# Refinements of Singular Value and Unitarily Invariant Norm Inequalities

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***Abstract*—In this paper, by combining the Kantorovich constant and scalar inequalities of the weighted algebraic mean for sector matrices, we derive an inequality that relates the inverse of the real part of a matrix to the real part of its inverse within the Cartesian decomposition for sector matrices. Additionally, this inequality is used to enhance two existing singular value inequalities for sector matrices under certain conditions. Furthermore, leveraging the convexity of relevant functions, we provide two refinements of unitarily invariant norm inequalities for matrices.**

***Index Terms*—inequality, sector matrix, Kantorovich constant, singular value, unitarily invariant norm**

## I. Introduction

In this paper, let $M, m, M', m'$ represent scalars. Denote by $M_n$ the space of $n \times n$ complex matrices, with the identity matrix of this space being $I_n$. The Cartesian decomposition of $A$ is given as follows

$$A = \mathcal{R}(A) + i\mathcal{I}(A),$$

where $\mathcal{R}(A)$ and $\mathcal{I}(A)$ denote the real and imaginary parts of $A$, respectively.

Here and in what follows, a norm $\|\cdot\|$ is said to be unitarily invariant norm if $\|UAV\| = \|A\|$ for any $A \in M_n$ with all unitary matrices $U, V \in M_n$. The *Ky Fan k-norm*$\|\cdot\|_{(k)}$ is expressed as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j\,(A), k = 1, \cdots, n,$$

where $s_j\,(A)\,(j = 1, 2, \cdots, n)$ represents $j$-th largest singular of $A$ with $s_1\,(A) \geq s_2\,(A) \geq \cdots \geq s_n\,(A)$. These singular values correspond to the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, which are listed in decreasing order and repeated according to multiplicity. The *Schattenp-norm* $\|\cdot\|_p$ is defined as

$$\|A\|_p = (\sum_{j=1}^{n} s_j^p\,(A))^{1/p} = (\operatorname{tr}|A|^p)^{1/p}\,, \quad 1 \leq p < \infty.$$

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It is well known that both the *Ky Fan k-norm*$\|\cdot\|_{(k)}$ and the *Schattenp-norm* $\|\cdot\|_p$ are unitarily invarian norms (See [1]).

The Kantorovich constant, denoted by $K(h') = \frac{(h'+1)^2}{4h'}$, satisfies the following properties:

(i) $K(1) = 1$;
(ii) $K(h') = K(\frac{1}{h'})$ for $h > 0$;
(iii) $K(h')$ is monotonic, increasing on $[1, +\infty)$ and decreasing on $(0, 1]$ (See [2]).

For $\alpha \in [0, \frac{\pi}{2})$, we define a sector $S_\alpha$ as

$$S_\alpha = \{z \in C : \mathcal{R}(z) > 0, |\mathcal{I}(z)| \leq \tan\alpha\mathcal{R}(z)\}.$$

The numerical range of $A \in M_n$ is given by the set on the complex plane [3]:

$$W(A) = \{x^*Ax | x \in C^n, x^*x = 1\}.$$

If $W(A) \subset S_\alpha$, for some $\alpha \in [0, \frac{\pi}{2})$, then $A$ is called a sector matrix [4]. Clearly, if $W(A) \subset S_\alpha$, then $\mathcal{R}(A)$ is positive definite. Furthermore, if $A$ and $B$ are sector matrices, then $A+B$ and $(A+B)^{-1}$ are also sector matrices. Subsequently, Numerous influential studies have investigated inequalities involving sector matrices, particularly those related to singular values, unitarily invariant norms and determinants. Notable references include [5-9].

If $A, B \in M_n$ are positive definite, we adopt the notation as introduced in [10]

$$A\nabla_v B = (1-v)A + vB,$$

$$A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}, 0 \leq v \leq 1,$$

when $v = \frac{1}{2}$, we denote $A\nabla B$ and $A\sharp B$ in place of $A\nabla_{\frac{1}{2}}B$ and $A\sharp_{\frac{1}{2}}B$, respectively.

It is clear that

$$A\nabla_v B \geq A\sharp_v B, 0 \leq v \leq 1. \tag{1}$$

Garg and Aujla [11] derived that if $A, B \in M_n$ are positive semidefinite, then

$$\prod_{j=1}^{l} s_j(A+B) \leq \prod_{j=1}^{l} s_j(I_n + A) \prod_{j=1}^{l} s_j(I_n + B) \tag{2}$$

and

$$\prod_{j=1}^{l} s_j(I_n + A + B) \leq \prod_{j=1}^{l} s_j(I_n + A) \prod_{j=1}^{l} s_j(I_n + B), \tag{3}$$

where $1 \leq l \leq n$.

For $A, B \in M_n$ with $W(A), W(B) \subset S_\alpha$, Nasiri and Furuichi [12] obtained

$$\prod_{j=1}^{l} s_j(A+B)^{-1} \leq \frac{\sec^{4l}(\alpha)}{4^l} \prod_{j=1}^{l} s_j(I_n + A^{-1}) \prod_{j=1}^{l} s_j(I_n + B^{-1}) \tag{4}$$

and

$$\prod_{j=1}^{l} s_j(I_n+(A+B)^{-1}) \le \sec^{2l}(\alpha)\prod_{j=1}^{l} s_j(I_n+\frac{\sec^2(\alpha)}{4}A^{-1}) \prod_{j=1}^{l} s_j(I_n+\frac{\sec^2(\alpha)}{4}B^{-1}), \quad (5)$$

where $1 \le l \le n$.

Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then, the function

$$\Psi(r) = ||A^r XB^{2-r} + A^{2-r}XB^r||$$

is convex on $[0, 2]$. $\Psi(r)$ is decreasing on $[0, 1]$ and increasing on $[1, 2]$, moreover, $\Psi(r) = \Psi(2 - r)$ for $r \in [0, 2]$, consequently $\Psi(1) \le \Psi(r)$, which implies that

$$2\|AXB\| \le \left\|A^r XB^{2-r} + A^{2-r}XB^r\right\|, \quad (6)$$

where $0 \le r \le 2$.

Zhan [13] showed that for $A, B, X \in M_n$, if $A$ and $B$ are positive semidefinite, then

$$\left\|A^r XB^{2-r} + A^{2-r}XB^r\right\| \le \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|, \quad (7)$$

where $\frac{1}{2} \le r \le \frac{3}{2}$ and $-2 < t \le 2$. Combining (6) with (7), the following result is obvious

$$2\|AXB\| \le \left\|A^r XB^{2-r} + A^{2-r}XB^v\right\| \le \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|. \quad (8)$$

Later, Fu and He [14] demonstrated a stronger version of inequality (8) as follows:

$$2\|AXB\| + 2\Big(\int_{\frac{1}{2}}^{\frac{3}{2}} ||A^r XB^{2-r} + A^{2-r}XB^r||dr - 2||AXB||\Big) \le \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|. \quad (9)$$

Recently, by utilizing the convexity of the function $\Psi(r)$, Xue and Hu [15] obtained a refined version of inequality (9), which can be expressed as:

$$4\Big(\int_{\frac{1}{2}}^{\frac{3}{2}} ||A^r XB^{2-r} + A^{2-r}XB^r||dr - ||AXB|| - \frac{1}{2}||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}||\Big) + 2||AXB|| \le \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|. \quad (10)$$

Bhatia and Davis [16] derived that for $A, B, X \in M_n$, if $A$ and $B$ are positive semidefinite, then

$$||A^{\frac{1}{2}}XB^{\frac{1}{2}}|| \le ||\frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2}|| \le ||\frac{AX + XB}{2}||, \quad (11)$$

where $0 \le \nu \le 1$.

Setting

$$\psi(\nu) = ||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||,$$

then inequality (11) can be simply rewritten as

$$\psi(\frac{1}{2}) \le \psi(\nu) \le \psi(0).$$

Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then, the function $\psi(\nu)$ is convex on $[0, 1]$. $\psi(\nu)$ is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$, moreover, $\psi(\nu) = \psi(1 - \nu)$ for $\nu \in [0, 1]$. Using the convexity of $\psi(\nu)$, Zou and He [1] gave a stronger version of $\psi(\frac{1}{2}) \le \psi(0)$ as follows:

$$\psi(\frac{1}{2}) + 2\Big(\int_0^1 \psi(\nu)d\nu - \psi(\frac{1}{2})\Big) \le \psi(0), 0 \le \nu \le 1. \quad (12)$$

Additional information on convex functions can be found in the recent paper [17] and references therein.

This note, building on the preceding discussions, focuses on refining singular value inequalities for sector matrices and inequalities involving unitarily invariant norms. The structure of the note is as follows. In Section 2, we introduce a new inequality for sector matrices and use it to derive two sharper singular value inequalities, which improve inequalities (4) and (5) under certain conditions. Section 3 is dedicated to providing two refinements of inequalities (10) and (12) by utilizing the properties of convexity. Finally, Section 4 provides concluding remarks.

## II. SINGULAR VALUE INEQUALITIES FOR SECTOR MATRICES

Before presenting our main results, we first summarize four lemmas that will be instrumental in the proof of the results in this paper.

**Lemma 1** ([4]). Let $A \in M_n$ with $W(A) \subset S_\alpha$. Then

$$\mathcal{R}(A^{-1}) \le \mathcal{R}^{-1}(A) \le \sec^2(\alpha)\mathcal{R}(A^{-1}).$$

**Lemma 2** ([18]). Let $A, B \in M_n$ be positive definite such that $0 < mI_n \le A \le m'I_n < M'I_n \le B \le MI_n$ or $0 < mI_n \le B \le m'I_n < M'I_n \le A \le MI_n$. Then

$$K^r(h')(A^{-1}\#_v B^{-1}) \le A^{-1}\nabla_v B^{-1},$$

where $0 \le v \le 1$, $h' = \frac{M'}{m'}$, $K(h') = \frac{(h'+1)^2}{4h'}$ and $r = \min\{v, 1 - v\}$.

**Lemma 3** ([3]). Let $A \in M_n$ with $W(A) \subset S_\alpha$. Then

$$s_j(A) \le \sec^2(\alpha)s_j(\mathcal{R}(A)), 1 \le j \le n.$$

**Lemma 4** ([3]). Let $A \in M_n$. Then

$$s_j(\mathcal{R}(A)) \le s_j(A), 1 \le j \le n.$$

We are now prepared to present the first theorem in this note.

**Theorem 1.** Let $A, B \in M_n$ with $W(A), W(B) \subset S_\alpha$ such that $0 < mI_n \le \mathcal{R}(A) \le m'I_n < M'I_n \le \mathcal{R}(B) \le MI_n$ or $0 < mI_n \le \mathcal{R}(B) \le m'I_n < M'I_n \le \mathcal{R}(A) \le MI_n$. Then

$$\mathcal{R}(A\nabla_v B)^{-1} \le \frac{\sec^2(\alpha)}{K^r(h')}\mathcal{R}(A^{-1}\nabla_v B^{-1}), \quad (13)$$

where $0 \le v \le 1$, $h' = \frac{M'}{m'}$, $K(h') = \frac{(h'+1)^2}{4h'}$ and $r = \min\{v, 1-v\}$.

**Proof.** Compute

$$\begin{aligned}
&\mathcal{R}((1-v)A+vB)^{-1} \\
&\le \mathcal{R}^{-1}((1-v)A+vB) \\
&\quad (\text{by Lemma 1}) \\
&= ((1-v)\mathcal{R}(A)+v\mathcal{R}(B))^{-1} \\
&\le (\mathcal{R}(A)\sharp_v\mathcal{R}(B))^{-1} \quad (\text{by (1)}) \\
&= \mathcal{R}^{-1}(A)\sharp_v\mathcal{R}^{-1}(B) \\
&\le \frac{1}{K^r(h')}((1-v)\mathcal{R}^{-1}(A)+v\mathcal{R}^{-1}(B)) \\
&\quad (\text{by Lemma 2}) \\
&\le \frac{\sec^2(\alpha)}{K^r(h')}((1-v)\mathcal{R}(A^{-1})+v\mathcal{R}(B^{-1})) \\
&\quad (\text{by Lemma 1}) \\
&= \frac{\sec^2(\alpha)}{K^r(h')}\mathcal{R}((1-v)A^{-1}+vB^{-1}).
\end{aligned}$$

This completes the proof.

In the following, we will apply Theorem 1 to derive two singular value inequalities for sector matrices, which provide improved versions of inequalities (4) and (5).

**Theorem 2.** Let $A, B \in M_n$ with $W(A), W(B) \subset S_\alpha$ such that $0 < mI_n \le \mathcal{R}(A) \le m'I_n < M'I_n \le \mathcal{R}(B) \le MI_n$ or $0 < mI_n \le \mathcal{R}(B) \le m'I_n < M'I_n \le \mathcal{R}(A) \le MI_n$. Then

$$\begin{aligned}
&\prod_{j=1}^{l} s_j(A+B)^{-1} \\
&\le \frac{\sec^{4l}(\alpha)}{4^l K^{\frac{l}{2}}(h')} \prod_{j=1}^{l} s_j(I_n+A^{-1}) \\
&\quad \prod_{j=1}^{l} s_j(I_n+B^{-1})
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\prod_{j=1}^{l} s_j(I_n+(A+B)^{-1}) \\
&\le \sec^{2l}(\alpha) \prod_{j=1}^{l} s_j(I_n+\frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}A^{-1}) \\
&\quad \prod_{j=1}^{l} s_j(I_n+\frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}B^{-1}),
\end{aligned} \tag{15}$$

where $0 \le v \le 1$, $1 \le l \le n$, $h' = \frac{M'}{m'}$, $K(h') = \frac{(h'+1)^2}{4h'}$ and $r = \min\{v, 1-v\}$.

**Proof.** For $A, B \in M_n$ with $W(A), W(B) \subset S_\alpha$, we demonstrate that $(A+B)^{-1}$ is a sector matrix.

Compute

$$\begin{aligned}
&\prod_{j=1}^{l} s_j(A+B)^{-1} \\
&\le \sec^{2l}(\alpha) \prod_{j=1}^{l} s_j(\mathcal{R}(A+B)^{-1}) \\
&\quad (\text{by Lemma 3}) \\
&\le \frac{\sec^{4l}(\alpha)}{4^l K^{\frac{l}{2}}(h')} \prod_{j=1}^{l} s_j(\mathcal{R}(A^{-1}+B^{-1})) \\
&\quad (\text{by (13)}) \\
&= \frac{\sec^{4l}(\alpha)}{4^l K^{\frac{l}{2}}(h')} \prod_{j=1}^{l} s_j(\mathcal{R}(A^{-1})+\mathcal{R}(B^{-1})) \\
&\le \frac{\sec^{4l}(\alpha)}{4^l K^{\frac{l}{2}}(h')} \prod_{j=1}^{l} s_j(I_n+\mathcal{R}(A^{-1})) \\
&\quad \prod_{j=1}^{l} s_j(I_n+\mathcal{R}(B^{-1})) \\
&\quad (\text{by (2)}) \\
&= \frac{\sec^{4l}(\alpha)}{4^l K^{\frac{l}{2}}(h')} \prod_{j=1}^{l} s_j(\mathcal{R}(I_n+A^{-1})) \\
&\quad \prod_{j=1}^{l} s_j(\mathcal{R}(I_n+B^{-1})) \\
&\le \frac{\sec^{4l}(\alpha)}{4^l K^{\frac{l}{2}}(h')} \prod_{j=1}^{l} s_j(I_n+A^{-1}) \\
&\quad \prod_{j=1}^{l} s_j(I_n+B^{-1}) \\
&\quad (\text{by Lemma 4}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\prod_{j=1}^{l} s_j(I_n+(A+B)^{-1}) \\
&\le \sec^{2l}(\alpha) \prod_{j=1}^{l} s_j(\mathcal{R}(I_n+(A+B)^{-1})) \\
&\quad (\text{by Lemma 3}) \\
&\le \sec^{2l}(\alpha) \prod_{j=1}^{l} s_j(I_n+\frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}\mathcal{R}(A^{-1}+B^{-1})) \\
&\quad (\text{by (13)}) \\
&= \sec^{2l}(\alpha) \prod_{j=1}^{l} s_j(I_n+\frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}\mathcal{R}(A^{-1}) \\
&\quad +\frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}\mathcal{R}(B^{-1}))
\end{aligned}$$

$$\leq \sec^{2l}(\alpha)\prod_{j=1}^{l} s_j(I_n + \frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}\mathcal{R}(A^{-1}))$$

$$\prod_{j=1}^{l} s_j(I_n + \frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}\mathcal{R}(B^{-1}))$$

$$(\text{by}\,(3))$$

$$= \sec^{2l}(\alpha)\prod_{j=1}^{l} s_j(\mathcal{R}(I_n + \frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}A^{-1}))$$

$$\prod_{j=1}^{l} s_j(\mathcal{R}(I_n + \frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}B^{-1}))$$

$$\leq \sec^{2l}(\alpha)\prod_{j=1}^{l} s_j(I_n + \frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}A^{-1})$$

$$\prod_{j=1}^{l} s_j(I_n + \frac{\sec^2(\alpha)}{4K^{\frac{1}{2}}(h')}B^{-1})$$

$$(\text{by}\,\text{Lemma}\,4).$$

This completes the proof.

**Remark 1.** Under the conditions such that $0 < mI_n \leq \mathcal{R}(A) \leq m'I_n < M'I_n \leq \mathcal{R}(B) \leq MI_n$ or $0 < mI_n \leq \mathcal{R}(B) \leq m'I_n < M'I_n \leq \mathcal{R}(A) \leq MI_n$, because of $K(h') = \frac{(h'+1)^2}{4h'} \geq 1$ with $h' \geq 1$, inequalities (14) and (15) are sharper than inequalities (4) and (5).

## III. UNITARILY INVARIANT NORM INEQUALITIES

In this section, we leverage the convexity of the functions $\Psi(r)$ and $\psi(\nu)$ to derive two unitarily invariant norm inequalities for matrices, which yield enhanced versions of inequalities (10) and (12), respectively. To begin our discussion, we first present the following two lemmas.

**Lemma 5** ([19])**.** Let $f$ be a real valued convex function on the interval $[a, b]$ which contains $(x_1, x_2)$. Then for $x_1 \leq x \leq x_2$, we have

$$f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}.$$

**Lemma 6** ([20])**.** Let $\varphi$ be a real valued convex function on the interval $[a, b]$. Then

$$\varphi(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b \varphi(t)dt \leq \frac{\varphi(a) + \varphi(b)}{2}.$$

**Theorem 3.** Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then

$$2||AXB||$$

$$+4(\int_{\frac{1}{2}}^{\frac{3}{2}} ||A^r XB^{2-r} + A^{2-r}XB^r||dr$$

$$-\frac{3}{10}||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}||$$

$$-\frac{1}{4}||A^{\frac{4}{5}}XB^{\frac{6}{5}} + A^{\frac{6}{5}}XB^{\frac{4}{5}}|| - \frac{9}{10}||AXB||)$$

$$\leq \frac{2}{t+2}\left\|A^2X + tAXB + XB^2\right\|,$$

where $\frac{1}{2} \leq r \leq \frac{3}{2}$, $-2 < t \leq 2$.

**Proof.** If $\frac{1}{2} \leq r \leq \frac{3}{4}$, applying Lemma 5 and utilizing the convexity of the function $\Psi(r)$, we have

$$\Psi(r) \leq \frac{\Psi(\frac{3}{4}) - \Psi(\frac{1}{2})}{\frac{3}{4} - \frac{1}{2}}r - \frac{\frac{1}{2}\Psi(\frac{3}{4}) - \frac{3}{4}\Psi(\frac{1}{2})}{\frac{3}{4} - \frac{1}{2}},$$

which is equivalent to

$$\Psi(r) \leq (3-4r)\Psi(\frac{1}{2}) + 2(2r-1)\Psi(\frac{3}{4}).$$

So

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \Psi(r)dr$$

$$\leq \Psi(\frac{1}{2})\int_{\frac{1}{2}}^{\frac{3}{4}}(3-4r)dr + 2\Psi(\frac{3}{4})\int_{\frac{1}{2}}^{\frac{3}{4}}(2r-1)dr,$$

that is

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \Psi(r)dr \leq \frac{1}{8}[\Psi(\frac{1}{2}) + \Psi(\frac{3}{4})]. \tag{16}$$

If $\frac{3}{4} \leq r \leq \frac{4}{5}$ , applying Lemma 5 and the convexity of the function $\Psi(r)$, we obtain

$$\Psi(r) \leq \frac{\Psi(\frac{4}{5}) - \Psi(\frac{3}{4})}{\frac{4}{5} - \frac{3}{4}}r - \frac{\frac{3}{4}\Psi(\frac{4}{5}) - \frac{4}{5}\Psi(\frac{3}{4})}{\frac{4}{5} - \frac{3}{4}},$$

which is equivalent to

$$\Psi(r) \leq 4(4-5r)\Psi(\frac{3}{4}) + 5(4r-3)\Psi(\frac{4}{5}).$$

So

$$\int_{\frac{3}{4}}^{\frac{4}{5}} \Psi(r)dr$$

$$\leq 4\Psi(\frac{3}{4})\int_{\frac{3}{4}}^{\frac{4}{5}}(4-5r)dr + 5\Psi(\frac{4}{5})\int_{\frac{3}{4}}^{\frac{4}{5}}(4r-3)dr,$$

that is

$$\int_{\frac{3}{4}}^{\frac{4}{5}} \Psi(r)dr \leq \frac{1}{40}[\Psi(\frac{3}{4}) + \Psi(\frac{4}{5})]. \tag{17}$$

If $\frac{4}{5} \leq r \leq 1$, applying Lemma 5 and the convexity of the function $\Psi(r)$, we deduce

$$\Psi(r) \leq \frac{\Psi(1) - \Psi(\frac{4}{5})}{1 - \frac{4}{5}}r - \frac{\frac{4}{5}\Psi(1) - \Psi(\frac{4}{5})}{1 - \frac{4}{5}},$$

which is equivalent to

$$\Psi(r) \leq 5(1-r)\Psi(\frac{4}{5}) + (5r-4)\Psi(1).$$

So

$$\int_{\frac{4}{5}}^{1} \Psi(r)dr$$

$$\leq 5\Psi(\frac{4}{5})\int_{\frac{4}{5}}^{1}(1-r)dr + \Psi(1)\int_{\frac{4}{5}}^{1}(5r-4)dr,$$

that is

$$\int_{\frac{4}{5}}^{1} \Psi(r)dr \leq \frac{1}{10}[\Psi(\frac{4}{5}) + \Psi(1)]. \tag{18}$$

If $1 \le r \le \frac{6}{5}$, similarly, we have

$$\Psi(r) \le \frac{\Psi(\frac{6}{5}) - \Psi(1)}{\frac{6}{5} - 1} r - \frac{\Psi(\frac{6}{5}) - \frac{6}{5}\Psi(1)}{\frac{6}{5} - 1},$$

which is equivalent to

$$\Psi(r) \le (6 - 5r)\Psi(1) + 5(r - 1)\Psi(\frac{6}{5}).$$

So

$$\int_1^{\frac{6}{5}} \Psi(r)dr \le \Psi(1) \int_1^{\frac{6}{5}} (6 - 5r)dr + 5\Psi(\frac{6}{5}) \int_1^{\frac{6}{5}} (r - 1)dr,$$

that is

$$\int_1^{\frac{6}{5}} \Psi(r)dr \le \frac{1}{10}[\Psi(1) + \Psi(\frac{6}{5})]. \tag{19}$$

If $\frac{6}{5} \le r \le \frac{5}{4}$, similarly, we obtain

$$\Psi(r) \le \frac{\Psi(\frac{5}{4}) - \Psi(\frac{6}{5})}{\frac{5}{4} - \frac{6}{5}} r - \frac{\frac{6}{5}\Psi(\frac{5}{4}) - \frac{5}{4}\Psi(\frac{6}{5})}{\frac{4}{5} - \frac{6}{5}},$$

which is equivalent to

$$\Psi(r) \le 5(5 - 4r)\Psi(\frac{6}{5}) + 4(5r - 6)\Psi(\frac{5}{4}).$$

So

$$\int_{\frac{6}{5}}^{\frac{5}{4}} \Psi(r)dr \le 5\Psi(\frac{6}{5}) \int_{\frac{6}{5}}^{\frac{5}{4}} (5 - 4r)dr + 4\Psi(\frac{5}{4}) \int_{\frac{6}{5}}^{\frac{5}{4}} (5r - 6)dr,$$

that is

$$\int_{\frac{6}{5}}^{\frac{5}{4}} \Psi(r)dr \le \frac{1}{40}[\Psi(\frac{6}{5}) + \Psi(\frac{5}{4})]. \tag{20}$$

If $\frac{5}{4} \le r \le \frac{3}{2}$, similarly, we deduce

$$\Psi(r) \le \frac{\Psi(\frac{3}{2}) - \Psi(\frac{5}{4})}{\frac{3}{2} - \frac{5}{4}} r - \frac{\frac{5}{4}\Psi(\frac{3}{2}) - \frac{3}{2}\Psi(\frac{5}{4})}{\frac{3}{2} - \frac{5}{4}},$$

which is equivalent to

$$\Psi(r) \le 2(3 - 2r)\Psi(\frac{5}{4}) + (4r - 5)\Psi(\frac{3}{2}).$$

So

$$\int_{\frac{5}{4}}^{\frac{3}{2}} \Psi(r)dr \le 2\Psi(\frac{5}{4}) \int_{\frac{5}{4}}^{\frac{3}{2}} (3 - 2r)dr + \Psi(\frac{3}{2}) \int_{\frac{5}{4}}^{\frac{3}{2}} (4r - 5)dr,$$

that is

$$\int_{\frac{5}{4}}^{\frac{3}{2}} \Psi(r)dr \le \frac{1}{8}[\Psi(\frac{5}{4}) + \Psi(\frac{3}{2})]. \tag{21}$$

It follows from (16)-(21) and $\Psi(\frac{1}{2}) = \Psi(\frac{3}{2})$, $\Psi(\frac{3}{4}) = \Psi(\frac{5}{4})$, $\Psi(\frac{4}{5}) = \Psi(\frac{6}{5})$ that

$$4\int_{\frac{1}{2}}^{\frac{3}{2}} \Psi(r)dr \le \Psi(\frac{1}{2}) + \frac{6}{5}\Psi(\frac{3}{4}) + \Psi(\frac{4}{5}) + \frac{4}{5}\Psi(1),$$

which is equivalent to

$$\Psi(1) + 4[\int_{\frac{1}{2}}^{\frac{3}{2}} \Psi(r)dr - \frac{3}{10}\Psi(\frac{3}{4}) - \frac{1}{4}\Psi(\frac{4}{5}) - \frac{9}{20}\Psi(1)] \le \Psi(\frac{1}{2}).$$

Thus

$$\begin{aligned}
&2||AXB|| \\
&+4(\int_{\frac{1}{2}}^{\frac{3}{2}} ||A^rXB^{2-r} + A^{2-r}XB^r||dr \\
&-\frac{3}{10}||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\
&-\frac{1}{4}||A^{\frac{4}{5}}XB^{\frac{6}{5}} + A^{\frac{6}{5}}XB^{\frac{4}{5}}|| - \frac{9}{10}||AXB||) \\
&\quad \le ||A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}}||.
\end{aligned}$$

By (7), we get

$$\begin{aligned}
&2||AXB|| \\
&+4(\int_{\frac{1}{2}}^{\frac{3}{2}} ||A^rXB^{2-r} + A^{2-r}XB^r||dr \\
&-\frac{3}{10}||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\
&-\frac{1}{4}||A^{\frac{4}{5}}XB^{\frac{6}{5}} + A^{\frac{6}{5}}XB^{\frac{4}{5}}|| - \frac{9}{10}||AXB||) \\
&\quad \le \frac{2}{t+2}||A^2X + tAXB + XB^2||.
\end{aligned}$$

This completes the proof.

**Remark 2.** Theorem 3 is sharper than inequality (10).

By the convexity of the function $\Psi(r)$, it follows that

$$\Psi(\frac{4}{5}) \le \frac{4}{5}\Psi(\frac{3}{4}) + \frac{1}{5}\Psi(1).$$

Thus

$$\begin{aligned}
&4(\int_{\frac{1}{2}}^{\frac{3}{2}} ||A^rXB^{2-r} + A^{2-r}XB^r||dr - \frac{9}{10}||AXB|| \\
&-\frac{3}{10}||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\
&-\frac{1}{4}||A^{\frac{4}{5}}XB^{\frac{6}{5}} + A^{\frac{6}{5}}XB^{\frac{4}{5}}||) \\
&-4(\int_{\frac{1}{2}}^{\frac{3}{2}} ||A^rXB^{2-r} + A^{2-r}XB^r||dr \\
&-\frac{1}{2}||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| - ||AXB||) \\
&= \frac{4}{5}||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| + \frac{2}{5}||AXB|| \\
&-||A^{\frac{4}{5}}XB^{\frac{6}{5}} + A^{\frac{6}{5}}XB^{\frac{4}{5}}|| \\
&\quad \ge 0.
\end{aligned}$$

Consequently, Theorem 3 is a refinement of inequality (10).

**Theorem 4.** Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then

$$\psi(\frac{1}{2})+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{1}{2}))$$
$$+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{1}{8}))+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{1}{4}))$$
$$+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{3}{8}))$$
$$\leq \psi(0),$$

where $\psi(\nu)=||A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu||$, $0\leq\nu\leq1$.

**Proof.** If $0\leq\nu\leq\frac{1}{8}$, by Lemma 5, we have

$$\psi(\nu)\leq 8(\psi(\frac{1}{8})-\psi(0))\nu+\psi(0).$$

So

$$\int_0^{\frac{1}{8}}\psi(\nu)d\nu\leq 8(\psi(\frac{1}{8})-\psi(0))\int_0^{\frac{1}{8}}\nu d\nu+\int_0^{\frac{1}{8}}\psi(0)d\nu,$$

which is equivalent to

$$\int_0^{\frac{1}{8}}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(\frac{1}{8})+\psi(0)). \tag{22}$$

If $\frac{1}{8}\leq\nu\leq\frac{1}{4}$, by Lemma 5, we obtain

$$\psi(\nu)\leq 8(\psi(\frac{1}{4})-\psi(\frac{1}{8}))(\nu-\frac{1}{8})+\psi(\frac{1}{8}).$$

So

$$\int_{\frac{1}{8}}^{\frac{1}{4}}\psi(\nu)d\nu\leq 8(\psi(\frac{1}{4})-\psi(\frac{1}{8}))\int_{\frac{1}{8}}^{\frac{1}{4}}(\nu-\frac{1}{8})d\nu+\int_{\frac{1}{8}}^{\frac{1}{4}}\psi(\frac{1}{8})d\nu,$$

which is equivalent to

$$\int_{\frac{1}{8}}^{\frac{1}{4}}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(\frac{1}{4})+\psi(\frac{1}{8})). \tag{23}$$

If $\frac{1}{4}\leq\nu\leq\frac{3}{8}$, by Lemma 5, we deduce

$$\psi(\nu)\leq 8(\psi(\frac{3}{8})-\psi(\frac{1}{4}))(\nu-\frac{1}{4})+\psi(\frac{1}{4}).$$

So

$$\int_{\frac{1}{4}}^{\frac{3}{8}}\psi(\nu)d\nu\leq 8(\psi(\frac{3}{8})-\psi(\frac{1}{4}))\int_{\frac{1}{4}}^{\frac{3}{8}}(\nu-\frac{1}{4})d\nu+\int_{\frac{1}{4}}^{\frac{3}{8}}\psi(\frac{1}{4})d\nu,$$

which is equivalent to

$$\int_{\frac{1}{4}}^{\frac{3}{8}}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(\frac{3}{8})+\psi(\frac{1}{4})). \tag{24}$$

If $\frac{3}{8}\leq v\leq\frac{1}{2}$, by Lemma 5, we have

$$\psi(\nu)\leq 8(\psi(\frac{1}{2})-\psi(\frac{3}{8}))(\nu-\frac{3}{8})+\psi(\frac{3}{8}).$$

So

$$\int_{\frac{3}{8}}^{\frac{1}{2}}\psi(\nu)d\nu\leq 8(\psi(\frac{1}{2})-\psi(\frac{3}{8}))\int_{\frac{3}{8}}^{\frac{1}{2}}(\nu-\frac{3}{8})d\nu+\int_{\frac{3}{8}}^{\frac{1}{2}}\psi(\frac{3}{8})d\nu,$$

which is equivalent to

$$\int_{\frac{3}{8}}^{\frac{1}{2}}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(\frac{1}{2})+\psi(\frac{3}{8})). \tag{25}$$

If $\frac{1}{2}\leq v\leq\frac{5}{8}$, similarly, we get

$$\int_{\frac{1}{2}}^{\frac{5}{8}}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(\frac{5}{8})+\psi(\frac{1}{2})). \tag{26}$$

If $\frac{5}{8}\leq\nu\leq\frac{3}{4}$, similarly, we get

$$\int_{\frac{5}{8}}^{\frac{3}{4}}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(\frac{3}{4})+\psi(\frac{5}{8})). \tag{27}$$

If $\frac{3}{4}\leq\nu\leq\frac{7}{8}$, similarly, we get

$$\int_{\frac{3}{4}}^{\frac{7}{8}}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(\frac{7}{8})+\psi(\frac{3}{4})). \tag{28}$$

If $\frac{7}{8}\leq\nu\leq 1$, similarly, we get

$$\int_{\frac{7}{8}}^{1}\psi(\nu)d\nu\leq\frac{1}{16}(\psi(1)+\psi(\frac{7}{8})). \tag{29}$$

It follows from (22)-(29) and $\psi(0)=\psi(1)$, $\psi(\frac{1}{8})=\psi(\frac{7}{8})$, $\psi(\frac{1}{4})=\psi(\frac{3}{4})$, $\psi(\frac{3}{8})=\psi(\frac{5}{8})$ that

$$\begin{aligned}\int_0^1\psi(\nu)d\nu &= \int_0^{\frac{1}{8}}\psi(\nu)d\nu+\int_{\frac{1}{8}}^{\frac{1}{4}}\psi(\nu)d\nu\\ &\quad+\int_{\frac{1}{4}}^{\frac{3}{8}}\psi(\nu)d\nu+\int_{\frac{3}{8}}^{\frac{1}{2}}\psi(\nu)d\nu\\ &\quad+\int_{\frac{1}{2}}^{\frac{5}{8}}\psi(\nu)d\nu+\int_{\frac{5}{8}}^{\frac{3}{4}}\psi(\nu)d\nu\\ &\quad+\int_{\frac{3}{4}}^{\frac{7}{8}}\psi(\nu)d\nu+\int_{\frac{7}{8}}^{1}\psi(\nu)d\nu\\ &\leq \frac{1}{8}(\psi(0)+2\psi(\frac{1}{8})+2\psi(\frac{1}{4})\\ &\quad+\psi(\frac{1}{2})+2\psi(\frac{3}{8})),\end{aligned}$$

which is equivalent to

$$\begin{aligned}8\int_0^1\psi(\nu)d\nu &\leq \psi(0)+2\psi(\frac{1}{8})+2\psi(\frac{1}{4})\\ &\quad+\psi(\frac{1}{2})+2\psi(\frac{3}{8}).\end{aligned}$$

Thus

$$\psi(\frac{1}{2})+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{1}{2}))$$
$$+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{1}{8}))+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{1}{4}))$$
$$+2(\int_0^1 \psi(\nu)d\nu-\psi(\frac{3}{8}))$$
$$\leq \psi(0).$$

This completes the proof.

**Remark 3.** Theorem 4 is sharper than inequality (12).

By Lemma 6, we have

$$\psi(\frac{1}{8}) \leq 4\int_{0}^{\frac{1}{4}} \psi(\nu)d\nu, \tag{30}$$

$$\psi(\frac{3}{8}) \leq 4\int_{\frac{1}{4}}^{\frac{1}{2}} \psi(\nu)d\nu \tag{31}$$

and

$$\psi(\frac{1}{4}) \leq 2\int_{0}^{\frac{1}{2}} \psi(\nu)d\nu. \tag{32}$$

It follows from (30)-(32) that

$$\psi(\frac{1}{8}) + \psi(\frac{3}{8}) + \psi(\frac{1}{4}) \leq 6\int_{0}^{\frac{1}{2}} \psi(\nu)d\nu. \tag{33}$$

By Lemma 6, we have

$$\psi(\frac{5}{8}) \leq 4\int_{\frac{1}{2}}^{\frac{3}{4}} \psi(\nu)d\nu, \tag{34}$$

$$\psi(\frac{7}{8}) \leq 4\int_{\frac{3}{4}}^{1} \psi(\nu)d\nu \tag{35}$$

and

$$\psi(\frac{3}{4}) \leq 2\int_{\frac{1}{2}}^{1} \psi(\nu)d\nu. \tag{36}$$

It follows from (34)-(36) that

$$\psi(\frac{5}{8}) + \psi(\frac{7}{8}) + \psi(\frac{3}{4}) \leq 6\int_{\frac{1}{2}}^{1} \psi(\nu)d\nu. \tag{37}$$

It follows from (33) and (37), $\psi(\frac{1}{8}) = \psi(\frac{7}{8})$, $\psi(\frac{1}{4}) = \psi(\frac{3}{4})$, $\psi(\frac{3}{8}) = \psi(\frac{5}{8})$ that

$$6\int_{0}^{1} \psi(\nu)d\nu - 2\psi(\frac{1}{8}) - 2\psi(\frac{3}{8}) - 2\psi(\frac{1}{4}) \geq 0.$$

Consequently, Theorem 4 is a refinement of inequality (12).

## IV. Conclusion

Sector matrices have potential applications in various fields such as image processing, numerical analysis, computational fluid dynamics and optimization problems. For instance, sector matrices can be used for efficient convolution operations in image filtering, particularly when performing computations on image blocks or specific regions, thus reducing computational complexity. Similarly, unitarily invariant norm inequalities are widely applied in areas such as quantum mechanics, signal processing, data analysis and optimization theory. In particular, in the context of quantum entanglement measures, these inequalities play a crucial role in maintaining the consistency of entanglement properties across different reference frames. As a result, studying sector matrices and unitarily invariant norm inequalities holds considerable theoretical and practical significance. This paper investigates the inverse of the matrix real part and the real part of the inverse matrix in the context of Cartesian decomposition for sector matrices. This is done using the Kantorovich constant and scalar inequalities related to the weighted algebraic mean for sector matrices. The inequality derived in this work leads to refinements of two existing singular value inequalities for sector matrices under specific conditions. Additionally, by utilizing the convexity of the functions $\Psi(r)$ and $\psi(\nu)$, we introduce two new unitarily invariant norm inequalities for matrices, which enhance and extend several previously known results. Future research will further explore these topics.

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