Characterizing Regular and Intra-regular Semigroups in Terms of Bipolar Complex Fuzzy Bi-Ideals

Pannawit Khamrot, Thiti Gaketem

Abstract—**In 2022, Rehman et al. established the concept of bipolar complex fuzzy sets and proved the properties of bipolar complex fuzzy ideals in semigroups. This research gives the concept of bipolar complex fuzzy bi-ideals in semigroups. We prove the basic properties of bipolar complex fuzzy bi-ideals and study the relationship between bipolar complex fuzzy ideals and bipolar complex fuzzy bi-ideals in semigroups. Finally, we characterize a regular and an intra-regular semigroup in terms of bipolar complex fuzzy bi-ideals.**

Index Terms—**BCF sets, BCF ideals, BCF bi-ideals, regular, intra-regular**

I. Introduction

THEORY of semigroups is an algebraic structure that was applied in computer science, coding theory, graph theory, medical science, formal languages, and many more. The bi-ideal in semigroups studied in 1952 by Good and Hughes [1]. The theory of bipolar complex fuzzy sets is an extension of bipolar fuzzy sets. It is studied in the structure of real numbers positive, negative, and imaginary numbers positive, and negative with generalizations of bipolar fuzzy set.

The concept of fuzzy sets by Zadeh in 1975, [2]. After that, it has applications in several areas like medical science, image processing, decision-making methods, etc. After, Kuroki [3] studied fuzzy subsemigroups and types of fuzzy ideals in semigroups. Jun and Song [4] present fuzzy interior ideals in semigroups. In 1994 Zhang [5] developed the notion of fuzzy set go to bipolar fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$, and used them for modeling and decision analysis. In 2000, Lee [6] used the term bipolar valued fuzzy sets and applied it to algebraic structures. The theory of complex fuzzy sets is interesting by Ramot et al. [7]. It is a tool for dealing with uncertainty and complex information. Tamir et al. [8] studied the complex fuzzy set in structure cartesian by transforming the range from the unit circle to the complex plane. Al-Husban [9] discussed complex fuzzy groups. Hu et al. [10] developed the complex fuzzy set in orthogonality and application to signal detection. The complex intuitionistic fuzzy soft sets introduced by Kumar and Bajaj [11]. Moreover, research in types bipolar fuzzy ideals, such as Kang [12], studied bipolar fuzzy subsemigroups in semigroups. Chinnadurau and Arulmozhi [13] discussed the bipolar fuzzy ideal in order Γ-semigroups, and Khamrot and Siripitukdet [14] explained generalized bipolar fuzzy subsemigroups in semigroups. Gaketem and Khamrot [15] studied bipolar weakly interior ideals in semigroups. Mahmood [16] introduced bipolar soft set. Gaketem et al. [17] expand cubic bipolar fuzzy subsemigroups and ideals in semigroups. In the same year, Rehman et al. [18] presented bipolar complex fuzzy sets and bipolar complex fuzzy ideals in semigroups. Recently, Khamrot et al. [19] presented the concept of bipolar complex fuzzy interior ideals and we prove relations between bipolar complex ideals and bipolar complex interior ideals in semigroups.

In this study, we give details of bipolar complex fuzzy bi-ideals in semigroups and discuss the properties of bipolar complex fuzzy bi-ideals in semigroups. The remainder of this paper is organized in the following. In Section 3, we study the connection biploar complex fuzzy ideals and bipolar complex fuzzy bi-ideals in semigroups. In Section 4, we characterize a regular and an intra-regular semigroup in terms of bipolar complex fuzzy bi-ideals. The conclusions are presented in Section 5.

II. Preliminaries

In this topic, we will survey some basic definitions and theorems of semigroups, fuzzy sets, bipolar fuzzy sets, and bipolar complex fuzzy sets, which will be helpful in the next topic. This paper will denote a semigroup (SG) by $\mathfrak{X}$.

By a *subsemigroup* (SSG) of $\mathfrak{X}$ we mean a non-empty subset $\mathfrak{M}$ of $\mathfrak{X}$ such that $\mathfrak{M}^2 \subseteq \mathfrak{M}$.
A non-empty subset $\mathfrak{M}$ of $\mathfrak{X}$ is called a *left ideal [LID]* (right ideal [RID]) of $\mathfrak{X}$ if $\mathfrak{X}\mathfrak{M} \subseteq \mathfrak{M}$ ($\mathfrak{M}\mathfrak{X} \subseteq \mathfrak{M}$). By an *ideal* (ID) $\mathfrak{M}$ of $\mathfrak{X}$ we mean a LID and a RID of $\mathfrak{X}$. A *generalized bi-ideal* (GBID) of $\mathfrak{M}$ is a non-empty subset of $\mathfrak{X}$ such that $\mathfrak{M}\mathfrak{X}\mathfrak{M} \subseteq \mathfrak{M}$. An SSG $\mathfrak{M}$ of $\mathfrak{X}$ is called an *bi-ideal* (BID) of $\mathfrak{X}$ if $\mathfrak{M}\mathfrak{X}\mathfrak{M} \subseteq \mathfrak{M}$. A *regular* of $\mathfrak{X}$ if for each $\mathfrak{k} \in \mathfrak{X}$, there exists $\mathfrak{r} \in \mathfrak{X}$ such that $\mathfrak{k} = \mathfrak{k}\mathfrak{r}\mathfrak{k}$. A *left* (right) regular of $\mathfrak{X}$ if for each $\mathfrak{k} \in \mathfrak{X}$, there exists $\mathfrak{r} \in \mathfrak{X}$ such that $\mathfrak{k} = \mathfrak{r}\mathfrak{k}^2$ ($\mathfrak{k} = \mathfrak{k}^2\mathfrak{r}$). An *intra-regular* of $\mathfrak{X}$ if for each $\mathfrak{k} \in \mathfrak{X}$, there exist $\mathfrak{r}, \mathfrak{j} \in \mathfrak{X}$ such that $\mathfrak{k} = \mathfrak{r}\mathfrak{t}^2\mathfrak{j}$.

For any $\mathfrak{k}_i \in [0, 1]$, $i \in \mathcal{J}$, define

$$\bigvee_{i\in\mathcal{J}} \mathfrak{k}_i := \sup_{i\in\mathcal{J}}\{\mathfrak{k}_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{J}} \mathfrak{k}_i := \inf_{i\in\mathcal{J}}\{\mathfrak{k}_i\}.$$

We see that for any $\mathfrak{k}_1, \mathfrak{k}_2 \in [0, 1]$, we have

$$\mathfrak{k}_1 \vee \mathfrak{k}_2 = \max\{\mathfrak{k}_1, \mathfrak{k}_2\} \quad \text{and} \quad \mathfrak{h}_1 \wedge \mathfrak{k}_2 = \min\{\mathfrak{k}_1, \mathfrak{k}_2\}.$$

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A fuzzy set Υ of a non-empty set $\mathfrak{T}$ is a function $\Upsilon : \mathfrak{T} \to [0,1]$.

Definition 2.1. *A* bipolar fuzzy set *(shortly, BF set) Υ on $\mathfrak{T}$ is an object having the form*

$$\Upsilon := \{(\mathfrak{T}, \Upsilon^P(\mathfrak{k}), \Upsilon^N(\mathfrak{k})) \mid \mathfrak{k} \in \mathfrak{T}\},$$

where $\Upsilon^P : \mathfrak{T} \to [0,1]$ and $\Upsilon^N : \mathfrak{T} \to [-1,0]$.

Remark 2.2. *For the sake of simplicity we shall use the symbol $\Upsilon = (\mathfrak{X}; \Upsilon^P, \Upsilon^N)$ for the BF set $\Upsilon = \{(\mathfrak{X}, \Upsilon^P(\mathfrak{k}), \Upsilon^N(\mathfrak{k})) \mid \mathfrak{k} \in \mathfrak{X}\}$.*

Definition 2.3. *[20] A BF set $\Upsilon = (\mathfrak{X}; \Upsilon^P, \Upsilon^N)$ on $\mathfrak{X}$ is called a*

(1) BF subsemigroup *(BFSSG) on $\mathfrak{X}$ if $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_2)$ and $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^n(\mathfrak{k}_2)$ for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$.*
(2) BF left ideal *(BF LID) on $\mathfrak{X}$ if $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_2)$ and $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_2)$ for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$.*
(3) BF right ideal *on $\mathfrak{X}$ if $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_1)$ and $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_1)$ for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$.*
(4) *A BF set $\Upsilon = (\mathfrak{X}; \Upsilon^P, \Upsilon^N)$ on $\mathfrak{X}$ is called a* BF ideal *(BF ID) on $\mathfrak{X}$ if it is both a BF LID and a BF RID on $\mathfrak{X}$.*
(5) BF interior ideal *(BF IID) on $\mathfrak{X}$ if $\Upsilon = (\mathfrak{X}; \Upsilon^P, \Upsilon^N)$ is a BF subsemigroup on $\mathfrak{X}$, $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_2)$ and $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_2)$ for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$.*
(6) BF bi-ideal *(BF BID) on $\mathfrak{X}$ if $\Upsilon = (\mathfrak{X}; \Upsilon^P, \Upsilon^N)$ is a BF subsemigroup on $\mathfrak{X}$, $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3)$ and $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^P(\mathfrak{k}_3)$ for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$.*

Definition 2.4. *[18] A* bipolar complex fuzzy set *(shortly, BCF set) Υ^{RI} on $\mathfrak{X}$ is an object having the form*
$\Upsilon^{RI} := \{(\mathfrak{X}, \Upsilon^P(\mathfrak{k}) = \Upsilon^{RP}(\mathfrak{k}) + \iota\Upsilon^{IP}(\mathfrak{k}), \Upsilon^N(\mathfrak{k}) = \Upsilon^{RN}(\mathfrak{k}) + \iota\Upsilon^{IN}(\mathfrak{k})) \mid \mathfrak{k} \in \mathfrak{X}\}$, is called the positive supportive grade and negative supportive grade respectively, where $\Upsilon^{RP}, \Upsilon^{IP} : \mathfrak{X} \to [0,1]$, and $\Upsilon^{RN}, \Upsilon^{IN} : \mathfrak{X} \to [-1,0]$.

Remark 2.5. *For the sake of simplicity we shall use the symbol $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ for the BCF set $\Upsilon^{RI} = \{(\mathfrak{X}, \Upsilon^{RP}(\mathfrak{k}) + \iota\Upsilon^{IP}(\mathfrak{k}), \Upsilon^{RN}(\mathfrak{k}) + \iota\Upsilon^{IN}(\mathfrak{k})) \mid \mathfrak{k} \in \mathfrak{X}\}$.*

Definition 2.6. *[18] A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ is called a* BCF subsemigroup *(BCF SSG) on $\mathfrak{X}$ if for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$,*

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_2) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_2)$ *and* $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_2)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^N(\mathfrak{k}_2) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_2)$ *and* $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_2)$.

Definition 2.7. *[18] A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ is called a* BCF left ideal *(BCF LID) on $\mathfrak{X}$ if for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$,*

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_2) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{RP}(\mathfrak{k}_2)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_2) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{RN}(\mathfrak{k}_2)$ *and* $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{IN}(\mathfrak{k}_2)$.

Definition 2.8. *[18] A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ is called a* BCF right ideal *(BCF RID) on $\mathfrak{X}$ if for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$,*

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_1) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{RP}(\mathfrak{k}_1)$ *and* $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{IP}(\mathfrak{k}_1)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^N(\mathfrak{k}_1) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{RN}(\mathfrak{k}_1))$ *and* $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{IN}(\mathfrak{k}_1)$.

A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ is called a *BCF ID* on $\mathfrak{X}$ if it is both a BCF LID and a BCF BCF RID on $\mathfrak{X}$.

Definition 2.9. *[19] A BCF subsemigroup $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ is called a* BCF interior ideal *(BCF IID) on $\mathfrak{X}$ if for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$,*

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_2) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_2)$ *and* $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_2)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_2) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_2)$ *and* $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_2)$.

Next, we review the definition of the characteristic bipolar complex fuzzy function.

Let $\mathfrak{M}$ be a non-empty subset of $\mathfrak{X}$. The characteristic bipolar complex fuzzy set (shortly, CBCF set) $\chi^{RI}_{\mathfrak{M}} = (\mathfrak{X}; \chi^P_{\mathfrak{M}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{X}; \chi^{RP}_{\mathfrak{M}} + \iota\chi^{IP}_{\mathfrak{M}}, \chi^{RN}_{\mathfrak{M}} + \iota\chi^{IN}_{\mathfrak{M}})$ is defined as follows:

$$\chi^{RP}_{\mathfrak{M}} + \iota\chi^{IP}_{\mathfrak{M}}(\mathfrak{k}) = \begin{cases} 1 + \iota 1 & \text{if} \quad \mathfrak{k} \in \mathfrak{M} \\ 0 + \iota 0 & \text{if} \quad \mathfrak{k} \notin \mathfrak{M}, \end{cases}$$

$$\chi^{RN}_{\mathfrak{M}} + \iota\chi^{IN}_{\mathfrak{M}}(\mathfrak{k}) = \begin{cases} -1 - \iota 1 & \text{if} \quad \mathfrak{k} \in \mathfrak{M} \\ 0 + \iota 0 & \text{if} \quad \mathfrak{k} \notin \mathfrak{M}. \end{cases}$$

for all $\mathfrak{k} \in \mathfrak{X}$ and $\chi^{RI}_{\mathfrak{M}}$ is a characteristic bipolar complex fuzzy set.

In the following theorem, we give a relationship between a subsemigroup (left ideal, right ideal, ideal) and the BCF function which is proved easily.

Theorem 2.10. *[18] Let $\mathfrak{M}$ be a non-empty subset on $\mathfrak{X}$. Then $\mathfrak{M}$ is a SSG (LID, RID, ID) of $\mathfrak{X}$ if and only if $\chi^{RI}_{\mathfrak{M}} = (\mathfrak{X}; \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{M}}) = (\mathfrak{X}; \chi^{RP}_{\mathfrak{M}} + \iota\chi^{IP}_{\mathfrak{M}}, \chi^{RN}_{\mathfrak{M}} + \iota\chi^{IN}_{\mathfrak{M}})$ is a BCF SSG (LID, RID, ID) on $\mathfrak{X}$.*

Definition 2.11. *[18] A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ with $\pi, \eta \in [0,1]$ and $\varrho, \sigma \in [-1,0]$. Define the set*

(1) *$\mathcal{P}(\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi, \eta)) = \{\mathfrak{k} \in \mathfrak{X} \mid \Upsilon^{RP}(\mathfrak{k}) \geq \pi, \Upsilon^{IP}(\mathfrak{k}) \geq \eta\}$ is called **positive** (π, η)**-cut** of a CBF set of $\mathfrak{X}$.*
(2) *$\mathcal{N}(\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho, \sigma)) = \{h \in \mathfrak{X} \mid \Upsilon^{RN}(\mathfrak{k}) \leq \varrho, \Upsilon^{IN}(\mathfrak{k}) \leq \sigma\}$ is called **negative** (ϱ, σ)**-cut** of a CBF set of $\mathfrak{X}$.*
(3) *$\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\varpi, \eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho, \sigma))) = \mathcal{P}(\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi, \eta)) \cap \mathcal{N}(\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho, \sigma))$ is called $((\varpi, \eta), (\varrho, \sigma))$**-cut** of a CBF set on $\mathfrak{X}$.*

In the following theorems, we give a relationship between a SSG (LID, RID, ID) and the $((\pi, \eta), (\varrho, \sigma))$-cut of a BCF set which proved easily.

Theorem 2.12. *[18] A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF subsemigroup (left ideal, right ideal, ideal) of a semigroup $\mathfrak{X}$ if and only if the non-empty subset $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi, \eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho, \sigma)))$ is a SSG (LID, RID, ID) on $\mathfrak{X}$ for all $\pi, \eta \in [0,1]$ and $\varrho, \sigma \in [-1,0]$.*

Next, we study intersection and product of BCF sets as define.
Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ and $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ are BCF sets of $\mathfrak{X}$. Define

(1) $(\Upsilon^{RI} \cap \psi^{RI})(\mathfrak{k}) = \Upsilon^{RP}(h) \wedge \psi^{RP}(\mathfrak{k})$, $\Upsilon^{IP}(\mathfrak{k}) \wedge \psi^{IP}(\mathfrak{k})$ and $\Upsilon^{RN}(\mathfrak{k}) \vee \psi^{RN}(h)$, $\Upsilon^{IN}(\mathfrak{k}) \vee \psi^{IN}(\mathfrak{k})$ for all $\mathfrak{k} \in \mathfrak{X}$.
(2) $\Upsilon^{RI}(\mathfrak{k}) \precsim \psi^{RI}(\mathfrak{k}) = \Upsilon^{RP}(\mathfrak{k}) \leq \psi^{RP}(\mathfrak{k})$, $\Upsilon^{IP}(\mathfrak{k}) \leq \psi^{IP}(\mathfrak{k})$ and $\Upsilon^{RN}(h) \geq \psi^{RN}(h)$, $\Upsilon^{IN}(h) \geq \psi^{IN}(\mathfrak{k})$ for all $\mathfrak{k} \in \mathfrak{X}$.
(3) $\Upsilon^{RI} \circledcirc \psi^{RI} = (\mathfrak{X}; \Upsilon^P \circ \psi^P, \Upsilon^N \circ \psi^N) = (\mathfrak{X}; \Upsilon^{RP} \circ \psi^{RP} + \iota\Upsilon^{IP} \circ \psi^{IP}, \Upsilon^{RN} \circ \psi^{RN} + \iota\Upsilon^{IN} \circ \psi^{IN})$ where; $(\Upsilon^{RP} \circ \psi^{RP})(\mathfrak{k}) = \begin{cases} \bigvee_{(\mathfrak{s},\mathfrak{t}) \in A_\mathfrak{k}} \{\Upsilon^{RP}(\mathfrak{s}) \wedge \psi^{RP}(\mathfrak{t})\} & \text{if } A_\mathfrak{k} \neq \emptyset \\ 0 & \text{if } A_\mathfrak{k} = \emptyset, \end{cases}$ $(\Upsilon^{IP} \circ \psi^{IP})(\mathfrak{k}) = \begin{cases} \bigvee_{(\mathfrak{s},\mathfrak{t}) \in A_\mathfrak{k}} \{\Upsilon^{IP}(\mathfrak{s}) \wedge \psi^{IP}(\mathfrak{t})\} & \text{if } A_\mathfrak{k} \neq \emptyset \\ 0 & \text{if } A_\mathfrak{k} = \emptyset, \end{cases}$ $(\Upsilon^{RN} \circ \psi^{RN})(\mathfrak{k}) = \begin{cases} \bigwedge_{(\mathfrak{s},\mathfrak{t}) \in A_\mathfrak{k}} \{\Upsilon^{RN}(\mathfrak{s}) \vee \psi^{RN}(\mathfrak{t})\} & \text{if } A_\mathfrak{k} \neq \emptyset \\ 0 & \text{if } A_\mathfrak{k} = \emptyset, \end{cases}$ $(\Upsilon^{IN} \circ \psi^{IN})(\mathfrak{k}) = \begin{cases} \bigwedge_{(\mathfrak{s},\mathfrak{t}) \in A_\mathfrak{k}} \{\Upsilon^{IN}(\mathfrak{s}) \vee \psi^{IN}(\mathfrak{t})\} & \text{if } A_\mathfrak{k} \neq \emptyset \\ 0 & \text{if } A_\mathfrak{k} = \emptyset. \end{cases}$

Obviously, the operation $\circledcirc$ is associative [18]. For $\mathfrak{k} \in \mathfrak{X}$, define $A_\mathfrak{k} := \{(\mathfrak{s}, \mathfrak{t}) \in \mathfrak{X} \times \mathfrak{X} \mid \mathfrak{k} = \mathfrak{s}\mathfrak{t}\}$.

Next, we study equivalent conditions are important properties for BCF subsemigroups of semigroups which are shown in the following theorems.

Theorem 2.13. *[18] A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF SSG of $\mathfrak{X}$ if and only if $\Upsilon^{RI} \circledcirc \Upsilon^{RI} \precsim \Upsilon^{RI}$.*

III. Bipolar complex fuzzy bi-ideals

In this part, we give the concepts of bipolar complex fuzzy bi-ideals in semigroups and we study important properties of bipolar complex fuzzy bi-ideals in semigroups.

Definition 3.1. *A BCF SSG $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ is called a* BCF bi-ideal *(BCF BID) on $\mathfrak{X}$ if for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$,*

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3)$ *and* $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^P(\mathfrak{k}_3)$ *and* $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^P(\mathfrak{k}_3)$.

The following example is a BCF BID of a semigroup.

Example 3.2. *Consider a semigroup $(\mathfrak{X}, \cdot)$ defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	d
b	a	a	a	a
c	b	a	a	a
d	d	a	a	a

A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ in $\mathfrak{X}$ as follows:
$\Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN} = \{(a, (0.8 + \iota0.8, -0.6 - \iota0.6)), (b, (0.5 + \iota0.5, -0.5 - \iota0.5)), (c, (0.6 + \iota0.6, -0.7 - \iota0.7)), ((d, 0.3 + \iota0.3, -0.1 - \iota0.1))\}$. *By routine calculation, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF BID of $\mathfrak{X}$.*

Definition 3.3. *A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ on $\mathfrak{X}$ is called a* BCF gneralized bi-ideal *(BCF GBID) on $\mathfrak{X}$ if for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$,*

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3)$ *and* $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^P(\mathfrak{k}_3)$ *and* $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^P(\mathfrak{k}_3)$.

It is clearly every BCF BID is a BCF GBID in semigroups. The following example is a BCF GBID of a semigroup.

Example 3.4. *Consider a semigroup $(\mathfrak{X}, \cdot)$ defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ in $\mathfrak{X}$ as follows:
$\Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN} = \{(a, (0.8 + \iota0.8, -0.8 - \iota0.8)), (b, (0 + \iota0, -0.1 - \iota0.1)), (c, (0.7 + \iota0.7, -0.7 - \iota0.7)), ((d, 0.4 + \iota0.4, -0.4 - \iota0.4))\}$. *By routine calculation, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF GBID of $\mathfrak{X}$. But it is not a BCF BID of $\mathfrak{X}$, since $\Upsilon^{RP} + \iota\Upsilon^{IP}(c^2) = \Upsilon^{RP} + \iota\Upsilon^{IP}(b) = 0 + \iota0 \ngeq 0.6 + \iota0.6 = \Upsilon^{RP} + \iota\Upsilon^{IP}(c) \wedge \Upsilon^{RP} + \iota\Upsilon^{IP}(c)$ and $\Upsilon^{RN} + \iota\Upsilon^{IN}(ca) = \Upsilon^{RN} + \iota\Upsilon^{IN}(b) = -0.4 - \iota0.4 \nleq -0.7 - \iota0.7 = \Upsilon^{RN} + \iota\Upsilon^{IN}(c) \vee \Upsilon^{RN} + \iota\Upsilon^{IN}(a)$. Then,*

(1) $\Upsilon^P(c^2) = \Upsilon^P(b) = 0 \ngeq 0.6 = \Upsilon^P(c) \wedge = \Upsilon^P(c) \Rightarrow \Upsilon^{RP}(c^2) = \Upsilon^{RP}(b) = 0 \ngeq 0.6 = \Upsilon^{RP}(c) \wedge \Upsilon^{RP}(c)$ *and* $\Upsilon^{IP}(c^2) = \Upsilon^{IP}(b) = 0 \ngeq 0.6 = \Upsilon^{IP}(c) \wedge \Upsilon^{IP}(c)$.
(2) $\Upsilon^N(ca) = \Upsilon^N(b) = -0.4 \nleq -0.7 = \Upsilon^N(c) \wedge \Upsilon^N(a) \Rightarrow \Upsilon^{RN}(ca) = \Upsilon^{RN}(b) = -0.4 \nleq -0.7 = \Upsilon^{RN}(c) \vee \Upsilon^{RN}(a)$ *and* $\Upsilon^{IN}(ca) = \Upsilon^{IN}(b) = -0.4 \nleq -0.7 = \Upsilon^{IN}(c) \vee \Upsilon^{IN}(a)$.

Thus, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is not a BCF SSG of $\mathfrak{X}$. By Definition 3.1, it is not a BCF BID of $\mathfrak{X}$.

Theorem 3.5. *In regular and intra-regular of $\mathfrak{X}$, the BCF BIDs and BCF GBIDs coincide.*

Proof: Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ be a BCF BID of a regular of $\mathfrak{X}$ and let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$. Since $\mathfrak{X}$ is regular, we see that there exists $\mathfrak{h} \in \mathfrak{X}$ such that $\mathfrak{k}_2 = \mathfrak{k}_2\mathfrak{h}\mathfrak{k}_2$. Thus,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^P(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h}\mathfrak{k}_2)) = \Upsilon^P(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h})\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_2) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{RP}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h}\mathfrak{k}_2)) = \Upsilon^{RP}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h})\mathfrak{k}_2) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_2)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{IP}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h}\mathfrak{k}_2)) = \Upsilon^{IP}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h})\mathfrak{k}_2) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_2)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^N(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h}\mathfrak{k}_2)) = \Upsilon^N(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h})\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^N(\mathfrak{k}_2) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{RN}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h}\mathfrak{k}_2)) = \Upsilon^{RN}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h})\mathfrak{k}_2) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_2)$ and

$\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{IN}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h}\mathfrak{k}_2)) = \Upsilon^{IN}(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{h})\mathfrak{k}_2) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_2)$.

Hence, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF SSG of $\mathfrak{X}$. By Definition 3.1, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF BID of $\mathfrak{X}$.

Similarly, we can prove the other cases also. ∎

Theorem 3.6. *Every BCF ID of $\mathfrak{X}$ is a BCF BID of $\mathfrak{X}$.*

Proof: Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ be a BCF ID of $\mathfrak{X}$ and let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$. Then $\Upsilon = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF LID and BCF RID of $\mathfrak{X}$. Thus,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_2) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{RP}(\mathfrak{k}_2)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{IP}(\mathfrak{k}_2)$,
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_2) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{RN}(\mathfrak{k}_2)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{IN}(\mathfrak{k}_2)$.

Hence,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_2) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_2)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_2)$,
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^N(\mathfrak{k}_2) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_2)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_2)$.

This show that $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF SSG of $\mathfrak{X}$.

Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$. Then,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \Upsilon^P((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \Upsilon^{RP}((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_3)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \Upsilon^{IP}((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_3)$,
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \Upsilon^N((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_3) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \Upsilon^{RN}((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_3)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \Upsilon^{IN}((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_3)$.

Thus,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_3)$,
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^N(\mathfrak{k}_3) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_3)$.

Therefore, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF BID of $\mathfrak{X}$. ∎

Corollary 3.7. *Every BCF ID of $\mathfrak{X}$ is a BCF GID of $\mathfrak{X}$*

Remark 3.8. *In example 3.2 we can show that the converse of the above theorem is not true in general. Consider $\Upsilon^{RP} + \iota\Upsilon^{IP}(ca) = \Upsilon^{RP} + \iota\Upsilon^{IP}(b) = 0.5 + \iota 0.5 \ngeq 0.8 + \iota 0.8 = \Upsilon^{RP} + \iota\Upsilon^{IP}(a)$ and $\Upsilon^{RN} + \iota\Upsilon^{IN}(ca) = \Upsilon^{RN} + \iota\Upsilon^{IN}(b) = -0.5 - \iota 0.5 \nleq 0.6 - \iota 0.6 = \Upsilon^{RN} + \iota\Upsilon^{IN}(a)$. Then,*

(1) $\Upsilon^P(ca) = \Upsilon^P(b) = 0.5 \ngeq 0.8 = \Upsilon^P(a) \Rightarrow \Upsilon^{RP}(ca) = \Upsilon^{RP}(b) = 0.5 \ngeq 0.8 = \Upsilon^{RP}(a)$ *and* $\Upsilon^{IP}(ca) = \Upsilon^{IP}(b) = 0.5 \ngeq 0.8 = \Upsilon^{IP}(a)$.
(2) $\Upsilon^N(ca) = \Upsilon^N(b) = -0.5 \nleq -0.6 = \Upsilon^N(a) \Rightarrow \Upsilon^{RN}(ca) = \Upsilon^{RN}(b) = -0.5 \nleq -0.6 = \Upsilon^{RN}(a)$ *and* $\Upsilon^{IN}(ca) = \Upsilon^{IN}(b) = -0.5 \nleq -0.6 = \Upsilon^{IN}(a)$.

Thus, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is not a BCF ID of $\mathfrak{X}$.

The following theorem shows that the BCF BIDs and BCF IDs coincide for some types of semigroups.

Theorem 3.9. *In regular of $\mathfrak{X}$, the BCF BIDs and BCF IDs coincide.*

Proof: Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ be a BCF BID of a regular of $\mathfrak{X}$ and let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{X}$. Since $\mathfrak{X}$ is regular, we have $\mathfrak{k}_1\mathfrak{k}_2 \in (\mathfrak{k}_1\mathfrak{X}\mathfrak{k}_1)\mathfrak{X} \subseteq \mathfrak{k}_1\mathfrak{X}\mathfrak{k}_1$ which that $\mathfrak{k}_1\mathfrak{k}_2 = \mathfrak{k}_1\mathfrak{h}\mathfrak{k}_1$ for some $\mathfrak{h} \in \mathfrak{X}$. Thus,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^P(\mathfrak{k}_1\mathfrak{h}\mathfrak{k}_1) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_1) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{h}\mathfrak{k}_1) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_1)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{IP}(\mathfrak{k}_1\mathfrak{h}\mathfrak{k}_1) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_1)$
(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^N(\mathfrak{k}_1\mathfrak{h}\mathfrak{k}_1) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^N(\mathfrak{k}_1) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{h}\mathfrak{k}_1) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_1)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2) = \Upsilon^{IN}(\mathfrak{k}_1\mathfrak{h}\mathfrak{k}_1) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_1)$.

Hence, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF RID of $\mathfrak{X}$. Similarly, we can prove that $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF LID of $\mathfrak{X}$. Thus, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF ID of $\mathfrak{X}$. ∎

Corollary 3.10. *In regular of $\mathfrak{X}$, the BCF GBIDs and BCF IDs coincide.*

The following theorems are basic properties.

Theorem 3.11. *Let $\mathfrak{M}$ be a non-empty subset on $\mathfrak{X}$. Then $\mathfrak{M}$ is a GBID of $\mathfrak{X}$ if and only if $\chi^{RI}_{\mathfrak{M}} = (\mathfrak{X}; \chi^P_{\mathfrak{M}}, \chi^N_{\mathfrak{M}}) = (\mathfrak{X}; \chi^{RP}_{\mathfrak{M}} + \iota\chi^{IP}_{\mathfrak{M}}, \chi^{RN}_{\mathfrak{M}} + \iota\chi^{IN}_{\mathfrak{M}})$ is a BCF GBID of $\mathfrak{X}$.*

Proof: Suppose that $\mathfrak{M}$ is a GBID on $\mathfrak{X}$ and let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$.

If $\mathfrak{k}_1, \mathfrak{k}_3 \in \mathfrak{K}$, then $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \in \mathfrak{K}$. Thus, $1 + \iota 1 = \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_3) = \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_2) = \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_3) = \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$ and $-1 - \iota 1 = \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_3) = \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_3) = \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) = \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$. Hence,

(1) $\chi^P_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^P_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^P_{\mathfrak{K}}(\mathfrak{k}_3) \Rightarrow \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_3)$ and $\chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_3)$,
(2) $\chi^N_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^N_{\mathfrak{K}}(\mathfrak{k}_2) \vee \chi^N_{\mathfrak{K}}(\mathfrak{k}_3) \Rightarrow \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1) \vee \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_3)$ and $\chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1) \vee \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_3)$.

If $\mathfrak{k}_1 \notin \mathfrak{K}$ or $\mathfrak{k}_3 \notin \mathfrak{K}$, then $0 + \iota 0 = \chi^P_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^P_{\mathfrak{K}}(\mathfrak{k}_3)$ and $0 + \iota 0 = \chi^N_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^N_{\mathfrak{K}}(\mathfrak{k}_3)$. Thus,

(1) $\chi^P_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^P_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^P_{\mathfrak{K}}(\mathfrak{k}_3) \Rightarrow \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_3)$ and $\chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_3)$,
(2) $\chi^N_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^N_{\mathfrak{K}}(\mathfrak{k}_2) \vee \chi^N_{\mathfrak{K}}(\mathfrak{k}_3) \Rightarrow \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1) \vee \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_3)$ and $\chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1) \vee \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_3)$.

Hence, $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{X}; \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{X}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ is a BCF GBID of $\mathfrak{X}$.

Conversely, suppose that $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{X}; \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{X}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ is a BCF GBID of $\mathfrak{X}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$ and $\mathfrak{k}_1, \mathfrak{k}_3 \in \mathfrak{K}$. Then $1 + \iota 1 = \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_3) = \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_2) = \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_3)$ and $-1 - \iota 1 = \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_3) = \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1) = \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_3)$. By assumption,

(1) $\chi^P_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^P_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^P_{\mathfrak{K}}(\mathfrak{k}_3) \Rightarrow \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^{RP}_{\mathfrak{K}}(\mathfrak{k}_3)$ and $\chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_1) \wedge \chi^{IP}_{\mathfrak{K}}(\mathfrak{k}_3)$
(2) $\chi^N_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^N_{\mathfrak{K}}(\mathfrak{k}_1) \vee \chi^N_{\mathfrak{K}}(\mathfrak{k}_3) \Rightarrow \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_1) \vee \chi^{RN}_{\mathfrak{K}}(\mathfrak{k}_3)$ and $\chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_1) \vee \chi^{IN}_{\mathfrak{K}}(\mathfrak{k}_3)$.

Thus, $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \in \mathfrak{K}$. Therefore, $\mathfrak{K}$ is a GBID on $\mathfrak{X}$. ■

Corollary 3.12. *Let $\mathfrak{M}$ be a non-empty subset on $\mathfrak{X}$. Then $\mathfrak{M}$ is a bi-ideal of $\mathfrak{X}$ if and only if $\chi^{RI}_{\mathfrak{M}} = (\mathfrak{X}; \chi^P_{\mathfrak{M}}, \chi^N_{\mathfrak{M}}) = (\mathfrak{X}; \chi^{RP}_{\mathfrak{M}} + \iota\chi^{IP}_{\mathfrak{M}}, \chi^{RN}_{\mathfrak{M}} + \iota\chi^{IN}_{\mathfrak{M}})$ is a BCF BID of $\mathfrak{X}$.*

Proof: It follows from Theorems 2.10 and 3.11. ■

Theorem 3.13. *A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF GBID of $\mathfrak{X}$ if and only if the non-empty subset $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ is a GBID of $\mathfrak{X}$ for all $\pi,\eta \in [0,1]$ and $\varrho,\sigma \in [-1,0]$.*

Proof: Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF GBID of $\mathfrak{X}$. Then $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF SSG of $\mathfrak{X}$. Thus by Theorem 2.12, $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ is a SSGs of $\mathfrak{X}$.
Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}, \pi,\eta \in [0,1]$ and $\varrho,\sigma \in [-1,0]$.

If $\mathfrak{k}_1 \in \mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ and $\mathfrak{k}_3 \in \mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$, then $\Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3) \geq \pi, \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_3) \geq \eta$ and $\Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3) \leq \varrho, \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_3) \leq \sigma$. By assumption,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_2) \wedge \Upsilon^{IP}(\mathfrak{k}_3)$

(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^N(\mathfrak{k}_3) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_3)$.

Thus, $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3$ is an element of $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$.

If $\mathfrak{k}_1 \notin \mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ or $\mathfrak{k}_3 \notin \mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$,

(1) $\Upsilon^P(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^P(\mathfrak{k}_1) \wedge \Upsilon^P(\mathfrak{k}_3) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_2) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_2) \wedge \Upsilon^{IP}(\mathfrak{k}_3)$

(2) $\Upsilon^N(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^N(\mathfrak{k}_1) \vee \Upsilon^N(\mathfrak{k}_3) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_3)$.

Thus, $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3$ is an element of $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$.
Hence, $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ is a genrelized bi-ideal of $\mathfrak{X}$.

Conversely, suppose that $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ is a GBID of $\mathfrak{X}$. By assumption, $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ is a SSGs of $\mathfrak{X}$. Thus by Theorem 2.12, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF SSG of $\mathfrak{X}$. Let $\mathfrak{k}_1, \mathfrak{k}_3 \in \mathfrak{X}, \pi,\eta \in [0,1]$ and $\varrho,\sigma \in [-1,0]$. By assumption, $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3$ is an element of $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$. Thus, $\Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \pi = \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$, $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \eta = \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$, $\Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \varrho = \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \sigma = \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_3)$. Hence, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF GBID of $\mathfrak{X}$. ■

Theorem 3.14. *A BCF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF BID of $\mathfrak{X}$ if and only if the non-empty subset $\mathcal{PN}((\Upsilon^{RP} + \iota\Upsilon^{IP}, (\pi,\eta)), (\Upsilon^{RN} + \iota\Upsilon^{IN}, (\varrho,\sigma)))$ is a bi-ideal of $\mathfrak{X}$ for all $\pi,\eta \in [0,1]$ and $\varrho,\sigma \in [-1,0]$.*

Proof: It follows from Theorems 2.12 and 3.13. ■

Some equivalent conditions are important properties for a BCF BID of a of $\mathfrak{X}$ which are shown in the following theorem.

Theorem 3.15. *A CBF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF GBID of $\mathfrak{X}$ if and only if $\Upsilon^{RI} \circledcirc \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI} \precsim \Upsilon^{RI}$ where $\mathfrak{X}^{RI} = (\mathfrak{X}; \mathfrak{X}^P, \mathfrak{X}^N) = (\mathfrak{X}^{RP} + \iota\mathfrak{X}^{IP}, \mathfrak{X}^{RN} + \iota\mathfrak{X}^{IN})$ is a BCF set of $\mathfrak{X}$.*

Proof: Assume that $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF GBID of $\mathfrak{X}$. Let $\mathfrak{k} \in \mathfrak{X}$. Then $(\Upsilon^{RI} \circledcirc \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI})(\mathfrak{k}) = ((\mathfrak{X}^{RI} \circ \Upsilon^{RI}) \circ \mathfrak{X}^{RI})(\mathfrak{k})$.

If $A_{\mathfrak{k}} = \emptyset$, then it is easy to verify that, $(\Upsilon^{RP} \circ \mathfrak{X}^{RP}) \circ \Upsilon^{RP}(\mathfrak{k}) \leq \Upsilon^{RP}(\mathfrak{k})$, $(\Upsilon^{IP} \circ \mathfrak{X}^{IP}) \circ \Upsilon^{IP}(\mathfrak{k}) \leq \Upsilon^{IP}(\mathfrak{k})$ and $(\Upsilon^{RN} \circ \mathfrak{X}^{RN}) \circ \Upsilon^{RN}(\mathfrak{k}) \geq \Upsilon^{RN}(\mathfrak{k})$, $(\Upsilon^{IN} \circ \mathfrak{X}^{IN}) \circ \Upsilon^{IN}(\mathfrak{k}) \geq \Upsilon^{IN}(\mathfrak{k})$.

If $A_{\mathfrak{k}} \neq \emptyset$, then

$$
\begin{aligned}
&(\Upsilon^{RP} \circ \mathfrak{X}^{RP}) \circ \Upsilon^{RP}(\mathfrak{k}) \\
&= \bigvee_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{(\Upsilon^{RP} \circ \mathfrak{X}^{RP})(\mathfrak{r}) \wedge \Upsilon^{RP}(\mathfrak{o})\} \\
&= \bigvee_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{ \bigvee_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RP}(\mathfrak{u}) \wedge \mathfrak{X}^{RP}(\mathfrak{t})\} \wedge \Upsilon^{RP}(\mathfrak{o})\} \\
&= \bigvee_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{ \bigvee_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RP}(\mathfrak{u}) \wedge 1\} \wedge \Upsilon^{RP}(\mathfrak{o})\} \\
&= \bigvee_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{ \bigvee_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RP}(\mathfrak{u})\} \wedge \Upsilon^{RP}(\mathfrak{o})\} \\
&= \bigvee_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \bigvee_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RP}(\mathfrak{u}) \wedge \Upsilon^{RP}(\mathfrak{o})\} \\
&\leq \bigvee_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{\Upsilon^{RP}(\mathfrak{u}\mathfrak{t}\mathfrak{o})\} \\
&= \Upsilon^{RP}(\mathfrak{k}).
\end{aligned}
$$

Thus, $(\Upsilon^{RP} \circ \mathfrak{X}^{RP}) \circ \Upsilon^{RP}(\mathfrak{k}) \leq \Upsilon^{RP}(\mathfrak{k})$. Similarly, we can show that $(\Upsilon^{IP} \circ \mathfrak{X}^{IP}) \circ \Upsilon^{IP}(\mathfrak{k}) \leq \Upsilon^{IP}(\mathfrak{k})$. And

$$
\begin{aligned}
&(\Upsilon^{RN} \circ \mathfrak{X}^{RN}) \circ \Upsilon^{RN}(\mathfrak{k}) \\
&= \bigwedge_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{(\Upsilon^{RN} \circ \mathfrak{X}^{RN})(\mathfrak{r}) \vee \Upsilon^{RN}(\mathfrak{o})\} \\
&= \bigwedge_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{ \bigwedge_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RN}(\mathfrak{u}) \vee \mathfrak{X}^{RN}(\mathfrak{o})\} \vee \Upsilon^{RN}(\mathfrak{o})\} \\
&= \bigwedge_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{ \bigwedge_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RN}(\mathfrak{u}) \wedge -1\} \vee \Upsilon^{RN}(\mathfrak{o})\} \\
&= \bigwedge_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{ \bigwedge_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RN}(\mathfrak{u})\} \vee \Upsilon^{RN}(\mathfrak{o})\} \\
&= \bigwedge_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \bigwedge_{(\mathfrak{u},\mathfrak{t}) \in A_{\mathfrak{r}}} \{\Upsilon^{RN}(\mathfrak{u}) \vee \Upsilon^{RN}(\mathfrak{o})\} \\
&\geq \bigwedge_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}}} \{\Upsilon^{RN}(\mathfrak{u}\mathfrak{t}\mathfrak{o})\} \\
&= \Upsilon^{RN}(\mathfrak{k}).
\end{aligned}
$$

Thus, $(\Upsilon^{RN} \circ \mathfrak{X}^{RN}) \circ \Upsilon^{RN}(\mathfrak{k})$. Similarly, we can show that $(\Upsilon^{IN} \circ \mathfrak{X}^{IN}) \circ \Upsilon^{IN}(\mathfrak{k})$. Hence, $\Upsilon^{RI} \circledcirc \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI} \precsim \Upsilon^{RI}$.

Conversely, suppose that $\Upsilon^{RI} \circledcirc \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI} \precsim \Upsilon^{RI}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{X}$. Then by assumption, $(\Upsilon^{RP} \circ \mathfrak{X}^{RP}) \circ \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$, $(\Upsilon^{IP} \circ \mathfrak{X}^{IP}) \circ \Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$, $(\Upsilon^{RN} \circ \mathfrak{X}^{RN}) \circ \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$ and $(\Upsilon^{IN} \circ \mathfrak{X}^{IN}) \circ \Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$. Thus,

$$
\begin{aligned}
&\Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq (\Upsilon^{RP} \circ \mathfrak{X}^{RP}) \circ \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \\
&= \bigvee_{(\mathfrak{r},\mathfrak{o}) \in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \{(\Upsilon^{RP} \circ \mathfrak{X}^{RP}(\mathfrak{r})) \wedge \Upsilon^{RP}(\mathfrak{o})\}
\end{aligned}
$$

$= \bigvee_{(\mathfrak{r},\mathfrak{o})\in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \{ \bigvee_{(\mathfrak{u},\mathfrak{t})\in A_{\mathfrak{r}}} \{\Upsilon^{RP}(\mathfrak{u}) \wedge \mathfrak{X}^{RP}(\mathfrak{t})\} \wedge \Upsilon^{RP}(\mathfrak{o})\}$
$= \bigvee_{(\mathfrak{r},\mathfrak{o})\in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \{ \bigvee_{(\mathfrak{u},\mathfrak{t})\in A_{\mathfrak{r}}} \{\Upsilon^{RP}(\mathfrak{u}) \wedge 1\} \wedge \Upsilon^{RP}(\mathfrak{t})\}$
$= \bigvee_{(\mathfrak{r},\mathfrak{o})\in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \bigvee_{(\mathfrak{u},\mathfrak{t})\in A_{\mathfrak{r}}} \{\Upsilon^{RP}(\mathfrak{u}) \wedge \Upsilon^{RP}(\mathfrak{t})\}$
$\geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$.

Hence, $\Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$. Similarly, we can show that $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_3)$.
Therefore, $\Upsilon^{P}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{P}(\mathfrak{k}_1) \wedge \Upsilon^{P}(\mathfrak{k}_3) \Rightarrow \Upsilon^{RP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{RP}(\mathfrak{k}_1) \wedge \Upsilon^{RP}(\mathfrak{k}_3)$ and $\Upsilon^{IP}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \geq \Upsilon^{IP}(\mathfrak{k}_1) \wedge \Upsilon^{IP}(\mathfrak{k}_3)$.

$\Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq (\Upsilon^{RN} \circ \mathfrak{X}^{RN}) \circ \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$
$= \bigwedge_{(\mathfrak{r},\mathfrak{o})\in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \{(\Upsilon^{RN} \circ \mathfrak{X}^{RN}(\mathfrak{r})) \vee \Upsilon^{RN}(\mathfrak{o})\}$
$= \bigwedge_{(\mathfrak{r},\mathfrak{o})\in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \{ \bigwedge_{(\mathfrak{u},\mathfrak{t})\in A_{\mathfrak{r}}} \{\Upsilon^{RN}(\mathfrak{u}) \vee \mathfrak{X}^{RN}(\mathfrak{t})\} \vee \Upsilon^{RN}(\mathfrak{o})\}$
$= \bigwedge_{(\mathfrak{r},\mathfrak{o})\in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \{ \bigwedge_{(\mathfrak{u},\mathfrak{t})\in A_{\mathfrak{r}}} \{\Upsilon^{RN}(\mathfrak{u}) \vee -1\} \vee \Upsilon^{RN}(\mathfrak{t})\}$
$= \bigwedge_{(\mathfrak{r},\mathfrak{o})\in A_{\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3}} \bigwedge_{(\mathfrak{u},\mathfrak{t})\in A_{\mathfrak{r}}} \{\Upsilon^{RN}(\mathfrak{u}) \vee \Upsilon^{RN}(\mathfrak{t})\}$
$\leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3)$.

Hence, $\Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3)$. Similarly, we can show that $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_3)$.
Therefore, $\Upsilon^{N}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{N}(\mathfrak{k}_1) \vee \Upsilon^{N}(\mathfrak{k}_3) \Rightarrow \Upsilon^{RN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{RN}(\mathfrak{k}_1) \vee \Upsilon^{RN}(\mathfrak{k}_3)$ and $\Upsilon^{IN}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \leq \Upsilon^{IN}(\mathfrak{k}_1) \vee \Upsilon^{IN}(\mathfrak{k}_3)$.
Consequently, $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF GBID of $\mathfrak{X}$. ∎

Theorem 3.16. *A CBF set $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF BID of $\mathfrak{X}$ if and only if $\Upsilon^{RI} \circledcirc \Upsilon^{RI} \precsim \Upsilon^{RI}$ and $\Upsilon^{RI} \circledcirc \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI} \precsim \Upsilon^{RI}$ where $\mathfrak{X}^{RI} = (\mathfrak{X}; \mathfrak{X}^P, \mathfrak{X}^N) = (\mathfrak{X}^{RP} + \iota\mathfrak{X}^{IP}, \mathfrak{X}^{RN} + \iota\mathfrak{X}^{IN})$ is a BCF set of $\mathfrak{X}$.*

Proof: It follows from Theorems 2.13 and 3.15. ∎

IV. Characterizations of regular and intra-regular semigroups.

In this topic, we will characterize a regular and intra-regular of $\mathfrak{X}$ using in terms of BCF bi-ideals and BCF ideals.

This lemmas is a tool of characterization a regular and intra-regular of $\mathfrak{X}$ in terms of BCF BIDs.

Lemma 4.1. *[18] If $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ is a BCF RID and $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ is a BCF LID of $\mathfrak{X}$, then $\Upsilon^{RI} \circledcirc \psi^{RI} \precsim \Upsilon^{RI} \cap \psi^{RI}$.*

Lemma 4.2. *[18] Let $\mathfrak{K}$ and $\mathfrak{L}$ be a non-empty subsets of $\mathfrak{X}$. Then*
(1) $\chi^{RI}_{\mathfrak{K}} \circledcirc \chi^{RI}_{\mathfrak{L}} = \chi^{RI}_{\mathfrak{K}\mathfrak{L}}$.
(2) $\chi^{RI}_{\mathfrak{K}} \cap \chi^{RI}_{\mathfrak{L}} = \chi^{RI}_{\mathfrak{K}\cap\mathfrak{L}}$.

The following theorem shows an equivalent conditional statement for an intra-regular of $\mathfrak{X}$.

Theorem 4.3. *Let $\mathfrak{X}$ be a SG. Then the following are equivalent:*
(1) *$\mathfrak{X}$ is intra-regular,*
(2) *$\Upsilon^{RI} \cap \psi^{RI} \precsim \Upsilon^{RI} \circledcirc \psi^{RI}$, for every BCF LID $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ of $\mathfrak{X}$ and every BCF RID $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ of $\mathfrak{X}$.*

Theorem 4.4. *Let $\mathfrak{X}$ be a SG. Then the following are equivalent:*
(1) *$\mathfrak{X}$ is intra-regular,*
(2) *$\Upsilon^{RI} \cap \psi^{RI} \precsim \Upsilon^{RI} \circledcirc \psi^{RI} \circledcirc \mathfrak{X}^{RI}$, for every BCF GBID $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ of $\mathfrak{X}$ and every BCF LID $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ of $\mathfrak{X}$,*
(3) *$\Upsilon^{RI} \cap \psi^{RI} \precsim \Upsilon^{RI} \circledcirc \psi^{RI} \circledcirc \mathfrak{X}^{RI}$, for every BCF BID $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ of $\mathfrak{X}$ and every BCF LID $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ of $\mathfrak{X}$.*

Proof: (1) ⇒ (2) Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ and $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ be a BCF GBID and a BCF LID of $\mathfrak{X}$ respectively. Let $\mathfrak{k} \in \mathfrak{X}$. Then there exist $\mathfrak{r}, \mathfrak{e} \in \mathfrak{X}$ such that $\mathfrak{k} = \mathfrak{r}\mathfrak{k}^2\mathfrak{e}$. Also, $\mathfrak{k} = \mathfrak{r}\mathfrak{k}^2\mathfrak{e} = \mathfrak{r}(\mathfrak{r}\mathfrak{k}^2\mathfrak{e})\mathfrak{k}\mathfrak{e} = (\mathfrak{r}^2\mathfrak{k})(\mathfrak{k}\mathfrak{e}\mathfrak{k}\mathfrak{e})$. Thus,

$(\Upsilon^{RP} \circ \psi^{RP} \circ \mathfrak{X}^{RP})(\mathfrak{k})$
$= \bigvee_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{\Upsilon^{RP}(\mathfrak{u}) \wedge (\psi^{RP} \circ \mathfrak{X}^{RP})(\mathfrak{o})\}$
$\geq \bigvee_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{\Upsilon^{RP}(\mathfrak{r}\mathfrak{k}^2\mathfrak{e}) \wedge (\psi^{RP} \circ \mathfrak{X}^{RP})(\mathfrak{k}\mathfrak{e}\mathfrak{k}\mathfrak{e})\}$
$= \bigvee_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{\Upsilon^{RP}(\mathfrak{r}\mathfrak{k}^2\mathfrak{e}) \wedge \bigvee_{(\mathfrak{w},\mathfrak{c})\in A_{\mathfrak{k}\mathfrak{e}\mathfrak{k}\mathfrak{e}}} (\psi^{RP}(\mathfrak{w}) \wedge \mathfrak{X}^{RP})(\mathfrak{c})\}$
$\geq \Upsilon^{RP}(\mathfrak{k}) \wedge (\psi^{RP}(\mathfrak{k}\mathfrak{e}\mathfrak{k}) \wedge \mathfrak{X}^{RP}(\mathfrak{e}))$
$= \Upsilon^{RP}(\mathfrak{k}) \wedge (\psi^{RP}(\mathfrak{k}\mathfrak{e}\mathfrak{k}) \wedge 1) = \Upsilon^{RP}(\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k}\mathfrak{e}\mathfrak{k})$
$\geq \Upsilon^{RP}(\mathfrak{k}) \wedge (\psi^{RP}(\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k})) \geq \Upsilon^{RP}(\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k})$
$= (\Upsilon^{RP} \wedge \psi^{RP})(\mathfrak{k})$.

Hence, $(\Upsilon^{RP} \circ \psi^{RP} \circ \mathfrak{X}^{RP})(\mathfrak{k}) \geq (\Upsilon^{RP} \wedge \psi^{RP})(\mathfrak{k})$. Similarly, we can show that $(\Upsilon^{IP} \circ \psi^{IP} \circ \mathfrak{X}^{IP})(\mathfrak{k}) \geq (\Upsilon^{IP} \wedge \psi^{IP})(\mathfrak{k})$. And

$(\Upsilon^{RN} \circ \psi^{RN} \circ \mathfrak{X}^{RN})(\mathfrak{k})$
$= \bigwedge_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{\Upsilon^{RN}(\mathfrak{u}) \wedge (\psi^{RN} \circ \mathfrak{X}^{RN})(\mathfrak{o})\}$
$\leq \bigwedge_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{\Upsilon^{RP}(\mathfrak{r}\mathfrak{k}^2\mathfrak{e}) \vee (\psi^{RN} \circ \mathfrak{X}^{RN})(\mathfrak{k}\mathfrak{e}\mathfrak{k}\mathfrak{e})\}$
$= \bigwedge_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{\Upsilon^{RN}(\mathfrak{r}\mathfrak{k}^2\mathfrak{e}) \vee \bigwedge_{(\mathfrak{w},\mathfrak{c})\in A_{\mathfrak{k}\mathfrak{e}\mathfrak{k}\mathfrak{e}}} (\psi^{RN}(\mathfrak{w}) \vee \mathfrak{X}^{RN})(\mathfrak{c})\}$
$\leq \Upsilon^{RN}(\mathfrak{k}) \vee (\psi^{RN}(\mathfrak{k}\mathfrak{e}\mathfrak{k}) \vee \mathfrak{X}^{RN}(\mathfrak{e}))$
$= \Upsilon^{RN}(\mathfrak{k}) \vee (\psi^{RN}(\mathfrak{k}\mathfrak{e}\mathfrak{k}) \vee -1) = \Upsilon^{RN}(\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k}\mathfrak{e}\mathfrak{k})$
$\leq \Upsilon^{RN}(\mathfrak{k}) \vee (\psi^{RN}(\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k})) \leq \Upsilon^{RN}(\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k})$
$= (\Upsilon^{RN} \vee \psi^{RN})(\mathfrak{k})$.

Hence, $(\Upsilon^{RN} \circ \psi^{RN} \circ \mathfrak{X}^{RN})(\mathfrak{k}) \leq (\Upsilon^{RN} \vee \psi^{RN})(\mathfrak{k})$. Similarly, we can show that $(\Upsilon^{IN} \circ \psi^{IN} \circ \mathfrak{X}^{IN})(\mathfrak{k}) \leq (\Upsilon^{IN} \vee \psi^{IN})(\mathfrak{k})$. Therefore, $\Upsilon^{RI} \cap \psi^{RI} \precsim \Upsilon^{RI} \circledcirc \psi^{RI} \circledcirc \mathfrak{X}^{RI}$.
(2) ⇒ (3) Since every BCF GBID is BCF BID in of $\mathfrak{X}$.
(3) ⇒ (1) Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ and $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ be a BCF GBID and a BCF LID of $\mathfrak{X}$ respectively. Since every BCF RID of $\mathfrak{X}$ is a BCF BID, we have ψ^{RI} is also a BFC BID of $\mathfrak{X}$. Thus, by hypothesis $\Upsilon^{RI} \cap \psi^{RI} \precsim \Upsilon^{RI} \circledcirc (\psi^{RI} \circledcirc \mathfrak{X}^{RI}) \precsim \Upsilon^{RI} \circledcirc \psi^{RI}$. By Theorem 4.3, $\mathfrak{X}$ is intra-regular. ∎

Theorem 4.5. *Let $\mathfrak{X}$ be a SG. Then the following are equivalent:*
(1) *$\mathfrak{X}$ is intra-regular,*

(2) $\Upsilon^{RI} \cap \psi^{RI} \precsim \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI} \circledcirc \psi^{RI}$, *for every BCF GBID* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *of* $\mathfrak{X}$ *and every BCF RID* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$,

(3) $\Upsilon^{RI} \cap \psi^{RI} \precsim \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI} \circledcirc \psi^{RI}$, *for every BCF BID* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *of* $\mathfrak{X}$ *and every BCF RID* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$.

Proof: (1) $\Rightarrow$ (2) Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ and $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ be a BCF GBID and a BCF RID of $\mathfrak{X}$ respectively. Let $\mathfrak{k} \in \mathfrak{X}$. Then there exist $\mathfrak{r}, \mathfrak{e} \in \mathfrak{X}$ such that $\mathfrak{k} = \mathfrak{r}\mathfrak{k}^2\mathfrak{e}$. Also, $\mathfrak{k} = \mathfrak{r}\mathfrak{k}^2\mathfrak{e} = \mathfrak{r}\mathfrak{k}(\mathfrak{r}\mathfrak{k}^2\mathfrak{e})\mathfrak{e} = (\mathfrak{r}\mathfrak{k}\mathfrak{r}\mathfrak{k})(\mathfrak{k}\mathfrak{e}^2)$. Thus,

$$
\begin{aligned}
&(\mathfrak{X}^{RP} \circ \Upsilon^{RP} \circ \psi^{RP})(\mathfrak{k}) \\
&= \bigvee_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\mathfrak{X}^{RP} \circ \Upsilon^{RP})(\mathfrak{u}) \wedge \psi^{RP}(\mathfrak{o})\} \\
&\geq \bigvee_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\mathfrak{X}^{RP} \circ \Upsilon^{RP})(\mathfrak{r}\mathfrak{k}\mathfrak{r}\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k}\mathfrak{e}^2)\} \\
&= \bigvee_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{ \bigvee_{(\mathfrak{w},\mathfrak{c}) \in A_{\mathfrak{r}\mathfrak{k}\mathfrak{r}\mathfrak{k}}} (\mathfrak{X}^{RP}(\mathfrak{w}) \wedge \Upsilon^{RP}(\mathfrak{c})) \wedge \psi^{RP}(\mathfrak{k}\mathfrak{e}^2) \\
&\geq (\mathfrak{X}^{RP}(\mathfrak{r}) \wedge \Upsilon^{RP}(\mathfrak{k}\mathfrak{r}\mathfrak{k})) \wedge \psi^{RP}(\mathfrak{k}) \\
&= (1 \wedge \Upsilon^{RP}(\mathfrak{k}\mathfrak{r}\mathfrak{k})) \wedge \psi^{RP}(\mathfrak{k}) = \Upsilon^{RP}(\mathfrak{k}\mathfrak{r}\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k}) \\
&\geq (\Upsilon^{RP}(\mathfrak{k}) \wedge \Upsilon^{RP}(\mathfrak{k})) \wedge \psi^{RP}(\mathfrak{k}) = \Upsilon^{RP}(\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k}) \\
&= (\Upsilon^{RP} \wedge \psi^{RP})(\mathfrak{k}).
\end{aligned}
$$

Hence, $(\mathfrak{X}^{RP} \circ \Upsilon^{RP} \circ \psi^{RP})(\mathfrak{k}) \geq (\Upsilon^{RP} \wedge \psi^{RP})(\mathfrak{k})$. Similarly, we can show that $(\mathfrak{X}^{IP} \circ \Upsilon^{IP} \circ \psi^{IP})(\mathfrak{k}) \geq (\Upsilon^{IP} \wedge \psi^{IP})(\mathfrak{k})$. And

$$
\begin{aligned}
&(\mathfrak{X}^{RN} \circ \Upsilon^{RN} \circ \psi^{RN})(\mathfrak{k}) \\
&= \bigwedge_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\mathfrak{X}^{RN} \circ \Upsilon^{RN})(\mathfrak{u}) \vee \psi^{RN}(\mathfrak{o})\} \\
&\leq \bigwedge_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\mathfrak{X}^{RN} \circ \Upsilon^{RN})(\mathfrak{r}\mathfrak{k}\mathfrak{r}\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k}\mathfrak{e}^2)\} \\
&= \bigwedge_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{ \bigwedge_{(\mathfrak{w},\mathfrak{c}) \in A_{\mathfrak{r}\mathfrak{k}\mathfrak{r}\mathfrak{k}}} (\mathfrak{X}^{RN}(\mathfrak{w}) \vee \Upsilon^{RN}(\mathfrak{c})) \vee \psi^{RN}(\mathfrak{k}\mathfrak{e}^2) \\
&\leq (\mathfrak{X}^{RN}(\mathfrak{r}) \vee \Upsilon^{RN}(\mathfrak{k}\mathfrak{r}\mathfrak{k})) \vee \psi^{RN}(\mathfrak{k}) \\
&= (0 \wedge \Upsilon^{RN}(\mathfrak{k}\mathfrak{r}\mathfrak{k})) \vee \psi^{RN}(\mathfrak{k}) = \Upsilon^{RN}(\mathfrak{k}\mathfrak{r}\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k}) \\
&\leq (\Upsilon^{RN}(\mathfrak{k}) \vee \Upsilon^{RN}(\mathfrak{k})) \vee \psi^{RN}(\mathfrak{k}) = \Upsilon^{RN}(\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k}) \\
&= (\Upsilon^{RN} \wedge \psi^{RN})(\mathfrak{k}).
\end{aligned}
$$

Hence, $(\mathfrak{X}^{RN} \circ \Upsilon^{RN} \circ \psi^{RN})(\mathfrak{k}) \leq (\Upsilon^{RN} \vee \psi^{RN})(\mathfrak{k})$. Similarly, we can show that $(\mathfrak{X}^{IN} \circ \Upsilon^{IN} \circ \psi^{IN})(\mathfrak{k}) \leq (\Upsilon^{IN} \vee \psi^{IN})(\mathfrak{k})$. Therefore, $\Upsilon^{RI} \cap \psi^{RI} \precsim \mathfrak{X}^{RI} \circledcirc \Upsilon^{RI} \circledcirc \psi^{RI}$.
(2) $\Rightarrow$ (3) Since every BCF GBID is BCF BID of $\mathfrak{X}$.

(3) $\Rightarrow$ (1) Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ and $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ be a BCF GBID and a BCF RID of $\mathfrak{X}$ respectively. Since every BCF LID of $\mathfrak{X}$ is a BCF BID, we have ψ^{RI} is also a BFC BID of $\mathfrak{X}$. Thus, by hypothesis $\Upsilon^{RI} \cap \psi^{RI} \precsim (\mathfrak{X}^{RI} \circledcirc \Upsilon^{RI}) \circledcirc \psi^{RI} \precsim \Upsilon^{RI} \circledcirc \psi^{RI}$. By Theorem 4.3, $\mathfrak{X}$ is intra-regular. ■

Lemma 4.6. *[21] Let* $\mathfrak{X}$ *be a SG. Then* $\mathfrak{X}$ *is regular and intra-regular if and only if* $\mathfrak{B} = \mathfrak{B}^2$ *for every bi-ideal* $\mathfrak{B}$ *of* $\mathfrak{X}$

Lemma 4.7. *[18] For a SG* $\mathfrak{X}$, *the following conditions are equivalent:*

(1) $\mathfrak{X}$ *is regular and intra-regular,*

(2) $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$ *for every BCF LID* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *and every BCF RID* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$.

Theorem 4.8. *For a semigroup* $\mathfrak{X}$, *the following conditions are equivalent:*

(1) $\mathfrak{X}$ *is regular and intra-regular,*

(2) $\mathfrak{B} = \mathfrak{B} \circledcirc \mathfrak{B}$ *for every BCF BID* $\mathfrak{B}$ *of* $\mathfrak{X}$,

(3) $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$ *for every BCF BIDs* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *and* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$,

(4) $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$ *for every BCF BID* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *and every BCF LID* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$,

(5) $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$ *for every BCF BID* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *and every BCF RID* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$.

Proof: (1) $\Rightarrow$ (3) Let $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ and $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ be BCF BIDs and $\mathfrak{k} \in \mathfrak{X}$. Then there exist $\mathfrak{r}, \mathfrak{e}, \mathfrak{t} \in \mathfrak{X}$ such that $\mathfrak{k} = \mathfrak{k}\mathfrak{r}\mathfrak{k}$ and $\mathfrak{k} = \mathfrak{e}\mathfrak{k}^2\mathfrak{t}$. Also, $\mathfrak{k} = \mathfrak{k}\mathfrak{r}\mathfrak{k} = \mathfrak{k}\mathfrak{r}\mathfrak{k}\mathfrak{r}\mathfrak{k} = \mathfrak{r}\mathfrak{k}(\mathfrak{e}\mathfrak{k}^2\mathfrak{t})\mathfrak{r}\mathfrak{k} = (\mathfrak{k}\mathfrak{r}\mathfrak{e}\mathfrak{k})(\mathfrak{k}\mathfrak{t}\mathfrak{r}\mathfrak{k})$. Thus,

$$
\begin{aligned}
&(\Upsilon^{RP} \circ \psi^{RP})(\mathfrak{k}) = \bigvee_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\Upsilon^{RP})(\mathfrak{u}) \wedge \psi^{RP}(\mathfrak{o})\} \\
&\geq \Upsilon^{RP}(\mathfrak{r}\mathfrak{k}\mathfrak{e}\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k}\mathfrak{t}\mathfrak{r}\mathfrak{k}) \\
&\geq (\Upsilon^{RP}(\mathfrak{k}) \wedge \Upsilon^{RP}(\mathfrak{k})) \wedge (\psi^{RP}(\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k})) \\
&= \Upsilon^{RP}(\mathfrak{k}) \wedge \psi^{RP}(\mathfrak{k}) = (\Upsilon^{RP} \wedge \psi^{RP})(\mathfrak{k}).
\end{aligned}
$$

Hence, $(\Upsilon^{RP} \circ \psi^{RP})(\mathfrak{k}) \geq (\Upsilon^{RP} \wedge \psi^{RP})(\mathfrak{k})$. Similarly, we can show that $(\Upsilon^{IP} \circ \psi^{IP})(\mathfrak{k}) \geq (\Upsilon^{IP} \wedge \psi^{IP})(\mathfrak{k})$. And

$$
\begin{aligned}
&(\Upsilon^{RN} \circ \psi^{RN})(\mathfrak{k}) = \bigwedge_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\Upsilon^{RN})(\mathfrak{u}) \vee \psi^{RN}(\mathfrak{o})\} \\
&\leq \Upsilon^{RN}(\mathfrak{r}\mathfrak{k}\mathfrak{e}\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k}\mathfrak{t}\mathfrak{r}\mathfrak{k}) \\
&\leq (\Upsilon^{RN}(\mathfrak{k}) \vee \Upsilon^{RP}(\mathfrak{k})) \vee (\psi^{RN}(\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k})) \\
&= \Upsilon^{RN}(\mathfrak{k}) \vee \psi^{RN}(\mathfrak{k}) = (\Upsilon^{RN} \vee \psi^{RN})(\mathfrak{k}).
\end{aligned}
$$

Hence, $(\Upsilon^{RN} \circ \psi^{RN})(\mathfrak{k}) \leq (\Upsilon^{RN} \vee \psi^{RN})(\mathfrak{k})$. Similarly, we can show that $(\Upsilon^{IN} \circ \psi^{IN})(\mathfrak{k}) \leq (\Upsilon^{IN} \vee \psi^{IN})(\mathfrak{k})$. Therefore, $\Upsilon^{RI} \cap \psi^{RI} \precsim \Upsilon^{RI} \circledcirc \psi^{RI}$. In the same way we can show that $\Upsilon^{RI} \cap \psi^{RI} \precsim \psi^{RI} \circledcirc \Upsilon^{RI}$. Thus, $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$.

(3) $\Rightarrow$ (2)$\Upsilon^{RI} = \Upsilon^{RI} \cap \Upsilon^{RI} = (\Upsilon^{RI} \circledcirc \Upsilon^{RI}) \cap (\Upsilon^{RI} \circledcirc \Upsilon^{RI}$.

(3) $\Rightarrow$ (4) Since every BCF LID is BCF BID in $\mathfrak{X}$.

(4) $\Rightarrow$ (1) By Lemma 4.7.

(3) $\Rightarrow$ (5) Since every BCF RID is BCF BID in $\mathfrak{X}$.

(5) $\Rightarrow$ (1) By Lemma 4.7.

(2) $\Rightarrow$ (1) Let $\mathfrak{B}$ be a BID of $\mathfrak{X}$ and $\mathfrak{k} \in \mathfrak{B}$. Then by Corollary 3.12, $\chi^{RI}_{\mathfrak{M}} = (\mathfrak{X}; \chi^P_{\mathfrak{B}}, \chi^N_{\mathfrak{B}}) = (\mathfrak{X}; \chi^{RP}_{\mathfrak{B}} + \iota\chi^{IP}_{\mathfrak{B}}, \chi^{RN}_{\mathfrak{B}} + \iota\chi^{IN}_{\mathfrak{B}})$ is a BCF BID of $\mathfrak{X}$. By hypothesis, $(\chi^{RP}_{\mathfrak{B}} \circledcirc \chi^{IP}_{\mathfrak{B}})(\mathfrak{k}) = 1 + \iota 1$ and $(\chi^{RN}_{\mathfrak{B}} \circledcirc \chi^{IN}_{\mathfrak{B}})(\mathfrak{k}) = -1 - \iota 1$. Then

$$\bigvee_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\Upsilon^{RP})(\mathfrak{u}) \wedge \psi^{RP}(\mathfrak{o})\} = (\Upsilon^{RP} \circ \psi^{RP})(\mathfrak{k}) = 1,$$

$$\bigvee_{(\mathfrak{u},\mathfrak{o}) \in A_\mathfrak{k}} \{(\Upsilon^{RI})(\mathfrak{u}) \wedge \psi^{RI}(\mathfrak{o})\} = (\Upsilon^{IP} \circ \psi^{IP})(\mathfrak{k}) = \iota 1$$

and

$$\bigwedge_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{(\Upsilon^{RN})(\mathfrak{u}) \vee \psi^{RN}(\mathfrak{o})\} = (\Upsilon^{RN} \circ \psi^{RN})(\mathfrak{k}) = -1,$$

$$\bigwedge_{(\mathfrak{u},\mathfrak{o})\in A_{\mathfrak{k}}} \{(\Upsilon^{IN})(\mathfrak{u}) \vee \psi^{IN}(\mathfrak{o})\} = (\Upsilon^{IN} \circ \psi^{IN})(\mathfrak{k}) = -\iota 1.$$

Thus, $\mathfrak{k} \in \mathfrak{B}\mathfrak{B}$. Hence, $\mathfrak{B} \subseteq \mathfrak{B}\mathfrak{B}$. Clearly $\mathfrak{B}\mathfrak{B} \subseteq \mathfrak{B}$. By Lemma 4.6 $\mathfrak{X}$ is regular and intra-regular. ■

Corollary 4.9. *For a SG $\mathfrak{X}$, the following conditions are equivalent:*

(1) *$\mathfrak{X}$ is regular and intra-regular,*
(2) $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$ *for every BCF GIDs* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *and* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$,
(3) $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$ *for every BCF GBID* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *and every BCF LID* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$,
(4) $\Upsilon^{RI} \cap \psi^{RI} \precsim (\Upsilon^{RI} \circledcirc \psi^{RI}) \cap (\psi^{RI} \circledcirc \Upsilon^{RI})$ *for every BCF GBID* $\Upsilon^{RI} = (\mathfrak{X}; \Upsilon^P, \Upsilon^N) = (\mathfrak{X}; \Upsilon^{RP} + \iota\Upsilon^{IP}, \Upsilon^{RN} + \iota\Upsilon^{IN})$ *and every BCF RID* $\psi^{RI} = (\mathfrak{X}; \psi^P, \psi^N) = (\mathfrak{X}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ *of* $\mathfrak{X}$.

V. CONCLUSION

This paper gives the concept of bipolar complex fuzzy bi-ideals in semigroups. We investigate the properties of bipolar complex fuzzy bi-ideals and between relation bi-ideals and bipolar complex fuzzy bi-ideals. Additionally, we find conditions bipolar complex fuzzy ideals and bipolar complex fuzzy bi-ideals coincide and bipolar complex fuzzy generalized bi-ideals and bipolar complex fuzzy bi-ideals coincide. Finally, we characterize a regular and an intra-regular semigroup in the bipolar complex fuzzy bi-ideal. Further, we study bipolar complex fuzzy quasi-ideals in semigroups or algebraic systems.

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