12}$$

where $c_{nm}, (n = 0, 1, \cdots, 2^k - 1, m = 0, 1, \cdots, M)$ are unknown.

Step 2: Taking advantage of (11) and (12), we get

$$J^{\beta}D^{\beta}y(t) \simeq J^{\beta}(C^T\Psi(t)) = C^T P(t, \beta).$$

Due to the relations between J^{α} and D^{α} together with the initial value condition of (2), we re-express $y(t)$ as

$$y(t) \simeq C^T P(t, \beta) + \lambda. \tag{13}$$

Step 3: By substituting (12)-(13) into the equation of (2), we have

$$C^T\Psi(t) = a(t)+r(t)(C^T P(t,\beta)+\lambda)+k(t)(C^T P(t,\beta)+\lambda)^2. \tag{14}$$

Step 4: We take the points $\frac{2i-1}{2^{k+1}(M+1)}$ as collocation points. After putting these nodes into (14), the following system of matrix equations can be gotten

$$\begin{aligned} C^T\Psi(t_j) &= a(t_j) + r(t_j)(C^T P(t_j, \beta) + \lambda) + k(t_j) \\ &\quad \times (C^T P(t_j, \beta) + \lambda)^2,\ j = 1, 2, \cdots, 2^k(M + 1). \end{aligned} \tag{15}$$

The unknown coefficient C^T can be acquired by solving the above nonlinear equations. One of the classical methods is the Newton iteration method. Thereafter, the approximate solution can be obtained by inserting C^T into (13).

IV. Numerical examples

In this part, we shall solve several important FREDs which have been solved in [14], [15], [12] and [7], by utilizing our method introduced in Section III. As a contrast, the solutions we obtained have higher accuracy..

Example 1. Firstly, consider the following FRDE [14]:

$$\begin{cases} D^{\beta}y(t) = 1 - y^2(t), \\ y(0) = 0. \end{cases} \tag{16}$$

By the result of [14], $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$ is the exact solution for (16) if $\beta = 1$.

Using the algorithm described in Section III, we obtain numerical solutions. In Figure 1, the numerical solutions are shown with different fractional order β. To provide a clear impression of the error, Figure 2 shows the absolute error graph when $\beta = 1$. We carried out this numerical experiment by choosing different values of k and M. Table I gives the absolute errors for different values of k and M with the special case of $\beta = 1$. Furthermore, the absolute errors arising in the method are compared with the existing results from [9], [7], [8] in Table II.

Example 2. We consider the following FRDE, which is from [15]:

$$\begin{cases} D^{\beta}y(t) = (\frac{t^{\beta+1}}{\Gamma(\beta+2)})^2 + t - y(t)^2, \\ y(0) = 0. \end{cases} \tag{17}$$

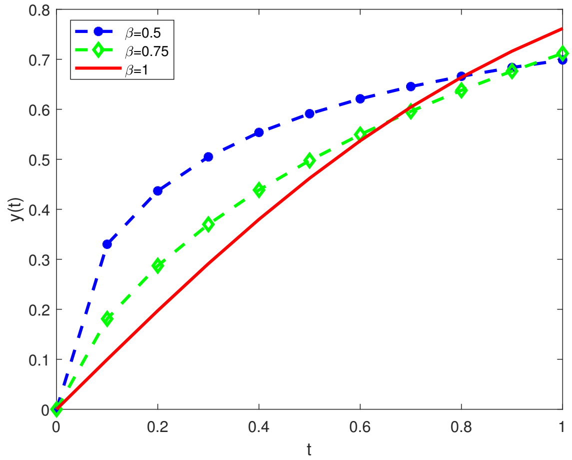

Fig. 1. Numerical solutions with different values of β in Example 1

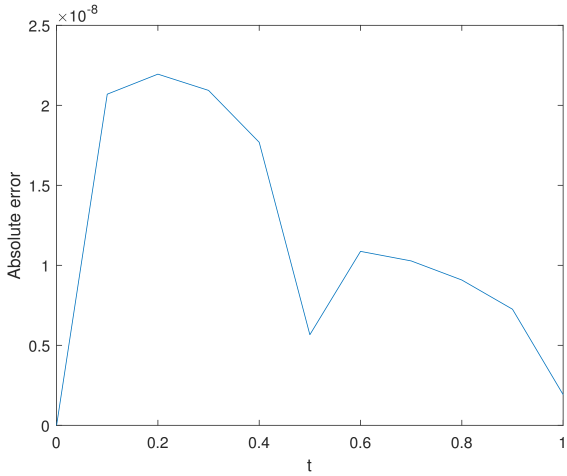

Fig. 2. Absolute error for Example 1

TABLE I
Absolute error at different parameters for Example 1

t	$k = 1, M = 5$	$k = 1, M = 6$	$k = 2, M = 4$
0.1	9.6743e-08	2.0696e-08	1.3335e-08
0.2	9.9965e-08	2.1949e-08	1.4507e-08
0.3	8.9148e-08	2.0932e-08	2.1650e-08
0.4	8.7907e-08	1.7689e-08	1.8073e-08
0.5	1.0842e-07	5.6627e-09	8.6511e-09
0.6	1.0712e-08	1.0872e-08	1.4801e-09
0.7	1.4968e-09	1.0275e-08	2.0432e-09
0.8	9.7704e-09	9.0817e-09	9.9654e-09
0.9	1.7352e-09	7.2565e-09	8.5933e-09
1.0	8.1716e-09	1.9171e-09	5.0215e-09

By the definition of fractional derivatives, it is not a difficult task to check that $y(t) = \frac{t^{\beta+1}}{\Gamma(\beta+2)}$ is the exact solution.

In general, we provide the numerical solutions for various values of β in Figure 3. The case of $\beta = 0.8$ is taken to consider the absolute error. The description of the absolute errors is shown in Figure 4. The comparison results of errors for $\beta = 0.8$ are also illustrated in Table III. To show the effectiveness of the presented method, we also give the results of errors by using various methods when $\beta = 0.5$

TABLE II
ABSOLUTE ERROR AT DIFFERENT METHODS FOR EXAMPLE 1

t	IDM[9]	IRKHSM[8]	GBCM[7]	Our method
0.1	1.00e-11	9.05e-06	2.51e-08	1.33e-08
0.2	0.00e-10	1.72e-05	4.64e-08	1.45e-08
0.3	2.50e-09	2.38e-05	5.60e-08	2.17e-08
0.4	5.61e-08	2.85e-05	1.08e-08	1.81e-08
0.5	6.03e-07	3.11e-05	1.46e-07	8.65e-09
0.6	4.09e-06	3.17e-05	2.42e-07	1.48e-09
0.7	2.01e-05	3.07e-05	1.17e-06	2.04e-09
0.8	7.78e-05	2.81e-05	1.57e-06	9.97e-09
0.9	2.50e-04	2.32e-05	1.16e-06	8.59e-09
1.0	6.99e-04	1.19e-05	1.04e-06	5.02e-09

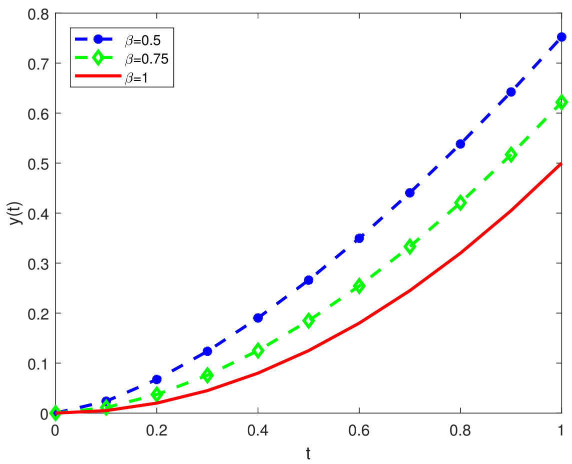

Fig. 3. Numerical solutions at different values of β for Example 2

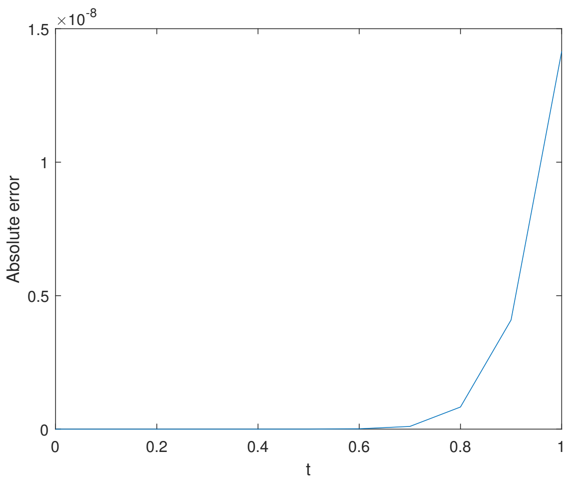

Fig. 4. Plot of absolute error with $\beta = 0.8$ for Example 2

in Table IV.

Example 3. Let us take the following FRDE as the last example [12]:

$$\begin{cases} D^{\beta}y(t) = -y(t)^2 + 2y(t) + 1, \\ y(0) = 0, \end{cases} \tag{18}$$

In the case of $\beta = 1$, the exact solution of (18) is expressed by

$$y(t) = 1 - \sqrt{2}\frac{\sqrt{2}\tanh(\sqrt{2}t) - 1}{\tanh(\sqrt{2}t) - \sqrt{2}}.$$

TABLE III
ABSOLUTE ERRORS AT DIFFERENT METHODS FOR EXAMPLE 2 WITH $\beta = 0.8$

t	WG[16]	FBW[17]	Our method
0.1	7.76e-06	2.00e-06	4.23e-16
0.2	2.75e-06	2.94e-06	8.05e-16
0.3	2.54e-06	2.86e-06	8.47e-16
0.4	1.30e-07	1.51e-06	1.92e-15
0.5	3.10e-08	3.97e-04	2.03e-14
0.6	1.32e-06	3.60e-04	1.24e-11
0.7	8.68e-07	3.24e-04	1.01e-10
0.8	4.04e-07	2.85e-04	8.28e-10
0.9	3.39e-06	2.40e-04	4.09e-09
1	5.19e-06	1.76e-04	1.42e-08

TABLE IV
ABSOLUTE ERRORS FOR EXAMPLE 2 WITH $\beta = 0.5$

t	GBCM [7]	FBW[17]	Our method
0.1	8.6736e-16	1.42121e-09	0
0.2	1.0131e-15	1.70884e-09	1.3014e-18
0.3	1.4794e-14	1.95085e-08	1.3878e-17
0.4	8.7930e-14	1.47103e-08	2.7756e-17
0.5	2.4547e-13	1.13654e-08	5.5511e-17
0.6	5.0115e-13	8.98023e-09	1.7375e-14
0.7	8.6287e-13	7.46806e-09	4.9905e-14
0.8	1.3342e-12	6.97788e-09	8.0380e-14
0.9	1.9169e-12	7.78465e-09	1.0436e-13
1	2.6106e-12	1.02211e-08	1.1580e-13

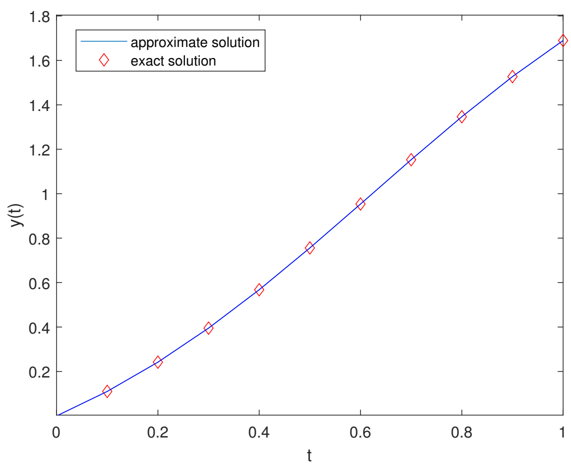

Fig. 5. Numerical solutions and exact solution for Example 3

We will continue to use graphs and tables to illustrate the numerical results. The closeness between the numerical and exact solutions is shown in Figures 5 and 6. For the purpose of showing the accuracy of the method, the global errors are given in Figure 7.

The absolute errors are stated with different k in Table V. Moreover, the errors generated by other methods are also displayed in Table VI.

V. CONCLUSIONS

The purpose of this article was to introduce an efficient numerical method using Bernstein wavelets to solve FRDEs.

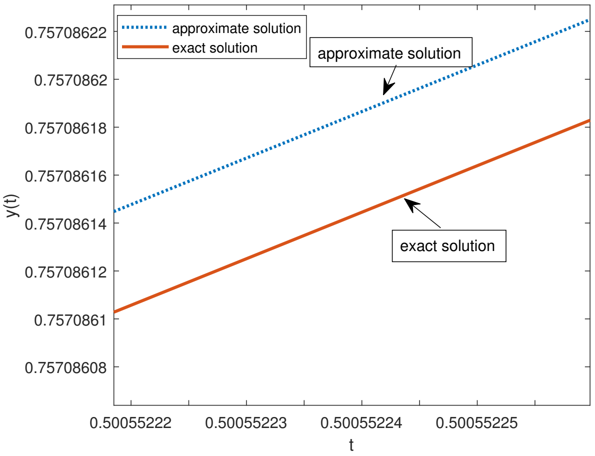

Fig. 6. An enlarged view of Figure 5 for Example 3

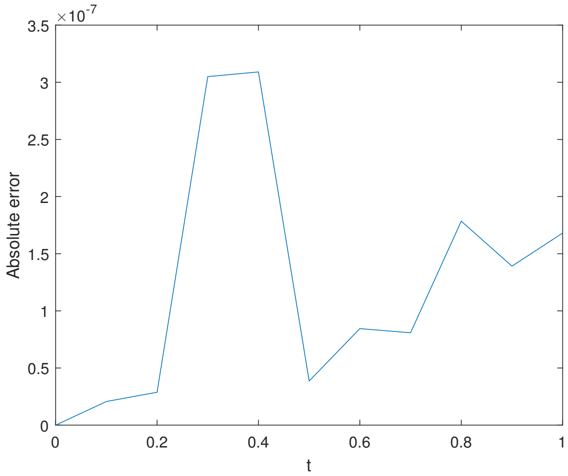

Fig. 7. Graph of absolute error at $\beta = 1$ for Example 3

TABLE V
COMPARISON OF ABSOLUTE ERRORS AT DIFFERENT PARAMETERS FOR EXAMPLE 3

t	$k = 1, M = 4$	$k = 2, M = 4$
0.1	1.2334e-05	2.0688e-08
0.2	1.3405e-05	2.8782e-08
0.3	1.5415e-05	3.0496e-07
0.4	1.8256e-05	3.0905e-07
0.5	4.2972e-06	3.8631e-08
0.6	6.9503e-06	8.4466e-08
0.7	5.9286e-06	8.0813e-08
0.8	5.4922e-06	1.7847e-07
0.9	6.6581e-06	1.3913e-07
1.0	1.3200e-05	1.6808e-07

The combination of the Bernstein wavelets operational matrix and the collocation method can convert the requested problem into nonlinear algebraic equations. In the numerical experiments, we show the effectiveness and accuracy of the method through graphs and tables. The numerical results indicate that a higher-accuracy solution can be obtained using the Bernstein wavelets method.

TABLE VI
THE ABSOLUTE ERRORS FOR EXAMPLE 3 WITH $\beta = 1$

t	[12]	[14]	[18]	[19]	Our method
0.2	1.55e-05	1.20e-05	2.90e-05	9.23e-05	2.88e-08
0.4	1.71e-05	3.03e-04	2.50e-03	7.35e-05	3.09e-07
0.5	1.93e-05	1.55e-03	4.40e-03	7.62e-05	3.86e-08
0.6	1.90e-05	4.69e-03	5.50e-03	7.56e-05	8.45e-08
0.8	1.92e-05	1.88e-02	3.80e-03	3.94e-05	1.78e-07
1.0	9.99e-06	3.43e-02	3.40e-03	7.12e-05	1.68e-07

REFERENCES

[1] W. T. Reid, *Riccati differential equations*, ser. Mathematics in Science and Engineering. Academic Press, New York-London, vol. 86, 1972.

[2] G. A. Einicke, L. B. White, and R. R. Bitmead, "The use of fake algebraic Riccati equations for co-channel demodulation," *IEEE Trans. Signal Process.*, vol. 51, no. 9, pp. 2288–2293, 2003.

[3] P. Rahimkhani and Y. Ordokhani, "Numerical investigation of distributed-order fractional optimal control problems via Bernstein wavelets," *Optimal Control Appl. Methods*, vol. 42, no. 1, pp. 355–373, 2021.

[4] B.-Q. Tang and X.-F. Li, "A new method for determining the solution of Riccati differential equations," *Appl. Math. Comput.*, vol. 194, no. 2, pp. 431–440, 2007.

[5] A. G. Atta, "Spectral collocation approach with shifted Chebyshev third-kind series approximation for nonlinear generalized fractional Riccati equation," *Int. J. Appl. Comput. Math.*, vol. 10, no. 2, pp. 59, 2024.

[6] B. Yuttanan, M. Razzaghi, and T. N. Vo, "Fractional-order generalized Legendre wavelets and their applications to fractional Riccati differential equations," *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 24, no. 1, pp. 57–69, 2023.

[7] R. Jin and L. Wang, "Generalized bell collocation method to solve fractional riccati differential equations," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 1, pp. 138–144, 2023.

[8] M. G. Sakar, "Iterative reproducing kernel Hilbert spaces method for Riccati differential equations," *J. Comput. Appl. Math.*, vol. 309, pp. 163–174, 2017.

[9] O. A. Taiwo and J. A. Osilagun, "Approximate solution of generalized Riccati differential equations by iterative decomposition algorithm," *Int. J. Eng. Innovative Tech.*, vol. 1, no. 2, pp. 53–56, 2012.

[10] W. Dahmen, A. Kunoth, and R. Schneider, "Wavelet least squares methods for boundary value problems," *SIAM J. Numer. Anal.*, vol. 39, no. 6, pp. 1985–2013, 2002.

[11] R. Dai and J. E. Cochran, Jr., "Wavelet collocation method for optimal control problems," *J. Optim. Theory Appl.*, vol. 143, no. 2, pp. 265–278, 2009.

[12] K. Rabiei and M. Razzaghi, "Fractional-order Boubaker wavelets method for solving fractional Riccati differential equations," *Appl. Numer. Math.*, vol. 168, pp. 221–234, 2021.

[13] S. Kumar, R. Kumar, C. Cattani, and B. Samet, "Chaotic behaviour of fractional predator-prey dynamical system," *Chaos Solitons Fractals*, vol. 135, pp. 109811, 2020.

[14] Z. Odibat and S. Momani, "Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order," *Chaos Solitons Fractals*, vol. 36, no. 1, pp. 167–174, 2008.

[15] A. H. Bhrawy, Y. A. Alhamed, D. Baleanu, and A. A. Al-Zahrani, "New spectral techniques for systems of fractional differential equations using fractional-order generalized Laguerre orthogonal functions," *Fract. Calc. Appl. Anal.*, vol. 17, no. 4, pp. 1137–1157, 2014.

[16] H. B. Jebreen and I. Dassios, "A biorthogonal hermite cubic spline galerkin method for solving fractional riccati equation," *Mathematics*, vol. 10, no. 9, pp. 1461, 2022.

[17] P. Rahimkhani, Y. Ordokhani, and E. Babolian, "Fractional-order Bernoulli wavelets and their applications," *Appl. Math. Model.*, vol. 40, no. 17-18, pp. 8087–8107, 2016.

[18] F. Mabood, A. I. M. Ismail, and I. Hashim, "Application of optimal homotopy asymptotic method for the approximate solution of Riccati equation," *Sains Malaysiana*, vol. 42, no. 6, pp. 863–867, 2013.

[19] M. G. Sakar, A. Akgül, and D. Baleanu, "On solutions of fractional Riccati differential equations," *Adv. Difference Equ.*, vol. 2017, pp. 39, 2017.