The Mathematical Modelling of Multi-Stage Inventory Supply Chain to Determine Product Demand with Ramp-Type

M. BALASUBRAMANIAN and R. KAMALI

***Abstract*- To maximize floor consumption orders in a two-stage supply chain with a ramp-type demand pattern, we create a unique inventory management model in this study. Different stages of inventory control are incorporated into the two-stage technique to improve overall system efficiency. The ramp-type demand, characterized by a gradual increase over time, is a common phenomenon in various industries and poses unique challenges to traditional inventory management practices. In the first stage, the model focuses on strategic decisions about order quantity and frequency at the central warehouse or distribution center. A dynamic optimization framework determines the optimal replenishment policies that balance the trade-off between holding costs and stockouts. The second stage involves tactical decisions at the retail or floor level, considering the unique characteristics of ramp-type demand. A refined ordering strategy is developed to accommodate the changing demand pattern and minimize stockouts during periods of rapid demand escalation. The study aims to identify the best replenishment strategies for reducing overall inventory costs and halting item deterioration. To determine the best course of action, an algorithm is created. Lastly, the suggested paradigm is demonstrated with numerical examples. A study on sensitivity is carried out, and several managerial implications are presented.**

***Index Term* - Inventory, Deteriorating Products, Order quantity, Ramp type demand, Warehouses.**

I. INTRODUCTION

In real-life various demand patterns, we have a comprehensive mathematical approach for a multi-stage supply chain inventory model specifically tailored to address the challenges posed by deteriorating products and ramp-type demand patterns. Deterioration of products is a critical aspect in several industries, and the dynamic nature of ramp-type demand adds another layer of complexity to traditional inventory management strategies. Our research aimed to bridge this gap by developing a robust mathematical framework that accounts for deteriorating products and ramp-type demand's unique characteristics. Furthermore, the model's ability to handle ramp-type demand significantly contributes to inventory management. Ramp-type demand is prevalent in various industries, and its gradual increase poses challenges for traditional inventory models. Our approach incorporates forecasting techniques and responsive ordering strategies to adapt to the changing demand pattern, thereby minimizing stockouts and improving overall system performance.

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M. Balasubramanian is a Research Scholar, Department of Mathematics, School of Basic Sciences, Vels Institute of Science, Technology & Advanced Studies, Pallavaram, Chennai – 600117. India. (phone: +91 7871112535; e-mail: srmbalu@gmail.com).

R. Kamali is an Assistant Professor, Department of Mathematics, School of Basic Sciences, Vels Institute of Science, Technology & Advanced Studies, Pallavaram, Chennai–600117. India. (phone: +91 9894290691; e-mail: kamali.sbs@velsuniv.ac.in).

A mathematical model for inventory management in supply chain networks that considers supply chain topology, inventory policies, and various demand scenarios was proposed by Zhang and Wang [1]. The objective of the approach is to reduce the overall inventory expense while maintaining a high standard of customer service. A unique mathematical model for inventory control in a multi-level supply chain network was created by Zhu and Cui [2]. The objective of the approach is to reduce the overall inventory expense while maintaining a high standard of customer service. A unique mathematical model for inventory control in a multi-level supply chain network was created by Zhu and Cui [2]. The model aims to maximize total inventory cost while meeting customer demand by considering the synchronization of inventory decisions across various supply chain tiers. A novel mathematical model for inventory management in a two-echelon supply chain that takes lead time variability and demand uncertainty into account was published by Li et al. [3]. The concept aims to provide excellent customer service while minimizing inventory costs. An enhanced mathematical model for inventory control in a closed-loop supply chain that takes the reverse logistics procedure into account was presented by Xu et al. [4]. The approach seeks to minimize the supply chain's harmful environmental effects while optimizing the overall cost of inventories. A stochastic mathematical model for inventory management in a multi-echelon supply chain that takes lead time variability and demand uncertainty into account was created by Gao et al. [5]. The objective of the concept is to reduce the overall cost of inventory while maintaining a high standard of customer service. A mathematical model for inventory control in a dual-channel supply chain that takes the competition between direct and indirect channels into account was created by Chen et al. [6]. The model seeks to optimize the inventory allocation between the two channels to maximize the supply chain's overall profit. Cheng et al. [7] developed a mathematical model for inventory management within a multi-tiered supply chain, incorporating considerations for supplier reliability in inventory decision-making processes. The model's primary objective is to minimize the total inventory cost across the supply chain. Xie et al. [8] introduced a novel mathematical framework for inventory control within a three-tiered supply chain involving numerous retailers. Their model considers uncertainties in demand, variability in lead times, and the risk of stockouts. The primary objective is to minimize overall inventory costs while maintaining a high level of service for customers. Lee and Kim [9] developed a mathematical formulation for inventory management within a supply chain comprising a single supplier and multiple retailers. Their model also addresses demand uncertainty and lead time variability, intending to minimize total inventory costs while upholding superior customer service. Chen and Liu [10] proposed a mathematical model for inventory management in a two-tiered supply chain considering demand uncertainty and lead time variability. Their model seeks to optimize inventory policies to maximize the anticipated profit of the supply chain.

Das et al. [11] concentrated two distribution center supply shows beneath plausibility/need/validity measures. In their study, Lee and Hsu [12] presented a two-warehouse generation model that effectively decomposes inventory items based on time-dependent demands. Jagg et al. [13] developed a two-warehouse fractional backlogging inventory model to analyze the demand for goods with a linear slope under inflationary conditions. Muniappan et al. [14] created a financial part-measuring generation demonstration for falling apart things beneath two-level exchange credits. Muniappan and Uthayakumar [15] examined ideal recharging methodologies utilizing scientific strategies. Using Logarithmic and AGM strategies, Teerapabolarn and Khamrod [16] investigated stock models joining backorders and flawed things. Muniappan et al. [17] inspected a financial arranged amount show for breaking down things, calculating in expansion and the time esteem of cash, with contemplations for time-dependent weakening rates and deferred instalments.

Furthermore, Muniappan et al. [18] concocted a generation stock demonstration for vendor-buyer coordination, joining amount rebates, back requesting, and revamping for items with settled lifetimes. Ravithammal et al. [19] [20] dove into a deterministic generation stock show for buyer-manufacturer flow, centering on amount rebates and backlogged deficiencies for items with settled lifetimes. They also concentrated on a coordinated generation stock framework for perishable things, analyzing settled and direct backorders.

Most of the studies mentioned above consider time-varying demand patterns, either increasing or decreasing, but in reality, demand tends to stabilize during the mature stage of a product's lifecycle once it gains market acceptance. This stabilization, termed 'ramp-type,' has been addressed in literature since Ritchie [21]. Deng et al. [22] explored inventory models for deteriorating items under ramp-type demand. Mandal [23] presented an EOQ inventory model for Weibull-distributed deteriorating items with ramp-type demand and shortages. Yogendra Kumar Rajoria et al. [24] studied an inventory model for decaying items with ramp demand patterns under inflation and partial backlogging. Rekha Rani Chaudhary and Vikas Sharma [25] investigated a model for Weibull-deteriorating items with demand rates dependent on prices and inflation. Skouri et al. [26] investigated supply chain models for deteriorating products that exhibit ramp-type demand rates while also considering the allowable payment delays. YUN HUANG and GEORGE Q. HUANG [27] delved into a numerical study to understand the influence of different parameters on the decisions and profits of the supply chain and its constituent members. YUN HUANG and GEORGE Q. HUANG [28] devised that when product cost exceeds a certain echelon, the chain members' profits will increase as the market becomes more sensitive to the retail price.

The abovementioned literature emphasizes the lack of an EOQ model that simultaneously handles deterioration, ramp-type demand, and dual warehouses. Thus, this study aims to examine these variables and create a thorough model for determining the economic order quantity and cutting down on overall expenses.

In the subsequent section, the notations and assumptions are specified. The mathematical formulation of the suggested model is presented in Section 3. In Section 4, numerical and sensitivity assessments are given. Ultimately, the investigation culminates with a concise overview of the results.

II. NOTATIONS AND ASSUMPTIONS

In this paper, we used the following notations:

NOTATIONS

Q	: Quantity ordered and replacement
D (t)	: Rate of demand $t \geq 0$
r	: Replenishment Cost per order
T	: Replenishment cycle length
O_w	: Warehouse (owned)
R_w	: Warehouse (rented)
w	: Storage Capacity (in the owned warehouse)
p1	: Price (per unit item)
p	: Price for Purchase/item
h_0	: Holding expenses/unit in OW, time
h_r	: Cost, unit, and period of holding in RW, $h_r \geq h_o$
θ_1	: The rate of deterioration in OW, where $\theta_1 < 0$
θ_2	: Rate of deterioration in RW, where $\theta_2 < 0$ and $\theta_1 > \theta_2$
$I_1(t)$	: The amount of inventory in OW at time t
$I_2(t)$	: The amount of inventory in RW at time t
T_w	: The moment in RW when the inventory level drops to zero
t_1	: The moment in OW when the inventory level drops to zero.

ASSUMPTIONS

$D(t) = a[t - (t-\mu)H(t-\mu)]$ Where a and μ are constants $\mu > 0$ d $H(t-\mu)$ is a Heaviside unit function of time defined as $\mathrm{H}(t-\mu) = \begin{cases} 1, & t \geq \mu \\ 0, & t < \mu \end{cases}$.

We operate under the assumption that shortages are not acceptable. The in-house warehouse (stage 1) has a restricted capacity of w units, while the rented warehouse (stage 2) has unlimited capacity. Priority is given to consuming items from Stage 1 before moving on to items from Stage 2.

III. MATHEMATICAL FORMULATION

In the Mathematical formulation for the proposed model, we designed differential equations that minimize the total cost over a planning horizon, considering order quantities, holding costs, fixed ordering costs, and unit costs at each stage of the supply chain. The model incorporates inventory balance equations, initial inventory levels, and constraints reflecting ramp-type demand patterns, deterioration rates, and non-negativity of decision variables; based on the above description, we have the following model.

$$\frac{dI_1(t)}{dt} + \theta_1 I_1(t) = -at, \ 0 \leq t \leq \mu \quad (1)$$

$$\frac{dI_1(t)}{dt} + \theta_1 I_1(t) = -a\mu, \ \mu \leq t \leq t_1 \quad (2)$$

With the boundary condition $I_1(0) = I_0$,

$I_1(\mu) = d_1$ and $I(t_1) = 0$.
From (1) and (2), we have

$$I_1(t) = \frac{a}{\theta_1^2}(\theta_1\mu - 1)\left(e^{\theta_1(\mu - t)} - 1\right) + d_1 e^{\theta_1(\mu - t)}, \quad 0 \le t \le \mu \tag{3}$$

$$I_1(t) = \frac{a\mu}{\theta_1}\left(e^{\theta_1(t_1 - t)} - 1\right), \mu \le t \le t_1 \tag{4}$$

For each cycle in OW is

$$Q_1 = I_1(0)\ at\ 0 \le t \le \mu +\ I_1(0)\ and\ \mu \le t \le t_1$$
$$= \frac{a}{\theta_1^2}(a\mu - 1)\left(e^{\theta_1\mu} - 1\right) + d_1 e^{\theta_1\mu} + \frac{a\mu}{\theta_1^2}\left(e^{\theta_1 t_1} - 1\right) \tag{5}$$

We described, $I_2(t)$ w.r.t the following differential equation

$$\frac{dI_2(t)}{dt} + \theta_2 I_2(t) = -at,\ 0 \le t \le \mu \tag{6}$$

$$\frac{dI_2(t)}{dt} + \theta_2 I_2(t) = -a\mu,\ \mu \le t \le T_w \tag{7}$$

With the boundary condition $I_2(0) = I_0,\ I_2(\mu) = d_1$ and $I(T_w) = 0$.
From (6) and (7), we have

$$I_2(t) = \frac{a}{\theta_2^2}(\theta_2\mu - 1)\left(e^{\theta_2(\mu - t)} - 1\right) + d_1 e^{\theta_2(\mu - t)}, \quad 0 \le t \le \mu \tag{8}$$

$$I_2(t) = \frac{a\mu}{\theta_2}\left(e^{\theta_2(T_w - t)} - 1\right), \mu \le t \le T_w \tag{9}$$

In $0 \le t \le T_w$, the variation of $I_{01}(t)$ w.r.t the following differential equation

$$\frac{dI_{01}(t)}{dt} = -\theta_1 I_{01}(t), 0 \le t \le T_w \tag{10}$$

with initial condition $I_{01}(t) = w$. From (10), we have

$$I_{01}(t) = we^{-\theta_1 t}, 0 \le t \le T_w \tag{11}$$

The RW is (for each cycle)

$$Q_2 = I_2(0)\ , \quad 0 \le t \le \mu +\ I_2(0)\ and\ \mu \le t \le t_1 + I_{01}(0)$$
$$= \frac{a}{\theta_2^2}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - 1\right) + d_1 e^{\theta_2\mu} + \frac{a\mu}{\theta_2}\left(e^{\theta_2 T_w} - 1\right) + w \tag{12}$$

The following elements make up the overall cost of inventory per unit of time:

(i) The ordering cost is calculated annually as r/T.

(ii) The cycle's degradation cost (DC) is [0, t_1].

$$DC = \frac{p}{T}\left\{I_0 - \left(\left(\int_0^{\mu} atdt + \int_{\mu}^{t_1} a\mu dt\right)\right)\right\}$$
$$= \frac{p}{T}\left\{\frac{a}{\theta_1^2}\left[(a\mu - 1)\left(e^{\theta_1\mu} - 1\right) + \mu\left(e^{\theta_1\mu} - 1\right)\right] - a\mu\left(t_1 - \frac{\mu}{2}\right) + d_1 e^{\theta_1\mu}\right\}$$

(iii) The cost of annually owning stocks in the RW is

$$HC_R = \frac{h_r}{T}\left\{\int_0^{\mu} I_2(t)dt + \int_{\mu}^{T_w} I_2(t)dt\right\}$$
$$= \frac{h_r}{T}\left\{\frac{a}{\theta_2^3}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - \theta_2\mu - 1\right) + \frac{a\mu}{\theta_2^2}\left[e^{\theta_2(T_w - \mu)} + \theta_2(\mu - T_w) - 1\right] + \frac{d_1}{\theta_2}\left(e^{\theta_2\mu} - 1\right)\right\}$$

(iv) The OW's annual stock holding cost is

$$HC_O = \frac{h_o}{T}\left\{\int_0^{\mu} I_1(t)dt + \int_{\mu}^{t_1} I_1(t)dt + \int_0^{T_w} I_{01}(t)dt\right\}$$
$$= \frac{h_o}{T}\left\{\frac{a}{\theta_1^3}(\theta_1\mu - 1)\left(e^{\theta_1\mu} - \theta_1\mu - 1\right) + \frac{a\mu}{\theta_1^2}\left[e^{\theta_1(t_1 - \mu)} + \theta_1(\mu - t_1) - 1\right] + \frac{d_1}{\theta_1}\left(e^{\theta_1\mu} - 1\right) - \frac{w}{\theta_1}\left(e^{-\theta_1 T_w} - 1\right)\right\}$$

(v) Annual purchasing cost in OW & RW is $PC_O = \frac{pQ_1}{T}$ and $PC_R = \frac{pQ_2}{T}$ respectively.

Hence total purchasing cost is $PC = \frac{p(Q_1 + Q_2)}{T}$ (both warehouses)

$$= \frac{p}{T}\left\{\frac{a}{\theta_1^2}(a\mu - 1)\left(e^{\theta_1\mu} - 1\right) + \mu\left(e^{\theta_1 t_1} - 1\right) + \frac{a}{\theta_2^2}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - 1\right) + \frac{a\mu}{\theta_2}\left(e^{\theta_2 T_w} - 1\right) + d_1\left(e^{\theta_2\mu} + e^{\theta_1\mu}\right) + w\right\}$$

Therefore, the total cost of inventory/unit of time is provided by

$$TC(T, t_1, T_w) = \frac{1}{T}\{OC + DC + HC_R + HC_O + PC\}$$
$$= \frac{1}{T}\Bigg\{r + p\left\{\frac{a}{\theta_1^2}\left[(a\mu - 1)\left(e^{\theta_1\mu} - 1\right) + \mu\left(e^{\theta_1\mu} - 1\right)\right] - a\mu\left(t_1 - \frac{\mu}{2}\right) + d_1 e^{\theta_1\mu}\right\} + h_r\left\{\frac{a}{\theta_2^3}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - \theta_2\mu - 1\right) + \frac{a\mu}{\theta_2^2}\left[e^{\theta_2(T_w - \mu)} + \theta_2(\mu - T_w) - 1\right] + \frac{d_1}{\theta_2}\left(e^{\theta_2\mu} - 1\right)\right\} + h_o\left\{\frac{a}{\theta_1^3}(\theta_1\mu - 1)\left(e^{\theta_1\mu} - \theta_1\mu - 1\right) + \frac{a\mu}{\theta_1^2}\left[e^{\theta_1(t_1 - \mu)} + \theta_1(\mu - t_1) - 1\right] + \frac{d_1}{\theta_1}\left(e^{\theta_1\mu} - 1\right) - \frac{w}{\theta_1}\left(e^{-\theta_1 T_w} - 1\right)\right\} + p\left\{\frac{a}{\theta_1^2}\left[(a\mu - 1)\left(e^{\theta_1\mu} - 1\right) + \mu\left(e^{\theta_1 t_1} - 1\right)\right] + \frac{a}{\theta_2^2}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - 1\right) + \frac{a\mu}{\theta_2}\left(e^{\theta_2 T_w} - 1\right) + d_1\left(e^{\theta_2\mu} + e^{\theta_1\mu}\right) + w\right\}\Bigg\}$$

Q_1^*, Q_2^*. Are prerequisites for minimizing the minimum order quantity in each warehouse and the total inventory cost per time $TC(T^*, \eta_1^*, \eta_2^*)$

$$t_1 = \eta_1 T, 0 < \eta_1 < 1 and\ T_w = \eta_2 T, 0 < \eta_2 < 1 \tag{13}$$

$$TC(T, t_1, T_w) = \frac{1}{T}\Bigg\{r + h_r\left\{\frac{a}{\theta_2^3}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - \theta_2\mu - 1\right) + \frac{a\mu}{\theta_2^2}\left[e^{\theta_2(\eta_2 T - \mu)} + \theta_2(\mu - \eta_2 T) - 1\right] + \frac{d_1}{\theta_2}\left(e^{\theta_2\mu} - 1\right)\right\} + h_o\left\{\frac{a}{\theta_1^3}(\theta_1\mu - 1)\left(e^{\theta_1\mu} - \theta_1\mu - 1\right) + \frac{a\mu}{\theta_1^2}\left[e^{\theta_1(\eta_1 T - \mu)} + \theta_1(\mu - \eta_1 T) - 1\right] + \frac{d_1}{\theta_1}\left(e^{\theta_1\mu} - 1\right) - \frac{w}{\theta_1}\left(e^{-\theta_1\eta_2 T} - 1\right)\right\} + p\left\{2\left[\frac{a}{\theta_1^2}(a\mu - 1)\left(e^{\theta_1\mu} - 1\right) + \mu\left(e^{\theta_1\eta_1 T} - 1\right)\right] + \frac{a}{\theta_2^2}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - 1\right) + \frac{a\mu}{\theta_2}\left(e^{\theta_2\eta_2 T} - 1\right) - a\mu\left(\eta_1 T - \frac{\mu}{2}\right) + d_1\left(2e^{\theta_2\mu} + e^{\theta_1\mu}\right) + w\right\}\Bigg\}$$
$$= \frac{1}{T}\left\{r + \varepsilon_7 + \frac{h_r a\mu}{\theta_2^2}\left[e^{\theta_2(\eta_2 T - \mu)} + \theta_2(\mu - \eta_2 T) - 1\right] + \frac{h_o a\mu}{\theta_1^2}\left[e^{\theta_1(\eta_1 T - \mu)} + \theta_1(\mu - \eta_1 T) - 1\right] - \frac{wh_o}{\theta_1}\left(e^{-\theta_1\eta_2 T}\right) + 2p\mu\left(e^{\theta_1\eta_1 T} - 1\right) + \frac{pa\mu}{\theta_2}\left(e^{\theta_2\eta_2 T} - 1\right) - a\mu\left(\eta_1 T - \frac{\mu}{2}\right)\right\} \tag{14}$$

where $\varepsilon_1 = \frac{h_r a}{\theta_2^3}(\theta_2\mu - 1)\left(e^{\theta_2\mu} - \theta_2\mu - 1\right)$, $\varepsilon_2 = \frac{h_r d_1}{\theta_2}\left(e^{\theta_2\mu} - 1\right)$, $\varepsilon_3 = \frac{h_o a}{\theta_1^3}(\theta_1\mu - 1)\left(e^{\theta_1\mu} - \theta_1\mu - 1\right)$, $\varepsilon_4 = \frac{h_r d_1}{\theta_2}\left(e^{\theta_2\mu} - 1\right)$, $\varepsilon_5 = \frac{2pa}{\theta_1^2}(a\mu - 1)\left(e^{\theta_1\mu} - 1\right)$, $\varepsilon_6 =$

$p\left\{\frac{a}{\theta_2^2}(\theta_2\mu-1)\left(e^{\theta_2\mu}-1\right)+d_1\left(2e^{\theta_2\mu}+e^{\theta_1\mu}\right)+w\right\}$,
$\varepsilon_7=\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6$
For optimality, $\frac{\partial TC}{\partial T}=0$, $\frac{\partial TC}{\partial \eta_1}=0$ and $\frac{\partial TC}{\partial \eta_2}=0$

Now, $\frac{\partial TC}{\partial T}=\frac{1}{-T^2}\Big\{r+\varepsilon_7+\frac{h_r a\mu}{\theta_2^2}\left[e^{\theta_2(\eta_2T-\mu)}+\theta_2(\mu-\eta_2T)-1\right]+\frac{h_o a\mu}{\theta_1^2}\left[e^{\theta_1(\eta_1T-\mu)}+\theta_1(\mu-\eta_1T)-1\right]-\frac{wh_o}{\theta_1}\left(e^{-\theta_1\eta_2T}-1\right)+2p\mu\left(e^{\theta_1\eta_1T}-1\right)+\frac{pa\mu}{\theta_2}\left(e^{\theta_2\eta_2T}-1\right)-a\mu\left(\eta_1T-\frac{\mu}{2}\right)\Big\}+\frac{1}{T}\Big\{\frac{h_r a\mu}{\theta_2^2}\left[\theta_2\eta_2e^{\theta_2(\eta_2T-\mu)}+\theta_2\eta_2\right]+\frac{h_o a\mu}{\theta_1^2}\left[\theta_1\eta_1e^{\theta_1(\eta_1T-\mu)}-\theta_1\eta_1\right]+\frac{wh_o}{\theta_1}\left(\theta_1\eta_2e^{-\theta_1\eta_2T}\right)+2p\mu\left(\theta_1\eta_1e^{\theta_1\eta_1T}\right)+\frac{pa\mu}{\theta_2}\left(\theta_2\eta_2e^{\theta_2\eta_2T}\right)-a\mu\eta_1\Big\}$

$\{-a\mu[h_r\eta_2^2+h_o\eta_1^2]+h_ow\theta_1\eta_2^2-p\mu[2\theta_1^2\eta_1^2+a\theta_2\eta_2^2]\}T^2+a\mu^2[h_r\eta_2+h_o\eta_1]T+\frac{a\mu^2}{2}+r+\varepsilon_7=0$
(15)

$\frac{\partial TC}{\partial \eta_1}=\frac{1}{T}\left\{\frac{h_o a\mu}{\theta_1}Te^{\theta_1(\eta_1T-\mu)}-\theta_1T+2p\mu\theta_1Te^{\theta_1\eta_1T}-a\mu T\right\}$
$\{h_oa\mu T+2p\mu\theta_1^2T\}\eta_1+\frac{h_oa\mu}{\theta_1}-h_oa\mu^2+2p\mu\theta_1-\theta_1-a\mu=0$, i.e., $b_1\eta_1+b_2=0$
(16)

where $b_1=h_oa\mu T+2p\mu\theta_1^2T$, $b_2=\frac{h_oa\mu}{\theta_1}-h_oa\mu^2+2p\mu\theta_1-\theta_1-a\mu$

$\frac{\partial TC}{\partial \eta_2}=\frac{1}{T}\left\{\frac{h_r a\mu}{\theta_2}Te^{\theta_2(\eta_2T-\mu)}-\theta_2T+h_owTe^{-\theta_1\eta_2T}+p\mu aTe^{\theta_2\eta_2T}\right\}\{h_ra\mu T-h_ow\theta_1T+p\mu a\theta_2T\}\eta_2+\frac{h_ra\mu}{\theta_2}-h_ra\mu^2+p\mu a-\theta_2+h_ow=0$, i.e., $c_1\eta_2+c_2=0$ (17)

where $c_1=h_ra\mu T-h_ow\theta_1T+p\mu a\theta_2T$ and $c_2=\frac{h_ra\mu}{\theta_2}-h_ra\mu^2+p\mu a-\theta_2+h_ow$

By algorithm, we find the total minimum inventory cost (for both warehouses)
$TC^*(T^*,t_1^*,T_w^*)$, $t_1^*=T^*\eta_1^*$, $T_w^*=T^*\eta_2^*$ and
the order quantity $Q^*=Q_1^*+Q_2^*$. (for both warehouses)

IV. SOLUTION PROCEDURE ALGORITHM

Step1. Please provide the input values.
Step2. Substituting the values into equation (15) and find $T_{(1)}$.
Step3. Utilise $T_{(1)}$ to determine $\eta_{1(1)},\eta_{2(1)}$ Using equation (16) and (17).
Step4. Use equation (13) to find $t_{1(1)}$ and $T_{w(1)}$.
Step5. Use equation (14) to calculate $TC(T_{(1)},t_{1(1)},T_{w(1)})$
Step6. Repeat steps 2 to 5 until $TC\left(T_{(n)-1},t_{1(n)-1},T_{w(n)-1}\right)\le TC\left(T_{(n)},t_{1(n)},T_{w(n)}\right)$. Set $T^*=T_{(n)-1}$, $t_1{}^*=t_{1(n)-1}$, $T_w{}^*=T_{w(n)-1}$ and proceed to step 7.
Step7. Determine $Q^*=Q_1^*+Q_2^*$ and $TC^*(T^*,t_1^*,T_w^*)$

V. NUMERICAL EXAMPLES

For the different values of the decision variable
Example 1
r = 100, a = 100, $h_r=0.3$, $h_o=0.2$, $\theta_1=0.6$, $\theta_2=0.3$, $\eta_1=0.3$, $\eta_2=0.2$, $\mu=0.1$, w = 50, p = 0.2, $d_1=20$.
We obtain
$T^*=28.4498$, $t_1^*=3.6947$, $T_w^*=8.9167$, $Q^*=891$, $TC^*(T^*,t_1^*,T_w^*)=12.9911$
Example 2
Let r = 200, a = 150, $h_r=0.5$, $h_o=0.3$, $\theta_1=0.8$, $\theta_2=0.5$, $\eta_1=0.5$, $\eta_2=0.1$, $\mu=0.1$, w = 100, p = 0.2, $d_1=50$.
We obtain
$T^*=11.3027$, $t_1^*=2.3407$, $T_w^*=3.1167$, $Q^*=692$, $TC^*(T^*,t_1^*,T_w^*)=104.5328$

The table I demonstrates how adjusting demand, holding, quality, and capacity parameters affects reorder intervals and total costs. Optimal inventory strategies involve balancing these variables to minimize costs while ensuring a stable and efficient supply chain.

DECISION VARIABLES ANALYSIS:
r -(restocking rate): As r increases from 200 to 400, T*, Tw*, and t_1* remain constant, indicating the model may stabilize around certain reorder intervals regardless of r. However, the total cost (TC*) varies significantly, suggesting r directly influences total costs.

a - (demand rate): An increase in ‘a’ leads to larger reorder cycles (T*), which implies that a higher demand necessitates less frequent, bulkier orders. This adjustment reduces total costs (TC*), showing economies of scale for demand.

HOLDING COSTS
h_r and h_o: Both regular and overtime holding costs significantly impact T* and TC*. When ho is at its highest value (0.15), T* sharply increases, resulting in substantial cost (TC*=309.42), suggesting that high overtime costs necessitate fewer, larger orders to minimize holding expenses.

ORDER QUALITY AND SUPPLY CHAIN PARAMETERS
θ_1 and θ_2 (quality levels): Higher quality levels (θ_1 and θ_2) reduce TC* by improving the efficiency of the inventory system, though they influence T* only moderately. This finding implies that quality improvements lower costs without dramatically affecting order frequency
η_1 and η_2 (service rate factors): Increasing η_1 results in a minor reduction in TC*, indicating a potential cost efficiency associated with higher service rates. Similarly, increases in η_2 lead to reduced total costs, affirming the importance of service rate enhancements.

WAREHOUSE CAPACITY (w)
As warehouse capacity www increases, TC* decreases, implying economies of scale with larger storage space. This effect suggests that optimizing warehouse capacity can lead to cost savings due to better inventory flow and reduced need for frequent restocking.

DETERIORATION RATE
d_1- (deterioration rate): Higher deterioration rates slightly increase TC*, reflecting the expected impact of higher spoilage or wastage costs.

PRODUCTION RATE ($p^{\cdot}$)
Increased production rates (from 0.2 to 0.4) result in higher reorder intervals (T^*) and increased total costs (TC^*), which may reflect the additional costs associated with more frequent replenishments or handling.

VI. CONCLUSION

The proposed model takes into account the complexity of modern supply chains, recognizing the interdependencies across various stages from production to distribution. By incorporating essential variables such as order quantity, frequency, and holding costs, we aimed to refine replenishment strategies at each stage. Additionally, we introduced deterioration factors to address the time-sensitive nature of certain products, ensuring the model's applicability to industries dealing with perishable goods. Through optimization, we minimized the expected total annual cost while determining the cycle length T^*, t_1^* and T_w^*. To help managers determine whether renting a warehouse would be a feasible option, an algorithm was developed to determine the best replenishment policy. We demonstrated the use of the algorithm with numerical examples. The model could be modified for the next research to account for various demand patterns and shortages. Furthermore, possible expansions comprise adding time-varying degradation rates and various cost elements such as transit costs between phases.

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M. Balasubramanian serves as an Assistant Professor in the Department of Mathematics at SRM Arts and Science College, located in SRM Nagar, Kattankulathur, Chengalpattu, Tamil Nadu, India.

TABLE I. SHOWS THE EFFECT OF CHANGE VARIOUS DECISION PARAMETER

Decision variables		T^*	T_w^*	t_1^*	Q^*	TC^* (T^*, t_1^*, T_w^*)
r	200	28.4469	8.9167	3.6947	891	16.506
	300	28.4469	8.9167	3.6947	891	2.0021
	400	28.4469	8.9167	3.6947	891	23.536
a	50	15.8433	3.7024	3.9524	279	8.1215
	75	20.2979	5.5985	3.7810	441	4.9711
	100	28.4496	8.9167	3.6947	891	12.991
h_r	0.2	18.0757	5.3431	3.6947	573	0.4172
	0.3	28.4496	8.9167	3.6947	891	12.991
	0.4	55.55.3	17.595	3.6947	6944	169.43
h_o	0.15	67.7598	21.000	5.4319	19028	309.42
	0.20	28.4496	8.9167	3.6947	891	12.991
	0.25	18.5228	6.1282	2.6485	498	3.8153
θ_1	0.7	22.3495	6.2941	3.9685	687	4.1174
	0.8	19.1266	4.8636	4.1805	714	1.6046
	0.9	17.1581	3.9630	4.3504	847	1.5744
θ_2	0.3	23.5945	8.9167	3.6947	912	20.210
	0.4	27.5215	8.5455	3.6947	1187	26.878
	0.5	27.6579	8.6000	3.6947	1909	46.380
η_1	0.4	24.3853	8.9167	3.6947	891	15.156
	0.5	21.3372	8.9167	3.6947	891	17.321
	0.6	18.9664	8.9167	3.6947	591	19.486
η_2	0.1	37.9327	8.9167	3.6947	891	9.7433
	0.15	32.5138	8.9167	3.6947	891	11.367
	0.2	28.4496	8.9167	3.6947	891	12.991
w	55	24.8246	7.4667	3.6947	726	8.2311
	60	22.4079	6.5000	3.6947	652	6.3230
	65	20.6817	5.8095	3.6947	613	5.6671
d_1	30	28.4469	8.6947	3.6947	912	16.747
	35	28.4469	8.6947	3.6947	923	18.625
	40	28.4469	8.6947	3.6947	933	20.502
$\dot{p}$	0.2	28.4496	8.9167	3.6947	891	12.991
	0.3	32.7927	10.666	3.6756	1222	22.556
	0.4	38.5945	13.000	3.6567	2048	47.910

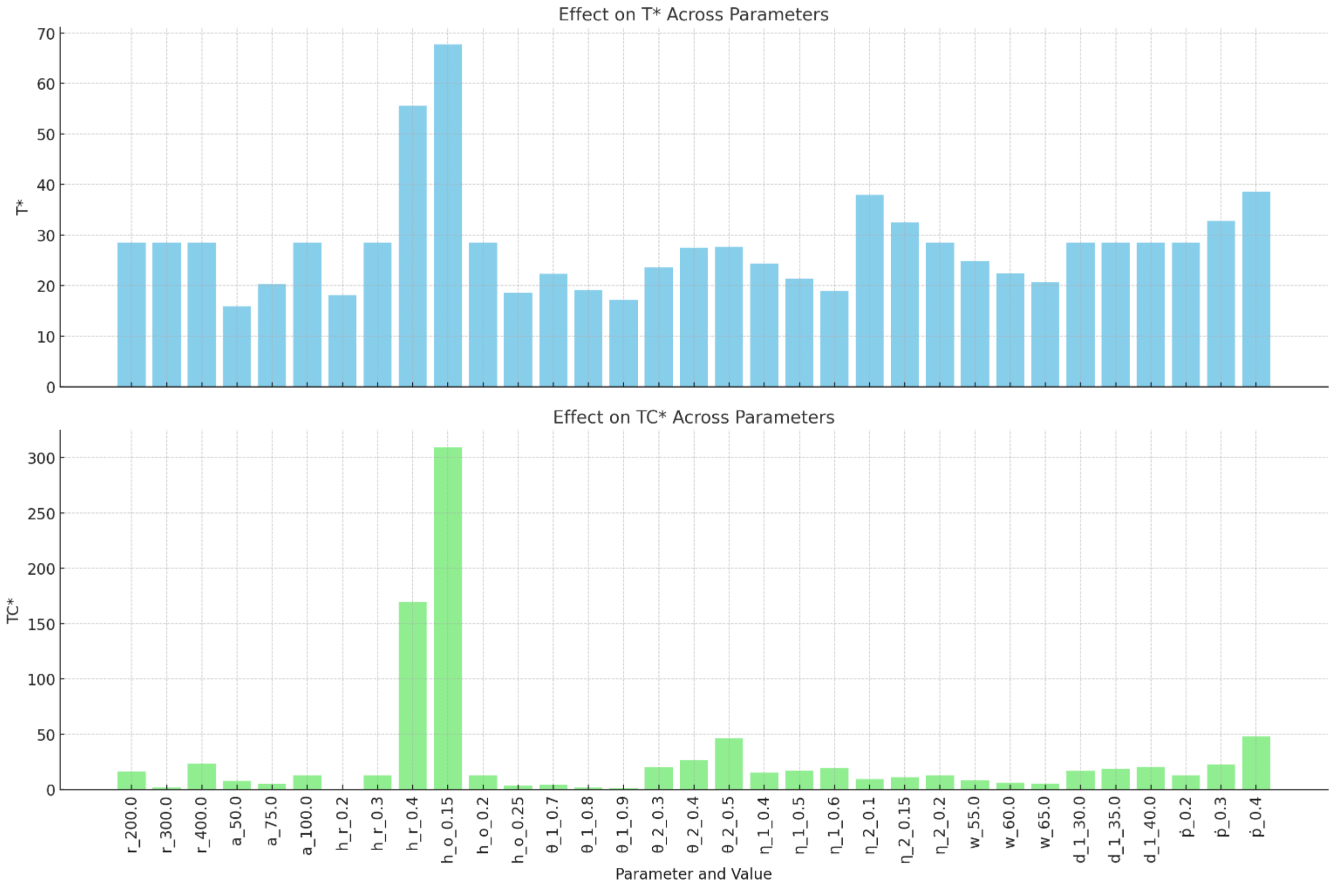

Fig 1. DIAGRAMETIC ANALYSIS OF TABLE I

TABLE II. A COMPARATIVE ANALYSIS WITH EXISTING INVENTORY MODELS

Aspect	Ramp-Type Demand Model	Existing Inventory Models
Demand Pattern	Gradual increase over time (ramp-type demand), posing unique challenges.	Assumes constant or stochastic demand patterns with fewer complexities in escalation.
Stages of Inventory Control	Two-stage model:	Typically, single or simpler multi-stage models, depending on the application.
	Strategic decisions at central warehouses (order quantity, frequency).	May focus on centralized or decentralized inventory but lacks ramp-specific strategies.
	Tactical decisions at retail or floor levels for handling ramp-type demand.	Tactical adjustments are often based on static or probabilistic demand forecasts.
Optimization Framework	Dynamic optimization to balance holding costs and stockouts.	May use static optimization or simpler heuristics.
Ordering Strategy	Refined strategy to minimize stockouts during rapid demand escalation.	General ordering policies may not address rapid escalation effectively.
Cost Efficiency	Aims to reduce overall inventory costs and halt item deterioration.	Typically focuses on minimizing costs but may not account for ramp-type complexities.
Flexibility	Highly adaptable to fluctuating demand patterns and periods of escalation.	Limited flexibility in handling dynamic, ramp-type demand patterns.
Implementation	Requires advanced modelling, dynamic frameworks, and sensitivity analysis for validation.	Relatively simpler to implement but may lack precision for dynamic demand scenarios.
Practical Implications	Suitable for industries with escalating demand patterns (e.g., seasonal products, new product launches).	More suited for stable demand environments or those with predictable variations.