New Results on Almost Periodic Dynamics of a Delayed Nicholson's Blowflies Model with Feedback Control

Nan Sun, Lian Duan, and Dong Xie

Abstract—**This paper considers a delayed almost periodic Nicholson's blowflies model with feedback control. Through employing the almost periodic differential equations theory and Lyapunov functional approach, some principles are set up for ensuring the persistence, existence and global asymptotic stability of positive almost periodic solutions of the considered model. Furthermore, a numerical example is provided for validating the theoretical analysis. The achieved results are novel and serve as an addition to the existing ones.**

Index Terms—**Nicholson's blowflies model, feedback control, persistence, almost periodicity, global asymptotic stability.**

I. Introduction

THE classical time delay Nicholson's blowflies model has drawn significant interest from theoretical and mathematical biologists, and has received thorough investigation. Gurney et al. [1] initially advanced the model to depict the dynamical behaviours of Nicholson's blowflies, which was described by the subsequent delay differential equation

$$r'(t) = -\lambda r(t) + ar(t-\iota)e^{-br(t-\iota)}, \tag{1}$$

in which $r(t)$ represents population magnitude at time t, λ means death rate on a daily basis of the adult organisms, a denotes the maximum daily productivity of the organisms' eggs, $\frac{1}{b}$ indicates the population level where the reproduction rate is the highest, and ι stands for the duration of one generation. In recent years, model (1) with its kinds of generalized forms have faced challenges from biology and physiology communities (see, e.g., [2]–[5] and references therein).

In mathematical biology theory, real world ecosystems are known to be constantly disturbed by unforeseeable forces that lead to changes in biological parameters, when an ecosystem encounters unpredictable disturbances, whether it can maintain persistence over a limited duration is an interesting problem. Moreover, the prominent biologist V. Volterra utilized the feedback control theory to elucidate the equilibrium between two fish populations in a confined pond in 1931, and consequently, the study of types of feedback control ecosystems and their applications has become an important research topic. For instance, Zhu et al. studied the dynamic behaviours in logistic model with feedback control and Allee effect in [6]. Chen in [7], [8] established the criteria for global stability of the unique interior equilibrium in Leslie-Gower predator-prey model with feedback controls. Wang and Fan in [9], [10] studied the permanence behavior of two kinds of Nicholson's blowflies models with feedback control. Hence, studying the population dynamics of feedback control ecosystems holds great importance.

On the flip side, researchers have discovered that the parameters related to biology or the environment in biomathematical models usually vary in time, environments that vary periodically or almost periodically are regarded as significant selective forces on ecosystems within a fluctuating setting. For instance, repetition is needed in numerous cognitive and biological processes (e.g., memorization, movement, heartbeat, respiration, and chewing). Besides, almost periodicity is a very important dynamic phenomenon in non-autonomous systems, with regard to long-term dynamical behaviors of non-autonomous biomathematical models, the periodic parameters of the model itself are frequently subject to certain disturbances, meaning that the parameters are periodic within a slight error. Therefore, much attention has been focused on investigating non-autonomous almost periodic Nicholson's blowflies models and many excellent contributions have been documented in the academic literature (see [11]–[16] and the references therein). Especially, Chen in [16] studied the question of positive almost periodic solutions for the Nicholson's blowflies model with multiple time-varying delays and feedback control, and established the relevant criteria for almost periodic solutions, ensuring their global exponential stability and boundedness. However, the magnitude of bounds is unknown and the delay effects on the dynamics of the considered model were not considered. Inspired by the aforementioned discussions and filling a gap in this research area, we will take into account the below delayed Nicholson's blowflies model with feedback control and almost periodic coefficients described by

$$\begin{cases} r_1'(t) = & -\lambda(t)r_1(t) \\ & +a(t)r_1(t-\iota(t))e^{-b(t)r_1(t-\iota(t))} \\ & -\alpha(t)r_1(t)r_2(t-\theta(t)), \\ r_2'(t) = & -\beta(t)r_2(t)+\delta(t)r_1(t-\zeta(t)), \end{cases} \tag{2}$$

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where $r_2(t)$ means the indirect control variable at time t, $\alpha(t)$ indicates inhibition level of $r_2(t)$ on population number $r_1(t)$ at time t, $\beta(t)$ denotes inhibition level of $r_2(t)$ at time t, $\delta(t)$ represents controllability of $r_2(t)$ at time t, $\theta(t)$ and $\zeta(t)$ denote feedback regulatory delay and maturation delay, respectively. The other time dependent biological parameters have the same biological implications as described in (1). The primary objective of the research article is to thoroughly examine the dynamical behaviors, including persistence, existence and global asymptotic stability of almost periodic solution for model (2).

In order to simplify the notations, considering the function h defined on $\mathbb{R}$ that is both bounded and continuous, we denote

$$h^- = \inf_{t\in\mathbb{R}} h(t), \qquad h^+ = \sup_{t\in\mathbb{R}} h(t).$$

Throughout the research, we denote $\nu = \max\{\iota^+, \theta^+, \zeta^+\}$, and always suppose that

$$\min\{\lambda^-, a^-, \iota^-, b^-, \alpha^-, \theta^-, \beta^-, \delta^-, \zeta^-\} > 0.$$

From the biological viewpoint, only the positive solutions of model (2) have practical significance and thus it is reasonable to consider the following initial conditions

$$r_{t_0} = \psi,\ \psi = (\psi_1, \psi_2) \in C_+ \times C_+,\ \psi_1(0) > 0,\ \psi_2(0) > 0. \tag{3}$$

Moreover, define $\mathbb{R}(\mathbb{R}_+)$ as the set of all (nonnegative) real numbers, $C = C([-\nu, 0], \mathbb{R})$ is the space of continuous functions supplemented with the supremum norm $\|\cdot\|$, also, define $C_+ = C([-\nu, 0], \mathbb{R}_+)$. Provided that $r_1(t),\ r_2(t)$ are both continuous within the interval $[t_0 - \nu, \rho)$, where $t_0, \rho \in \mathbb{R}$, we then define $r_t = (r_t^1, r_t^2) \in C \times C,\ r_t^1(\varpi) = r_1(t + \varpi),\ r_t^2(\varpi) = r_2(t + \varpi)$ for all $\varpi \in [-\nu, 0]$, and we signify the solution of the admissible initial value problem (2)-(3) by $r_t(t_0, \psi)(r(t; t_0, \psi))$. The interval $[t_0, \eta(\psi))$ represents the maximal right-interval of existence of $r_t(t_0, \psi)$.

The paper is sketched as described below. Section 2 provides several preliminaries. Section 3 formulates novel conditions for ensuring the permanence and global asymptotic stability of almost periodic model (2). Section 4 validates the theoretical analysis through a numerical example.

II. Preliminaries

This section provides the definitions and lemmas that are outlined below.

Definition 1. *Model (2) is regarded as permanent, if positive constants m_i and M_i can be found fulfilling the condition that*

$$m_i \le \liminf_{t\to+\infty} r_i(t) \le \limsup_{t\to+\infty} r_i(t) \le M_i, \quad for\ i = 1, 2.$$

Lemma 1 (see [17]). *If $p > 0, q > 0$ and $\frac{\mathrm{d}r}{\mathrm{d}t} \ge q - pr$, when $t \ge t''$ and $r(t'') > 0$, one has*

$$\liminf_{t\to+\infty} r(t) \ge \frac{q}{p}.$$

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Definition 2 (see [18]). *Suppose $h(t) : \mathbb{R} \to \mathbb{R}$ is continuous for $t \in \mathbb{R}$. $h(t)$ is characterized as almost periodic on $\mathbb{R}$ provided that, for any given positive constant ϵ, the set $E(h, \epsilon) = \{\xi : |h(t + \xi) - h(t)| < \epsilon,$ for all $t \in \mathbb{R}\}$ satisfies the property as being relatively dense, namely, for any given positive constant ϵ, a positive real number $k = k(\epsilon)$ can be found ensuring in any interval with length $k(\epsilon)$, one can find a number $\xi = \xi(\epsilon)$ within this interval satisfying $|h(t + \xi) - h(t)| < \epsilon$, for all $t \in \mathbb{R}$.*

Definition 3 (see [18]). *Suppose $\mu \in \mathbb{R}^n$ and $P(t)$ is a $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system*

$$\mu'(t) = P(t)\mu(t), \tag{4}$$

is defined as having an exponential dichotomy on $\mathbb{R}$ when one can find positive constants l, d, projection Q and the fundamental solution matrix $U(t)$ of (4) such that

$$\|U(t)QU^{-1}(\omega)\| \le le^{-d(t-\omega)}, \qquad for\ t \ge \omega,$$
$$\|U(t)(I - Q)U^{-1}(\omega)\| \le le^{-d(\omega-t)}, \qquad for\ t \le \omega,$$

in which I stands for the identity matrix.

Lemma 2 (see [18]). *Provided that the linear system (4) possesses an exponential dichotomy, it follows that the almost periodic system*

$$\mu'(t) = P(t)\mu(t) + q(t)$$

possesses a unique almost periodic solution $\mu(t)$, as well as

$$\mu(t) = \int_{-\infty}^{t} U(t)QU^{-1}(\omega)q(\omega)d\omega - \int_{t}^{+\infty} U(t)(I - Q)U^{-1}(\omega)q(\omega)d\omega.$$

Lemma 3 (see [18]). *Suppose $o_i(t)$ is an almost periodic function on $\mathbb{R}$ as well as*

$$M[o_i] = \lim_{W\to+\infty} \frac{1}{W}\int_t^{t+W} o_i(\omega)d\omega > 0, \quad i = 1, 2, \ldots, n.$$

Then the linear system

$$\mu'(t) = \mathrm{diag}(-o_1(t), -o_2(t), \ldots, -o_n(t))u(t)$$

has an exponential dichotomy on $\mathbb{R}$.

Lemma 4 (see Lemma 2.2 in [19]). *The solution $r_t(t_0, \psi) \in C_+$ for all $t \in [t_0, \eta(\psi))$, the set of $\{r_t(t_0, \psi) : t \in [t_0, \eta(\psi))\}$ is bounded, and $\eta(\psi) = +\infty$. Furthermore, $r_i(t; t_0, \psi) > 0$ for all $t \ge t_0,\ i = 1, 2$.*

For the purpose of deriving the following main results, we always assume that

(H1) $\lambda(t), a(t), \iota(t), b(t), \alpha(t), \theta(t), \beta(t), \delta(t), \zeta(t) :$ $\mathbb{R} \to (0, +\infty)$ are almost periodic functions.

III. Main Results

This part states and proves the main results of the present article. Firstly, we present the following notations which can simplify our further discussions.

Denote

$$M_1=\frac{a^+}{\lambda^-b^-e},\qquad M_2=\frac{\delta^+M_1}{\beta^-},$$
$$m_1=\min\left\{\frac{1}{b^+}\ln\frac{a^-}{\lambda^++\alpha^+M_2},\frac{a^-M_1e^{-b^+M_1}}{\lambda^++\alpha^+M_2}\right\},$$
$$m_2=\frac{\delta^-m_1}{\beta^+}.$$

Theorem 1. *Assuming that $(H1)$ is satisfied, it is further supposed that*

(H2) $\lambda^++\alpha^+M_2<a^-$

holds, then model (2) *is permanent.*

Proof: Presume that $(r_1(t),r_2(t))$ is an arbitrary positive solution to question (2)-(3). It follows from the first equation of model (2), combined with the statement $\sup\limits_{s\geq 0} se^{-b^-s}=\frac{1}{b^-e}$ that

$$\begin{aligned}r_1'(t)&\leq-\lambda(t)r_1(t)+a(t)r_1(t-\iota(t))e^{-b(t)r_1(t-\iota(t))}\\&\leq-\lambda^-r_1(t)+a^+r_1(t-\iota(t))e^{-b^-r_1(t-\iota(t))}\\&\leq-\lambda^-r_1(t)+\frac{a^+}{b^-e}.\end{aligned}$$

According to Lemma 1, we get

$$\limsup_{t\to+\infty}r_1(t)\leq\frac{a^+}{\lambda^-b^-e}:=M_1. \tag{5}$$

Then, we know that for any constant $\varepsilon>0$ that is sufficiently small, it is possible for one to find $T_1>t_0$ for which

$$r_1(t)<M_1+\varepsilon,\qquad t\in(T_1,+\infty).$$

Hence, the above estimate and the second equation of model (2) lead to

$$r_2'(t)<-\beta^-r_2(t)+\delta^+(M_1+\varepsilon),\qquad t\in(T_1+\nu,+\infty).$$

By Lemma 1 again, it is clear that

$$\limsup_{t\to+\infty}r_2(t)\leq\frac{\delta^+(M_1+\varepsilon)}{\beta^-}.$$

Letting $\varepsilon\to0$, the inequality mentioned above implies

$$\limsup_{t\to+\infty}r_2(t)\leq\frac{\delta^+M_1}{\beta^-}:=M_2. \tag{6}$$

For the given positive constant $\varepsilon>0$ as above, we know from (6) that one can find $T_2>T_1+\nu$ for which

$$r_2(t)\leq M_2+\varepsilon,\qquad t\in[T_2,+\infty). \tag{7}$$

We now proceed to demonstrate

$$\liminf_{t\to+\infty}r_1(t)>0. \tag{8}$$

Otherwise, we have $\liminf\limits_{t\to+\infty}r_1(t)=0$. For every $t\geq t_0$, we establish the following definition

$$\chi(t)=\max\{\varsigma:\varsigma\leq t,r_1(\varsigma)=\min_{t_0\leq s\leq t}r_1(s)\}.$$

Note that $\chi(t)\to+\infty$ as $t\to+\infty$ and

$$\lim_{t\to+\infty}r_1(\chi(t))=0. \tag{9}$$

As defined by $\chi(t)$, we find $r_1(\chi(t))=\min\limits_{t_0\leq s\leq t}r_1(s)$, and $r_1'(\chi(t))\leq0$. Subsequently, the first equation of model (2) and (7) yield

$$\begin{aligned}0\geq&r_1'(\chi(t))\\=&-\lambda(\chi(t))r_1(\chi(t))\\&+a(\chi(t))r_1(\chi(t)-\iota(\chi(t)))e^{-b(\chi(t))r_1(\chi(t)-\iota(\chi(t)))}\\&-\alpha(\chi(t))r_1(\chi(t))r_2(\chi(t)-\theta(\chi(t)))\\\geq&-\lambda^+r_1(\chi(t))\\&+a^-r_1(\chi(t)-\iota(\chi(t)))e^{-b^+r_1(\chi(t)-\iota(\chi(t)))}\\&-\alpha^+(M_2+\varepsilon)r_1(\chi(t)),\qquad t\geq\max\{\chi(t),T_2+\nu\},\end{aligned}$$

a direct calculation produces

$$\begin{aligned}&\big(\lambda^++\alpha^+(M_2+\varepsilon)\big)r_1(\chi(t))\\&\geq a^-r_1(\chi(t)-\iota(\chi(t)))e^{-b^+r_1(\chi(t)-\iota(\chi(t)))},\end{aligned}$$

which, in conjunction with (9), results in

$$\lim_{t\to+\infty}r_1(\chi(t)-\iota(\chi(t)))=0,$$

as well as

$$\begin{aligned}&\lambda^++\alpha^+(M_2+\varepsilon)\\&\geq\frac{a^-r_1(\chi(t)-\iota(\chi(t)))}{r_1(\chi(t))}e^{-b^+r_1(\chi(t)-\iota(\chi(t)))}\\&\geq\frac{a^-r_1(\chi(t)-\iota(\chi(t)))}{r_1(\chi(t)-\iota(\chi(t)))}e^{-b^+r_1(\chi(t)-\iota(\chi(t)))}\\&=a^-e^{-b^+r_1(\chi(t)-\iota(\chi(t)))}.\end{aligned}$$

Considering that ε is arbitrary and letting $t\to+\infty$, we can find that

$$\lambda^++\alpha^+M_2\geq a^-,$$

which contradicts with $(H2)$. Hence, (8) holds.

Set

$$w=\liminf_{t\to+\infty}r_1(t),\qquad d(r)=re^{-b^+r},$$
$$k=\min\{d(w),d(M_1)\}.$$

From the first equation of (2) and (7), one has

$$\begin{aligned}r_1'(t)\geq&-\lambda^+r_1(t)+a^-r_1(t-\iota(t))e^{-b^+r_1(t-\iota(t))}\\&-\alpha^+r_1(t)(M_2+\varepsilon)\\\geq&-\lambda^+r_1(t)+a^-k-\alpha^+r_1(t)(M_2+\varepsilon)\\=&-(\lambda^++\alpha^+(M_2+\varepsilon))r_1(t)+a^-k,\\&\qquad\text{for } t\in[T_2+\nu,+\infty).\end{aligned}$$

By Lemma 1, and letting $\varepsilon\to0$, the inequality mentioned above leads to

$$\liminf_{t\to+\infty}r_1(t)\geq\frac{a^-k}{\lambda^++\alpha^+M_2}. \tag{10}$$

If $k=d(w)$, we obtain from (10) that

$$w\geq\frac{a^-}{\lambda^++\alpha^+M_2}we^{-b^+w},$$

that is,

$$w\geq\frac{1}{b^+}\ln\frac{a^-}{\lambda^++\alpha^+M_2}.$$

If $k = d(M_1)$, it follows from (10) that

$$w \geq \frac{a^-}{\lambda^+ + \alpha^+ M_2} M_1 e^{-b^+ M_1}.$$

The above discussion produces

$$\liminf_{t\to+\infty} r_1(t) \geq \min\left\{\frac{1}{b^+}\ln\frac{a^-}{\lambda^+ + \alpha^+ M_2}, \frac{a^- M_1 e^{-b^+ M_1}}{\lambda^+ + \alpha^+ M_2}\right\} := m_1. \tag{11}$$

For the same given small enough positive constant ε, (11) implies that one can find $T_3 > T_2 + \nu$ such that

$$r_1(t) > m_1 - \varepsilon, \qquad t \in [T_3, +\infty),$$

which combined with the second equation of (2), results in

$$r_2'(t) > -\beta^+ r_2(t) + \delta^-(m_1 - \varepsilon), \qquad t \in [T_3 + \nu, +\infty).$$

By Lemma 1 once more, we conclude that

$$\liminf_{t\to+\infty} r_2(t) \geq \frac{\delta^-(m_1 - \varepsilon)}{\beta^+}.$$

As $\varepsilon \to 0$, the inequality above yields

$$\liminf_{t\to+\infty} r_2(t) \geq \frac{\delta^- m_1}{\beta^+} := m_2. \tag{12}$$

Finally, (5), (6), (11) and (12) mean that system (2) is permanent, and the proof is completed. ∎

Theorem 2. *Suppose that (H1) and (H2) are valid, and assume further that*

(H3) $\frac{1}{b^-} \leq m_1 \leq \frac{1}{\lambda^+}(a^- M_1 e^{-b^+ M_1} - \alpha^+ M_1 M_2)$;

(H4) $\max\left\{\frac{1}{\lambda^-}\left(\frac{a^+}{e^2} + \alpha^+(M_1 + M_2)\right), \frac{\delta^+}{\beta^-}\right\} < 1$.

Then there is a unique positive almost periodic solution of model (2) *within the region*

$$L^* = \{r | r \in AP(\mathbb{R};\mathbb{R}) \times AP(\mathbb{R};\mathbb{R}), m_i \leq r_i(t) \leq M_i, t \in \mathbb{R}, i = 1, 2\}.$$

Proof: The proof closely mirrors the one for Theorem 3.1 presented in [16], and we give the proof as an appendix for the convenience of reading. ∎

Subsequently, we examine the asymptotic stability of almost periodic solution $z^*(t) = (r_1^*(t), r_2^*(t))$ for the model (2).

Theorem 3. *Assuming the requirements of Theorem 2 are satisfied, and additionally supposing that*

(H5) *the delay functions $\iota(t), \zeta(t), \theta(t)$ are continuously differentiable, moreover there exist non-negative constants $\iota^*, \zeta^*, \theta^*$ satisfying*

$$0 \leq \iota'(t) \leq \iota^* < 1, \quad 0 \leq \zeta'(t) \leq \zeta^* < 1, \quad 0 \leq \theta'(t) \leq \theta^* < 1.$$

(H6)

$$\Lambda_1 \triangleq 2\lambda^- - \frac{a^+}{e^2} - \alpha^+ M_1 - 2\alpha^+ M_2 - \frac{a^+}{e^2}\frac{1}{1-\iota^*} - \frac{\delta^+}{1-\zeta^*} > 0,$$
$$\Lambda_2 \triangleq 2\beta^- - \delta^+ - \frac{\alpha^+ M_1}{1-\theta^*} > 0.$$

Then the almost periodic solution of model (2) *is global asymptotically stable.*

Proof: Suppose $z^*(t) = (r_1^*(t), r_2^*(t))$ is the positive almost periodic solution of model (2), additionally, $z(t) = (r_1(t), r_2(t))$ is another solution of model (2). Denote

$$\Omega_1(t) = r_1(t) - r_1^*(t), \qquad \Omega_2(t) = r_2(t) - r_2^*(t),$$

as well as

$$\begin{aligned} n(t) =& a(t)(r_1(t-\iota(t))e^{-b(t)r_1(t-\iota(t))} \\ &- r_1^*(t-\iota(t))e^{-b(t)r_1^*(t-\iota(t))}) \\ &- \alpha(t)(r_1(t)r_2(t-\theta(t)) - r_1^*(t)r_2^*(t-\theta(t))), \end{aligned}$$

then (2) reduces to

$$\begin{cases} \Omega_1'(t) &= -\lambda(t)\Omega_1(t) + n(t), \\ \Omega_2'(t) &= -\beta(t)\Omega_2(t) + \delta(t)\Omega_1(t-\zeta(t)). \end{cases} \tag{13}$$

Based on the mean value theorem and the fact $\sup\limits_{s\geq 1}|\frac{1-s}{e^s}| = \frac{1}{e^2}$ that

$$\begin{aligned} &|He^{-H} - Je^{-J}| \\ &= \left|\frac{1-(H+\sigma(J-H))}{e^{H+\sigma(J-H)}}\right||H-J| \\ &\leq \frac{1}{e^2}|H-J|, \qquad H, J \in [1, +\infty), \quad 0 < \sigma < 1. \end{aligned}$$

Then

$$\begin{aligned} |n(t)| \leq& a(t)|r_1(t-\iota(t))e^{-b(t)r_1(t-\iota(t))} \\ &- r_1^*(t-\iota(t))e^{-b(t)r_1^*(t-\iota(t))}| \\ &+ \alpha(t)\Big(|r_1(t)||r_2(t-\theta(t)) - r_2^*(t-\theta(t))| \\ &+ |r_2^*(t-\theta(t))||r_1(t) - r_1^*(t)|\Big) \\ \leq& \frac{a^+}{e^2}|\Omega_1(t-\iota(t))| + \alpha^+(M_1|\Omega_2(t-\theta(t))| \\ &+ M_2|\Omega_1(t)|). \end{aligned} \tag{14}$$

Consider the Lyapunov function given below

$$\Pi(t) = \Pi_1(t) + \Pi_2(t),$$

in which

$$\begin{aligned} \Pi_1(t) =& \Omega_1^2(t) + \Omega_2^2(t), \\ \Pi_2(t) =& \frac{a^+}{e^2}\frac{1}{1-\iota^*}\int_{t-\iota(t)}^t \Omega_1^2(s)ds \\ &+ \frac{\delta^+}{1-\zeta^*}\int_{t-\zeta(t)}^t \Omega_1^2(s)ds \\ &+ \frac{\alpha^+ M_1}{1-\theta^*}\int_{t-\theta(t)}^t \Omega_2^2(s)ds. \end{aligned}$$

According to (13) and (H5), by computing the derivatives of

$\Pi_1(t)$ as well as $\Pi_2(t)$, it is easy to find that

$$\begin{aligned}\frac{\mathrm{d}\Pi_1(t)}{\mathrm{d}t} =&2\Omega_1(t)\Omega_1'(t)+2\Omega_2(t)\Omega_2'(t)\\ =&2\Omega_1(t)[-\lambda(t)\Omega_1(t)+n(t)]\\ &+2\Omega_2(t)[-\beta(t)\Omega_2(t)+\delta(t)\Omega_1(t-\zeta(t))]\\ \leq&-2\lambda^-\Omega_1^2(t)+2|\Omega_1(t)||n(t)|-2\beta^-\Omega_2^2(t)\\ &+2\delta^+|\Omega_2(t)||\Omega_1(t-\zeta(t))|\\ \leq&-2\lambda^-\Omega_1^2(t)+2|\Omega_1(t)|\bigg[\frac{a^+}{e^2}|\Omega_1(t-\iota(t))|\\ &+\alpha^+(M_1|\Omega_2(t-\theta(t))|+M_2|\Omega_1(t)|)\bigg]\\ &-2\beta^-\Omega_2^2(t)+2\delta^+|\Omega_2(t)||\Omega_1(t-\zeta(t))|\\ \leq&-2\lambda^-\Omega_1^2(t)+\frac{a^+}{e^2}(\Omega_1^2(t)+\Omega_1^2(t-\iota(t)))\\ &+\alpha^+M_1(\Omega_1^2(t)+\Omega_2^2(t-\theta(t)))\\ &+2\alpha^+M_2\Omega_1^2(t)-2\beta^-\Omega_2^2(t)\\ &+\delta^+(\Omega_2^2(t)+\Omega_1^2(t-\zeta(t)))\\ =&\bigg(-2\lambda^-+\frac{a^+}{e^2}+\alpha^+M_1+2\alpha^+M_2\bigg)\Omega_1^2(t)\\ &+\frac{a^+}{e^2}\Omega_1^2(t-\iota(t))+\delta^+\Omega_1^2(t-\zeta(t))\\ &+(-2\beta^-+\delta^+)\Omega_2^2(t)+\alpha^+M_1\Omega_2^2(t-\theta(t)),\end{aligned}\tag{15}$$

and

$$\begin{aligned}\frac{\mathrm{d}\Pi_2(t)}{\mathrm{d}t} =&\frac{a^+}{e^2}\frac{1}{1-\iota^*}[\Omega_1^2(t)-\Omega_1^2(t-\iota(t))(1-\iota'(t))]\\ &+\frac{\delta^+}{1-\zeta^*}[\Omega_1^2(t)-\Omega_1^2(t-\zeta(t))(1-\zeta'(t))]\\ &+\frac{\alpha^+M_1}{1-\theta^*}[\Omega_2^2(t)-\Omega_2^2(t-\theta(t))(1-\theta'(t))]\\ =&\frac{a^+}{e^2}\frac{1}{1-\iota^*}\Omega_1^2(t)-\frac{a^+}{e^2}\frac{1-\iota'(t)}{1-\iota^*}\Omega_1^2(t-\iota(t))\\ &+\frac{\delta^+}{1-\zeta^*}\Omega_1^2(t)-\delta^+\frac{1-\zeta'(t)}{1-\zeta^*}\Omega_1^2(t-\zeta(t))\\ &+\frac{\alpha^+M_1}{1-\theta^*}\Omega_2^2(t)-\alpha^+M_1\frac{1-\theta'(t)}{1-\theta^*}\Omega_2^2(t-\theta(t))\\ \leq&\frac{a^+}{e^2}\frac{1}{1-\iota^*}\Omega_1^2(t)-\frac{a^+}{e^2}\Omega_1^2(t-\iota(t))\\ &+\frac{\delta^+}{1-\zeta^*}\Omega_1^2(t)-\delta^+\Omega_1^2(t-\zeta(t))\\ &+\frac{\alpha^+M_1}{1-\theta^*}\Omega_2^2(t)-\alpha^+M_1\Omega_2^2(t-\theta(t)).\end{aligned}\tag{16}$$

Then based on (15) and (16), we conclude that

$$\begin{aligned}\frac{\mathrm{d}\Pi(\mathrm{t})}{\mathrm{d}t} \leq&\bigg(-2\lambda^-+\frac{a^+}{e^2}+\alpha^+M_1+2\alpha^+M_2\\ &+\frac{a^+}{e^2}\frac{1}{1-\iota^*}+\frac{\delta^+}{1-\zeta^*}\bigg)\Omega_1^2(t)\\ &+\bigg(-2\beta^-+\delta^++\frac{\alpha^+M_1}{1-\theta^*}\bigg)\Omega_2^2(t)\\ =&-\Lambda_1\Omega_1^2(t)-\Lambda_2\Omega_2^2(t)\\ <&0.\end{aligned}$$

Accordingly

$$\Pi(t)+\Lambda_1\int_0^t\Omega_1^2(s)ds+\Lambda_2\int_0^t\Omega_2^2(s)ds\leq\Pi(0),\quad t\geq0.\tag{17}$$

Based on Lemma 4, we come to the conclusion that the solution of model (2) is bounded on $[0,+\infty)$, this indicates that $\frac{\mathrm{d}\Omega_1(t)}{\mathrm{d}t}$ as well as $\frac{\mathrm{d}\Omega_2(t)}{\mathrm{d}t}$ are also bounded on $[0,+\infty)$, thus $\Omega_1(t)$ as well as $\Omega_2(t)$ are uniformly continuous on $[0,+\infty)$. (17) also means that $\Omega_1,\Omega_2\in L[0,+\infty)$, According to Barbalatt's Lemma [20], it is possible for us to deduce that

$$\lim_{t\to+\infty}\Omega_1(t)=0,\qquad\lim_{t\to+\infty}\Omega_2(t)=0.$$

Then we can assert that $z^*(t)$ is global asymptotically stable. The proof has come to an end. ■

IV. A NUMERICAL EXAMPLE

In this part, a numerical example is provided to confirm the effectiveness of the suggested theoretical results.

Example 1 Take into account the following delayed Nicholson's blowflies model with feedback control:

$$\begin{cases} r_1'(t)= & -(19.99+\cos^2 t)r_1(t)+(11\\ & +0.0001|\sin\sqrt{2}t|)e^{e-1}r_1(t-e)e^{-r_1(t-e)}\\ & -(0.05+0.05|\sin t|)r_1(t)r_2(t-\frac{1}{e}),\\ r_2'(t)= & -(1+0.1\cos^4 t)r_2(t)\\ & +(0.4+0.001|\sin t|)r_1(t-\frac{1}{e}).\end{cases}\tag{18}$$

Clearly, we have $\lambda^-=19.99,\lambda^+=20.99,a^-=11e^{e-1},a^+=11.0001e^{e-1},b^-=1,b^+=1,\alpha^-=0.05,\alpha^+=0.1,\beta^-=1,\beta^+=1.1,\delta^-=0.4,\delta^+=0.401$. By calculating, we have

$$\begin{aligned}&M_1=\frac{a^+}{\lambda^-b^-e}=1.129,\qquad M_2=\frac{\delta^+M_1}{\beta^-}=0.453,\\ &m_1=\min\left\{\frac{1}{b^+}\ln\frac{a^-}{\lambda^++\alpha^+M_2},\frac{a^-M_1e^{-b^+M_1}}{\lambda^++\alpha^+M_2}\right\}=1.064,\\ &m_2=\frac{\delta^-m_1}{\beta^+}=0.387,\\ &\lambda^++\alpha^+M_2=21.035<a^-=11e^{e-1},\\ &\frac{1}{b^-}<m_1<\frac{1}{\lambda^+}(a^-M_1e^{-b^+M_1}-\alpha^+M_1M_2)=1.065,\\ &\frac{1}{\lambda^-}\bigg(\frac{a^+}{e^2}+\alpha^+(M_1+M_2)\bigg)=0.423<1,\\ &\frac{\delta^+}{\beta^-}=0.401<1,\\ &\Lambda_1=22.778>0,\qquad\Lambda_2=1.486>0,\end{aligned}$$

which implies that the criteria stated in Theorems 2 and 3 are met. Thus, we conclude that model (18) is permanent, and possesses a unique almost periodic solution that is globally asymptotically stable(see Fig. 1).

Remark 1. *In [9], Wang and Fan established the sufficient criteria for the persistence of a time delayed discrete Nicholson's blowflies model with feedback control, as well as in [10], the same authors studied the dynamics of the delayed periodic Nicholson's blowflies model with feedback control on time scales. The persistence and global exponential convergence of a generalized Nicholson's blowflies model*

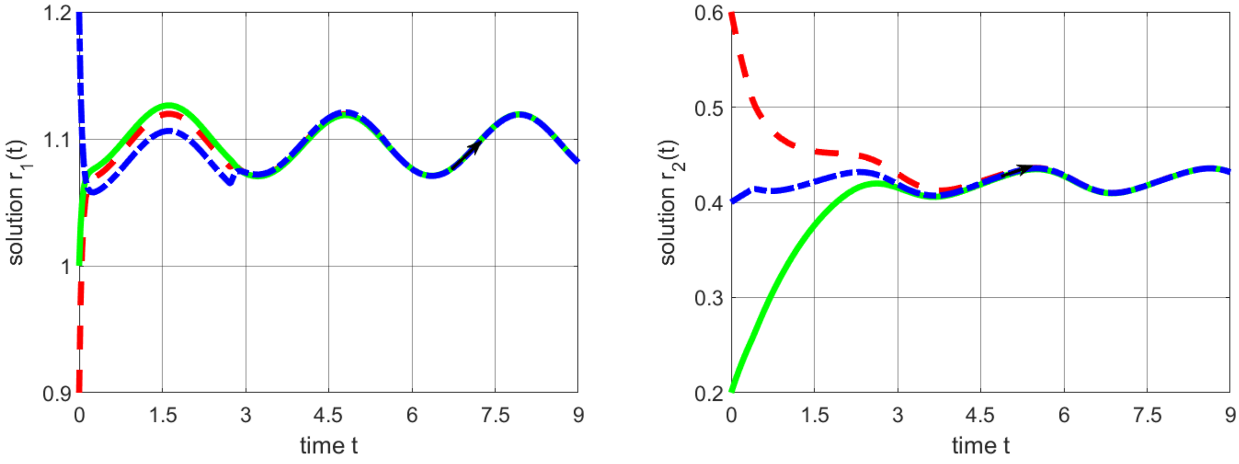

Fig. 1. Simulated trajectories of the model (18) with initial values $(\psi_1(s), \psi_2(s)) = (0.9, 0.2), (1, 0.4), (1.2, 0.6), s \in [-e, 0]$.

with feedback control and multiple time-varying delays are addressed in [19]. However, one can find that the approaches employed in this paper vary from those mentioned in aforementioned literature, and the almost periodic dynamics were not considered in [9], [10], [19]. On the other hand, Chen in [16] obtained the existence of boundedness of solutions, but we established the boundedness precisely expressed by the model's parameters, which is more convenient for estimation and applications. Consequently, the theoretical results derived in the present paper extend and enrich the existing findings.

Appendix A
Proof of Theorem 2

Proof: For every $\psi_i \in AP(\mathbb{R};\mathbb{R})$, we shall take the nonlinear almost periodic differential equation into consideration as follows

$$\begin{cases} r_1'(t) = & -\lambda(t)r_1(t) \\ & +a(t)\psi_1(t-\iota(t))e^{-b(t)\psi_1(t-\iota(t))} \\ & -\alpha(t)\psi_1(t)\psi_2(t-\theta(t)), \\ r_2'(t) = & -\beta(t)r_2(t) + \delta(t)\psi_1(t-\zeta(t)). \end{cases} \tag{19}$$

Since $M[\lambda] > 0, M[\beta] > 0$, from Lemma 3, it can be seen that the linear system

$$\begin{cases} r_1'(t) = -\lambda(t)r_1(t), \\ r_2'(t) = -\beta(t)r_2(t), \end{cases}$$

has an exponential dichotomy on $\mathbb{R}$, therefore, following Lemma 2, it can be concluded that model (19) admits an almost periodic solution $r^{\psi}(t) = (r^{\psi_1}(t), r^{\psi_2}(t))$ expressed by

$$\begin{cases} r^{\psi_1}(t) = & \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)du} \\ & \times\Big(a(s)\psi_1(s-\iota(s))e^{-b(s)\psi_1(s-\iota(s))} \\ & -\alpha(s)\psi_1(s)\psi_2(s-\theta(s))\Big)ds, \\ r^{\psi_2}(t) = & \int_{-\infty}^{t} e^{-\int_s^t \beta(u)du}\Big(\delta(s)\psi_1(s-\zeta(s))\Big)ds. \end{cases} \tag{20}$$

Next, we establish a mapping $\Gamma : L^* \to L^*$

$$\Gamma(\psi(t)) = r^{\psi}(t), \qquad \text{for all } \psi \in L^*.$$

Clearly, L^* is a closed subset of $AP(\mathbb{R};\mathbb{R}) \times AP(\mathbb{R};\mathbb{R})$. For each $\psi \in L^*$, due to (20) and $\sup\limits_{s\geq 0} se^{-b^- s} = \frac{1}{b^- e}$, one has

$$\begin{cases} r^{\psi_1}(t) \leq & \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)du} \\ & \times a(s)\psi_1(s-\iota(s))e^{-b(s)\psi_1(s-\iota(s))}ds \\ \leq & \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)du} a^+ \frac{1}{b^- e} ds \\ \leq & \frac{a^+}{\lambda^- b^- e} = M_1, \\ r^{\psi_2}(t) \leq & \int_{-\infty}^{t} e^{-\int_s^t \beta(u)du} \delta^+ M_1 ds \\ \leq & \frac{\delta^+ M_1}{\beta^-} = M_2. \end{cases} \tag{21}$$

By $(H3)$, and $\min\limits_{1\leq s\leq \kappa} se^{-s} = \kappa e^{-\kappa}$, together with (20) yield

$$\begin{cases} r^{\psi_1}(t) \geq & \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)du}\Big(\frac{a^-}{b^+} b^+\psi_1(s-\iota(s)) \\ & \times e^{-b^+\psi_1(s-\iota(s))} - \alpha^+ M_1 M_2\Big)ds \\ \geq & \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)du}\Big(\frac{a^-}{b^+} b^+ M_1 e^{-b^+ M_1} \\ & -\alpha^+ M_1 M_2\Big)ds \\ \geq & \frac{1}{\lambda^+}(a^- M_1 e^{-b^+ M_1} - \alpha^+ M_1 M_2) \geq m_1, \\ r^{\psi_2}(t) \geq & \int_{-\infty}^{t} e^{-\int_s^t \beta(u)du} \delta^- m_1 ds \\ \geq & \frac{\delta^- m_1}{\beta^+} = m_2. \end{cases} \tag{22}$$

(21) and (22) mean that Γ maps L^* into L^*.

Now, we check that the contraction condition required by the Banach fixed point theorem is satisfied. By the mean value theorem and with the help of the fact $\sup\limits_{s\geq 1}|\frac{1-s}{e^s}| = \frac{1}{e^2}$, we get

$$\begin{aligned} & |He^{-H} - Je^{-J}| \\ = & \left|\frac{1-(H+\sigma(J-H))}{e^{H+\sigma(J-H)}}\right||H-J| \\ \leq & \frac{1}{e^2}|H-J|, \qquad H, J \in [1, +\infty), \quad 0 < \sigma < 1. \end{aligned} \tag{23}$$

For each $\phi, \varphi \in L^*$, we have

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} |\Gamma(\phi_1(t)) - \Gamma(\varphi_1(t))| \\
=&\sup_{t\in\mathbb{R}} \bigg| \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)du} \bigg\{ a(s)\bigg(\phi_1(s-\iota(s)) \\
&\times e^{-b(s)\phi_1(s-\iota(s))} - \varphi_1(s-\iota(s)) e^{-b(s)\varphi_1(s-\iota(s))}\bigg) \\
&-\alpha(s)\bigg(\phi_1(s)\phi_2(s-\theta(s)) - \varphi_1(s)\varphi_2(s-\theta(s))\bigg)\bigg\} ds \bigg| \\
\leq& \frac{a^+}{\lambda^- e^2}\|\phi-\varphi\| \\
&+\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)du} \alpha(s)\bigg(|\phi_1(s)||\phi_2(s-\theta(s)) \\
&-\varphi_2(s-\theta(s))| + |\varphi_2(s-\theta(s))||\phi_1(s)-\varphi_1(s)|\bigg) ds \\
\leq& \frac{a^+}{\lambda^- e^2}\|\phi-\varphi\| + \frac{\alpha^+}{\lambda^-}(M_1+M_2)\|\phi-\varphi\| \\
=& \frac{1}{\lambda^-}\bigg(\frac{a^+}{e^2} + \alpha^+(M_1+M_2)\bigg)\|\phi-\varphi\|,
\end{aligned} \tag{24}
$$

and

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} |\Gamma(\phi_2(t)) - \Gamma(\varphi_2(t))| \\
=&\sup_{t\in\mathbb{R}} \bigg| \int_{-\infty}^{t} e^{-\int_s^t \beta(u)du} \delta(s)\bigg(\phi_1(s-\zeta(s)) \\
&-\varphi_1(s-\zeta(s))\bigg) ds \bigg| \\
\leq& \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \beta(u)du} \delta^+ |\phi_1(s-\zeta(s)) - \varphi_1(s-\zeta(s))| ds \\
\leq& \frac{\delta^+}{\beta^-}\|\phi-\varphi\|.
\end{aligned} \tag{25}
$$

Thus

$$
\begin{aligned}
&\|\Gamma\phi - \Gamma\varphi\| \\
\leq& \max\bigg\{ \frac{1}{\lambda^-}\bigg(\frac{a^+}{e^2} + \alpha^+(M_1+M_2)\bigg), \frac{\delta^+}{\beta^-}\bigg\}\|\phi-\varphi\|,
\end{aligned}
$$

which together with (H4) implies that the operator Γ is a contraction, hence Γ has a unique fixed point $\psi^* \in L^*$, namely $\Gamma\psi^* = \psi^*$. Consequently, ψ^* stands as a unique almost periodic solution of (2) and (3) within L^*. The proof is now concluded. ■

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