Fixed Point Theorems for (Beta-Psi one-Psi two) Contractive Condition on Partially E-Cone Metric Spaces

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***Abstract*—In this paper, we introduce fixed point theorems within the context of $(\beta-\psi_1-\psi_2)$ contractive conditions applied to partially E-cone metric spaces. This research work explores the existence and uniqueness of fixed points for mappings satisfying these specific contractive conditions, shedding light on the behavior of such mappings in the context of partially ordered cone metric spaces. Our findings contribute to the ongoing discourse on fixed point theory and its applications in metric spaces, while also identifying avenues for future research aimed at bridging existing gaps in knowledge and understanding within this domain.**

***Index Terms*—contraction mapping, fixed point theorems, continuous mapping, partially E-cone metric spaces, β-admissible.**

I. Introduction

Partially E-cone metric spaces are an interesting generalization of traditional metric space concept. The genesis of E-cone metric spaces involving using a specific cone in Banach space [46] to define the metric, structure that provide non-negative scalar which support the definition of distance in a reacher setting than real number alone [4], [5], [9], [17], [20], [28], [35], [39], [45]. Partially E-cone metric spaces represent an intriguing development in the field of metric geometry integrating the element of cone metric spaces with the partial ordering. Introduce as a generalization of metric spaces where distance is measure not in the usual set of non-negative real number but within a cone in the Banach space [46], these spaces offer a further extend in mathematical structure, the essence of partially E-cone metric spaces lies in their ability to incorporate and order relation into the framework of cone metric spaces. This innovative approach allows for more flexibility and broader application than traditional metric spaces. The idea of metric spaces is functional in mathematical analysis and topology, offering a structural way to define distance and continuity, offering new prospective particularly beneficial in areas like fixed point theory and theoretical economics where order relationship are crucial. For a foundational discussion on E-cone metric spaces, one can refer to the work of Kadalburg, Radenovic and Rakocevic who provide comprehensive studies in this areas, particularly in their paper "Revisiting cone metric spaces and fixed point theorem of contractive mapping" [13], [14], [29], [36], [47], and reference therein. In 2007, Huang and Zhang [40] presented the concept of cone metric space. Basile et al. [16] introduced the concept of semi-interior point in 2017 by embedding a non-solid cone and taking into account fixed point results in E-metric spaces. Mehmood et al. [13] and Huang [11] in 2019 obtained some fixed point theorem in the context of embedding cone in E-metric space. Partially E-cone metric spaces have recently been further explore to enhance their theoretical foundation and broader their applicative reach. These spaces are a sophisticated adaptation of metric spaces are a utilising a cone within a Banach space [46] coupled with a partial ordering to define distance. These framework not only enriches the classical notion of distance and conversion but also integrate on ordering structured that pivotal for mathematical model involving optimization and hierarchy sensitive processes. A recent comprehensive review an advancement in this field can be seen at the work of Aydi and Karapinar [10], where the author developed into fixed point theorem that are foundational for mathematical analysis and algorithm in such structured space. We defined the idea of partially E-cone metric spaces using (β-ψ_1-ψ_2)-contractive conditions in this paper.

Manuscript received May 3, 2024; revised October 11, 2024.

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II. Preliminaries

Definition 1. *[40] A vector space over the real numbers, called an ordered space E, with a partial order relation "$\preceq$" such that*

(i) $r \preceq t \Rightarrow r+s \preceq t+s, \quad \forall\, r,s,t \in E.$
(ii) $\forall\, \alpha \in \mathbb{R}^+$ *and* $\forall\, r \in E$ *with* $r \succeq 0_E, \alpha r \succeq 0_E.$

Furthermore, E is known as a normed ordered space if it has a norm of $\|.\|$.

Definition 2. *[11] Assume that E is a real normed space, 0_E is a zero element in E, and E^+ is a convex and non-empty closed subset of E. Then, E^+ is known as a positive cone. if it satisfies*

(i) $r \in E^+$ *and* $a \geq 0 \Rightarrow ar \in E^+$*;*
(ii) $r \in E^+$ *and* $-r \in E^+ \Rightarrow r = 0_E.$

Definition 3. *[40] Let E be a real normed space and E^+ a positive cone in E. We say $\preceq$ is a partial ordering relation on E if*

$$r, s \in E, r \preceq s \;\Leftrightarrow\; s - r \in E^+.$$

Clearly,

$$r \in E^+ \;\Leftrightarrow\; 0_E \preceq r.$$

Definition 4. *[13] Let E be a real normed space and E^+ a positive cone in E. Then E^+ is called:*

(i) *a solid cone if* $intE^{+} \neq \phi$;
(ii) *a normal cone if there exists an* $K > 0$ *such that*

$$0_E \preceq r \preceq s \text{ imply } \|r\| \leq K\|s\|, \text{ for all } r, s \in E.$$

The least positive number satisfying the above is called the normal constant of E^{+}.

Definition 5. *[42] The cone* E^{+} *is called regular if every increasing sequence which is bounded from above is convergent.*
That is, if $\{s_n\}_{n\geq 1}$ *is sequence such that*

$$s_1 \preceq s_2 \preceq ... \preceq s_n \preceq ... \preceq r, \text{ for some } r \in E^{+},$$

then there is $s \in E^{+}$ *such that*

$$\|s_n - s\| \to 0 \ (n \to \infty).$$

Similarly, the cone E^{+} *is called regular if and only if every decreasing sequence that has a lower bound is convergent. Regular cones are known to be normal cones.*

Definition 6. *[30] Let* $X \neq \phi$ *and consider an ordered space* E *over the real scalars. An* E*-valued function* $d^E : X \times X \to E$ *is an ordered* E*-metric in* X *such that* $\forall\ r, s$ *and* $t \in X$, *we have*

(i) $0_E \preceq d^E(r,s), d^E(r,s) = 0_E \ \Leftrightarrow \ r = s$;
(ii) $d^E(r,s) = d^E(s,r)$;
(iii) $d^E(r,s) \preceq d^E(r,t) + d^E(t,s)$.

Then the pair $d^E(X,d)$ *is called* E*-metric space.*

Example 7. *[34] Let* $E = \mathbb{R}^2,\ E^{+} = \{(r,s) \in E : r, s \geq 0\},\ X = \mathbb{R}$ *and* $d : X \times X \to E$ *be define by* $d^E(r,s) = (|r-s|, \alpha|r-s|)$, *where* $\alpha \geq 0$ *ia a constant. Then* (X, d^E) *is an* E*-cone metric space.*

Definition 8. *[1] Let* $X \neq \phi$ *and* E *be an ordered space over the real scalars ordered by its positive cone with the assumption that* $(E^{+})^{\oslash} \neq \phi$. *A partially* E*-cone metric on* X *is a function* $p^E : X \times X \to E^{+}$ *such that* $\forall\ r, s, t \in X$;

$(p_1):\ 0_E \preceq p^E(r,r) \preceq p^E(r,s)$.
$(p_2):\ p^E(r,r) = p^E(r,s)$ *if and only if* $r = s$
$(p_3):\ p^E(r,s) = p^E(s,r)$,
$(p_4):\ p^E(r,s) \preceq p^E(r,t) + p^E(r,s) - p^E(t,t)$.

A pair (X, p^E) *is called partially* E*-cone metric space where* $X \neq \phi$ *and* p^E *is a partially* E*-cone metric on the set* X.
Clearly, if $p^E(r,s) = 0_E \ \Rightarrow r = s$, *[from* (p_1) *and* (p_2)*].*
But if $r = s$, *then* $p^E(r,s)$ *may not be equal to* 0_E.

Definition 9. *[1] Let* (X, p^E) *be a partially* E*-cone metric and* E *be an ordered space with the assumption that* $(E^{+})^{\oslash} \neq \phi$. *Consider a sequence* $\{r_n\}$ *in* X *and* $r \in X$. *Then*

(i) *A sequence* $\{r_n\}$ *is said to be e-converges to* r *if for every* $0_E \lll e$, *there exist a natural number* c *such that*

$$p^E(r_n, r) \lll e,\ \forall\ n \geq c.$$

In this case, we write $\lim_{n\to\infty} r_n = r$ *or* $r_n \xrightarrow{e} r$.

(ii) *A sequence* $\{r_n\}$ *is said to be e-Cauchy sequence if for every* $0_E \lll e$, *there exists a natural number* c *such that*

$$p^E(r_n, r_m) \lll e,\ \forall\ n, m \geq c.$$

(iii) (X, p^E) *is e-complete if every e-Cauchy sequence is e-convergent.*

Definition 10. *[23] Consider a partial ordered set* $(X, \preceq)$ *and a mapping* $F : X \to X$. *Then* F *is called nondecreasing w.r.t.* $\preceq$ *if*

$$r, s \in X, \quad r \preceq s \quad \Rightarrow Fr \preceq Fs.$$

Definition 11. *[23] Consider a partially ordered set* $(X, \preceq)$, *then a sequence* $\{r_n\}$ *is called a nondecreasing w.r.t.* $\preceq$ *if* $r_n \preceq r_{n+1},\ \forall\ n \in \mathbb{N}$.

Lemma 12. *[24] A partial* E*-Cone metric space* (X, p^E) *with coefficient* $k > 1$ *and let us consider* $\{r_n\} \to r$ *and* $\{s_n\} \to s$. *Then*

$$\begin{aligned}
\frac{1}{k^2}p^E(r,s) - &\frac{1}{k}p^E(r,r) - p^E(s,s) \\
&\leq \liminf_{n\to\infty} p^E(r_n, s_n) \\
&\leq \limsup_{n\to\infty} p^E(r_n, s_n) \\
&\leq kp^E(r,r) + k^2p^E(s,s) + k^2p^E(r,s).
\end{aligned}$$

Definition 13. *[23] Let* $X \neq \phi$, *suppose* $F : X \to X$ *and* $\beta : X \times X \to [0,1)$ *are mappings. Then* F *is called* β*-admissible if for all* $r, s \in X$,

$$\beta(r,s) \geq 1 \Rightarrow \beta(Fr, Fs) \geq 1.$$

We also state that F *is* R_β*-admissible (or* L_β*-admissible) if* $r, s \in X$,

$$\beta(r,s) \geq 1 \Rightarrow \beta(r, Fs) \geq 1 \ \big(or\ \beta(Fr, s) \geq 1\big).$$

Definition 14. *[44] A function* $\psi : [0,\infty) \to [0,\infty)$ *such that the following properties are met,*

(i) $\psi(p)$ *is continuous and nondecreasing,*
(ii) $\psi(p) = 0 \ \Leftrightarrow \ p = 0$.

Then the function ψ *is called an altering distance function.*

III. Main Result

Two new ideas in ordered partially E-cone metric space and based fixed point findings are presented in this section.
Result-I Type-I $(\beta - \psi_1 - \psi_2)$-contractive mapping in ordered partially E-cone metric space is a new contractive mapping that we introduce to support our initial finding.

Definition 15. *Let* (X, p^E) *be an ordered partially* E*-cone metric space with the coefficient* $k \geq 1$. *A mapping* $F : X \to X$ *is said to be* $(\beta - \psi_1 - \psi_2)$*-contractive mapping of type-I, if there exist two altering distance functions* ψ_1, ψ_2 *and* $\beta : X \times X \to [0,\infty)$ *such that*

$$\begin{aligned}
\beta(r, Fr)\beta(s, Fs)\psi_1\big(ud^E(Fr, Fs)\big)& \\
\leq \psi_1\big(\Delta_u^F(r,s)\big) - \psi_2\big(\Delta_u^F(r,s)\big)&
\end{aligned} \tag{1}$$

for all comparable $r, s \in X$, *where*

$$\Delta_u^F(r,s) = \max\left\{\begin{array}{c} p^E(r,s), p^E(r,Fr), \\ p^E(s,Fs), \\ \frac{p^E(r,Fs)+p^E(s,Fr)}{4k}, \\ \frac{p^E(r,Fr)p^E(s,Fs)}{1+p^E(r,s)}, \\ \frac{p^E(r,Fr)p^E(s,Fs)}{1+p^E(Fr,Fs)} \end{array}\right\} \tag{2}$$

First result is as follows:

Theorem 16. *Let $(X, \preceq, p^E)$ be a p^E-complete ordered partially E-cone metric space with the coefficient $k \geq 1$. Let $F : X \to X$ be a $(\beta - \psi_1 - \psi_2)$-contractive mapping of type-I. Suppose that the following assertions hold:*

(i) *F is β-admissible and L_β-admissible (or R_β-admissible);*
(ii) *There exists $r_1 \in X$ such that $r_1 \preceq Fr_1$ and $\beta(r_1, Fr_1) \geq 1$;*
(iii) *F is continuous and nondecreasing, w.r.t. $\preceq$ and if $F^n r_1 \to t$ then $\beta(t,t) \geq 1$.*

Then F has a fixed point.

Proof: By supposition (2), a sequence $\{r_n\} \in X$ defined by $r_{n+1} = Fr_n,\ \forall\ n \geq 1$. We have $r_2 = Fr_1 \preceq Fr_2 = r_3$ since $r_1 \preceq Fr_1$ and F is nondecreasing. Also, $r_3 = Fr_2 \preceq Fr_3 = r_4$ since $r_2 \preceq Fr_2$ and F is nondecreasing. By induction, we get

$$r_1 \preceq r_2 \preceq r_3 \cdots \preceq r_n \preceq r_{n+1} \preceq \cdots$$

Then r is the fixed point of F with $r = r_n$, and if $r_n = r_{n+1}$ for some $n \in \mathbb{N}$, then this completes the proof. Hence, for some $n \in \mathbb{N}$, we may suppose $r_n \neq r_{n+1}$. Since F is β-admissible, we deduce

$$\begin{aligned} &\beta(r_1, Fr_1) = \beta(r_1, r_2) \geq 1 \\ \Rightarrow &\beta(Fr, Fr_2) = \beta(r_2, r_3) \geq 1. \end{aligned}$$

By induction on n we get

$$1 \leq \beta(r_n, r_{n+1}) \text{ and } 1 \leq \beta(r_{n+1}, r_{n+2}), \quad \forall n \in \mathbb{N} \tag{3}$$

Hence, by (1) $\forall\ n \in \mathbb{N}$ we get

$$\begin{aligned} \psi_1\big(p^E(r_{n+1}, r_{n+2})\big) &\leq \beta(r_n, Fr_{n+1})\beta(r_{n+1}, Fr_{n+2}) \\ &\qquad \psi_1\big(up^E(Fr_n, Fr_{n+1})\big) \\ &\leq \psi_1\big(\Delta_u^F(r_n, r_{n+1})\big) - \\ &\qquad \psi_2(\Delta_u^F(r_n, r_{n+1})\big) \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\Delta_u^p(r_n, r_{n+1}) \\ &= \max \left\{ \begin{array}{c} p^E(r_n, r_{n+1}), p^E(r_n, Fr_n), \\ p^E(r_{n+1}, Fr_{n+1}), \\ \frac{p^E(r_n, Fr_{n+1}) + p^E(r_{n+1}, Fr_n)}{4k}, \\ \frac{p^E(r_n, Fr_n)p^E(r_{n+1}, Fr_{n+1})}{1 + p^E(r_n, r_{n+1})}, \\ \frac{p^E(r_n, Fr_n)p^E(r_{n+1}, Fr_{n+1})}{1 + p^E(Fr_n, Fr_{n+1})} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2}), \\ \frac{p^E(r_n, r_{n+2}) + p^E(r_{n+1}, r_{n+1})}{4k}, \\ \frac{p^E(r_n, r_{n+1})p^E(r_{n+1}, r_{n+2})}{1 + p^E(r_n, r_{n+1})}, \\ \frac{p^E(r_n, r_{n+1})p^E(r_{n+1}, r_{n+2})}{1 + p^E(r_{n+1}, r_{n+2})} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2}), \\ \frac{\left\{ \begin{array}{l} up^E(r_n, r_{n+1}) \\ \quad + up^E(r_{n+1}, r_{n+2}) \\ \quad\quad + 2kp^E(r_{n+1}, r_{n+2}) \end{array} \right\}}{4k}, \\ \frac{p^E(r_n, r_{n+1})p^E(r_{n+1}, r_{n+2})}{1 + p^E(r_n, r_{n+1})}, \\ \frac{p^E(r_n, r_{n+1})p^E(r_{n+1}, r_{n+2})}{1 + p^E(r_{n+1}, r_{n+2})} \end{array} \right\} \\ &< \max \left\{ p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2}) \right\} \end{aligned} \tag{5}$$

From (4) and (5) we get

$$\begin{aligned} &\psi_1\big(p^E(r_{n+1}, r_{n+2})\big) \\ &\quad \leq \psi_1\left(\max\left\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\right\}\right) \\ &\quad - \psi_2\left(\max\left\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\right\}\right). \end{aligned} \tag{6}$$

Suppose that

$$\begin{aligned} \max\left\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\right\} \\ = p^E(r_{n+1}, r_{n+2}). \end{aligned}$$

Then (4) implies that

$$\begin{aligned} &\psi_1\big(p^E(r_{n+1}, r_{n+2})\big) \\ &\quad \leq \psi_1\big(p^E(r_{n+1}, r_{n+2})\big) - \psi_2\big(p^E(r_{n+1}, r_{n+2})\big) \\ &\quad < \psi_1\big(p^E(r_{n+1}, r_{n+2})\big) \end{aligned}$$

which is a contradiction. Therefore we get

$$\begin{aligned} \max\left\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\right\} \\ = p^E(r_n, r_{n+1}). \end{aligned}$$

and so

$$\begin{aligned} &\psi_1\big(p^E(r_{n+1}, r_{n+2})\big) \\ &\quad \leq \psi_1\big(p^E(r_n, r_{n+1})\big) - \psi_2\big(p^E(r_n, r_{n+1})\big). \end{aligned}$$

Thus the sequence $\left\{p^E(r_n, r_{n+1})\right\}$ is nondecreasing. As a result of its lower boundary, there exists $0 \leq \gamma$ such that

$$\lim_{n \to \infty} p^E(r_n, r_{n+1}) = \gamma.$$

Applying the properties of functions ψ_1 and ψ_2 we obtain

$$\begin{aligned}\psi_1(\gamma) &\leq \liminf\ \psi_1\big(p^E(r_{n+1}, r_{n+2})\big)\\ &\leq \limsup\ \psi_1\big(p^E(r_{n+1}, r_{n+2})\big)\\ &\leq \limsup\ \Big[\psi_1\big(p^E(r_n, r_{n+1})\big)\\ &\qquad - \psi_2\big(p^E(r_n, r_{n+1})\big)\Big]\\ &\leq \limsup\ \psi_1\big(p^E(r_n, r_{n+1})\big)\\ &\qquad - \liminf\ \psi_2\big(p^E(r_n, r_{n+1})\big)\\ &\leq \psi_1(\gamma) - \psi_2(\gamma)\\ &< \psi_1(\gamma)\end{aligned}$$

which is not possible for $\gamma > 0$. Thus,

$$\gamma = \lim_{n\to\infty} p^E(r_n, r_{n+1}) = 0. \tag{7}$$

Now, we need to prove that $\{r_n\}$ is a p^E Cauchy in (X, p^E). Let us consider to the contrary there exists $\epsilon > 0$ such that, for $k > 0$, there exist $n(q) > m(q) > k$ for which we can find the subsequences $\{r_{n(q)}\}$ and $\{r_{m(q)}\}$ of $\{r_n\}$ and

$$p^E(r_{n(q)}, r_{m(q)}) \geq \epsilon, \tag{8}$$

and $n(q)$ is the smallest index so that the above statement holds: that is

$$p^E(r_{m(q)}, r_{n(q)-1}) \geq \epsilon. \tag{9}$$

Then we get

$$\begin{aligned}\epsilon &\leq p^E(r_{n(q)}, r_{m(q)})\\ &\leq kp^E(r_{m(q)}, r_{n(q)-1}) + kp^E(r_{n(q)-1}, r_{n(q)})\\ &< k\epsilon + kp^E(r_{n(q)-1}, r_{n(q)})\end{aligned} \tag{10}$$

Taking the limit supremum for (9) as $n \to \infty$, we get

$$\begin{aligned}\frac{\epsilon}{k} &\leq \liminf_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1})\\ &\leq \limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \leq \epsilon.\end{aligned} \tag{11}$$

Also, from (10) and (11), we get

$$\epsilon \leq \liminf_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \leq k\epsilon.$$

We deduce,

$$\begin{aligned}&p^E(r_{m(q)+1}, r_{n(q)})\\ &\leq kp^E(r_{m(q)+1}, r_{m(q)}) + kp^E(r_{m(q)}, r_{n(q)})\\ &\leq kp^E(r_{m(q)+1}, r_{m(q)}) + k^2p^E(r_{m(q)}, r_{n(q)-1})\\ &\qquad + k^2p^E(r_{n(q)-1}, r_{n(q)})\\ &\leq kp^E(r_{m(q)+1}, r_{m(q)}) + k^2\epsilon + k^2p^E(r_{n(q)-1}, r_{n(q)})\end{aligned} \tag{12}$$

Then by taking upper limit as $n \to \infty$ in (12), we get

$$\limsup_{n\to\infty} p^E(r_{m(q)+1}, r_{n(q)}) \leq k^2\epsilon.$$

Finally,

$$\begin{aligned}&p^E(r_{m(q)+1}, r_{n(q)-1})\\ &\leq kp^E(r_{m(q)+1}, r_{m(q)}) + kp^E(r_{m(q)}, r_{n(q)-1})\\ &\leq kp^E(r_{m(q)+1}, r_{m(q)}) + k\epsilon.\end{aligned} \tag{13}$$

Also, by taking upper limit as $n \to \infty$ in (13), we get

$$\limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \leq k\epsilon.$$

Hence,

$$\begin{aligned}\frac{\epsilon}{k} &\leq \liminf_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1})\\ &\leq \limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \leq \epsilon.\end{aligned} \tag{14}$$

Similarly,

$$\limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)}) \leq k\epsilon, \tag{15}$$

$$\frac{\epsilon}{k} \leq \limsup_{n\to\infty} p^E(r_{m(q)+1}, r_{n(q)}), \tag{16}$$

and

$$\limsup_{n\to\infty} p^E(r_{m(q)+1}, r_{n(q)-1}) \leq k\epsilon. \tag{17}$$

As F is L_β-admissible (or R_β-admissible) and by (3), we get $\beta(r_{m(q)}, r_{m(q)+1}) \geq 1$ and $\beta(r_{n(q)}, r_{n(q)+1}) \geq 1$.

By using (1) we get

$$\begin{aligned}&\psi_1\big(kp^E(r_{m(q)+1})\big)\\ &\leq \beta(r_{m(q)}, r_{m(q)+1})\beta(r_{n(q)}, r_{n(q)+1})\\ &\qquad \psi_1\big(kp^E(Fr_{m(q)}, r_{n(q)-1})\big)\\ &\leq \psi_1\big(\Delta_u^F(r_{m(q)}, r_{n(q)-1})\big)\\ &\qquad - \psi_2\big(\Delta_u^F(r_{m(q)}, r_{n(q)-1})\big)\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\Delta_u^F(r_{m(q)}, r_{n(q)-1})\\ &= \max\left\{\begin{array}{c} p^E(r_{m(q)}, r_{n(q)-1}),\\ p^E(r_{m(q)}, Fr_{m(q)}),\\ p^E(r_{n(q)-1}, Fr_{n(q)-1}),\\ \frac{p^E(r_{m(q)}, Fr_{n(q)-1}) + p^E(r_{n(q)-1}, Fr_{m(q)})}{4k},\\ \frac{p^E(r_{m(q)}, Fr_{m(q)})p^E(r_{n(q)-1}, Fr_{n(q)-1})}{1+d^E(r_{m(q)}, r_{n(q)-1})},\\ \frac{p^E(r_{m(q)}, Fr_{m(q)})p^E(r_{n(q)-1}, Fr_{n(q)-1})}{1+d^E(Fr_{m(q)}, Fr_{n(q)-1})} \end{array}\right\}\\ &= \max\left\{\begin{array}{c} p^E(r_{m(q)}, r_{n(q)-1}),\\ p^E(r_{m(q)}, r_{m(q)+1}),\\ p^E(r_{n(q)-1}, r_{n(q)}),\\ \frac{p^E(r_{m(q)}, r_{n(q)}) + p^E(r_{n(q)-1}, r_{m(q)+1})}{4k},\\ \frac{p^E(r_{m(q)}, r_{m(q)+1})p^E(r_{n(q)-1}, r_{n(q)})}{1+d^E(r_{m(q)}, r_{n(q)-1})},\\ \frac{p^E(r_{m(q)}, r_{m(q)+1})p^E(r_{n(q)-1}, r_{n(q)})}{1+d^E(r_{m(q)+1}, r_{n(q)})} \end{array}\right\}\end{aligned} \tag{19}$$

By taking the upper limit as $n \to \infty$ in (19) and using (7), (14), (15) and (17) we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\Delta_u^F(r_{m(q)},r_{n(q)-1})\\
&=\max\left\{\begin{array}{c}
\limsup\limits_{n\to\infty}p^E(r_{m(q)},r_{n(q)-1}),\\
\limsup\limits_{n\to\infty}p^E(r_{m(q)},r_{m(q)+1}),\\
\limsup\limits_{n\to\infty}p^E(r_{n(q)-1},r_{n(q)}),\\
\frac{\left\{\begin{array}{c}\limsup\limits_{n\to\infty}p^E(r_{m(q)},r_{n(q)})\\ +\limsup\limits_{n\to\infty}p^E(r_{n(q)-1},r_{m(q)+1})\end{array}\right\}}{4k},\\
\frac{\left\{\begin{array}{c}\limsup\limits_{n\to\infty}p^E(r_{m(q)},r_{m(q)+1})\\ \limsup\limits_{n\to\infty}p^E(r_{n(q)-1},r_{n(q)})\end{array}\right\}}{1+\limsup\limits_{n\to\infty}d^E(r_{m(q)},r_{n(q)-1})},\\
\frac{\left\{\begin{array}{c}\limsup\limits_{n\to\infty}p^E(r_{m(q)},r_{m(q)+1})\\ \limsup\limits_{n\to\infty}p^E(r_{n(q)-1},r_{n(q)})\end{array}\right\}}{1+\limsup\limits_{n\to\infty}d^E(r_{m(q)+1},r_{n(q)})}
\end{array}\right\}\\
&=\max\left\{\begin{array}{c}
\limsup\limits_{n\to\infty}p^E(r_{m(q)},r_{n(q)-1}),0,0,\\
\frac{\left\{\begin{array}{c}\limsup\limits_{n\to\infty}p^E(r_{m(q)},r_{n(q)})\\ +\limsup\limits_{n\to\infty}p^E(r_{n(q)-1},r_{m(q)+1})\end{array}\right\}}{4k},\\
0,0
\end{array}\right\}\\
&\le\max\left\{\epsilon,\frac{\epsilon}{2}\right\}\\
&=\epsilon
\end{aligned}\tag{20}$$

Next, taking the upper limit as $n\to\infty$ in (18) and using (16) and (20) we get

$$\begin{aligned}
\psi_1\left(k\frac{\epsilon}{k}\right)&\le\psi_1\left(\limsup_{n\to\infty}kp^E(r_{m(q)+1},r_{n(q)})\right)\\
&\le\psi_1\left(\limsup_{n\to\infty}\Delta_u^F(r_{m(q)},r_{n(q)-1})\right)\\
&\quad-\psi_2\left(\liminf_{n\to\infty}\left(\Delta_u^F(r_{m(q)},r_{n(q)-1})\right)\right)\\
&\le\psi_1(\epsilon)\\
&\quad-\psi_2\left(\liminf_{n\to\infty}\left(\Delta_u^F(r_{m(q)},r_{n(q)-1})\right)\right)
\end{aligned}$$

which implies that

$$\psi_2\left(\liminf_{n\to\infty}\left(\Delta_u^F(r_{m(q)},r_{n(q)-1})\right)\right)=0$$

or

$$\liminf_{n\to\infty}\Delta_u^F(r_{m(q)},r_{n(q)-1})=0.$$

Therefore by using (18) we obtain,

$$\liminf_{n\to\infty}p^E(r_{m(q)},r_{n(q)-1})=0$$

which conflicts with (14). Therefore, $\{r_n\}$ is a p^E-Cauchy sequence in the partially $E-$cone metric space (X,p^E). As (X,p^E) is a p^E-complete, then (X,p^E) is a p^E-complete partially E-cone metric space. So from the completeness it follows that $t\in X$ exist such that,

$$\lim_{n\to\infty}p^E(r_n,t)=0.$$

Therefore, by using (7), the condition

$$p^E(r_n,r_n)\le p^E(t,r_n)$$

and

$$\lim_{n\to\infty}p^E(r_n,r_n)=0$$

we get

$$\lim_{n\to\infty}p^E(r_n,t)=\lim_{n\to\infty}p^E(r_n,r_n)=p^E(t,t)=0.$$

We obtain

$$p^E(t,Ft)\le kp^E(t,Fr_n)+kp^E(Fr_n,Ft).\tag{21}$$

So taking limit as $n\to\infty$ in (21) and using the continuity of F we get

$$\begin{aligned}
&p^E(t,Ft)\\
&\quad\le k\lim_{n\to\infty}p^E(t,r_{n+1})+k\lim_{n\to\infty}(Fr_n,Ft)\\
&\quad=kp^E(Ft,Ft)
\end{aligned}\tag{22}$$

Since $\beta(t,t)\ge1$ and by (1) we get

$$\begin{aligned}
&\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\beta(t,Ft)\beta(t,Ft)\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\psi_1\big(\Delta_u^F(t,t)\big)-\psi_2\big(\Delta_u^F(t,t)\big)
\end{aligned}$$

where

$$\begin{aligned}
&\Delta_u^F(t,t)\\
&\quad=\max\left\{\begin{array}{c}
p^E(t,t),p^E(t,Ft),p^E(t,Ft),\\
\frac{p^E(t,\tau t)+p^E(t,Ft)}{4k},\\
\frac{p^E(t,Ft),p^E(t,Ft)}{1+p^E(t,t)},\\
\frac{p^E(t,Ft),p^E(t,Ft)}{1+p^E(Ft,Ft)}
\end{array}\right\}\\
&\quad<p^E(t,Ft).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\beta(t,Ft)\beta(t,Ft)\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\psi_1\big(p^E(t,Ft)\big)-\psi_2\big(p^E(t,Ft)\big)
\end{aligned}\tag{23}$$

Since ψ_1 is nondecreasing $kp^E(Ft,Ft)\le p^E(t,Ft)$ and $kp^E(Ft,Ft)=p^E(t,Ft)$, which is possible only when $(p^E(t,Ft))=0$ and $Ft=t$. Hence, a fixed point of F is t.

We observe that the prior result is still applicable for F that is not always continuous. The outcome is as follows. ■

Theorem 17. *Let $(X,\preceq,p^E)$ be a p^E-complete ordered partially E-cone metric space with the coefficient $k\ge1$. Let $F:X\to X$ be a $(\beta-\psi_1-\psi_2)$-contractive mapping of type-I. Suppose that the following conditions hold:*

(i) *F is β-admissible and L_β-admissible (or R_β-admissible);*
(ii) *There exists $r_1\in X$ such that $r_1\preceq Fr_1$ and $\beta(r_1,Fr_1)\ge1$;*
(iii) *F is nondecreasing, w.r.t $\preceq$;*
(iv) *If a sequence $\{r_n\}$ in X such that $r_n\preceq r\ \forall\ n\in\mathbb{N},\beta(r_n,r_{n+1})\ge1$ and $r_n\to r\in X$, as $n\to\infty$, then $\beta(r_n,r)\ge1\ \forall\ n\in\mathbb{N}$.*

Then, F has a fixed point.

Proof: Along the same lines as Theorem 16, the sequence $\{r_n\}$ defined by $r_{n+1} = Fr_n, \forall n \in \mathbb{N}$ is a nondecreasing p^E-Cauchy sequence in the p^E-complete partially E-cone metric space (X, p^E). According to (X, p^E) completeness, there exist $t \in X$ such that $\lim_{n\to\infty} r_n = t$. Assuming on X, we deduce $r_n \preceq t, \forall n \in \mathbb{N}$. Thus it suffices to show that $Ft = t$. Then by using (16) and $\beta(r_n, r) \geq 1$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} &\psi_1\big(kp^E(r_n, Ft)\big) \\ &\quad \leq \beta(r_n, Fr_n)\beta(t, Ft)\psi_1\big(kp^E(Fr_n, Ft)\big) \\ &\quad \leq \psi_1\big(\Delta_u^F(r_n,t)\big) - \psi_2\big(\Delta_u^F(r_n,t)\big) \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\Delta_u^F(r_n,t) \\ &= \max\left\{\begin{array}{c} p^E(r_n,t), \\ p^E(r_n,Fr_n), p^E(t,Ft), \\ \frac{p^E(r_n,Ft)p^E(Fr_n,t)}{4k}, \\ \frac{p^E(r_n,Fr_n)p^E(t,Ft)}{1+p^E(r_n,t)}, \\ \frac{p^E(r_n,Fr_n)p^E(t,Ft)}{1+p^E(Fr_n,Ft)} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} p^E(r_n,t), \\ p^E(r_n,r_{n+1}), p^E(t,Ft), \\ \frac{p^E(r_n,Ft)+p^E(r_{n+1},t)}{4k}, \\ \frac{p^E(r_n,r_{n+1})p^E(t,t)}{1+p^E(r_n,t)}, \\ \frac{p^E(r_n,r_{n+1})p^E(t,t)}{1+p^E(r_{n+1},t)} \end{array}\right\} \end{aligned} \tag{25}$$

Now by taking the limit as $n \to \infty$ in (25) and using Lemma 12, we get

$$\begin{aligned} \frac{p^E(t,Ft)}{4k^2} &= \min\left\{p^E(t,Ft), \frac{\frac{p^E(t,Ft)}{k}}{4k}\right\} \\ &\leq \liminf_{n\to\infty} \Delta_u^F(r_n,t) \\ &\leq \limsup_{n\to\infty} \Delta_u^F(r_n,t) \\ &\leq \max\left\{p^E(t,Ft), \frac{kp^E(t,Ft)}{4k}\right\} \\ &= p^E(t,Ft) \end{aligned} \tag{26}$$

Also, using (24) and applying the upper limit as $n \to \infty$

$$\begin{aligned} &\psi_1\big(kp^E(r_{n+1}, Ft)\big) \\ &\qquad \leq \beta(r_n, Fr_n)\beta(t, Ft)\psi_1\big(kp^E(Fr_n, Ft)\big) \\ &\qquad \leq \psi_1\big(\Delta_u^F(r_n,t)\big) - \psi_2\big(\Delta_u^F(r_n,t)\big), \end{aligned}$$

and using Lemma 12, we get

$$\begin{aligned} \psi_1\big(p^E(t,Ft)\big) &= \psi_1\left(k\frac{1}{k}p^E(r_{n+1},Ft)\right) \\ &\leq \psi_1\left(k\limsup_{n\to\infty} p^E(r_{n+1},Ft)\right) \\ &\leq \psi_1\left(\limsup_{n\to\infty} \Delta_u^F(r_n,t)\right) \\ &\qquad - \psi_2\left(\liminf_{n\to\infty} \Delta_u^F(r_n,t)\right) \\ &\leq \psi_1\big(p^E(t,Ft)\big) - \psi_2\big(p^E(t,Ft)\big) \\ &< \psi_1\big(p^E(t,Ft)\big), \end{aligned}$$

a contradiction. Therefore $t = Ft$. Hence, a fixed point of F is t. ■

Result-II Type-II $(\beta - \psi_1 - \psi_2)$-contractive mapping in ordered partially E-cone metric space was introduced to support our second finding.

Definition 18. *Let (X, p^E) be a ordered partially E-cone metric space with coefficient $k \geq 1$. The mapping $F : X \to X$ is called a $(\beta - \psi_1 - \psi_2)$-contractive mapping of type-II, if there exist two altering distance functions ψ_1, ψ_2 and $\beta : X \times X \to [0, \infty)$ exists such that*

$$\begin{aligned} &\beta(r,Fr)\beta(s,Fs)\psi_1\big(kd^E(Fr,Fs)\big) \\ &\quad \leq \psi_1\big((\Delta_I)_u^F(r,s)\big) - \psi_2\big((\Delta_I)_u^F(r,s)\big) \end{aligned} \tag{27}$$

for all comparable $r, s \in X$, where

$$\begin{aligned} &(\Delta_I)_u^F(r,s) \\ &= \max\left\{\begin{array}{c} p(r,s), \\ p^E(s,Fs), p^E(r,Fr), \\ \frac{p^E(r,Fs)+p^E(s,Fr)}{4k}, \\ \frac{p^E(r,Fr)p^E(r,Fs)+p^E(s,Fs)p^E(s,Fr)}{1+k\left[p^E(r,Fr)+p^E(s,Fs)\right]}, \\ \frac{p^E(r,Fr)p^E(r,Fs)+p^E(s,Fs)p^E(s,Fr)}{1+p^E(r,Fs)+p^E(s,Fr)} \end{array}\right\} \end{aligned} \tag{28}$$

Theorem 19. *In replacement of the Type-I $(\beta - \psi_1 - \psi_2)$-contractive condition in Theorem 16, suppose that Type-II $(\beta - \psi_1 - \psi_2)$-contrative condition is satisfied. Then F has a fixed point.*

Proof: Let $r_1 \in X$ such that $r \preceq Fr_1$ and $\beta(r_1, Fr_1) \geq 1$. A sequence $\{r_n\}$ in X is defined by $r_{n+1} = Fr_n, \forall n \geq 1$. We have $r_2 = Fr_1 \preceq Fr_2 = r_3$ since $r_1 \preceq Fr_1$ and F is nondecreasing. Also, $r_3 = Fr_2 \preceq Fr_3 = r_4$ since $r_2 \preceq Fr_2$ and F is nondecreasing. We obtain by induction,

$$r_1 \preceq r_2 \preceq r_3 \cdots \preceq r_n \preceq r_{n+1} \preceq \cdots$$

If $r_n = r_{n+1}$ for some n $\in \mathbb{N}$, then $r = r_n$ is a fixed point of F and the proof is finished. So we may assume that $r_n \neq r_{n+1}$ for some $n \in \mathbb{N}$. Since F is β- admissible, we deduce

$$\begin{aligned} &\beta(r_1, Fr_1) = \beta(r_1, r_2) \geq 1 \\ \Rightarrow &\beta(Fr_1, Fr_1) = \beta(r_2, r_3) \geq 1 \end{aligned}$$

By induction on n we get

$$\beta(r_n, r_{n+1}) \geq 1 \text{ and } \beta(r_{n+1}, r_{n+2}) \geq 1 \tag{29}$$

for all $n \in \mathbb{N}$.
Therefore by using (29) $\forall n \in \mathbb{N}$, we get

$$\begin{aligned} &\psi_1\big(p^E(r_{n+1}, r_{n+2})\big) \\ &\quad \leq \beta(r_n, Fr_{n+1})\beta(r_{n+1}, Fr_{n+2})\psi_1\big(kp^E(Fr_n, Fr_{n+1})\big) \\ &\quad \leq \psi_1\big((\Delta_I)_u^F(r_n, r_{n+1})\big) - \psi_2\big((\Delta_I)_u^F(r_n, r_{n+1})\big) \end{aligned} \tag{30}$$

where

$$
\begin{aligned}
&(\Delta_I)_u^F(r_n, r_{n+1})\\
&= \max\left\{\begin{array}{c}
p^E(r_n, r_{n+1}),\\
p^E(r_n, Fr_n), p^E(r_{n+1}, Fr_{n+1}),\\
\frac{p^E(r_n, Fr_{n+1})+p^E(r_{n+1}, Fr_n)}{4k},\\
\frac{\left\{\begin{array}{l} p^E(r_n, Fr_n)p^E(r_n, Fr_{n+1})\\ \quad + p^E(r_{n+1}, Fr_{n+1})p^E(r_{n+1}, Fr_n)\end{array}\right\}}{1+k\left[p^E(r_n, Fr_n)+p^E(r_{n+1}, Fr_{n+1})\right]},\\
\frac{\left\{\begin{array}{l} p^E(r_n, Fr_n)p^E(r_n, Fr_{n+1})\\ \quad + p^E(r_{n+1}, Fr_{n+1})p^E(r_{n+1}, Fr_n)\end{array}\right\}}{1+p^E(r_n, Fr_{n+1})+p^E(r_{n+1}, Fr_n)}
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2}),\\
\frac{p^E(r_n, rn+2)+p^E(r_{n+1}, r_{n+1})}{4k},\\
\frac{\left\{\begin{array}{l} p^E(r_n, r_{n+1})p^E(r_n, r_{n+2})\\ \quad + p^E(r_{n+1}, r_{n+2})p^E(r_{n+1}, r_{n+1})\end{array}\right\}}{1+k\left[p^E(r_n, r_{n+1})+p^E(r_{n+1}, r_{n+2})\right]},\\
\frac{\left\{\begin{array}{l} p^E(r_n, r_{n+1})p^E(r_n, r_{n+2})\\ \quad + p^E(r_{n+1}, r_{n+2})p^E(r_{n+1}, r_{n+1})\end{array}\right\}}{1+p^E(r_n, r_{n+2})+p^E(r_{n+1}, r_{n+1})}
\end{array}\right\}\\
&\le \max\left\{\begin{array}{c}
p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2}),\\
\frac{\left\{\begin{array}{l} kp^E(r_n, r_{n+1}) + kp^E(r_{n+1}, r_{n+2})\\ \qquad + 2kp^E(r_{n+1}, r_{n+2})\end{array}\right\}}{4k},\\
\frac{\left\{\begin{array}{l} p^E(r_n, r_{n+1})p^E(r_n, r_{n+2})\\ \quad + p^E(r_{n+1}, r_{n+2})p^E(r_{n+1}, r_{n+1})\end{array}\right\}}{1+k\left[p^E(r_n, r_{n+1})+p^E(r_{n+1}, r_{n+2})\right]},\\
\frac{\left\{\begin{array}{l} p^E(r_n, r_{n+1})p^E(r_n, r_{n+2})\\ \quad + p^E(r_{n+1}, r_{n+2})p^E(r_{n+1}, r_{n+1})\end{array}\right\}}{1+p^E(r_n, r_{n+2})+p^E(r_{n+1}, r_{n+1})}
\end{array}\right\}\\
&< \max\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\}
\end{aligned} \tag{31}
$$

From (30) and (31) we get

$$
\begin{aligned}
&\psi_1\big(p^E(r_{n+1}, r_{n+2})\big)\\
&\le \psi_1\Big(\max\big\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\big\}\Big)\\
&- \psi_2\Big(\max\big\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\big\}\Big)
\end{aligned} \tag{32}
$$

Assume that

$$
\begin{aligned}
\max\big\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\big\}&\\
= p^E(r_{n+1}, r_{n+2}).&
\end{aligned}
$$

Then (30) implies that

$$
\begin{aligned}
&\psi_1\big(kp^E(r_{n+1}, r_{n+2})\big)\\
&\quad \le \psi_1\big(p^E(r_{n+1}, r_{n+2})\big) - \psi_2\big(p^E(r_{n+1}, r_{n+2})\big)\\
&\quad < \psi_1\big(p^E(r_{n+1}, r_{n+2})\big).
\end{aligned}
$$

which is contradiction. This implies that

$$
\begin{aligned}
\max\big\{p^E(r_n, r_{n+1}), p^E(r_{n+1}, r_{n+2})\big\}&\\
= p^E(r_n, r_{n+1})&
\end{aligned}
$$

and then

$$
\begin{aligned}
&\psi_1\big(p^E(r_{n+1}, r_{n+2})\big)\\
&\quad \le \psi_1\big(p^E(r_n, r_{n+1})\big) - \psi_2\big(p^E(r_n, r_{n+1})\big).
\end{aligned}
$$

Thus the sequence $\big\{p^E(r_n, r_{n+1})\big\}$ is nondecreasing. There exists $\gamma \ge 0$, as it is bounded from below, such that $\lim_{n\to\infty} p^E(r_n, r_{n+1}) = \gamma$. Then by applying the properties of functions ψ_1 and ψ_2 we obtain

$$
\begin{aligned}
\psi_1(\gamma) &\le \liminf \psi_1\big(p^E(r_{n+1}, r_{n+2})\big)\\
&\le \limsup \psi_1\big(p^E(r_{n+1}, r_{n+2})\big)\\
&\le \limsup \Big[\psi_1\big(p^E(r_n, r_{n+1})\big)\\
&\qquad - \psi_2\big(p^E(r_n, r_{n+1})\big)\Big]\\
&\le \limsup \psi_1\big(p^E(r_n, r_{n+1})\big)\\
&\qquad - \liminf \psi_2\big(p^E(r_n, r_{n+1})\big)\\
&\le \psi_1(\gamma) - \psi_2(\gamma)\\
&< \psi_1(\gamma).
\end{aligned}
$$

which is not possible for $\gamma > 0$. Thus,

$$
\gamma = \lim_{n\to\infty} p^E(r_n, r_{n+1}) = 0. \tag{33}
$$

Now, we need to prove that $\{r_n\}$ is a p^E Cauchy sequence in (X, p^E). Suppose to the contrary that there exist $\epsilon > 0$ such that, for $k > 0$, there exist $n(q) > m(q) > k$ for which we can find the subsequences $\{r_{n(q)}\}$ and $\{r_{m(q)}\}$ of $\{r_n\}$ and

$$
p^E(r_{n(q)}, r_{m(q)}) \ge \epsilon, \tag{34}
$$

and $n(q)$ is the smallest index so that the above statement holds; that is

$$
p^E(r_{m(q)}, r_{n(q)-1}) < \epsilon, \tag{35}
$$

Then we have

$$
\begin{aligned}
\epsilon &\le p^E(r_{n(q)}, r_{m(q)})\\
&\le kp^E(r_{m(q)}, r_{n(q)-1}) + kp^E(r_{n(q)-1}, r_{n(q)})\\
&< k\epsilon + kp^E(r_{n(q)-1}, r_{n(q)}).
\end{aligned} \tag{36}
$$

Applying the upper limit for (35) as $n \to \infty$, we get

$$
\begin{aligned}
\frac{\epsilon}{k} &\le \liminf_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1})\\
&\le \limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \le \epsilon
\end{aligned} \tag{37}
$$

Also, from (36) and (37), we obtain

$$
\epsilon \le \limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \le k\epsilon
$$

We deduce,

$$
\begin{aligned}
&p^E(r_{m(q)+1}, r_{n(q)})\\
&\le kp^E(r_{m(q)+1}, r_{m(q)}) + kp^E(r_{m(q)}, r_{n(q)})\\
&\le kp^E(r_{m(q)+1}, r_{m(q)}) + k^2p^E(r_{m(q)}, r_{n(q)-1})\\
&\qquad + k^2p^E(r_{n(q)-1}, r_{n(q)})\\
&\le kp^E(r_{m(q)+1}, r_{m(q)}) + k^2\epsilon\\
&\qquad + k^2p^E(r_{n(q)-1}, r_{n(q)}),
\end{aligned} \tag{38}
$$

by applying the upper limit as $n \to \infty$ in (38), we obtain

$$
\limsup_{n\to\infty} p^E(r_{m(q)+1}, r_{n(q)}) \le k^2\epsilon
$$

Finally,

$$
\begin{aligned}
&p^E(r_{m(q)+1}, r_{n(q)-1})\\
&\le kp^E(r_{m(q)+1}, r_{m(q)}) + kp^E(r_{m(q)}, r_{n(q)-1})\\
&\le kp^E(r_{m(q)+1}, r_{m(q)}) + k\epsilon.
\end{aligned} \tag{39}
$$

Also, by applying the upper limit as $n \to \infty$ in (39), we obtain

$$\limsup_{n\to\infty} p^E(r_{m(q)+1}, r_{n(q)-1}) \leq k\epsilon.$$

Hence,

$$\frac{\epsilon}{k} \leq \liminf_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \leq \limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)-1}) \leq \epsilon. \tag{40}$$

Similarly,

$$\limsup_{n\to\infty} p^E(r_{m(q)}, r_{n(q)}) \leq k\epsilon, \tag{41}$$

$$\frac{\epsilon}{k} \leq \liminf_{n\to\infty} p^E(r_{m(q)+1}, r_{n(q)}), \tag{42}$$

$$\limsup_{n\to\infty} p^E(r_{m(q)+1}, r_{n(q)-1}) \leq k\epsilon. \tag{43}$$

As F is L_β-admissible (or R_β-admissible) and by (29), we get $\beta(r_{m(q)}, r_{m(q)+1}) \geq 1$ and $\beta(r_{n(q)}, r_{n(q)+1}) \geq 1$.

By using (27) we get

$$\begin{aligned}\psi_1\big(kp^E(r_{m(q)+1}, r_{n(q)})\big) &\leq \beta(r_{m(q)}, r_{m(q)+1})\beta(r_{n(q)}, r_{n(q)+1}) \\ &\qquad \psi_1\big(kp^E(Fr_{m(q)}, Fr_{n(q)-1})\big) \\ &\leq \psi_1\big((\Delta_I)^F_u(r_{m(q)}, r_{n(q)-1})\big) \\ &\qquad - \psi_2\big((\Delta_I)^F_u(r_{m(q)}, r_{n(q)-1})\big)\end{aligned} \tag{44}$$

where

$$\begin{aligned}&(\Delta_I)^F_u(r_{m(q)}, r_{m(q)-1}) \\ &= \max\left\{\begin{array}{c} p^E(r_{m(q)}, r_{n(q)-1}), p^E(r_{m(q)}, Fr_{m(q)}), \\ p^E(r_{n(q)-1}, Fr_{n(q)-1}), \\ \dfrac{\left\{\begin{array}{l} p^E(r_{m(q)}, Fr_{n(q)-1}) \\ \quad + p^E(r_{n(q)-1}, Fr_{m(q)})\end{array}\right\}}{4k}, \\ \dfrac{\left[\begin{array}{l} p^E(r_{m(q)}, Fr_{m(q)}) \\ \quad p^E(r_{m(q)}, Fr_{n(q)-1}) \\ +p^E(r_{n(q)-1}, Fr_{n(q)-1}) \\ \quad p^E(r_{n(q)-1}, Fr_{m(q)})\end{array}\right]}{\left\{\begin{array}{l} 1 + kp^E(r_{m(q)}, Fr_{m(q)}) \\ \quad + kp^E(r_{n(q)-1}, Fr_{n(q)-1})\end{array}\right\}}, \\ \dfrac{\left[\begin{array}{l} p^E(r_{m(q)}, Fr_{m(q)}) \\ \quad p^E(r_{m(q)}, Fr_{n(q)-1}) \\ +p^E(r_{n(q)-1}, Fr_{n(q)-1}) \\ \quad p^E(r_{n(q)-1}, Fr_{m(q)})\end{array}\right]}{\left\{\begin{array}{l} 1 + p^E(r_{m(q)}, Fr_{n(q)-1}) \\ \quad + p^E(r_{n(q)-1}, Fr_{m(q)})\end{array}\right\}} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} p^E(r_{m(q)}, r_{n(q)-1}), p^E(r_{m(q)}, r_{m(q)+1}), \\ p^E(r_{n(q)-1}, r_{n(q)}), \\ \dfrac{\left\{\begin{array}{l} p^E(r_{m(q)}, r_{n(q)}) \\ \quad + p^E(r_{n(q)-1}, r_{m(q)+1})\end{array}\right\}}{4k}, \\ \dfrac{\left[\begin{array}{l} p^E(r_{m(q)}, r_{m(q)+1}) \\ \quad p^E(r_{m(q)}, r_{n(q)}) \\ +p^E(r_{n(q)-1}, r_{n(q)}) \\ \quad p^E(r_{n(q)-1}, r_{m(q)+1})\end{array}\right]}{\left\{\begin{array}{l} 1 + kp^E(r_{m(q)}, r_{m(q)+1}) \\ \quad + kp^E(r_{n(q)-1}, r_{n(q)})\end{array}\right\}}, \\ \dfrac{\left[\begin{array}{l} p^E(r_{m(q)}, r_{m(q)+1}) \\ \quad p^E(r_{m(q)}, r_{n(q)}) \\ +p^E(r_{n(q)-1}, r_{n(q)}) \\ \quad p^E(r_{n(q)-1}, r_{m(q)+1})\end{array}\right]}{\left\{\begin{array}{l} 1 + p^E(r_{m(q)}, r_{n(q)}) \\ \quad + p^E(r_{n(q)-1}, r_{m(q)+1})\end{array}\right\}} \end{array}\right\}\end{aligned} \tag{45}$$

By taking the upper limit as $n \to \infty$ in (45) and using

(33),(40),(41),(43) we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}(\Delta_I)_u^F(r_{m(q)},r_{n(q)-1})\\
&=\max\left\{\begin{array}{c}
\limsup_{n\to\infty} p^E(r_{m(q)},r_{n(q)-1}),\\
\limsup_{n\to\infty} p^E(r_{m(q)},r_{m(q)+1}),\\
\limsup_{n\to\infty} p^E(r_{n(q)-1},r_{n(q)}),\\
\frac{\left\{\begin{array}{c}\limsup_{n\to\infty} p^E(r_{m(q)},r_{n(q)})\\ +\limsup_{n\to\infty} p^E(r_{n(q)-1},r_{m(q)+1})\end{array}\right\}}{4k},\\
\frac{\left\{\begin{array}{c}\limsup_{n\to\infty} p^E(r_{m(q)},r_{m(q)+1})\\ \limsup_{n\to\infty} p^E(r_{m(q)},r_{n(q)})\\ +\limsup_{n\to\infty} p^E(r_{n(q)-1},r_{n(q)})\\ \limsup_{n\to\infty} p^E(r_{n(q)-1},r_{m(q)+1})\end{array}\right\}}{\left\{\begin{array}{c}1+k\limsup_{n\to\infty} p^E(r_{m(q)},r_{m(q)+1})\\ +k\limsup_{n\to\infty} p^E(r_{n(q)-1},r_{n(q)})\end{array}\right\}},\\
\frac{\left\{\begin{array}{c}\limsup_{n\to\infty} p^E(r_{m(q)},r_{m(q)+1})\\ \limsup_{n\to\infty} p^E(r_{m(q)},r_{n(q)})\\ +\limsup_{n\to\infty} p^E(r_{n(q)-1},r_{n(q)})\\ \limsup_{n\to\infty} p^E(r_{n(q)-1},r_{m(q)+1})\end{array}\right\}}{\left\{\begin{array}{c}1+\limsup_{n\to\infty} p^E(r_{m(q)},r_{n(q)})\\ +\limsup_{n\to\infty} p^E(r_{n(q)-1},r_{m(q)+1})\end{array}\right\}}
\end{array}\right\}\\
&=\max\left\{\begin{array}{c}
\limsup_{n\to\infty} p^E(r_{m(q)},r_{n(q)-1}),0,0\\
\frac{\left\{\begin{array}{c}\limsup_{n\to\infty} p^E(r_{m(q)},r_{n(q)})\\ +\limsup_{n\to\infty} p^E(r_{n(q)-1},r_{m(q)+1})\end{array}\right\}}{4k},\\
0,0
\end{array}\right\}\\
&\le\max\{\epsilon,\frac{\epsilon}{2}\}\\
&=\epsilon.
\end{aligned} \tag{46}$$

Then by applying the upper limit (44) as n $\to\infty$ and using (42) and (46) we obtain

$$\begin{aligned}
\psi_1\left(k\frac{\epsilon}{k}\right)&\le\psi_1\big(\limsup_{n\to\infty} kp^E(r_{m(q)+1},r_{n(q)})\big)\\
&\le\psi_1\big(\limsup_{n\to\infty}(\Delta_I)_u^F(r_{m(q)},r_{n(q)-1})\big)\\
&\quad-\psi_2\big(\liminf_{n\to\infty}(\Delta_I)_u^F(r_{m(q)},r_{n(q)-1})\big)\\
&\le\psi_1(\epsilon)\\
&\quad-\psi_2\big(\liminf_{n\to\infty}(\Delta_I)_u^F(r_{m(q)},r_{n(q)-1})\big),
\end{aligned}$$

which implies that

$$\psi_2\big(\liminf_{n\to\infty}(\Delta_I)_u^F(r_{m(q)},r_{n(q)-1})\big)=0$$

or

$$\liminf_{n\to\infty}(\Delta_I)_u^F(r_{m(q)},r_{n(q)-1})=0$$

Therefore by using (44) we obtain,

$$p^E(r_{m(q)},r_{n(q)-1})=0$$

which conflict with (35). Therefore,$\{r_n\}$ is a p^E-Cauchy sequence in partially E-cone metric space (X,p^E). As (X,p^E) is a p^E-complete, then (X,p^E) is a p^E-complete partially E-cone metric space. So from the completeness it follows that $t\in X$ exist such that,

$$\lim_{n\to\infty} p^E(r_n,t)=0.$$

Therefore, by using (33), the condition $p^E(r_n,r_n)\le p^E(t,r_n)$ and $\lim_{n\to\infty} p^E(r_n,r_n)=0$ we get

$$\lim_{n\to\infty} p^E(r_n,t)=\lim_{n\to\infty} p^E(r_n,r_n)=p^E(t,t)=0.$$

We obtain

$$p^E(t,Ft)\le kp^E(t,Fr_n)+kp^E(Fr_n,Ft). \tag{47}$$

So by using the continuity of F and applying limit as $n\to\infty$ in (47), we obtain

$$\begin{aligned}
&p^E(t,Ft)\\
&\quad\le k\lim_{n\to\infty} p^E(t,r_{n+1})+k\lim_{n\to\infty} p^E(Fr_n,Ft)\\
&\quad=kp^E(Ft,Ft).
\end{aligned}\tag{48}$$

As $\beta(t,t)\ge1$ and using (27) we obtain

$$\begin{aligned}
&\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\beta(t,Ft)\beta(t,Ft)\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\psi_1\big((\Delta_I)_u^F(t,t)\big)-\psi_2\big((\Delta_I)_u^F(t,t)\big)
\end{aligned}$$

here

$$\begin{aligned}
&(\Delta_I)_u^F(t,t)\\
&=\max\left\{\begin{array}{c}
p^E(t,t),p^E(t,Ft),p^E(t,Ft),\\
\frac{p^E(t,\tau t)+p^E(t,Ft)}{4k},\\
\frac{p^E(t,Ft)p^E(t,Ft)+p^E(t,Ft)p^E(t,Ft)}{1+k\left[p^E(t,Ft)+p^E(t,Ft)\right]}\\
\frac{p^E(t,Ft)p^E(t,Ft)+p^E(t,Ft)p^E(t,Ft)}{1+p^E(t,Ft)+p^E(t,Ft)}
\end{array}\right\}\\
&<p^E(t,Ft)
\end{aligned}$$

so

$$\begin{aligned}
&\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\beta(t,Ft)\beta(t,Ft)\psi_1\big(kp^E(Ft,Ft)\big)\\
&\quad\le\psi_1\big(p^E(t,Ft)\big)-\psi_2\big(p^E(t,Ft)\big).
\end{aligned}\tag{49}$$

Since ψ_1 is nondecreasing $kp^E(Ft,Ft)\le p^E(t,Ft)$ and $kp^E(Ft,Ft)=p^E(t,Ft)$, which is true only when $p^E(t,Ft)=0$ and $Ft=t$. Thus, t is a fixed point of F. ■

Similarly result can be design for $(\beta-\psi_1-\psi_2)$-contractive mapping of type-II in the line of Theorem 17. Now from our main result, we have the following consequences:

Corollary 20. *Let $(X,\preceq,p^E)$ be an p^E-complete ordered E-cone metric space. Assume that there exists ψ_1,ψ_2 and $\beta:X\times X\to[0,\infty)$. Suppose $F:X\to X$ is an increasing mapping with respect to $\preceq$ such that an element $r_0\in X$ exists with $r_0\preceq F^m(r_0)$, and satisfying $(\beta-\psi_1-\psi_2)$-contractive mapping of the form*

$$\begin{aligned}
&\beta(r,F^mr)\beta(r,F^mr)\psi_1\big(kd^E(F^mr,F^ms)\big)\\
&\quad\le\psi_1\big(\Delta_u^F(r,s)\big)-\psi_2\big(\Delta_u^F(r,s)\big)
\end{aligned}\tag{50}$$

for all comparable $r, s \in X$, where

$$\Delta_u^F(r,s) = \max\left\{\begin{array}{c} p^E(r,s), p^E(r,F^m r), \\ p^E(s,F^m s), \\ \frac{p^E(r,F^m s)+p^E(s,F^m r)}{4k}, \\ \frac{p^E(r,F^m r)p^E(s,F^m s)}{1+p^E(r,s)}, \\ \frac{p^E(r,F^m r)p^E(s,F^m s)}{1+p^E(F^m r,F^m s)} \end{array}\right\} \quad (51)$$

Consider the following statements hold:

(i) *F is $\beta-$admissible and $L_\beta-$admissible (or $R_\beta-$admissible)*
(ii) *$r_1 \in F$ exists such that $r_1 \preceq Fr_1$ and $\beta(r_1, Fr_1) \geq 1$;*
(iii) *If $F^n r_1 \to t$ then $\beta(t,t) \geq 1$ and F is continuous*

Then F has a fixed point.

Corollary 21. *Consider a p^E-complete ordered E-cone metric space $(X, \preceq, p^E)$ with coefficient $k \geq 1$. Assume that ψ_1, ψ_2 are mappings. Let $F : X \to X$ be a continuous and nondecreasing mapping satisfying*

$$\psi_1\big(kd^E(Fr,Fs)\big) \leq \psi_1\big(\Delta_u^F(r,s)\big) - \psi_2\big(\Delta_u^F(r,s)\big)$$

for all comparable $r, s \in X$, where

$$\Delta_u^F(r,s) = \max\left\{\begin{array}{c} p^E(r,s), p^E(r,Fr), \\ p^E(s,Fs), \\ \frac{p^E(r,Fs)+p^E(s,Fr)}{4k}, \\ \frac{p^E(r,Fr)p^E(s,Fs)}{1+p^E(r,s)}, \\ \frac{p^E(r,Fr)p^E(s,Fs)}{1+p^E(Fr,Fs)} \end{array}\right\}$$

If there exists $r_1 \in X$ such that $r_1 \preceq Fr_1$, then F has a fixed point.

Corollary 22. *Consider a p^E-complete ordered E-cone metric space $(X, \preceq, p^E)$ with the coefficient $k \geq 1$. Assume that ψ_1, ψ_2 are mappings, Let $F : X \to X$ be a continuous, nondecreasing mapping satisfying*

$$\psi_1\big(kd^E(Fr,Fs)\big) \leq \psi_1\big((\Delta_I)_u^F(r,s)\big) - \psi_2\big((\Delta_I)_u^F(r,s)\big)$$

for all comparable $r, s \in X$, where

$$(\Delta_I)_u^F(r,s) = \max\left\{\begin{array}{c} p^E(r,s), p^E(r,Fr), p^E(s,Fs), \\ \frac{p^E(r,Fs)+p^E(s,Fr)}{4k}, \\ \frac{p^E(r,Fr)p^E(r,Fs)+p^E(s,Fs)p^E(s,Fr)}{1+k\left[p^E(r,Fr)+p^E(s,Fs)\right]}, \\ \frac{p^E(r,Fr)p^E(r,Fs)+p^E(s,Fs)p^E(s,Fr)}{1+p^E(r,Fs)+p^E(s,Fr)} \end{array}\right\}$$

If $r_1 \in X$ exists such that $r \preceq Fr_1$, then F has a fixed point.

Example 23. *Let $X = [0, \infty)$ with the partial order $\preceq$ defined by*

$$r \preceq s \Leftrightarrow r \leq s$$

and the function $p^E : X \times X \to \mathbb{R}^+$ defined by $p^E(r,s) = \big(\max\{r,s\}\big)^2 \ \forall\ r, s \in X$. Clearly, (X, p^E) is an ordered partially E-cone metric space with $k = 1$.
A mapping $F : X \to X$ defined by

$$F(r) = \begin{cases} \frac{r}{\sqrt{3r+2}} & \text{if } r \in [0,2], \\ \frac{5r}{3} & \text{otherwise.} \end{cases},$$

and $\beta : X \times X \to [0, \infty)$ by

$$\beta(r,s) = \begin{cases} 1 & \text{if } r, s \in [0,2], \\ 0 & \text{otherwise.} \end{cases}$$

Now using control functions $\psi_1(p) = p$ and

$$\psi_2(p) = \begin{cases} \frac{p(3\sqrt{p}+1)}{3\sqrt{p}+2} & \text{if } p \in [0,2], \\ \frac{3p}{5} & \text{if } p \geq 2, \end{cases}$$

Then F is increasing and continuous, $0 \preceq F0$.
We will prove the following:

(i) *$F : X \to X$ is an $(\beta - \psi_1 - \psi_2)$-contractive mapping of type-I, with $\psi_1(p) = p\ \forall\ p \geq 0$;*
(ii) *F is β-admissible;*
(iii) *$r_1 \in X$ exists such that $r_1 \preceq Fr_1$ and $\beta(r_1, Fr_1) \geq 1$;*
(iv) *If a sequence $\{r_n\}_{n=1}^{\infty}$ in X such that $\beta(r_n, r_{n+1}) \geq 1$ and $r_n \to r$, as $n \to \infty$, then $\beta(r_n, r) \geq 1,\ \forall\ n \in \mathbb{N}$.*

Proof:

(i) Clearly F is $(\beta - \psi_1 - \psi_2)$-contractive mapping with $\psi_1(p) = p,\ \forall\ p \geq 0, since\ \forall\ r, s \in X,$

$$\begin{aligned} &\beta(r,Fr)\beta(s,Fs)\psi_1\big(kp^E(Fr,Fs)\big) \\ &= \psi_1\left(p^E\left(\frac{r}{\sqrt{3r+2}}, \frac{s}{\sqrt{3s+2}}\right)\right) \\ &= \psi_1\left(\left(\max\left\{\frac{r}{\sqrt{3r+2}}, \frac{s}{\sqrt{3s+2}}\right\}\right)^2\right). \end{aligned}$$

keeping generality intact if $0 \leq s \leq r \leq 2$, then

$$\begin{aligned} &\beta(r,Fr)\beta(s,Fs)\psi_1\big(kp^E(Fr,Fs)\big) \\ &= \psi_1\left(p^E\left(\frac{r}{\sqrt{3r+2}}, \frac{s}{\sqrt{3s+2}}\right)\right) \\ &= \psi_1\left(\left(\frac{r}{\sqrt{3r+2}}\right)^2\right) \\ &= \left(\frac{r^2}{3r+2}\right) \end{aligned}$$

and

$$\Delta_u^F(r,s) = \max\left\{\begin{array}{c} r^2, r^2, s^2, \\ \frac{r^2+\left(\max\left\{s, \frac{r}{\sqrt{3r+2}}\right\}\right)^2}{4}, \\ \frac{(r^2)(s^2)}{1+r^2}, \frac{(r^2)(s^2)}{1+\left(\frac{r}{\sqrt{3r+2}}\right)} \end{array}\right\} = r^2.$$

Then

$$\begin{aligned} &\beta(r,Fr)\beta(s,Fs)\psi_1\big(kp^E(Fr,Fs)\big) \\ &= \left(\frac{r^2}{3r+2}\right) \leq r^2 - \left(\frac{3r^3+r^2}{3r+2}\right) \\ &\leq \psi_1(r^2) - \psi_2(r^2) \\ &= \psi_1\big(\Delta_u^F(r,s)\big) - \psi_2\big(\Delta_u^F(r,s)\big). \end{aligned}$$

(ii) Let $(r,s) \in X \times X$ be such that $\beta(r,s) \geq 1$. From the definition of F and β we have both $Fr = \frac{r}{\sqrt{3r+2}}$, and $Fs = \frac{s}{\sqrt{3s+2}}$ are in $[0,2]$, so we have $\beta(Fr, Fs) = 1 \geq 1$. Then F is an β-admissible.
(iii) Assuming $r_1 = 1 \in X$, we get

$$\beta(r_1, Fr_1) = \beta(1, F1) = \beta(1, \frac{1}{\sqrt{5}}) = 1 \geq 1.$$

(iv) Consider a sequence $\{r_n\}$ in X such that $\beta(r_n, r_{n+1}) \geq 1$, for all $n \in \mathbb{N}$ and $r_n \to r \in X$ as $n \to \infty$. As $\beta(r_n, r_{n+1}) \geq 1,\ \forall\ n \in \mathbb{N}$ and by the definition of β, we obtain $r_n \in [0, 2],\ \forall\ n \in \mathbb{N}$ and $r \in [0, 2]$. Then $\beta(r_n, r) = 1 \geq 1$.

At this point, every hypothesis in Theorem 16 is true. Therefore, F has a fixed point.

It is noted that without for the β term, the contraction condition (1) is not true. For example at $r = 1$ and $s = 3$, we obtain

$$\begin{aligned}\psi_1\big(p^E(F1, F3)\big) &= \psi_1\left(F^E(\frac{1}{\sqrt{5}}, 5)\right)\\ &= \psi_1(25) = 25 \nleq 10 = 25 - 15\\ &= \psi_1(25) - \psi_2(25)\\ &= \psi_1\big(\Delta_u^F(1,3)\big) - \psi_2\big(\Delta_u^F(1,3)\big).\end{aligned}$$

■

Example 24. *Let $X = [0, \infty)$ with the partial order $\preceq$ defined by*

$$r \preceq s \Leftrightarrow r \leq s$$

and the function $p^E : X \times X \to \mathbb{R}^+$ defined by $p^E(r, s) = \big(\max\{r, s\}\big)^2,\ \forall\ r, s \in X$. Clearly, (X, p^E) is an ordered partially E-cone metric space with $k = 1$.
A mapping $F : X \to X$ defined by

$$F(r) = \begin{cases} \frac{r}{\sqrt{r^2+5}} & \textit{if } r \in [0, 2],\\ r + 2 & \textit{otherwise}. \end{cases},$$

and $\beta : X \times X \to [0, \infty)$ by

$$\beta(r, s) = \begin{cases} 1 & \textit{if } r, s \in [0, 2],\\ 0 & \textit{otherwise}. \end{cases}$$

Now using control functions $\psi_1(p) = p$ and

$$\psi_2(p) = \begin{cases} \frac{p(p+4)}{p+5} & \textit{if } p \in [0, 2],\\ \frac{(p+1)}{2} & \textit{if } p \geq 2, \end{cases}$$

Then F is increasing and continuous, $0 \preceq F0$.
We will prove the following:

(i) *$F : X \to X$ is an $(\beta - \psi_1 - \psi_2)$-contractive mapping of type-II, with $\psi_1(p) = p\ \forall\ p \geq 0$;*
(ii) *F is β-admissible;*
(iii) *There exist $r_1 = 0 \in X$ such that $r_1 \preceq Fr_1$ and $\beta(r_1, Fr_1) \geq 1$;*
(iv) *If a sequence $\{r_n\}_{n=1}^{\infty}$ in X such that $\beta(r_n, r_{n+1}) \geq 1$ and $r_n \to r$, as $n \to \infty$, then $\beta(r_n, r) \geq 1,\ \forall\ n \in \mathbb{N}$.*

Proof:

(i) Obviously F is $(\beta - \psi_1 - \psi_2)$-contractive mapping with $\psi_1(p) = p,\ \forall\ p \geq 0,$ *since* $\forall\ r, s \in X$,

$$\begin{aligned}&\beta(r, Fr)\beta(s, Fs)\psi_1\big(kp^E(Fr, Fs)\big)\\ &\quad= \psi_1\left(p^E\left(\frac{r}{\sqrt{r^2+5}}, \frac{s}{\sqrt{s^2+5}}\right)\right)\\ &\quad= \psi_1\left(\left(\max\left\{\frac{r}{\sqrt{r^2+5}}, \frac{s}{\sqrt{s^2+5}}\right\}\right)^2\right).\end{aligned}$$

keeping generality intact if $0 \leq y \leq x \leq 2$, then

$$\begin{aligned}&\beta(r, Fr)\beta(s, Fs)\psi_1\big(kp^E(Fr, Fs)\big)\\ &\quad= \psi_1\left(p^E\left(\frac{r}{\sqrt{r^2+5}}, \frac{s}{\sqrt{s^2+5}}\right)\right)\\ &\quad= \psi_1\left(\left(\frac{r}{\sqrt{r^2+5}}\right)^2\right)\\ &\quad= \left(\frac{r^2}{r^2+5}\right)\end{aligned}$$

and

$$\Delta_u^F(r, s)$$
$$= \max\left\{\begin{array}{c} r^2, s^2, r^2,\\ \frac{r^2 + \left(\max\left\{s, \frac{r}{\sqrt{r^2+5}}\right\}\right)^2}{4},\\ \frac{(r^2)(r^2) + (s^2)\left(\max\left\{s, \frac{r}{\sqrt{r^2+5}}\right\}\right)^2}{1 + (r^2+s^2)},\\ \frac{(r^2) + (s^2)\left(\max\left\{s, \frac{r}{\sqrt{r^2+5}}\right\}\right)^2}{1 + r^2 + s^2} \end{array}\right\} = r^2.$$

Then

$$\begin{aligned}&\beta(r, Fr)\beta(s, Fs)\psi_1\big(kp^E(Fr, Fs)\big)\\ &\quad= \left(\frac{r^2}{r^2+5}\right) \leq r^2 - \left(\frac{r^4 + 4r^2}{r^2+5}\right)\\ &\quad\leq \psi_1(r^2) - \psi_2(r^2)\\ &\quad= \psi_1\big(\Delta_u^F(r, s)\big) - \psi_2\big(\Delta_u^F(r, s)\big).\end{aligned}$$

(ii) Let $(r, s) \in X \times X$ be such that $\beta(r, s) \geq 1$. From the definition of F and β we have both $Fr = \frac{r}{\sqrt{r^2+5}}$, and $Fs = \frac{s}{\sqrt{s^2+5}}$ are in $[0, 2]$, so we have $\beta(Fr, Fs) = 1 \geq 1$. Then F is an β-admissible.
(iii) Taking $r_1 = 1 \in X$, we have

$$\beta(r_1, Fr_1) = \beta(1, F1) = \beta(1, \frac{1}{\sqrt{6}}) = 1 \geq 1.$$

(iv) Consider a sequence $\{r_n\}$ in X such that $\beta(r_n, r_{n+1}) \geq 1$, for all $n \in \mathbb{N}$ and $r_n \to r \in X$ as $n \to \infty$. Since $\beta(r_n, r_{n+1}) \geq 1,\ \forall\ n \in \mathbb{N}$ and by the definition of β, we obtain $r_n \in [0, 2],\ \forall\ n \in \mathbb{N}$ and $r \in [0, 2]$. Then $\beta(r_n, r) = 1 \geq 1$.

At this point, every hypothesis in Theorem 19 is true. Therefore, F has a fixed point.

It is noted that without for the β term, the contraction condition (1) is not true. For example at $r = 0$ and $s = 3$, we obtain

$$\begin{aligned}\psi_1\big(p^E(F0, F3)\big) &= \psi_1\big(F^E(0, 5)\big)\\ &= \psi_1(25) = 25 \nleq 12 = 25 - 13\\ &= \psi_1(25) - \psi_2(25)\\ &= \psi_1\big(\Delta_s^F(0,3)\big) - \psi_2\big(\Delta_s^F(0,3)\big).\end{aligned}$$

■

IV. An Application To Integral Equation

Consider the following integral equation

$$x(v) = \int_0^{\tau} K(v, q)h\big(q, x(q)\big)dq\ \forall\ p \in I = [0, \tau], \quad (52)$$

where $\tau > 0, h : I \times \mathbb{R} \to \mathbb{R}$ and $K : [0, \tau] \times [0, \tau] \to [0, \infty)$ are continuous functions.

In this section we prove an existence theorem for a solution of (52) using Theorem 17.
Assume the space

$$X = C(I, \mathbb{R}) = \{x : I \to \mathbb{R} \mid x \text{ is continuous on I}\}.$$

Define $p^E : X \times X \to \mathbb{R}^+$ by

$$p^E(x, y) = \sup_{p \in [0,\tau]} \Big(|x(p)| + |y(p)|\Big)^2 \ \forall\ x, y \in X.$$

Obviously, $(X, p^E, 2)$ is a complete partially E-cone metric space. Define an order relation $\preceq$ on X by

$$r \preceq s \text{ iff } r(p) \leq s(p),\ \forall\ p \in I.$$

Then $(X, \preceq)$ is a partially ordered set. Thus, $(X, p^E, 2, \preceq)$ is a complete ordered partially E-cone metric space.

Theorem 25. *Consider the following assertions hold:*

(i) $f : I \times \mathbb{R} \to \mathbb{R}$ *is continuous;*

(ii) *A nondecreasing function* $h(k, \cdot),\ \forall\ k \in [0, \tau]$, *i.e.,*

$$r, s \in \mathbb{R}, r \leq s \Rightarrow h(q, r) \leq h(q, s);$$

(iii) *consider*

$$|h(q, r)| + |h(q, s)| \leq \Theta_u(r, s)$$

where

$$\Theta_u(r, s) = \max \left\{ \begin{array}{c} (r+s)^2, \\ (r+Fr)^2, (s+Fs)^2, \\ \left(\frac{(r+Fs)^2+(s+Fs)^2}{4k}\right), \\ \left(\frac{(r+Fr)(s+Fs)}{1+(r+s)}\right)^2, \\ \left(\frac{(r+Fr)(s+Fs)}{1+(Fr+Fs)}\right)^2 \end{array} \right\}^{\frac{1}{2}}$$

for all $r, s \in X$ *with* $r \preceq s$ *and for every* $q \in I$;

(iv) *there exist* $x_0 \in C(I, \mathbb{R})$ *such that*

$$x_0(t) \leq \int_0^\tau K(p, q)h\big(p, r_0(q)\big)dq, \quad p, q \in I;$$

(v) $\displaystyle\sup_{p \in [0,\tau]} \int_0^\tau K(p, q)dq \leq \frac{1}{2}.$

Then $x^* \in C(I, \mathbb{R})$ *is a solution of the integral equation (52).*

Proof: A mapping $F : X \times X$ is defined as

$$Fx(v) = \int_0^\tau K(v, q)h\big(q, x(q)\big)dq\ \forall\ v \in [0, \tau].$$

It follows from (i)-(ii) that p is non-decreasing and continuous mapping with respect to $\preceq$. Again (iv), $x_0 \in F$ exist such that $x_0 \preceq Fx_0$.

For all $v \in [0, \tau]$, and condition (iii) and (v), we get,

$$\begin{aligned}
&\Big(|F(r)| + |F(s)|\Big)^2 \\
&= \left(\begin{array}{l} \left|\int_0^\tau K(v, q)h\big(q, r(q)\big)dq\right| \\ \quad + \left|\int_0^\tau K(v, q)h\big(q, s(q)\big)dq\right| \end{array} \right)^2 \\
&\leq \left(\begin{array}{l} \int_0^\tau K(v, q)\big|h\big(q, r(q)\big)\big|dq \\ \quad + \int_0^\tau K(v, q)\big|h\big(q, s(q)\big)\big|dq \end{array} \right)^2 \\
&= \left(\int_0^\tau K(v, q)\left(\big|h\big(q, r(q)\big) + h\big(q, s(q)\big)\big|\right)dq\right)^2 \\
&\leq \left(\int_0^\tau K(v, q)\big(\Theta_u(r, s)\big)dq\right)^2 \\
&\leq \left(\int_0^\tau K(v, q) \max \left\{ \begin{array}{c} (r, s)^2, \\ (r+Fr)^2, \\ (s+Fs)^2, \\ \frac{(r+Fs)^2+(s+Fs)^2}{4k}, \\ \left(\frac{(r+Fr)(s+Fs)}{1+(r+s)}\right)^2, \\ \left(\frac{(r+Fr)(s+Fs)}{1+(Fr+Fs)}\right)^2 \end{array} \right\}^{\frac{1}{2}} dq \right)^2 \\
&\leq \frac{1}{4} \max \left\{ \begin{array}{c} \big(|r| + |s|\big)^2, \\ \big(|r| + |Fr|\big)^2, \big(|s| + |Fs|\big)^2, \\ \frac{\big(|r|+|Fs|\big)^2+\big(|s|+|Fs|\big)^2}{4k}, \\ \left(\frac{\big(|r|+|Fr|\big)\big(|s|+|Fs|\big)}{1+\big(|r|+|s|\big)}\right)^2, \\ \left(\frac{\big(|r|+|Fr|\big)\big(|s|+|Fs|\big)}{1+\big(|Fr|+|Fs|\big)}\right)^2 \end{array} \right\}
\end{aligned}$$

Now, by considering the control functions $\psi_1, \psi_2 : [0, +\infty)$ into itself defined by:

$$\psi_1(v) = p, \quad \text{and} \quad \psi_2(p) = \frac{3p}{4},\ \text{for } p \geq 0,$$

we get

$$\begin{aligned}
&\psi_1\big(p^E(Fr, Fs)\big) \\
&\leq \psi_1 \left(\max \left\{ \begin{array}{c} p^E(r+s), \\ p^E(r+Fr), p^E(s, Fs), \\ \frac{p^E(r+Fs)+p^E(s+Fs)}{4k}, \\ \left(\frac{p^E(r+Fr)p^E(s+Fs)}{1+p^E(r+s)}\right), \\ \left(\frac{p^E(r+Fr)p^E(s+Fs)}{1+p^E(Fr+Fs)}\right) \end{array} \right\} \right) \\
&\quad - \psi_2 \left(\max \left\{ \begin{array}{c} p^E(r+s), \\ p^E(r+Fr), p^E(s, Fs), \\ \frac{p^E(r+Fs)+p^E(s+Fs)}{4k}, \\ \left(\frac{p^E(r+Fr)p^E(s+Fs)}{1+p^E(r+s)}\right), \\ \left(\frac{p^E(r+Fr)p^E(s+Fs)}{1+p^E(Fr+Fs)}\right) \end{array} \right\} \right)
\end{aligned}$$

Thus all the hypotheses of Corollary 21 are fulfilled. Therefore $x^* \in C(I, \mathbb{R})$ is a solution of the integral equation (52). ■

V. Conclusion

The investigation into fixed point theorems for $(\beta-\psi_1-\psi_2)$ contractive conditions in partially E-cone metric spaces offers valuable insights into the behavior of mappings within this specific framework. Through rigorous analysis and mathematical reasoning, this research has contributed to our understanding of fixed point existence and uniqueness in partially ordered cone metric spaces. The study has illuminated the significance of $(\beta-\psi_1-\psi_2)$ contractive conditions in establishing the existence of fixed points for mappings, providing a theoretical foundation for further exploration in this area. By elucidating the properties and implications of these contractive conditions, the research has enriched the discourse on fixed point theory and its applications in metric spaces. Despite the progress made in this study, notable gaps in research remain, particularly concerning the extension and generalization of results to broader classes of mappings and spaces. Future research endeavors should aim to address these gaps, thereby enhancing our understanding of Fixed Foint Theory in diverse contexts and uncovering new avenues for exploration.

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