Spatiotemporal Dynamics of a Nonautonomous Reaction-Diffusive 4-species Ratio-Dependent Predator-Prey Model

Changyou Wang, Qi Shang, and Lili Jia

***Abstract*—This article delves into the global stability of a 4-species ratio-dependent predator-prey model (RDPPM) with time-varying coefficients in a reaction-diffusion system. Firstly, employing the comparison principle of differential equations alongside the fixed point theory, we obtained some sufficient conditions for the existence of positive periodic solution in the degenerate model, deriving corresponding conditions for the existence of strictly positive space homogeneous periodic solution (SHPS) in the model being studied. Secondly, we apply the method of upper and lower solutions (UALS) for PDEs and Lyapunov stability theory to formulate additional sufficient conditions that ensure the global stability of the strictly positive SHPS. Finally, to validate the theoretical framework presented in this article, we offer two numerical examples that illustrate the practical application and veracity of our findings.**

***Index Terms*—Time-varying coefficients; Predators-prey model; Ratio-dependent; Periodic solution; Stability**

I. INTRODUCTION

This article investigates the following non-autonomous reaction-diffusion 4-species RDPPM

$$
\begin{aligned}
&\frac{\partial v_1(x,t)}{\partial t}-d_1(t)\Delta v_1(x,t)=v_1(x,t)[r_1(t)-a_{11}(t)v_1(x,t)-a_{12}(t)v_2(x,t)\\
&\qquad-\frac{a_{13}(t)v_3(x,t)}{b_{13}(t)v_3(x,t)+v_1(x,t)}-\frac{a_{14}(t)v_4(x,t)}{b_{14}(t)v_4(x,t)+v_1(x,t)}],\\
&\frac{\partial v_2(x,t)}{\partial t}-d_2(t)\Delta v_2(x,t)=v_2(x,t)[r_2(t)-a_{22}(t)v_2(x,t)-a_{21}(t)v_1(x,t)\\
&\qquad-\frac{a_{23}(t)v_3(x,t)}{b_{23}(t)v_3(x,t)+v_2(x,t)}-\frac{a_{24}(t)v_4(x,t)}{b_{24}(t)v_4(x,t)+v_2(x,t)}],\\
&\frac{\partial v_3(x,t)}{\partial t}-d_3(t)\Delta v_3(x,t)=v_3(x,t)[-r_3(t)+\frac{a_{31}(t)v_1(x,t)}{b_{13}(t)v_3(x,t)+v_1(x,t)}\\
&\qquad+\frac{a_{32}(t)v_2(x,t)}{b_{23}(t)v_3(x,t)+v_2(x,t)}-a_{34}(t)v_4(x,t)],\\
&\frac{\partial v_4(x,t)}{\partial t}-d_4(t)\Delta v_4(x,t)=v_4(x,t)[-r_4(t)+\frac{a_{41}(t)v_1(x,t)}{b_{14}(t)v_4(x,t)+v_1(x,t)}\\
&\qquad+\frac{a_{42}(t)v_2(x,t)}{b_{24}(t)v_4(x,t)+v_2(x,t)}-a_{43}(t)v_3(x,t)],
\end{aligned}
\tag{1.1}
$$

with the boundary and initial conditions

$$
\begin{aligned}
&\partial v_i(x,t)/\partial n=0,\ (x,t)\in\partial\Omega\times R^+,\\
&v_i(x,0)=v_{i0}(x)>0,\ x\in\Omega,\ i=1,2,3,4,
\end{aligned}
\tag{1.2}
$$

where Δ is a Laplace operator on Ω, $\partial/\partial n$ represents the outward normal derivation on $\partial\Omega$, Ω is a smooth bounded domain in R^n with the boundary $\partial\Omega$, $v_i(x,t)$ denotes the density of i-th species at point $x=(x_1,\cdots,x_n)$ and the time of t. As evident from Table I, the biological importance of the parameters in model (1.1) becomes apparent.

All coefficients in model (1.1) are continuous, positive, and periodic functions. Since Lotka [1] and Volterra [2] introduced the classic Lotka-Volterra model in the 1920s, many scholars have delved into extensive research on this topic [3-12]. The functional response, which characterizes the rate at which predators consume prey, is regarded as the core issue of the Lotka-Volterra predator-prey model. The incorporation of predator dependence into functional responses, with the response function being treated as ratio-dependent, was carried out by Arditi and Ginzburg in 1989 [13]. Subsequently, in 1999, Conser et al. [14] demonstrated that a predator-prey model incorp- orating a ratio-dependent function aligns more closely with practical scenarios. In 2000, the author applied Lyapunov stability theory to investigate an autonomous RDPPM without diffusion [15], deriving sufficient conditions for the persistence and extinction of solutions within the model. In 2009, M. Haque [16] established sufficient conditions for the global stability of solutions in a 2-species autonomous RDPPM that incorporated chemotaxis. In 2012, Bairagi et al. [17] studied a RDPPM with M-M harvesting rate, and obtained some sufficient conditions for the existence of heteroclinic bifurcations. In 2013, Gao and Li [18] studied a RDPPM with a strong Allee effect and show that the system exhibited a Bogdanov-Takens bifurcation. In 2015, Agrawal and Saleem [19] considered a 3-species RDPPM and obtained sufficient conditions for the existence of a

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Changyou Wang is a professor of College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225 China. (e-mail: wangchangyou417@163.com).

Qi Shang is a postgraduate student of College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225 China. (e-mail: shangqi0206@163.com).

Lili Jia is a professor of Department of Basic Teaching, Dianchi College, Kunming, Yunnan 650228 China. (Corresponding author, e-mail: lilijiadianchi@163.com).

chaotic attractor for the model. In 2018, Mandal [20] conducted a study on a stochastic forced RDPPM incorporating a strong Allee effect, revealing that the model exhibits either a stable equilibrium point or a limit cycle with two coexisting populations. In 2020, Wang et al. [21] studied the qualitative behavior of a RDPPM with feedback controls, and obtained sufficient conditions for the global stability for the periodic solution of the model. In 2021, Arancibia- Ibarra et al. [22] studied a RDPPM with predator intra- specific interactions and proved the existence and stability of two interior equilibrium points. In 2023, Yu et al. [23] studied a novel RDPPM with additional prey supply and obtained rich dynamic characteristics of the model. The above literature fully demonstrates the important application value of studying RDPPM. For more related work, please refer to references [24-28]. It is noteworthy that the RDPPM examined in the aforementioned literature lacks diffusion terms. Since animals instinctively gravitate towards food and water sources, incorporating diffusion terms into the RDPPM provides a more precise representation of population interaction dynamics. However, the methodologies presented in the previous literature are not immediately applicable to this newly formulated model.

In recent years, RDPPM with diffusion have garnered increasing attention. In 2013, Ko and Ahn [29-30] studied a 3-species reaction-diffusion RDPPM with two competing predators and a prey. They derived sufficient conditions that guarantee the persistence and global attractiveness of the model's solutions. Yang et al. [31] conducted a study in 2015 on a reaction-diffusion RDPPM, employing an L-G function response in their analysis. Utilizing the fixed point theory, they derived sufficient conditions for ensuring the existence of an attractor in the model. In 2017, Wang [32] investigated the dynamical characteristics of a predator-prey diffusion model that was subject to Neumann boundary conditions and characterized by a Holling Type III functional response. Employing coincidence theory and bifurcation methods, Wang obtained sufficient conditions that guarantee the existence of non- constant equilibrium solutions and periodic orbits within this model. Wu and Zhao [33] conducted an investigation in 2020 into a predator-prey diffusion model incorporating the Allee effect and threshold hunting, utilizing a Jacobian matrix to analyze the asymptotic stability of the model's equilibrium point. Subsequently, in 2022, Yan and Zhang [34] studied a predator-prey diffusion model characterized by a B-D functional response, deriving criteria for both the stability and instability of its positive constant equilibrium point. In 2023, Chen and Wu [35] used the Leray-Schauder degree theory and Poincare inequality to study the spatiotemporal behavior of a predator-prey diffusion model with a B-D response function. It is worth noting that the above models are autonomous reaction-diffusion RDPPM. Given that birth rates, death rates, and inter-population interactions are not static, non-autonomous reaction-diffusion RDPPM offer a more accurate simulation of species interactions in predator-prey systems. However, the methodologies employed in the above literature face significant challenges when applied to multi-species non-autonomous reaction-diffusion predator-prey models, and these difficulties are further compounded when studying non-autonomous reaction-diffusion RDPPM. Undoubtedly, the stability analysis of non-autonomous multi-species reaction-diffusion RDPPM is extremely challenging due to the intricate interactions among multiple species. Consequently, research findings in this area are scarce and infrequently published.

Drawing inspiration from the previous works mentioned, this article delves into the study of the global stability of a nonautonomous reaction-diffusion RDPPM with 4-species, as described by equations (1.1) and (1.2). The structure of the article is outlined as follows: In Section 2, we will outline some essential definitions and preliminary findings. Section 3 will focus on exploring the existence of strictly positive SHPS for the nonautonomous reaction-diffusion RDPPM. Subsequently, Section 4 will examine the global asymptotic stability of these strictly positive SHPS. In Section 5, we will present two numerical examples to substantiate the theoretical approach put forth in this article. Lastly, we will conclude by summarizing the main contributions of this work.

Remark 1.1: The article boasts the following innovations and accomplishments: (1) By integrating ratio-dependent functions and variable coefficients into prevalent population models, we have formulated a non-autonomous reaction-diffusion RDPPM that provides a more accurate portrayal of the complex population interactions observed in natural environments. (2) Using the comparison principle of differential equations, fixed point theory, and inequality techniques, we establish sufficient conditions for the existence of strictly positive SHPS for our model. (3) To address the issue of global asymptotic stability of the strictly positive SHPS, we construct a novel Lyapunov functional tailored to the specific characteristics of our model. Furthermore, by employing the method of UALS for parabolic PDEs, we derive sufficient conditions that ensure the global stability of the SHPS. (4) In comparison to the findings reported in references [29-34], the results obtained in this article offer broader applicability and facilitate the further utilization of the Lotka-Volterra predator-prey model.

II. Preliminary

In this section, we present some preliminary results along with the definition of UALS. For the definitions of SHPS and its globally asymptotically stable properties, please refer to reference [36].

TABLE I
THE BIOLOGICAL SIGNIFICANCE OF THE PARAMETERS IN MODEL (1.1)

Parameter	Definition	Parameter	Definition
$d_i(t),(i=1,2,3,4)$	The diffusivity rates	$a_{ij}(t),(i=1,2,j=3,4)$	The capturing rates of the predators
$r_i(t),(i=1,2)$	The intrinsic growth rate	$a_{ii}(t),(i=1,2)$	The interaction within prey species
$r_j(t),(j=3,4)$	The death rates	$a_{12}(t),a_{21}(t),a_{34}(t),a_{43}(t)$	The interference between two species
$a_{ij}(t),(i=1,2,j=3,4)$	The cinversion rates	$b_{ij}(t),(i=1,2,j=3,4)$	The interference within predator species

Definition 2.1. Assume that $\bar{V}(x,t)=(\bar{v}_1(x,t),\bar{v}_2(x,t),\bar{v}_3(x,t),\bar{v}_4(x,t))$, $\underline{V}(x,t)=(\underline{v}_1(x,t),\underline{v}_2(x,t),\underline{v}_3(x,t),\underline{v}_4(x,t))$, if $\bar{V}(x,t)\geq\underline{V}(x,t)$ and for $(x,t)\in\Omega\times R^+$

$$\partial\bar{v}_1(x,t)/\partial t-d_1(t)\Delta\bar{v}_1(x,t)\geq\bar{v}_1(x,t)[r_1(t)-a_{11}(t)\bar{v}_1(x,t)-a_{12}(t)\underline{v}_2(x,t)-\frac{a_{13}(t)\underline{v}_3(x,t)}{b_{13}(t)\underline{v}_3(x,t)+\bar{v}_1(x,t)}-\frac{a_{14}(t)\underline{v}_4(x,t)}{b_{14}(t)\underline{v}_4(x,t)+\bar{v}_1(x,t)}],$$

$$\partial\bar{v}_2(x,t)/\partial t-d_2(t)\Delta\bar{v}_2(x,t)\geq\bar{v}_2(x,t)[r_2(t)-a_{22}(t)\bar{v}_2(x,t)-a_{21}(t)\underline{v}_1(x,t)-\frac{a_{23}(t)\underline{v}_3(x,t)}{b_{23}(t)\underline{v}_3(x,t)+\bar{v}_2(x,t)}-\frac{a_{24}(t)\underline{v}_4(x,t)}{b_{24}(t)\underline{v}_4(x,t)+\bar{v}_2(x,t)}],$$

$$\partial\bar{v}_3(x,t)/\partial t-d_3(t)\Delta\bar{v}_3(x,t)\geq\bar{v}_3(x,t)[-r_3(t)+\frac{a_{31}(t)\bar{v}_1(x,t)}{b_{13}(t)\bar{v}_3(x,t)+\bar{v}_1(x,t)}+\frac{a_{32}(t)\bar{v}_2(x,t)}{b_{23}(t)\bar{v}_3(x,t)+\bar{v}_2(x,t)}-a_{34}(t)\underline{v}_4(x,t)],$$

$$\partial\bar{v}_4(x,t)/\partial t-d_4(t)\Delta\bar{v}_4(x,t)\geq\bar{v}_4(x,t)[-r_4(t)+\frac{a_{41}(t)\bar{v}_1(x,t)}{b_{14}(t)\bar{v}_4(x,t)+\bar{v}_1(x,t)}+\frac{a_{42}(t)\bar{v}_2(x,t)}{b_{24}(t)\bar{v}_4(x,t)+\bar{v}_2(x,t)}-a_{43}(t)\underline{v}_3(x,t)],$$

$$\partial\underline{v}_1(x,t)/\partial t-d_1(t)\Delta\underline{v}_1(x,t)\leq\underline{v}_1(x,t)[r_1(t)-a_{11}(t)\underline{v}_1(x,t)-a_{12}(t)\bar{v}_2(x,t)-\frac{a_{13}(t)\bar{v}_3(x,t)}{b_{13}(t)\bar{v}_3(x,t)+\underline{v}_1(x,t)}-\frac{a_{14}(t)\bar{v}_4(x,t)}{b_{14}(t)\bar{v}_4(x,t)+\underline{v}_1(x,t)}],$$

$$\partial\underline{v}_2(x,t)/\partial t-d_2(t)\Delta\underline{v}_2(x,t)\leq\underline{v}_2(x,t)[r_2(t)-a_{22}(t)\underline{v}_2(x,t)-a_{21}(t)\bar{v}_1(x,t)-\frac{a_{23}(t)\bar{v}_3(x,t)}{b_{23}(t)\bar{v}_3(x,t)+\underline{v}_2(x,t)}-\frac{a_{24}(t)\bar{v}_4(x,t)}{b_{24}(t)\bar{v}_4(x,t)+\underline{v}_2(x,t)}],$$

$$\partial\underline{v}_3(x,t)/\partial t-d_3(t)\Delta\underline{v}_3(x,t)\leq\underline{v}_3(x,t)[-r_3(t)+\frac{a_{31}(t)\underline{v}_1(x,t)}{b_{13}(t)\underline{v}_3(x,t)+\underline{v}_1(x,t)}+\frac{a_{32}(t)\underline{v}_2(x,t)}{b_{23}(t)\underline{v}_3(x,t)+\underline{v}_2(x,t)}-a_{34}(t)\bar{v}_4(x,t)],$$

$$\partial\underline{v}_4(x,t)/\partial t-d_4(t)\Delta\underline{v}_4(x,t)\leq\underline{v}_4(x,t)[-r_4(t)+\frac{a_{41}(t)\underline{v}_1(x,t)}{b_{14}(t)\underline{v}_4(x,t)+\underline{v}_1(x,t)}+\frac{a_{42}(t)\underline{v}_2(x,t)}{b_{24}(t)\underline{v}_4(x,t)+\underline{v}_2(x,t)}-a_{43}(t)\bar{v}_3(x,t)],$$

$$\partial\bar{v}_i(x,t)/\partial n\geq 0,\partial\underline{v}_i(x,t)/\partial n\leq 0,(x,t)\in\partial\Omega\times R^+,$$

and

$$\bar{v}_i(x,0)\geq v_{i0}(x),\underline{v}_i(x,0)\leq v_{i0}(x),x\in\bar{\Omega},i=1,2,3,4,$$

we called $\bar{V}(x,t),\underline{V}(x,t)$ are a pair of ordered UALS for models (1.1)-(1.2).

Lemma 2.1 ([37]). Assume that $\bar{V}(x,t),\underline{V}(x,t)$ are a pair of ordered UALS of models (1.1)-(2.2), then there exists a unique solution $V(x,t)$ of models (1.1)-(1.2) such that $\bar{V}(x,t)\geq V(x,t)\geq\underline{V}(x,t)$.

Lemma 2.2 ([38]). If the function $f(t):R^+\to R$ is uniformly continuous, and the limit $\lim_{t\to\infty}\int_0^t f(s)ds$ exists and is finite, then $\lim_{t\to+\infty}f(t)=0$.

Lemma 2.3 ([39]). Suppose that $S\subset R_n$ be convex and compact and mapping $T:S\to S$ is continuous, then there exists $x^*\in S$ such that $T(x^*)=x^*$.

III. EXISTENCE OF THE SHPS

Given the biological interpretation of the model (1.1), it is logical to focus solely on its positive solutions. Suppose that $\varphi(x)$ is ω-periodic function in R^+, we denote

$$\varphi^m=\sup\{\varphi(x),x\in R^+\},\varphi^l=\inf\{\varphi(x),x\in R^+\}.$$

Subsequently, to prove our main results, we first study the following ordinary differential equation (ODE) that corresponds to model (1.1)

$$\dot{v}_1(t)=v_1(t)[r_1(t)-a_{11}(t)v_1(t)-a_{12}(t)v_2(t)-\frac{a_{13}(t)v_3(t)}{b_{13}(t)v_3(t)+v_1(t)}-\frac{a_{14}(t)v_4(t)}{b_{14}(t)v_4(t)+v_1(t)}],$$

$$\dot{v}_2(t)=v_2(t)[r_2(t)-a_{22}(t)v_2(t)-a_{21}(t)v_1(t)-\frac{a_{23}(t)v_3(t)}{b_{23}(t)v_3(t)+v_2(t)}-\frac{a_{24}(t)v_4(t)}{b_{24}(t)v_4(t)+v_2(t)}],$$
$$\dot{v}_3(t)=v_3(t)[-r_3(t)+\frac{a_{31}(t)v_1(t)}{b_{13}(t)v_3(t)+v_1(t)}+\frac{a_{32}(t)v_2(t)}{b_{23}(t)v_3(t)+v_2(t)}-a_{34}(t)v_4(t)],$$
$$\dot{v}_4(t)=v_4(t)[-r_4(t)+\frac{a_{41}(t)v_1(t)}{b_{14}(t)v_4(t)+v_1(t)}+\frac{a_{42}(t)v_2(t)}{b_{24}(t)v_4(t)+v_2(t)}-a_{43}(t)v_3(t)]. \tag{3.1}$$

For the ODE (3.1), we set

$$M_1^*=\frac{r_1^m}{a_{11}^l},\ M_2^*=\frac{r_2^m}{a_{22}^l},\ M_3^{1*}=\frac{M_1(a_{31}^m+a_{32}^m-r_3^l)}{(r_3^l-a_{32}^m)b_{13}^l},$$
$$M_4^{1*}=\frac{M_1(a_{41}^m+a_{42}^m-r_4^l)}{(r_4^l-a_{42}^m)b_{14}^l},$$
$$m_1^*=\frac{r_1^l b_{13}^l b_{14}^l-a_{12}^m M_2 b_{13}^l b_{14}^l-a_{13}^m b_{14}^l-a_{14}^m b_{13}^l}{a_{11}^m b_{13}^l b_{14}^l},$$
$$m_2^*=\frac{r_2^l b_{23}^l b_{24}^l-a_{21}^m M_1 b_{23}^l b_{24}^l-a_{23}^m b_{24}^l-a_{24}^m b_{23}^l}{a_{22}^m b_{23}^l b_{24}^l},$$
$$m_3^{1*}=\frac{a_{31}^l m_1-(r_3^m+a_{34}^m M_4^1)m_1}{(r_3^m+a_{34}^m M_4^1)b_{13}^m},$$
$$m_4^{1*}=\frac{a_{41}^l m_1-(r_4^m+a_{43}^m M_3^1)m_1}{(r_4^m+a_{43}^m M_3^1)b_{14}^m},$$
$$M_3^{2*}=\frac{M_2(a_{31}^m+a_{32}^m-r_3^l)}{(r_3^l-a_{31}^m)b_{23}^l},$$
$$M_4^{2*}=\frac{M_2(a_{41}^m+a_{42}^m-r_4^l)}{(r_4^l-a_{41}^m)b_{24}^l},$$
$$m_3^{2*}=\frac{a_{32}^l m_2-(r_3^m+a_{34}^m M_4^2)m_2}{(r_3^m+a_{34}^m M_4^2)b_{23}^m},$$
$$m_4^{2*}=\frac{a_{42}^l m_2-(r_4^m+a_{43}^m M_3^2)m_2}{(r_4^m+a_{43}^m M_3^2)b_{24}^m}.$$

Definition 3.1. Suppose that there exist eight positive real numbers $Q_i, q_i, (i=1,2,3,4)$ and T, such that $q_i \le v_i(t) \le Q_i$, as $t>T$ for each positive solution $(v_1(t), v_2(t), v_3(t), v_4(t))$ of the ODE (3.1) subject to the positive initial values, then ODE (3.1) is called permanent.

Theorem 3.1. If it holds that

(H_1) $\frac{a_{31}^m+a_{32}^m-r_3^l}{r_3^l-a_{32}^m}>0$, (H_2) $\frac{a_{41}^m+a_{42}^m-r_4^l}{r_4^l-a_{42}^m}>0$,

(H_3) $r_1^l b_{13}^l b_{14}^l-a_{12}^m M_2 b_{13}^l b_{14}^l-a_{13}^m b_{14}^l-a_{14}^m b_{13}^l>0$,

(H_4) $r_2^l b_{23}^l b_{24}^l-a_{21}^m M_1 b_{23}^l b_{24}^l-a_{23}^m b_{24}^l-a_{24}^m b_{23}^l>0$,

(H_5) $a_{31}^l>r_3^m+a_{34}^m M_4^1$, (H_6) $a_{41}^l>r_4^m+a_{43}^m M_3^1$.

Then the ODE (3.1) is permanent.

Proof. When the ODE (3.1) satisfies the conditions $(H_1)-(H_6)$, we can choose some appropriate positive real numbers $M_i, m_i, M_j^1, m_j^1, (i=1,2, j=3,4)$ such that $0<m_i<m_i^*<M_i^*<M_i, 0<m_j^1<m_j^{1*}<M_j^{1*}<M_j^1$.

According to the first equation of ODE (3.1), it follows that

$$\dot{v}_1(t)\le v_1(t)[r_1(t)-a_{11}(t)v_1(t)]$$
$$\le v_1(t)[r_1^m-a_{11}^l v_1(t)]=a_{11}^l v_1(t)[-v_1(t)+\frac{r_1^m}{a_{11}^l}]$$
$$=a_{11}^l v_1(t)[-v_1(t)+M_1^*]<a_{11}^l v_1(t)[-v_1(t)+M_1].$$

Based on the comparison theorem of ODE, we can obtain

(1) When $0<v_1(t_0)<M_1$, if $t\ge t_0$, then $v_1(t)\le M_1$.

(2) When $v_1(t_0)\ge M_1$, for a enough large t, one has $v_1(t)\le M_1$. Otherwise, if $v_1(t)>M_1$, then there exists $\alpha>0$ such that $v_1(t)\ge M_1^*+\alpha$. Furthermore, one has

$$\dot{v}_1(t)\Big|_{v_1(t)>M_1}\le v_1(t)[r_1(t)-a_{11}(t)v_1(t)]$$
$$\le a_{11}^l v_1(t)[M_1^*-v_1(t)]<-a_{11}^l \alpha v_1(t),$$

thus, it holds that $v_1(t)<v_1(t_0)\exp(-a_{11}^l\alpha t)\to 0$ as $t\to+\infty$. The above inequality contradicts $v_1(t)>M_1$, so we can choose a adequacy large $T_1\ge t_0\ge 0$ such that

$$v_1(t)\le M_1 \ \text{as}\ t>T_1.$$

Similarly, according to the second equation of the ODE (3.1), it holds that there exist a sufficiently large $T_2\ge t_0\ge 0$ such that

$$v_2(t)\le M_2 \ \text{as}\ t>T_2.$$

Based on the third equation of the ODE (3.1), and using (3.3) and (3.4), it follows that

$$\dot{v}_3\le v_3(t)[-r_3^l+\frac{a_{31}^m M_1}{b_{13}^l v_3(t)+M_1}+a_{32}^m]$$
$$=v_3(t)[\frac{-r_3^l(b_{13}^l v_3(t)+M_1)+a_{31}^m M_1+a_{32}^m(b_{13}^l v_3(t)+M_1)}{b_{13}^l v_3(t)+M_1}]$$
$$=v_3(t)\frac{(r_3^l-a_{32}^m)b_{13}^l}{b_{13}^l v_3(t)+M_1}[-v_3(t)+\frac{M_1(a_{31}^m+a_{32}^m-r_3^l)}{(r_3^l-a_{32}^m)b_{13}^l}]$$
$$=v_3(t)\frac{(r_3^l-a_{32}^m)b_{13}^l}{b_{13}^l v_3(t)+M_1}[-v_3(t)+M_3^{1*}]$$
$$<v_3(t)\frac{(r_3^l-a_{32}^m)b_{13}^l}{b_{13}^l v_3(t)+M_1}[-v_3(t)+M_3^1].$$

Based on the comparison theorem and the preceding analysis, we can conclude that

(3) When $0 < v_3(t_0) < M_3^1$, if $t \ge t_0$, then $v_3(t) \le M_3^1$,

(4) When $v_3(t_0) \ge M_3^1$, for a enough large t, we have $v_3(t) \le M_3^1$.

Therefore, it follows that there exist a enough large $T_3 \ge t_0 \ge 0$ such that

$$v_3(t) \le M_3^1 \text{ as } t > T_3.$$

Similarly, according to the fourth equation of the ODE (3.1), and using (3.3) and (3.4), it follows that there exist a enough large $T_4 \ge t_0 \ge 0$ such that

$$v_4(t) \le M_4^1 \text{ as } t > T_4.$$

Next, we prove that $v_1(t), v_2(t), v_3(t), v_4(t)$ have positive lower bound. According to the first equation of ODE (3.1), it holds that

$$\dot{v}_1 \ge v_3(t)[r_1^l - a_{11}^m v_1(t) - a_{12}^m M_2 - \frac{a_{13}^m}{b_{13}^l} - \frac{a_{14}^m}{b_{14}^l}]$$
$$= v_3(t) a_{11}^m [-v_1(t) + \frac{r_1^l b_{13}^l b_{14}^l - a_{12}^m M_2 b_{13}^l b_{14}^l - a_{13}^m b_{14}^l - a_{14}^m b_{13}^l}{a_{11}^m b_{13}^l b_{14}^l}]$$
$$= v_1(t) a_{11}^m [-v_1(t) + m_1^*] \ge v_1(t) a_{11}^m [-v_1(t) + m_1].$$

Based on the comparison theorem of ODE, it holds that

(5) When $m_1 < v_1(t_0)$, if $t \ge t_0$, then $m_1 \le v_1(t)$,

(6) When $0 < v_1(t_0) \le m_1$, for a enough large t, we have $m_1 \le v_1(t)$. Otherwise, if $v_1(t) < m_1$, then there exists $\beta > 0$ such that $v_1(t) \le m_1^* - \beta$. Furthermore, we can obtain

$$\dot{v}_1(t)\Big|_{x_1(t) < m_1} \ge a_{11}^m v_1(t)[-v_1(t) + m_1^*] > a_{11}^m \beta v_1(t),$$

thus, it holds that $v_1(t) > v_1(t_0)\exp(a_{11}^m \beta t) \to +\infty$ as $t \to +\infty$.

The above inequality contradicts $v_1(t) < m_1$, so we can choose a adequacy large $T_1' \ge t_0 \ge 0$ such that

$$v_1(t) \ge m_1 \text{ as } t > T_1'.$$

Similarly, utilizing the second equation of ODE (3.1), we can demonstrate the existence of a sufficiently large constant T_2' such that

$$v_2(t) > m_2^* = \frac{r_2^l b_{23}^l b_{24}^l - a_{21}^m M_1 b_{23}^l b_{24}^l - a_{23}^m b_{24}^l - a_{24}^m b_{23}^l}{a_{22}^m b_{23}^l b_{24}^l}$$

as $t > T_2^*$.

According to the third equation of the ODE (3.1), and using (3.6) and (3.7), it follows that

$$\dot{v}_3 \ge v_3(t)[-r_3^m + \frac{a_{31}^l m_1}{b_{13}^m v_3(t) + m_1} - a_{34}^m M_4^1]$$
$$= v_3(t)[\frac{-(r_3^m + a_{34}^m M_4^1)(b_{13}^m v_3(t) + m_1) + a_{31}^l m_1}{b_{13}^m v_3(t) + m_1}]$$
$$= v_3(t)\frac{(r_3^m + a_{34}^m M_4^1) b_{13}^m}{b_{13}^m v_3(t) + m_1}[-v_3(t) + \frac{a_{31}^l m_1 - (r_3^m + a_{34}^m M_4^1) m_1}{(r_3^m + a_{34}^m M_4^1) b_{13}^m}]$$
$$= v_3(t)\frac{(r_3^m + a_{34}^m M_4^1) b_{13}^m}{b_{13}^m v_3(t) + m_1}[-v_3(t) + m_3^{1*}]$$
$$> v_3(t)\frac{(r_3^m + a_{34}^m M_4^1) b_{13}^m}{b_{13}^m v_3(t) + m_1}[-v_3(t) + m_3^1].$$

Utilizing the same method outlined above, along with the comparison theorem of ODEs, we can conclude that

(7) When $m_3^1 < v_3(t_0)$, if $t \ge t_0$, then $m_3^1 \le v_3(t)$,

(8) When $v_3(t_0) \ge m_3^1$, for a enough large t, we have $m_3^1 \le v_3(t)$.

Therefore, it follows that there exist a enough large $T_3' \ge t_0 \ge 0$ such that

$$v_3(t) \ge m_3^1 \text{ as } t > T_3'.$$

Analogously, according to the fourth equation of the ODE (3.1) we can choose a adequacy large positive constants T_4' such that

$$v_4(t) \ge m_4^1 > m_4^{1*} = \frac{a_{41}^l m_1 - (r_4^m + a_{43}^m M_3^1) m_1}{(r_4^m + a_{43}^m M_3^1)\, b_{14}^m},$$

as $t > T_4'$.

From (3.3)-(3.10), and set $T = \max_{1 \le i \le 4}\{T_i, T_i'\}$, then we have $m_i \le v_i(t) \le M_i, (i = 1, 2), m_j^1 \le v_j(t) \le M_j^1, \ (j = 3, 4)$ as $t > T$ for any positive solution $(v_1(t), v_2(t), v_3(t), v_4(t))$ of the ODE (3.1) subject to the positive initial values. Thus, we complete the proof of Theorem 3.1. □

Given the symmetry inherent in ODE (3.1), one can arrive at analogous conclusions by employing the previously outlined analysis and proof techniques.

Theorem 3.2. If it follows that,

$$(H_3)\, r_1^l b_{13}^l b_{14}^l - a_{12}^m M_2 b_{13}^l b_{14}^l - a_{13}^m b_{14}^l - a_{14}^m b_{13}^l > 0,$$
$$(H_4)\, r_2^l b_{23}^l b_{24}^l - a_{21}^m M_1 b_{23}^l b_{24}^l - a_{23}^m b_{24}^l - a_{24}^m b_{23}^l > 0,$$
$$(H_7)\, \frac{a_{31}^m + a_{32}^m - r_3^l}{r_3^l - a_{31}^m} > 0, \quad (H_8)\, \frac{a_{41}^m + a_{42}^m - r_4^l}{r_4^l - a_{41}^m} > 0,$$
$$(H_9)\, a_{32}^l > r_3^m + a_{34}^m M_4^2, \quad (H_{10})\, a_{42}^l > r_4^m + a_{43}^m M_3^2.$$

Then the ODE (3.1) is permanent.

Theorem 3.3. Supposed that the model (1.1) satisfies the conditions $(H_1) - (H_6)$, then the model (1.1) has a strictly positive SHPS $V(t) = (\hat{v}_1(t), \hat{v}_2(t), \hat{v}_3(t), \hat{v}_4(t))$.

Proof. Utilizing the theorem regarding the existence and uniqueness of ODE solutions, we are able to define a Poincaré operator $\varphi: R_+^4 \to R_+^4$ in the manner specified below

$$\varphi(V_0) = V(t, \omega, t_0, V_0),$$

where $V(t,\omega,t_0,V_0) = (v_1(t), v_2(t), v_3(t), v_4(t))$ be a positive solution of the ODE (3.1) subject to the initial values $V_0 = (v_1(t_0), v_2(t_0), v_3(t_0), v_4(t_0))$. And define

$$S = \left\{ (v_1, v_2, v_3, v_4) \in R_+^4 \middle| m_i \le v_i \le M_i, i = 1, 2, 3, 4 \right\},$$

it becomes evident that $S \subset R_+^4$ possesses the properties of being both compact and convex. By consulting Theorem 3.1 and invoking the continuity of the solution to ODE (3.1) with respect to initial values, it becomes apparent that the mapping φ is continuous from S to S. Additionally, applying Lemma 2.3 allows us to conclude that ODE (3.1) possesses a positive ω- periodic solution $(v_1^*(t), v_2^*(t), v_3^*(t), v_4^*(t))$. The fact that $(v_1^*(t), v_2^*(t), v_3^*(t), v_4^*(t))$ serves as a strictly positive SHPS for model (1.1) is readily apparent, thereby concluding the proof of Theorem 3.3. □

Utilizing the same analysis and proof techniques employed in Theorem 3.3, one can deduce analogous conclusions based on Theorem 3.2.

Theorem 3.4. If the conditions $(H_3), (H_4), (H_7)-(H_{10})$ are satisfied, then the model (1.1) has a strictly positive SHPS $V(t) = \left(\hat{v}_1(t), \hat{v}_2(t), \hat{v}_3(t), \hat{v}_4(t)\right)$.

IV. Stability of the SHPS for RDPPM (1.1)-(1.2)

In this section, we research the globally asymptotic stability of the SHPS for the 4-species nonautonomous reaction-diffusive periodic RDPPM, equations (1.1)-(1.2), employing the method of UALS for parabolic PDEs along- side Lyapunov stability theory. Consequently, we provide some readily verifiable sufficient conditions for this stability.

Theorem 4.1. Suppose that the reaction-diffusive nonautonomous ω- periodic RDPPM (1.1) satisfies assumptions $(H_1)-(H_6)$ and the following assumptions

$$(H_{11})\ d_{11}^l - a_{21}^m - \frac{(a_{13}^m + a_{31}^m b_{13}^m) M_3^1}{(b_{13}^l m_3^1 + m_1)^2} - \frac{(a_{14}^m + a_{41}^m b_{14}^m) M_4^1}{(b_{14}^l m_4^1 + m_1)^2} > 0,$$

$$(H_{12})\ d_{22}^l - a_{12}^m - \frac{(a_{23}^m + a_{32}^m b_{23}^m) M_3^1}{(b_{23}^l m_3^1 + m_2)^2} - \frac{(a_{24}^m + a_{42}^m b_{24}^m) M_4^1}{(b_{24}^l m_4^1 + m_2)^2} > 0,$$

$$(H_{13})\ -a_{43}^m + \frac{(a_{31}^l b_{13}^l - a_{13}^m) m_1}{(b_{13}^m M_3^1 + M_1)^2} + \frac{(a_{32}^l b_{23}^l - a_{23}^m) m_2}{(b_{23}^m M_3^1 + M_2)^2} > 0,$$

$$(H_{14})\ -a_{34}^m + \frac{(a_{41}^l b_{14}^l - a_{14}^m) m_1}{(b_{14}^m M_4^1 + M_1)^2} + \frac{(a_{42}^l b_{24}^l - a_{24}^m) m_2}{(b_{24}^m M_4^1 + M_2)^2} > 0.$$

then there is a strictly positive SHPS $(v_1^*(t), v_2^*(t), v_3^*(t), v_4^*(t))$ of models (1.1)-(1.2). And the the SHPS is globally asymptotically stable, i.e. , the solution $(v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))$ of RDPPM (1.1)-(1.2) with respect to any positive initial values satisfies

$$\lim_{t\to\infty} \left(v_i(x,t) - v_i^*(t) \right) = 0, i = 1,2,3,4,$$

uniformly for $x \in \bar{\Omega}$.

Proof. After obtaining existence results through Theorem 3.3, our focus shifts to examining the stability aspects. Let $l_i = \min_{x\in\bar{\Omega}} v_{i0}(x)$, $r_i = \max_{x\in\bar{\Omega}} v_{i0}(x)$, $i = 1,2,3,4$, then $0 < l_i \le v_{i0}(x) \le r_i$. Suppose that $(\bar{v}_1(t), \bar{v}_2(t), \bar{v}_3(t), \bar{v}_4(t))$ and $(\underline{v}_1(t), \underline{v}_2(t), \underline{v}_3(t), \underline{v}_4(t))$ are the solutions for (3.1) corresponding to initial values $(\bar{v}_1(0), \bar{v}_2(0), \bar{v}_3(0), \bar{v}_4(0)) = (r_1, r_2, r_3, r_4)$ and $(\underline{v}_1(0), \underline{v}_2(0), \underline{v}_3(0), \underline{v}_4(0)) = (l_1, l_2, l_3, l_4)$ respectively, then there are a pair of ordered UALS $(\bar{v}_1(t), \bar{v}_2(t), \bar{v}_3(t), \bar{v}_4(t))$ and $(\underline{v}_1(t), \underline{v}_2(t), \underline{v}_3(t), \underline{v}_4(t))$ for models (1.1)-(1.2). Thus, there exists a unique solution $(v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))$, $(x,t) \in \bar{\Omega} \times R^+$, for models (1.1)-(1.2) by Lemma 2.1, which satisfies

$$\begin{aligned}(\underline{v}_1(t), \underline{v}_2(t), \underline{v}_3(t), \underline{v}_4(t)) &\le (v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t)) \\ &\le (\bar{v}_1(t), \bar{v}_2(t), \bar{v}_3(t), \bar{v}_4(t))\end{aligned}$$

If we have

$$\lim_{t\to\infty} \bar{v}_i(t) - v_i^*(t) = \lim_{t\to\infty} \underline{v}_i(t) - v_i^*(t) = 0,$$

then (4.1) is obtained. Hence, to attain (4.2), it is imperative that we demonstrate that the solution $(v_1(t), v_2(t), v_3(t), v_4(t))$ for ODE (3.1) with respect to any positive initial value $(v_1(0), v_2(0), v_3(0), v_4(0)) = (v_{10}, v_{20}, v_{30}, v_{40})$ satisfy

$$\lim_{t\to\infty} \left(v_i(t) - v_i^*(t) \right) = 0, (i = 1,2,3,4).$$

From Theorem3.1, there exist positive constants $M_i, m_i, M_j^1, m_j^1, i = 1,2, j = 3,4$, and T, such that $m_i \le v_i(t) \le M_i, m_j^1 \le v_j(t) \le M_j^1, i = 1,2, j = 3,4,$ for all $t > T$.

Define the Lyapunov function as outlined below

$$U(t) = \sum_{i=1}^{4} [\left| \ln v_i(t) - \ln v^*{}_i(t) \right|], t > 0.$$

Suppose that D^+U denotes the right derivation on function $U(t)$, it follows that

$$D^{+}U(t)=\sum_{i=1}^{4}D^{+}[\left|\ln v_i(t)-\ln v_i^{*}(t)\right|]=\sum_{i=1}^{4}\operatorname{sgn}\{v_i(t)-v_i^{*}(t)\}(\frac{\dot v_i(t)}{v_i(t)}-\frac{\dot v_i^{*}(t)}{v^{*}{}_i(t)})$$

$$=\operatorname{sgn}\{v_1(t)-v_1^{*}(t)\}[-a_{11}(t)(v_1(t)-v_1^{*}(t))-a_{12}(t)(v_2(t)-v_2^{*}(t))-a_{13}(t)(\frac{v_3(t)}{b_{13}(t)v_3(t)+v_1(t)}-\frac{v_3^{*}(t)}{b_{13}(t)v_3^{*}(t)+v_1^{*}(t)})$$

$$-a_{14}(t)(\frac{v_4(t)}{b_{14}(t)v_4(t)+v_1(t)}-\frac{v_4^{*}(t)}{b_{14}(t)v_4^{*}(t)+v_1^{*}(t)})]+\operatorname{sgn}\{v_2(t)-v_2^{*}(t)\}[-a_{22}(t)(v_2(t)-v_2^{*}(t))-a_{21}(t)(v_1(t)-v_1^{*}(t))$$

$$-a_{23}(t)(\frac{v_3(t)}{b_{23}(t)v_3(t)+v_2(t)}-\frac{v_3^{*}(t)}{b_{23}(t)v_3^{*}(t)+v_2^{*}(t)})-a_{24}(t)(\frac{v_4(t)}{b_{24}(t)v_4(t)+v_2(t)}-\frac{v_4^{*}(t)}{b_{24}(t)v_4^{*}(t)+v_2^{*}(t)})$$

$$+\operatorname{sgn}\{v_3(t)-v_3^{*}(t)\}[a_{31}(t)(\frac{v_1(t)}{b_{13}(t)v_3(t)+v_1(t)}-\frac{v_1^{*}(t)}{b_{13}(t)v_3^{*}(t)+v_1^{*}(t)})+a_{32}(t)(\frac{v_2(t)}{b_{23}(t)v_3(t)+v_2(t)}-\frac{v_2^{*}(t)}{b_{23}(t)v_3^{*}(t)+v_2^{*}(t)})$$

$$-a_{34}(t)(v_4(t)-v_4^{*}(t))]+\operatorname{sgn}\{v_4(t)-v_4^{*}(t)\}[a_{41}(t)(\frac{v_1(t)}{b_{14}(t)v_4(t)+v_1(t)}-\frac{v_1^{*}(t)}{b_{14}(t)v_4^{*}(t)+v_1^{*}(t)})+a_{42}(t)(\frac{v_2(t)}{b_{24}(t)v_4(t)+v_2(t)}$$

$$-\frac{v_2^{*}(t)}{b_{24}(t)v_4^{*}(t)+v_2^{*}(t)})-a_{43}(t)(v_3(t)-v_3^{*}(t))]$$

$$=\operatorname{sgn}\{v_1(t)-v_1^{*}(t)\}[-a_{11}(t)(v_1(t)-v_1^{*}(t))-a_{12}(t)(v_2(t)-v_2^{*}(t))-a_{13}(t)\frac{v_1^{*}(t)(v_3(t)-v_3^{*}(t))-v_3^{*}(t)(v_1(t)-v_1^{*}(t))}{(b_{13}(t)v_3(t)+v_1(t))(b_{13}(t)v_3^{*}(t)+v_1^{*}(t))}$$

$$-a_{14}(t)\frac{v_1^{*}(t)(v_4(t)-v_4^{*}(t))-v_4^{*}(t)(v_1(t)-v_1^{*}(t))}{(b_{14}(t)v_4(t)+v_1(t))(b_{14}(t)v_4^{*}(t)+v_1^{*}(t))}]+\operatorname{sgn}\{v_2(t)-v_2^{*}(t)\}[-a_{22}(t)(v_2(t)-v_2^{*}(t))-a_{21}(t)(v_1(t)-v_1^{*}(t))$$

$$-a_{23}(t)\frac{v_2^{*}(t)(v_3(t)-v_3^{*}(t))-v_3^{*}(t)(v_2(t)-v_2^{*}(t))}{(b_{23}(t)v_3(t)+v_2(t))(b_{23}(t)v_3^{*}(t)+v_2^{*}(t))}-a_{24}(t)\frac{v_2^{*}(t)(v_4(t)-v_4^{*}(t))-v_4^{*}(t)(v_2(t)-v_2^{*}(t))}{(b_{24}(t)v_4(t)+v_2(t))(b_{24}(t)v_4^{*}(t)+v_2^{*}(t))}]$$

$$+\operatorname{sgn}\{v_3(t)-v_3^{*}(t)\}[a_{31}(t)b_{13}(t)\frac{v_3^{*}(t)(v_1(t)-v_1^{*}(t))-v_1^{*}(t)(v_3(t)-v_3^{*}(t))}{(b_{13}(t)v_3(t)+v_1(t))(b_{13}(t)v_3^{*}(t)+v_1^{*}(t))}+a_{32}(t)b_{23}(t)\frac{v_3^{*}(t)(v_2(t)-v_2^{*}(t))-v_2^{*}(t)(v_3(t)-v_3^{*}(t))}{(b_{23}(t)v_3(t)+v_2(t))(b_{23}(t)v_3^{*}(t)+v_2^{*}(t))}$$

$$-a_{34}(t)(v_4(t)-v_4^{*}(t))]+\operatorname{sgn}\{v_4(t)-v_4^{*}(t)\}[a_{41}(t)b_{14}(t)\frac{v_4^{*}(t)(v_1(t)-v_1^{*}(t)-v_1^{*}(t)(v_4(t)-v_4^{*}(t))}{(b_{14}(t)v_4(t)+v_1(t))(b_{14}(t)v_4^{*}(t)+v_1^{*}(t))}$$

$$+a_{42}(t)b_{24}(t)\frac{v_4^{*}(t)(v_2(t)-v_2^{*}(t))-v_2^{*}(t)(v_4(t)-v_4^{*}(t))}{(b_{24}(t)v_4(t)+v_2(t))(b_{24}(t)v_4^{*}(t)+v_2^{*}(t))}-a_{43}(t)(v_3(t)-v_3^{*}(t))]$$

$$\le\left|v_1(t)-v_1^{*}(t)\right|[-a_{11}(t)+a_{21}(t)+\frac{(a_{13}(t)+a_{31}(t)b_{13}(t))v_3^{*}(t)}{(b_{13}(t)v_3(t)+v_1(t)(b_{13}(t)v_3^{*}(t)+v_1^{*}(t))}+\frac{(a_{14}(t)+a_{41}(t)b_{14}(t))v_4^{*}(t)}{(b_{14}(t)v_4(t)+v_1(t))(b_{14}(t)v_4^{*}(t)+v_1^{*}(t))}]$$

$$+\left|v_2(t)-v_2^{*}(t)\right|[-a_{22}(t)+a_{12}(t)+\frac{(a_{23}(t)+a_{32}(t)b_{23}(t))v_3^{*}(t)}{(b_{23}(t)v_3(t)+v_2(t))(b_{23}(t)v_3^{*}(t)+v_2^{*}(t))}+\frac{(a_{24}(t)+a_{42}(t)b_{24}(t))v_4^{*}(t)}{(b_{24}(t)v_4(t)+v_2(t))(b_{24}(t)v_4^{*}(t)+v_2^{*}(t))}]$$

$$+\left|v_3(t)-v_3^{*}(t)\right|[a_{43}(t)-\frac{(a_{31}(t)b_{13}(t)-a_{13}(t))v_1^{*}(t)}{(b_{13}(t)v_3(t)+v_1(t)(b_{13}(t)v_3^{*}(t)+v_1^{*}(t))}-\frac{(a_{32}(t)b_{23}(t)-a_{23}(t))v_2^{*}(t)}{(b_{23}(t)v_3(t)+v_2(t))(b_{23}(t)v_3^{*}(t)+v_2^{*}(t))}]$$

$$+\left|v_4(t)-v_4^{*}(t)\right|[a_{34}(t)-\frac{(a_{41}(t)b_{14}(t)-a_{14}(t))v_1^{*}(t)}{(b_{14}(t)v_4(t)+v_1(t))(b_{14}(t)v_4^{*}(t)+v_1^{*}(t))}-\frac{(a_{42}(t)b_{24}(t)-a_{24}(t))v_2^{*}(t)}{(b_{24}(t)v_4(t)+v_2(t))(b_{24}(t)v_4^{*}(t)+v_2^{*}(t))}]$$

$$\le-\left|v_1(t)-v_1^{*}(t)\right|[a_{11}^{l}-a_{21}^{m}-\frac{(a_{13}^{m}+a_{31}^{m}b_{13}^{m})M_3^{1}}{(b_{13}^{l}m_3^{1}+m_1)^2}-\frac{(a_{14}^{m}+a_{41}^{m}b_{14}^{m})M_4^{1}}{(b_{14}^{l}m_4^{1}+m_1)^2}]$$

$$-\left|v_2(t)-v_2^{*}(t)\right|[a_{22}^{l}-a_{12}^{m}-\frac{(a_{23}^{m}+a_{32}^{m}b_{23}^{m})M_3^{1}}{(b_{23}^{l}m_3^{1}+m_2)^2}-\frac{(a_{24}^{m}+a_{42}^{m}b_{24}^{m})M_4^{1}}{(b_{24}^{l}m_4^{1}+m_2)^2}]$$

$$-\left|v_3(t)-v_3^{*}(t)\right|[-a_{43}^{m}+\frac{(a_{31}^{l}b_{13}^{l}-a_{13}^{m})m_1}{(b_{13}^{m}M_3^{1}+M_1)^2}+\frac{(a_{32}^{l}b_{23}^{l}-a_{23}^{m})m_2}{(b_{23}^{m}M_3^{1}+M_2)^2}]$$

$$-\left|v_4(t)-v_4^{*}(t)\right|[-a_{34}^{m}+\frac{(a_{41}^{l}b_{14}^{l}-a_{14}^{m})m_1}{(b_{14}^{m}M_4^{1}+M_1)^2}+\frac{(a_{42}^{l}b_{24}^{l}-a_{24}^{m})m_2}{(b_{24}^{m}M_4^{1}+M_2)^2}].$$

In view of conditions $(H_{11})-(H_{14})$, one has

$$\alpha=\min\{a_{11}^{l}-a_{21}^{m}-\frac{(a_{13}^{m}+a_{31}^{m}b_{13}^{m})M_{3}^{1}}{(b_{13}^{l}m_{3}^{1}+m_{1})^{2}}-\frac{(a_{14}^{m}+a_{41}^{m}b_{14}^{m})M_{4}^{1}}{(b_{14}^{l}m_{4}^{1}+m_{1})^{2}},$$
$$a_{22}^{l}-a_{12}^{m}-\frac{(a_{23}^{m}+a_{32}^{m}b_{23}^{m})M_{3}^{1}}{(b_{23}^{l}m_{3}^{1}+m_{2})^{2}}-\frac{(a_{24}^{m}+a_{42}^{m}b_{24}^{m})M_{4}^{1}}{(b_{24}^{l}m_{4}^{1}+m_{2})^{2}},$$
$$-a_{43}^{m}+\frac{(a_{31}^{l}b_{13}^{l}-a_{13}^{m})m_{1}}{(b_{13}^{m}M_{3}^{1}+M_{1})^{2}}+\frac{(a_{32}^{l}b_{23}^{l}-a_{23}^{m})m_{2}}{(b_{23}^{m}M_{3}^{1}+M_{2})^{2}},$$
$$-a_{34}^{m}+\frac{(a_{41}^{l}b_{14}^{l}-a_{14}^{m})m_{1}}{(b_{14}^{m}M_{4}^{1}+M_{1})^{2}}+\frac{(a_{42}^{l}b_{24}^{l}-a_{24}^{m})m_{2}}{(b_{24}^{l}M_{4}^{1}+M_{2})^{2}}\}>0.$$

Thus,

$$D^{+}U(t)\le-\alpha\sum_{i=1}^{4}\left|v_{i}(t)-v_{i}^{*}(t)\right|.$$

Integrating (4.4) from T to t ($T\ge t_{0}$), one has

$$U(t)+\alpha\int_{T}^{t}(\sum_{i=1}^{4}\left|x_{i}(t)-y_{i}(t)\right|)ds\le U(T)<+\infty.$$

Therefore,

$$\int_{T}^{t}(\sum_{i=1}^{4}\left|v_{i}(t)-v_{i}^{*}(t)\right|)ds\le\frac{U(T)}{\alpha}<+\infty.$$

By (4.5), we have

$$\sum_{i=1}^{4}(\left|v_{i}(t)-v_{i}^{*}(t)\right|)\in L^{1}(T,+\infty).$$

From the uniformity permanence of the model (1.1), $\sum_{i=1}^{4}(\left|v_{i}(t)-v_{i}^{*}(t)\right|)$ is uniformity continuous. By Lemma 2.2, we can obtain that

$$\lim_{t\to+\infty}\left|v_{i}(t)-v_{i}^{*}(t)\right|=0,(i=1,2,3,4).$$

This ends the proof of Theorem 4.1. □

Utilizing the analysis and proof techniques employed in Theorem 4.1, one can derive analogous conclusions from Theorems 3.2 and 3.4.

Theorem 4.2. Assume that the model (1.1) satisfies $(H_{3}),(H_{4}),(H_{7})-(H_{10})$ and the following conditions

$$(H_{15})\,a_{11}^{l}-a_{21}^{m}-\frac{(a_{13}^{m}+a_{31}^{m}b_{13}^{m})M_{3}^{2}}{(b_{13}^{l}m_{3}^{2}+m_{1})^{2}}-\frac{(a_{14}^{m}+a_{41}^{m}b_{14}^{m})M_{4}^{2}}{(b_{14}^{l}m_{4}^{2}+m_{1})^{2}}>0,$$

$$(H_{16})\,a_{22}^{l}-a_{12}^{m}-\frac{(a_{23}^{m}+a_{32}^{m}b_{23}^{m})M_{3}^{2}}{(b_{23}^{l}m_{3}^{2}+m_{2})^{2}}-\frac{(a_{24}^{m}+a_{42}^{m}b_{24}^{m})M_{4}^{2}}{(b_{24}^{l}m_{4}^{2}+m_{2})^{2}}>0,$$

$$(H_{17})-a_{43}^{m}+\frac{(a_{31}^{l}b_{13}^{l}-a_{13}^{m})m_{1}}{(b_{13}^{m}M_{3}^{2}+M_{1})^{2}}+\frac{(a_{32}^{l}b_{23}^{l}-a_{23}^{m})m_{2}}{(b_{23}^{m}M_{3}^{2}+M_{2})^{2}}>0,$$

$$(H_{18})-a_{34}^{m}+\frac{(a_{41}^{l}b_{14}^{l}-a_{14}^{m})m_{1}}{(b_{14}^{m}M_{4}^{2}+M_{1})^{2}}+\frac{(a_{42}^{l}b_{24}^{l}-a_{24}^{m})m_{2}}{(b_{24}^{m}M_{4}^{2}+M_{2})^{2}}>0.$$

then there is a strictly positive SHPS $(v_{1}^{*}(t),v_{2}^{*}(t),v_{3}^{*}(t),v_{4}^{*}(t))$ of models (1.1)-(1.2). And the SHPS exhibits global asymptotic stability, meaning that for any positive initial values, the solution $(v_{1}(x,t),v_{2}(x,t),v_{3}(x,t),v_{4}(x,t))$ to the reaction-diffusive RDP-PM, equations (1.1)-(1.2), satisfies

$$\lim_{t\to\infty}(v_{i}(x,t)-v_{i}^{*}(t))=0,i=1,2,3,4,$$

uniformly for $x\in\bar{\Omega}$.

V. Numerical simulations

Two examples are provided to substantiate the findings presented in this article. To verify the accuracy of Theorems 4.1 and 4.2, we utilize a 2-periodic function as the coefficient for the reaction-diffusive nonautonomous ω - periodic RDPPM (1.1)-(1.2).

Example 5.1. A 2-periodic reaction-diffusion model encompassing four species and incorporating functional responses that are dependent on ratios is considered here. Based on the conditions $(H_{1})-(H_{6})$ and $(H_{11})-(H_{14})$ outlined in Theorem 4.1, we have selected specific parameter values for the models (5.1)-(5.3) through a series of calculations. It is important to note that the choice of these parameters is not unique.

$$\partial v_{1}(x,t)/\partial t-\frac{\partial^{2}v_{1}(x,t)}{\partial x^{2}}=v_{1}(x,t)[(29+\cos\pi t)$$
$$-(27+\sin\pi t)v_{1}(x,t)-(0.8+0.2\sin\pi t)v_{2}(x,t)$$
$$-\frac{(0.075+0.025\sin\pi t)v_{3}(x,t)}{(1.05+0.05\sin\pi t)v_{3}(x,t)+v_{1}(x,t)}$$
$$-\frac{(0.065+0.035\sin\pi t)v_{4}(x,t)}{(1.15+0.15\sin\pi t)v_{4}(x,t)+v_{1}(x,t)}],$$

$$\partial v_{2}(x,t)/\partial t-\frac{\partial^{2}v_{2}(x,t)}{\partial x^{2}}=v_{2}(x,t)[(25+\cos\pi t)$$
$$-(22+\sin\pi t)v_{2}(x,t)-(0.7+0.3\sin\pi t)v_{1}(x,t)$$
$$-\frac{(0.085+0.015\sin\pi t)v_{3}(x,t)}{(1.04+0.04\sin\pi t)v_{3}(x,t)+v_{2}(x,t)}$$
$$-\frac{(0.075+0.025\sin\pi t)v_{4}(x,t)}{(1.06+0.06\sin\pi t)v_{4}(x,t)+v_{2}(x,t)}],$$

$$\partial v_{3}(x,t)/\partial t-\frac{\partial^{2}v_{3}(x,t)}{\partial x^{2}}=v_{3}(x,t)[-(2.05+0.05\cos\pi t)$$
$$+\frac{(3.2+0.2\sin\pi t)v_{1}(x,t)}{(1.05+0.05\sin\pi t)v_{3}(x,t)+v_{1}(x,t)}$$
$$+\frac{(0.75+0.25\sin\pi t)v_{2}(x,t)}{(1.04+0.04\sin\pi t)v_{3}(x,t)+v_{2}(x,t)}$$
$$-(0.085+0.015\sin\pi t)v_{4}(x,t)],$$

$$\partial v_{4}(x,t)/\partial t-\frac{\partial^{2}v_{4}(x,t)}{\partial x^{2}}=v_{4}(x,t)[-(2.1+0.1\cos\pi t)$$

$$+\frac{(3.1+0.1\sin \pi t)v_1(x,t)}{(1.15+0.15\sin \pi t)v_4(x,t)+v_1(x,t)}$$
$$+\frac{(0.65+0.35\sin \pi t)v_2(x,t)}{(1.06+0.06\sin \pi t)v_4(x,t)+v_2(x,t)}$$
$$-(0.065+0.035\sin \pi t)v_3(x,t)], \tag{5.1}$$

subject to the Neumman boundary conditions

$$\frac{\partial v_1}{\partial n}=\frac{\partial v_2}{\partial n}=\frac{\partial v_3}{\partial n}=\frac{\partial v_4}{\partial n}=0, t>0, x=0, 2\pi, \tag{5.2}$$

and initial values

$$\begin{aligned} &v_1(x,0)=1.07, v_2(x,0)=1.15, \\ &v_3(x,0)=0.85, v_4(x,0)=0.67, x\in(0,2\pi). \end{aligned} \tag{5.3}$$

By calculating, we have

$$M_1^* \approx 1.1538,\ M_1=1.1539,\ M_2^* \approx 1.1304, M_2=1.1305,$$
$$M_3^{1*} \approx 2.7693,\ M_3^1=2.7694, M_4^{1*} \approx 2.5385,\ M_4^1=2.5386,$$
$$m_1^* \approx 0.95248,\ m_1=0.9524, m_2^* \approx 0.98461,\ m_2=0.9846,$$
$$m_3^{1*} \approx 0.20315,\ m_3^1=0.2031, m_4^{1*} \approx 0.15993,\ m_4^1=0.1547,$$
$$\frac{a_{31}^m+a_{32}^m-r_3^l}{r_3^l-a_{32}^m}=2.4, \frac{a_{41}^m+a_{42}^m-r_4^l}{r_4^l-a_{42}^m}=2.2,$$
$$r_1^l b_{13}^l b_{14}^l-a_{12}^m M_2 b_{13}^l b_{14}^l-a_{13}^m b_{14}^l-a_{14}^m b_{13}^l \approx 26.6695,$$
$$r_2^l b_{23}^l b_{24}^l-a_{21}^m M_1 b_{23}^l b_{24}^l-a_{23}^m b_{24}^l-a_{24}^m b_{23}^l \approx 22.6461,$$
$$a_{11}^l-a_{21}^m-\frac{(a_{13}^m+a_{31}^m b_{13}^m)M_3^1}{(b_{13}^l m_3^1+m_1)^2}-\frac{(a_{14}^m+a_{41}^m b_{14}^m)M_4^1}{(b_{14}^l m_4^1+m_1)^2} \approx 7.3126,$$
$$a_{31}^l-r_3^m+a_{34}^m M_4^1 \approx 0.6461, a_{41}^l-r_4^m+a_{43}^m M_3^1 \approx 0.5231,$$
$$a_{22}^l-a_{12}^m-\frac{(a_{23}^m+a_{32}^m b_{23}^m)M_3^1}{(b_{23}^l m_3^1+m_2)^2}-\frac{(a_{24}^m+a_{42}^m b_{24}^m)M_4^1}{(b_{24}^l m_4^1+m_2)^2} \approx 14.8873,$$
$$-a_{43}^m+\frac{(a_{31}^l b_{13}^l-a_{13}^m)m_1}{(b_{13}^m M_3^1+M_1)^2}+\frac{(a_{32}^l b_{23}^l-a_{23}^m)m_2}{(b_{23}^m M_3^1+M_2)^2} \approx 0.0796,$$
$$-a_{34}^m+\frac{(a_{41}^l b_{14}^l-a_{14}^m)m_1}{(b_{14}^m M_4^1+M_1)^2}+\frac{(a_{42}^l b_{24}^l-a_{24}^m)m_2}{(b_{24}^m M_4^1+M_2)^2} \approx 0.0429,$$

The assumptions of Theorem 4.1 are clearly met by systems (5.1)-(5.3). Consequently, it is straightforward to deduce from Theorem 4.1 that system (5.1) possesses a strictly positive SHPS $(v_1^*(t), v_2^*(t), v_3^*(t), v_4^*(t))$, and the solution $(v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))$ for models (5.1)-(5.3) satisfies

$$\lim_{t\to\infty}\left(v_i(x,t)-v_i^*(t)\right)=0\ , i=1,2,3,4\ ,$$

uniformly for $x \in (0, 2\pi)$.

By employing the finite differences method in conjunction with the MATLAB 7.1 software package, we gained numerical solutions for the system described by equations (5.1-(5.3). These solutions are visually depicted in Figures 5.1-5.4. Upon analyzing Figures 5.1-5.4, it is evident that the system, as defined by equations (5.1)-(5.3), exhibit a strictly positive global asymptotic stability. Specifically, the densities of both prey and predator species within the system oscillate periodically, with a consistent period of 2. As time progresses and reaches sufficiently long duration, these species distribute homogeneously across the spatial domain. This finding underscores the effectiveness of theoretical method presented in this article, as implemented through MATLAB 7.1, in capturing the dynamic behavior of the reaction-diffusion RDPPM. To further validate the results presented in this paper, we conducted numerical simulations using various initial values. The findings indicate that, regardless of the positive initial value chosen, the solutions all converge to the same stable periodic solution. The projection plots and phase diagrams obtained, as shown in Figures 5.5-5.9, adequately illustrate the dynamical behavior of the solutions for the equations (5.1)-(5.3).

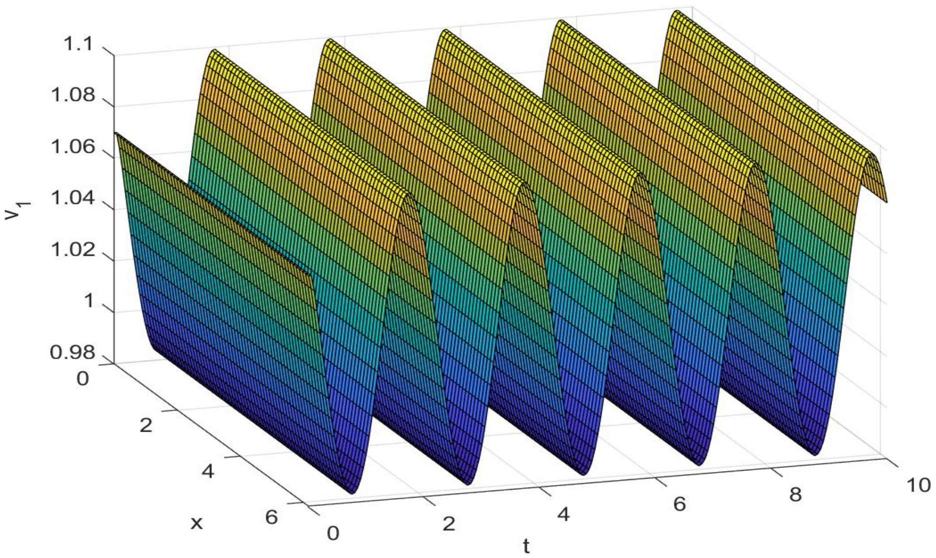

Fig.1. The evolution process concerning the density of species $v_1(x,t)$ within models (5.1)-(5.3).

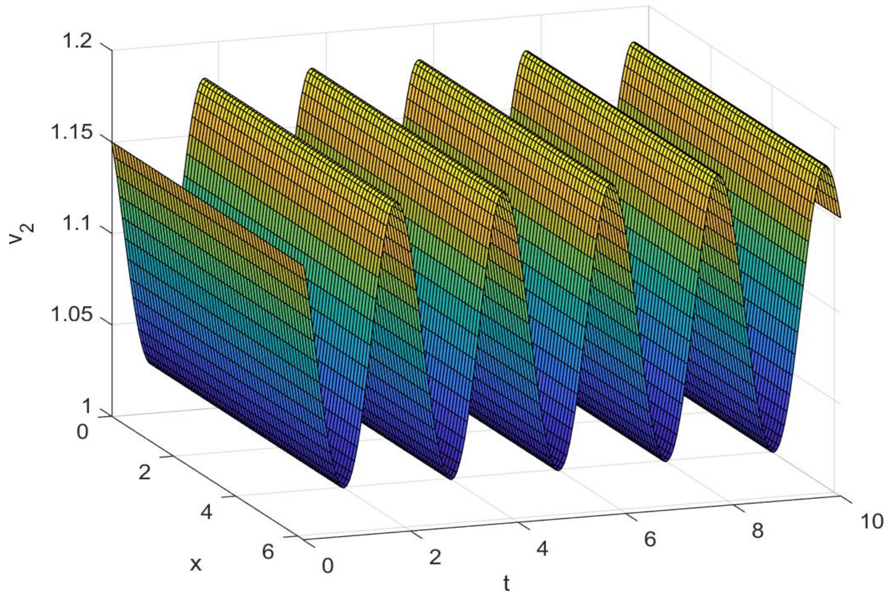

Fig. 2. The evolution process concerning the density of species $v_2(x,t)$ within models (5.1)-(5.3).

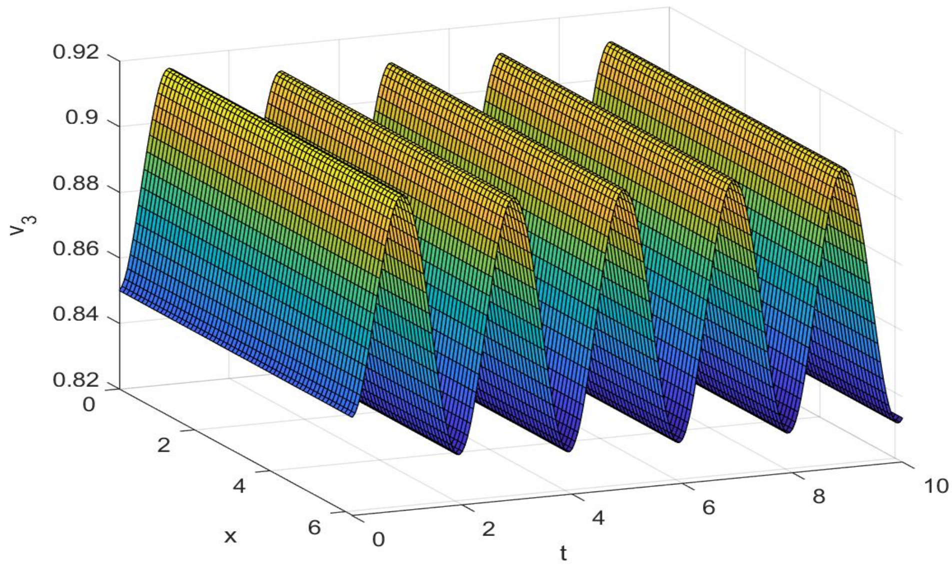

Fig.3. The evolution process concerning the density of species $v_3(x,t)$ within models (5.1)-(5.3).

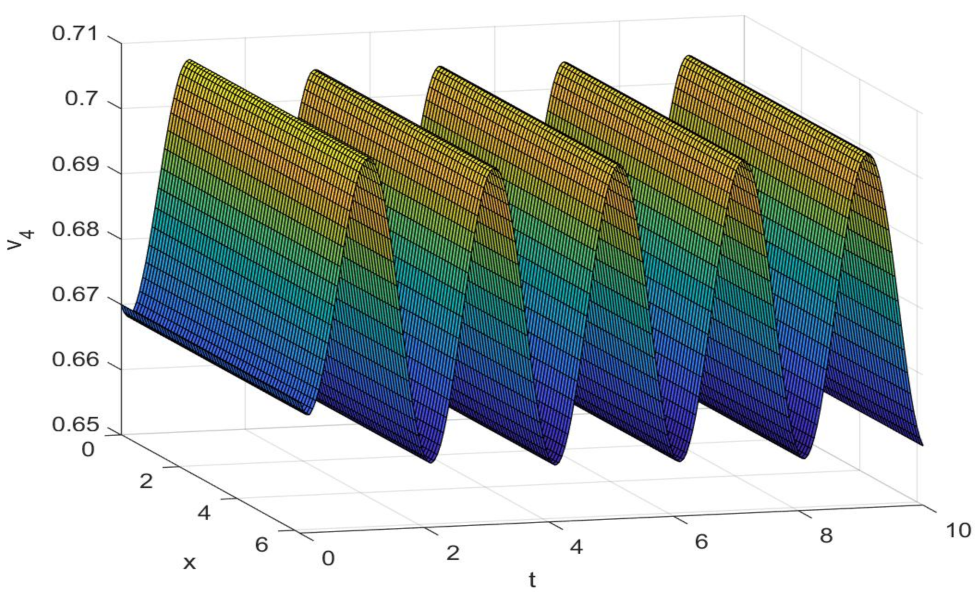

Fig. 4. The evolution process concerning the density of species $v_4(x,t)$ within models (5.1)-(5.3).

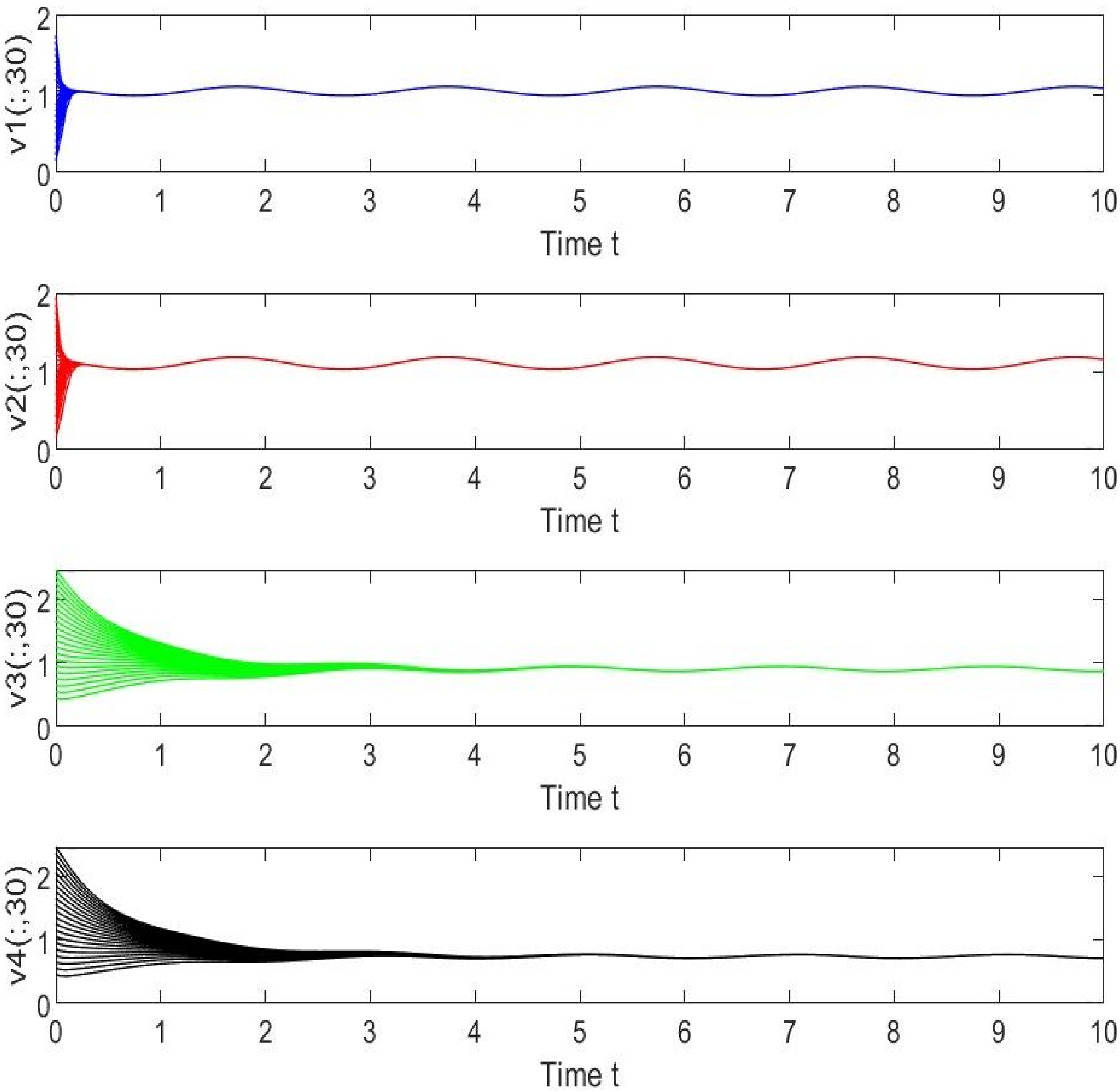

Fig. 5. As the spatial variable $x = 0.6\pi$, the plane projection of the numerical solutions of models (5.1)–(5.2) with different initial values.

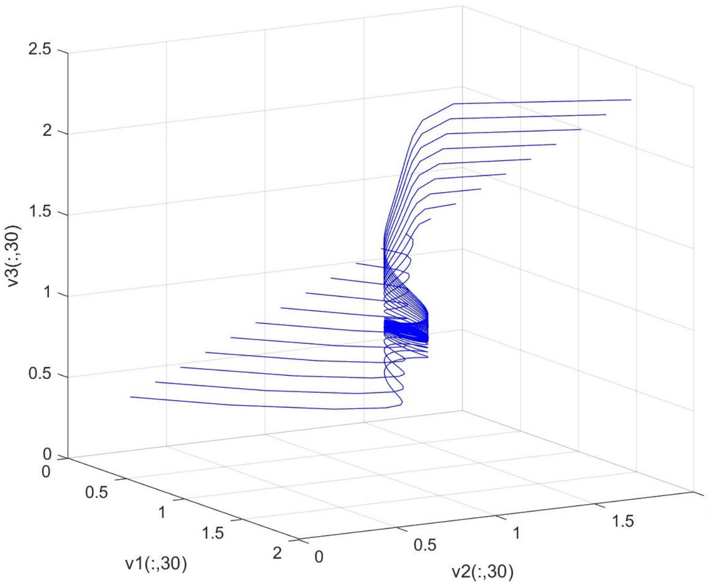

Fig. 6. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_2, v_3 in models (5.1)-(4.2) with different positive initial values.

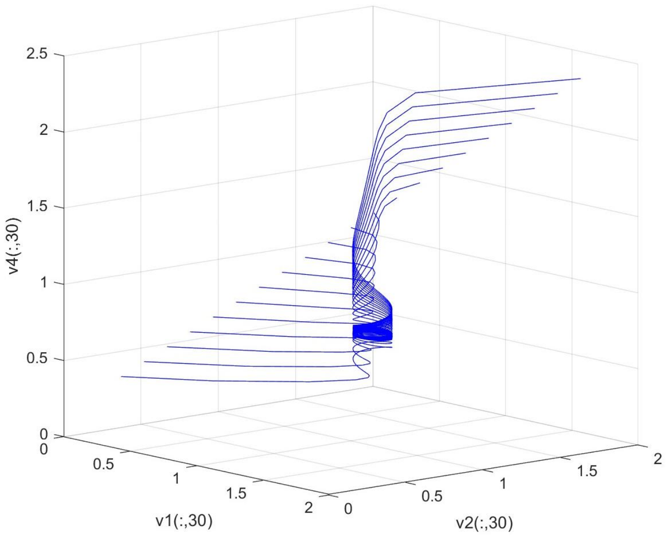

Fig. 7. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_2, v_3 in models (5.1)-(4.2) with different positive initial values.

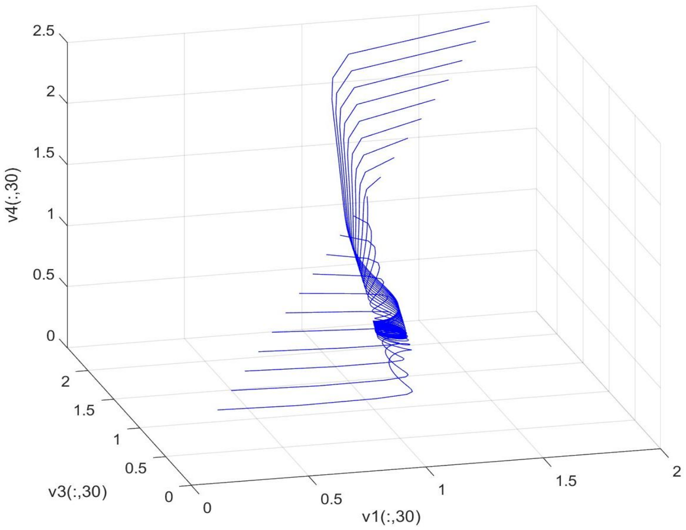

Fig. 8. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_2, v_3 in models (5.1)-(4.2) with different positive initial values.

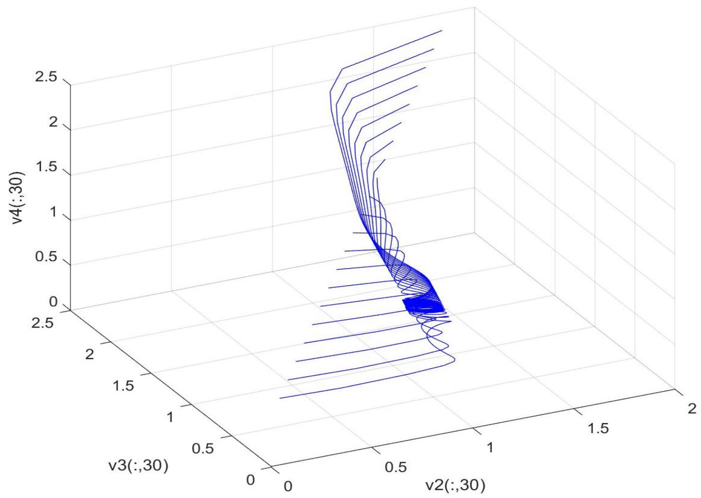

Fig. 9. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_2, v_3 in models (5.1)-(4.2) with different positive initial values.

Example 5.2. Take into consideration the periodic RDPPM that involves four species and incorporates reaction-diffusion processes. Based on the assumptions $(H_3), (H_4), (H_7)-(H_{10})$ and $(H_{15})-(H_{18})$ of Theorem 4.2, we have selected specific parameter values for the models (5.4)-(5.6) through a series of calculations. It is important to note that the choice of these parameters is not unique.

$$\begin{aligned}
&\frac{\partial v_1(x,t)}{\partial t}-\frac{\partial^2 v_1(x,t)}{\partial x^2}=v_1(x,t)[(29+\cos\pi t)\\
&\quad-(27+\sin\pi t)v_1(x,t)-(0.8+0.2\sin\pi t)v_2(x,t)\\
&\quad-\frac{(0.075+0.025\sin\pi t)v_3(x,t)}{(1.05+0.05\sin\pi t)v_3(x,t)+v_1(x,t)}\\
&\quad-\frac{(0.065+0.035\sin\pi t)v_4(x,t)}{(1.15+0.15\sin\pi t)v_4(x,t)+v_1(x,t)}],\\
&\frac{\partial v_2(x,t)}{\partial t}-\frac{\partial^2 v_2(x,t)}{\partial x^2}=v_2(x,t)[(25+\cos\pi t)\\
&\quad-(22+\sin\pi t)v_2(x,t)-(0.7+0.3\sin\pi t)v_1(x,t)\\
&\quad-\frac{(0.085+0.015\sin\pi t)v_3(x,t)}{(1.04+0.04\sin\pi t)v_3(x,t)+v_2(x,t)}\\
&\quad-\frac{(0.075+0.025\sin\pi t)v_4(x,t)}{(1.06+0.06\sin\pi t)v_4(x,t)+v_2(x,t)}],\\
&\frac{\partial v_3(x,t)}{\partial t}-\frac{\partial^2 v_3(x,t)}{\partial x^2}=v_3(x,t)[-(2.05+0.05\cos\pi t)\\
&\quad+\frac{(0.75+0.25\sin\pi t)v_1(x,t)}{(1.05+0.05\sin\pi t)v_3(x,t)+v_1(x,t)}\\
&\quad+\frac{(3.2+0.2\sin\pi t)v_2(x,t)}{(1.04+0.04\sin\pi t)v_3(x,t)+v_2(x,t)}\\
&\quad-(0.085+0.015\sin\pi t)v_4(x,t)],\\
&\frac{\partial v_4(x,t)}{\partial t}-\frac{\partial^2 v_4(x,t)}{\partial x^2}=v_4(x,t)[-(2.1+0.1\cos\pi t)\\
&\quad+\frac{(0.65+0.35\sin\pi t)v_1(x,t)}{(1.15+0.15\sin\pi t)v_4(x,t)+v_1(x,t)}\\
&\quad+\frac{(3.1+0.1\sin\pi t)v_2(x,t)}{(1.06+0.06\sin\pi t)v_4(x,t)+v_2(x,t)}\\
&\quad-(0.065+0.035\sin\pi t)v_3(x,t)],
\end{aligned}\tag{5.4}$$

with the boundary conditions

$$\frac{\partial v_1}{\partial n}=\frac{\partial v_2}{\partial n}=\frac{\partial v_3}{\partial n}=\frac{\partial v_4}{\partial n}=0, t>0, x=0, 2\pi, \tag{5.5}$$

and the initial conditions

$$\begin{aligned}v_1(x,0)=1.06, v_2(x,0)=1.16, v_3(x,0)=0.9,\\ v_4(x,0)=0.75, x\in(0,2\pi).\end{aligned}\tag{5.6}$$

By calculating, we have

$$\begin{aligned}
&M_1^*\approx 1.1538,\ M_1=1.1539,\ M_2^*\approx 1.1304,\ M_2=1.1305,\\
&M_3^{1*}\approx 2.7693,\ M_3^1=2.7694, M_4^{1*}\approx 2.5385,\ M_4^1=2.5386,\\
&m_1^*\approx 0.95248,\ m_1=0.9524, m_2^*\approx 0.98461, m_2=0.9846,\\
&m_3^{2*}\approx 0.2528,\ m_3^2=0.2527, m_4^{2*}\approx 0.1881,\ m_4^2=0.1880,\\
&\frac{a_{31}^m+a_{32}^m-r_3^l}{r_3^l-a_{31}^m}=2.4, \frac{a_{41}^m+a_{42}^m-r_4^l}{r_4^l-a_{41}^m}=2.2,\\
&r_1^l b_{13}^l b_{14}^l-a_{12}^m M_2 b_{13}^l b_{14}^l-a_{13}^m b_{14}^l-a_{14}^m b_{13}^l\approx 26.6695,\\
&r_2^l b_{23}^l b_{24}^l-a_{21}^m M_1 b_{23}^l b_{24}^l-a_{23}^m b_{24}^l-a_{24}^m b_{23}^l\approx 22.6461,\\
&a_{32}^l-r_3^m+a_{34}^m M_4^2\approx 0.6513, a_{42}^l-r_4^m+a_{43}^m M_3^2\approx 0.5287,\\
&a_{11}^l-a_{21}^m-\frac{(a_{13}^m+a_{31}^m b_{13}^m)M_3^2}{(b_{13}^l m_3^1+m_1)^2}-\frac{(a_{14}^m+a_{41}^m b_{14}^m)M_4^2}{(b_{14}^l m_4^1+m_1)^2}\approx 19.7046,\\
&a_{22}^l-a_{12}^m-\frac{(a_{23}^m+a_{32}^m b_{23}^m)M_3^2}{(b_{23}^l m_3^1+m_2)^2}-\frac{(a_{24}^m+a_{42}^m b_{24}^m)M_4^2}{(b_{24}^l m_4^1+m_2)^2}\approx 6.3202,\\
&-a_{43}^m+\frac{(a_{31}^l b_{13}^l-a_{13}^m)m_1}{(b_{13}^m M_3^2+M_1)^2}+\frac{(a_{32}^l b_{23}^l-a_{23}^m)m_2}{(b_{23}^m M_3^2+M_2)^2}\approx 0.0954,\\
&-a_{34}^m+\frac{(a_{41}^l b_{14}^l-a_{14}^m)m_1}{(b_{14}^m M_4^2+M_1)^2}+\frac{(a_{42}^l b_{24}^l-a_{24}^m)m_2}{(b_{24}^m M_4^2+M_2)^2}\approx 0.0961.
\end{aligned}$$

It is easy to show that models (5.3)-(5.4) satisfy the conditions of Theorem 4.2. It follows from Theorem 4.2 that there exists a strictly positive SHPS $(v_1^*(t), v_2^*(t), v_3^*(t), v_4^*(t))$ for (5.3), and the solution $(v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))$ for models (5.3)-(5.4) satisfies

$$\lim_{t\to\infty}\left(v_i(x,t)-v_i^*(t)\right)=0, i=1,2,3,4,$$

uniformly for $x\in(0,2\pi)$.

By employing the finite differences method in conjunction with the MATLAB 7.1 software package, we gained numerical solutions for the system described by equations (5.4)-(5.6). These solutions are visually depicted in Figures 5.10-5.13. Upon analyzing Figures 5.10-5.13, it is evident that the system exhibits global asymptotic stability. Specifically, the densities of both prey and predator species within the system oscillate periodically with a consistent period of 2. As time progresses and reaches a sufficiently long duration, these species distribute homogeneously across the spatial domain. This finding underscores the effectiveness of the theoretical results presented in this article.To further validate the results presented in this paper, we conducted numerical simulations using various initial values. The findings indicate that, regardless of the positive initial value chosen, the solutions all converge to the same stable periodic solution. The projection plots and phase diagrams gained, as shown in Figures 5.14-5.18, adequately illustrate the dynamical behavior of the solutions for the equations (5.4)-(5.6).

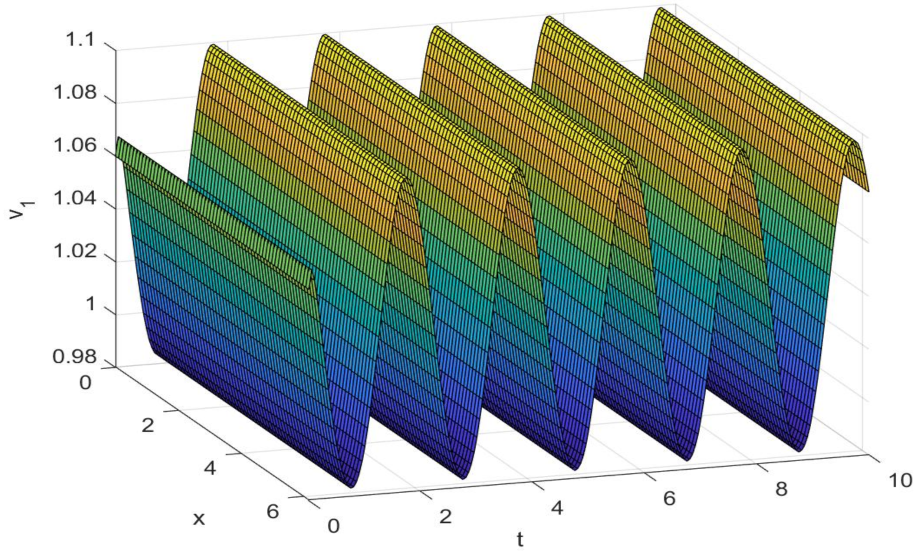

Fig. 10. The evolution process concerning the density of species $v_1(x,t)$ within models (5.4)-(5.6).

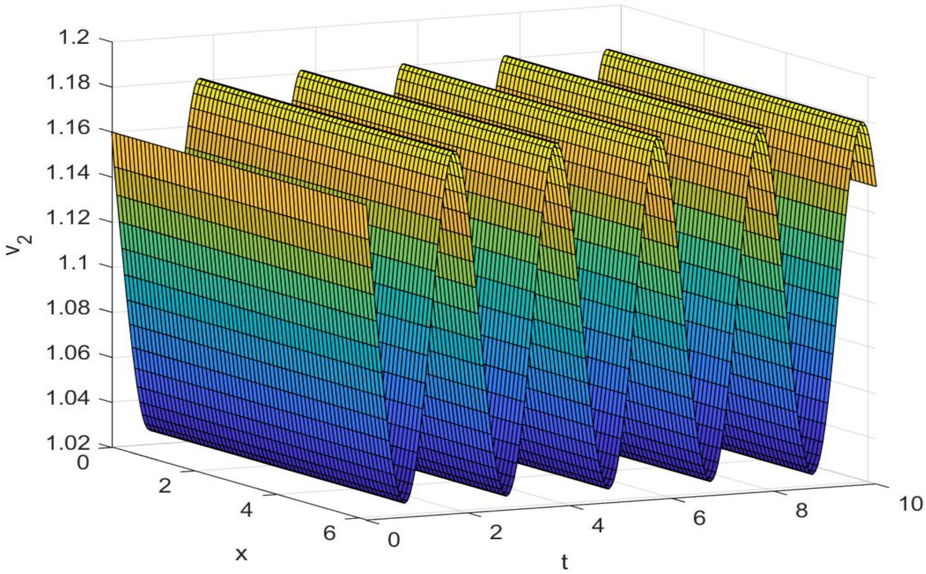

Fig. 11. The evolution process concerning the density of species $v_2(x,t)$ within models (5.4)-(5.6).

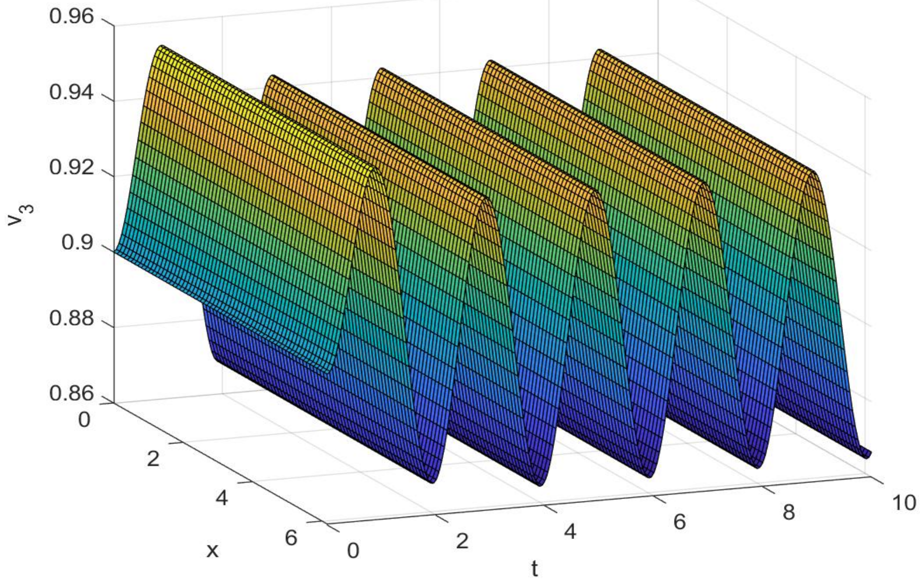

Fig. 12. The evolution process concerning the density of species $v_3(x,t)$ within models (5.4)-(5.6).

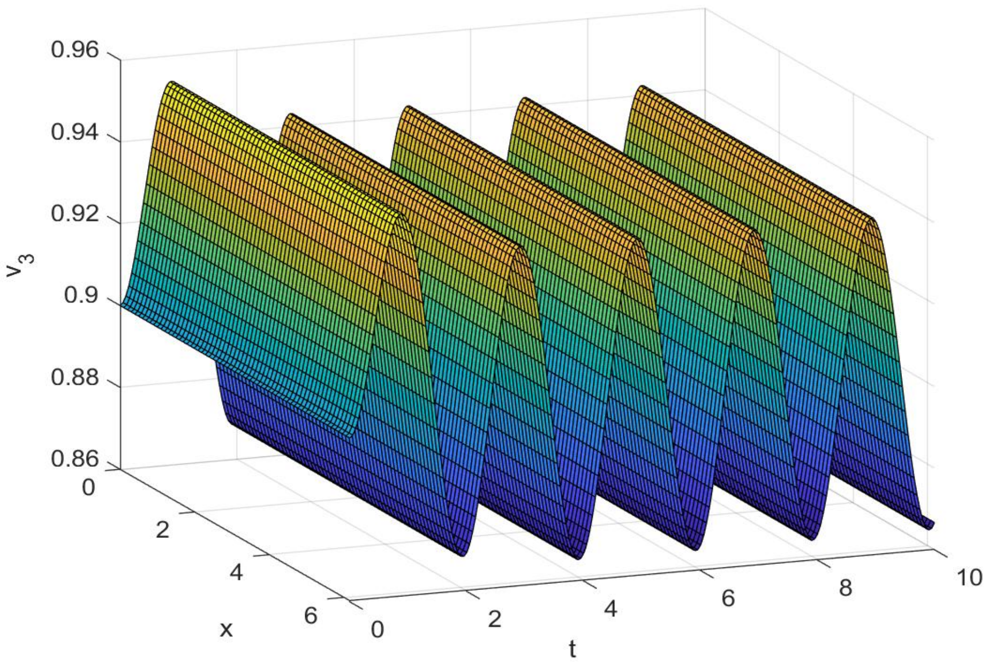

Fig. 13. The evolution process concerning the density of species $v_4(x,t)$ within models (5.4)-(5.6).

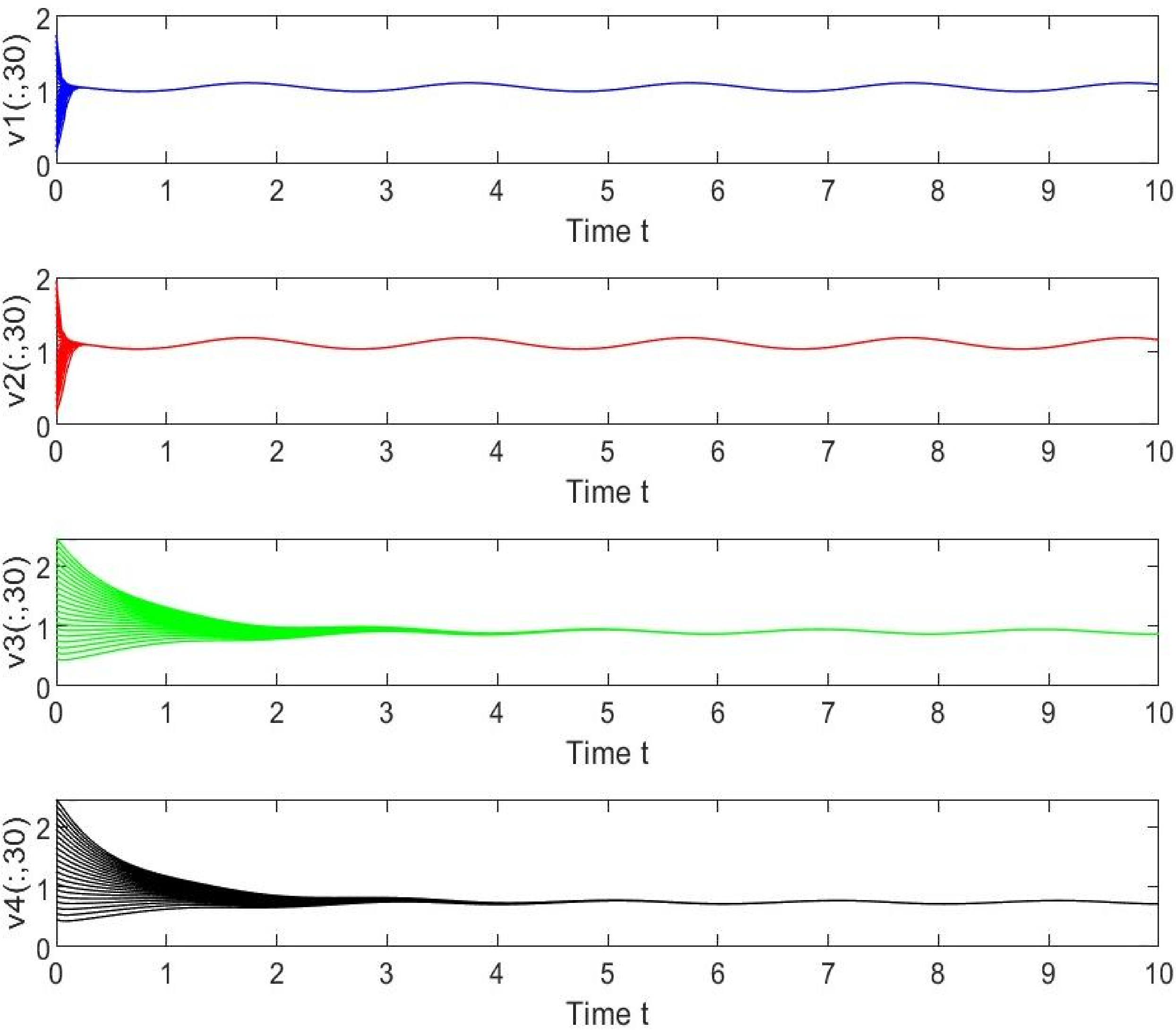

Fig. 14. As the spatial variable $x = 0.6\pi$, the plane projection of the numerical solutions of models (5.4)–(5.5) with different initial values.

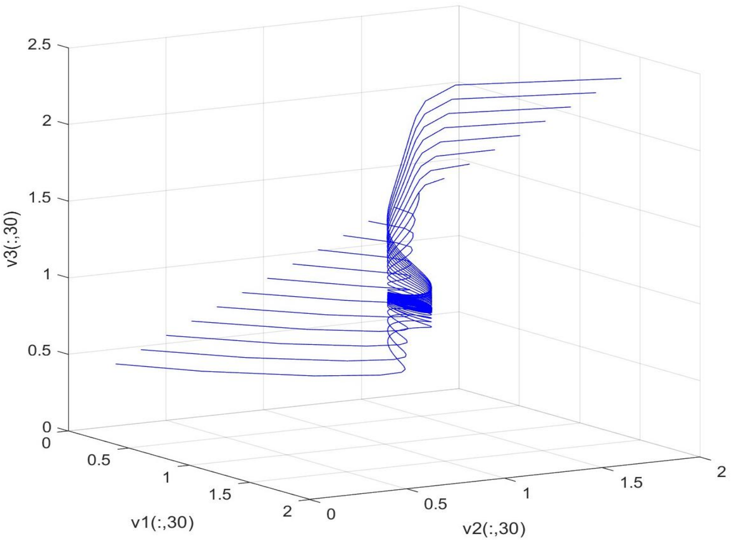

Fig. 15. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_2, v_3 in models (5.4)-(4.5) with different positive initial values.

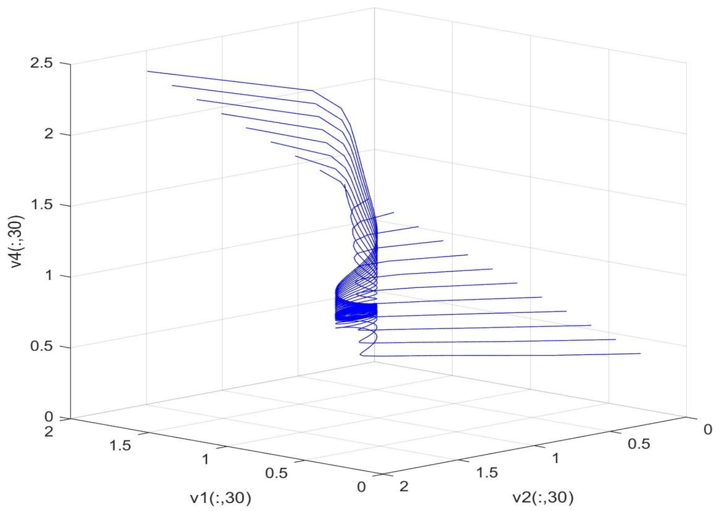

Fig. 16. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_2, v_4 in models (5.4)-(4.5) with different positive initial values.

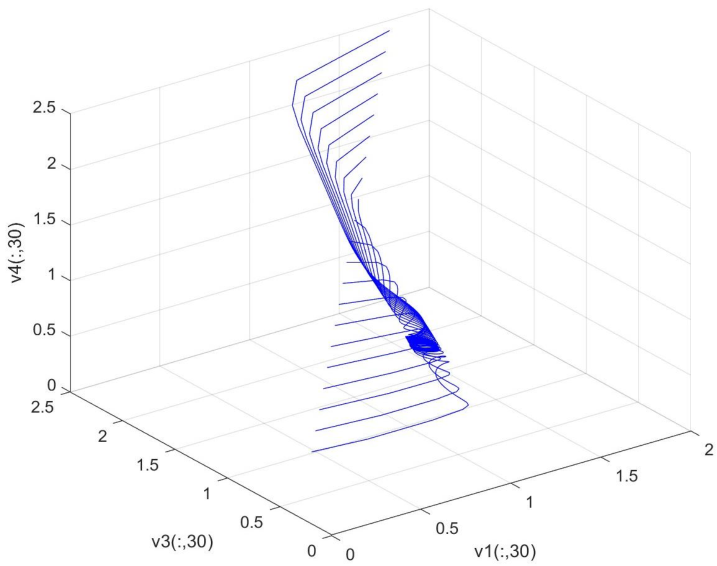

Fig. 17. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_3, v_4 in models (5.4)-(4.5) with different positive initial values.

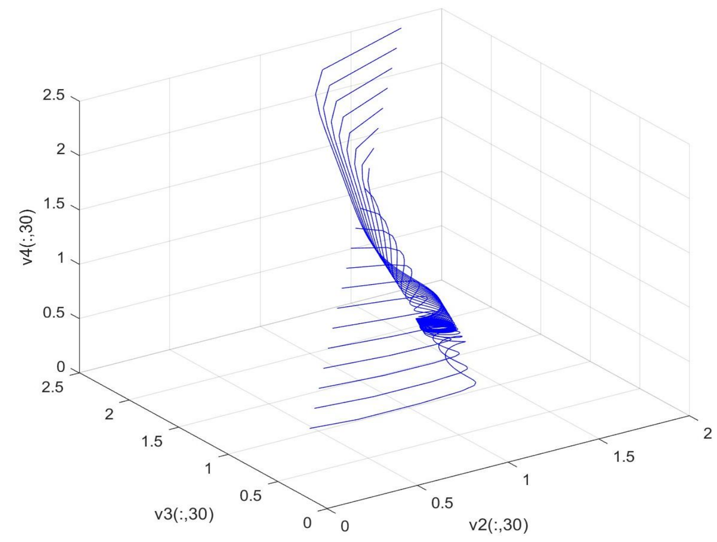

Fig. 18. As the spatial variable $x = 0.6\pi$, the changing patterns of population densities v_1, v_2, v_4 in models (5.4)-(4.5) with different positive initial values.

VI. Conclusion

The UALS method exhibits notable robustness when utilized for nonlinear nonautonomous reaction-diffusion equations, as demonstrated in this article. It has been broadly applied to tackle problems related to nonlinear PDEs across diverse fields, including chemistry, engineering, and mathematical physics. By employing an innovative strategy of constructing a Lyapunov function accompanied by a pair of ordered UALS, this method offers a significant reference for addressing stability concerns in nonlinear PDEs. Utilizing this approach, researchers can attain a deeper understanding of the dynamical characteristics of these intricate systems.

In this article, we delve into the study of a nonautonomous 4-species reaction-diffusion RDPPM. Our objectives include establishing sufficient conditions for the existence and stability of strictly positive SHPS of this model. Furthermore, we generalize and enhance the relevant conclusions presented in previous studies [29-34]. Notably, the sufficient conditions derived in this paper are notably straightforward, thereby facilitating the application and analysis of nonlinear multi-species reaction-diffusive predator-prey models. It is important to acknowledge that the impact of time delay on the model has not been considered in this article. However, in real-world ecosystems, time delays are ubiquitous and can potentially exert significant influence on the stability of the model. Therefore, our next objective is to investigate a nonautonomous multi-species RDPPM that incorporates time delay. By incorporating delay into our model, we aim to gain a more comprehensive understanding of the dynamical behavior of predator-prey interactions in the more complex ecosystems.

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Changyou Wang is Professor of College of Applied Mathematics, Chengdu University of Information Technology, Sichuan, China, since September 2017, and is Professor and Director of Institute of Applied Mathematics, Chongqing University of Posts and Telecommunications, Chongqing, China, from November 2011 to August 2017. He acts as a reviewer for Mathematical Reviews for American Mathematical Society, since 2014. His research area include delay reaction-diffusion equation, fractional-order differential equation, difference equation, biomathematics, control theory and control engineering, neural network, and digital image processing.

Qi Shang is a postgraduate student of School of Applied Mathematics, Chengdu University of Information Technology, Sichuan, China. Her research interests include stability theory of nonlinear systems and biomathematics.

Lili Jia is a professor and director of Department of Basic Teaching, Dianchi College, Yunnan, China, since December 2020. Her research interests include difference equation, differential equation, neural network, and biomathematics.