Least Squares Estimation for Uncertain Delay Differential Equations Based on Implicit Euler Scheme

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***Abstract*—Uncertain delay differential equations (UDDEs) are typically employed to portray systems affected by the Liu process with delays. This study centers on addressing the issue of estimating parameters for UDDEs from discrete observations. Initially, we establish the discrete form of UDDEs by utilizing the implicit Euler scheme. Subsequently, we outline the equations for acquiring parameter estimations for both the drift component and diffusion component. Lastly, we offer illustrative instances to showcase the approaches proposed in this manuscript.**

***Index Terms*—UDDEs; parameter estimation; Liu process; difference equation; implicit Euler scheme**

I. Introduction

To estimate the parameters is very important for modeling the stochastic models and many scholars devoted to study this problem. For instance, Zhang et al. ([26]) proposed a numerical method to identify the topology and estimate line parameters without the information of voltage angles. Maldonado et al. ([19]) used sequential Bayesian method to estimate the parameter in stochastic dynamic load models. Zhang et al. ([27]) studied the joint estimation of states and parameters of a special class of nonlinear bilinear systems. Ji and Kang ([11]) investigated new estimation methods for on-line parameter estimation for a class of nonlinear systems. Escobar et al. ([8]) offered several strategies to address the issue of parameter estimation in stochastic systems operating in continuous time. Ding ([7]) explored the properties of the least squares methods and the multi-innovation least squares methods. These strategies effectively consider both white and colored noise perturbations and employ traditional methodologies commonly used in this field. Shin and Park ([21]) applied generator-regularized continuous conditional generative adversarial network to estimate uncertain parameters. Amorino et al. ([1]) proposed a contrast function based on a pseudo likelihood approach and estimated the parameter for drift and diffusion coefficients of a stochastic McKean-Vlasov equation. Mehmood and Raja ([20]) investigated in evolutionary heuristics of weighted differential evolution to estimate the parameters of Hammerstein-Wiener model along with comparative evaluation from state-of-the-art counterparts. Brusa et al. ([5]) presented an evolutionary optimization approach to facilitate the process of maximum likelihood and approximate maximum likelihood estimation for discrete latent variable models. In practice, due to the uncertain communication environment such as population dynamics with a time lag, time delay is always unavoidable. Parameter estimation for stochastic delay differential equations have attracted an increasing interest during past few decades. Berezansky and Braverman ([4]) discussed the solution estimates for linear differential equations with delay. Benke and Pap ([3]) studied the weak convergence of the maximum likelihood estimator. Liu and Jia ([16]) applied the method of moments to estimate the parameters based on discrete observations of solutions. Zhu et al. ([28]) investigated the parameter identification of a reaction-diffusion rumor propagation system with time delay. Jamilla et al. ([10]) apply genetic algorithm with multi-parent crossover to obtain parameter estimates of three neutral delay differential equation models with a discrete delay.

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Stochastic differential equations may fail to model many time-varying systems such as stock prices. Therefore, the uncertainty theory was created by Liu ([14]) and perfected by Liu ([15]) based on the normality, duality, subadditivity and product axioms. Liu process is the uncertain process for dealing with dynamic systems in uncertain environments. In recent years, parameter estimation for uncertain differential equations (UDEs) has been discussed in some literature. For example, Li et al. ([12]) provided three methods to estimate the parameters in UDEs based on discrete observation data. Chen et al. ([6]) used the method of moments to estimate parameters of uncertain SIR model and designed a numerical algorithm to solve them. Liu ([17]) utilized generalized moment estimation to obtain the estimators. Yang et al. ([24]) applied α-path approach to obtain the estimators. Liu and Yang ([18]) proposed moment estimations for unknown parameters by Euler method approximation of high-order UDEs. Wei ([22]) applied contrast function to obtain the least squares estimators of uncertain Vasicek model and analyzed the consistency and asymptotic distribution. Wei ([23]) studied the parameter estimation for Ornstein-Uhlenbeck process driven by Liu process. Ye and Liu ([25]) proposed a method to test whether an uncertain differential equation fits the observed data or not. He et al. ([9]) derived an algorithm of parameter estimation for a special uncertain fractional differential equation. Li and Xia ([13]) proposed one novel estimation method named the estimating function technique of uncertain differential equations based on uncertain integrals.

Although the problem of parameter estimation for UDEs has been developed in recent years, the time lag factor has been considered in few literature. Moreover, the method used in literature to derive the difference equation is explicit difference method. However, this numerical method is numerically unstable. Motivated by the above considerations, in this paper, we study the parameter estimation for UDDEs based on implicit Euler scheme from discrete observations. We give the difference equation of UDDEs by using the implicit Euler scheme. Then, we provide the equations which can obtain the parameter estimators both for drift item and diffusion item and provide some examples to illustrate the methods used in this paper. The structure of this paper is organized as follows. Section 2 gives some definitions about uncertain variables and Liu process. Section 3 introduces the UDDEs considered in this paper and presents the equation applied to obtain the estimators. Some numerical examples are provided to illustrate the effective of the methods in Section 4. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Firstly, we give some definitions about uncertain variables and Liu process.

Definition 1: ([14], [15]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1: (Normality Axiom) $\mathcal{M}(\Gamma) = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$,

$$\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\Pi_{k=1}^{\infty} \Lambda_k\} = \min_{k \geq 1} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$.

An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

Definition 2: ([14]) For any real number x, let ξ be an uncertain variable and its uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}(\xi \leq x).$$

In particular, an uncertain variable ξ is called normal if it has an uncertainty distribution

$$\Phi(x) = (1 + \exp(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}))^{-1}, x \in \Re,$$

denoted by $\mathcal{N}(\mu, \sigma)$. If $\mu = 0$, $\sigma = 1$, ξ is called a standard normal uncertain variable.

Definition 3: ([15]) An uncertain process C_t is called a Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous, (ii) C_t has stationary and independent increments, (iii) the increment $C_{s+t} - C_s$ has a normal uncertainty distribution

$$\Phi_t(x) = (1 + \exp(\frac{-\pi x}{\sqrt{3}t}))^{-1}, x \in \Re.$$

Definition 4: ([2]) Suppose that C_t is a Liu process, h and w are two measurable real functions, τ stands for a nonnegative time delay. Then

$$dX_t = h(t, X_t, X_{t-\tau})dt + w(t, X_t, X_{t-\tau})dC_t \tag{1}$$

is called an uncertain delay differential equation.

Moreover, a real-valued function X_t^α is called the α-path of above uncertain differential equation if it solves the corresponding ordinary differential equation

$$dX_t^\alpha = h(t, X_t^\alpha, X_{t-\tau}^\alpha)dt + |w(t, X_t^\alpha, X_{t-\tau}^\alpha)|\Phi^{-1}(\alpha)dt,$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad \alpha \in (0,1).$$

Remark 1: The uncertain delay differential equation (1) has a unique solution if the coefficients $h(t,x,y)$ and $w(t,x,y)$ satisfy the following conditions

$$|h(t,x,y)| + |w(t,x,y)| \leq L(1 + |x| + |y|),$$

$$\begin{aligned}&|h(t,x,y) - h(t,x_1,y_1)| + |w(t,x,y) - w(t,x_1,y_1)| \\ &\leq L(|x - x_1| + |y - y_1|).\end{aligned}$$

III. Main Results and Proofs

The UDDEs considered in this paper is described as follows:

$$dX_t = h(t, X_t, X_{t-\tau}, \theta)dt + w(t, X_t, X_{t-\tau}, \beta)dC_t, \tag{2}$$

where θ and β are an unknown parameters, C_t is a Liu process and τ is a given delay time.

By applying the implicit Euler scheme, the Eq. (2) has the following difference form

$$\begin{aligned}X_{t_{i+1}} - X_{t_i} &= h(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}-\tau}, \theta)(t_{i+1} - t_i) \\ &\quad + w(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}-\tau}, \beta)(C_{t_{i+1}} - C_{t_i}),\end{aligned}$$

namely

$$\begin{aligned}&X_{t_{i+1}} - X_{t_i} - h(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}-\tau}, \theta)(t_{i+1} - t_i) \\ &= w(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}-\tau}, \beta)(C_{t_{i+1}} - C_{t_i}).\end{aligned} \tag{3}$$

As the diffusion term of Eq. (2) is usually regarded as the noise, the right term of Eq. (4) should be as small as possible. Then, given the observed data (t_i, x_{t_i}), $i = 1, 2, \cdots, n$. We can derive the estimator of θ by solving the optimization problem as follows:

$$\min \sum_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - h(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1}-\tau}, \theta)(t_{i+1} - t_i))^2. \tag{4}$$

Thus, the estimator of β can be obtained by solving the following equation

$$\begin{aligned}&\mathbb{E}[\sum_{i=1}^{n-1} (w(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1}-\tau}, \beta)(C_{t_{i+1}} - C_{t_i}))^2] \\ &= \sum_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - h(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1}-\tau}, \widehat{\theta}) \\ &(t_{i+1} - t_i))^2,\end{aligned}$$

where $\widehat{\theta}$ is the estimator of θ.

Since $\mathbb{E}[C_{t_{i+1}} - C_{t_i}] = 0$ and $Var[C_{t_{i+1}} - C_{t_i}] = (t_{i+1} - t_i)^2$, we have

$$\begin{aligned}
&\mathbb{E}[\sum_{i=1}^{n-1}(w(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1-\tau}}, \beta)(C_{t_{i+1}} - C_{t_i}))^2] \\
&= \sum_{i=1}^{n-1}\mathbb{E}[(w(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1-\tau}}, \beta)(C_{t_{i+1}} - C_{t_i}))^2] \\
&= \sum_{i=1}^{n-1} w^2(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1-\tau}}, \beta)\mathbb{E}[(C_{t_{i+1}} - C_{t_i})^2] \\
&= \sum_{i=1}^{n-1} w^2(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1-\tau}}, \beta)(t_{i+1} - t_i)^2.
\end{aligned}$$

Then, we can obtain the estimator of β by solving the equation

$$\begin{aligned}
&\sum_{i=1}^{n-1} w^2(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1-\tau}}, \beta)(t_{i+1} - t_i)^2 \\
&= \sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} - h(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1-\tau}}, \widehat{\theta})(t_{i+1} - t_i))^2
\end{aligned}$$

IV. Example

Example 1: Consider the following uncertain delay differential equation:

$$dX_t = \theta dt + \beta X_{t-0.5}dC_t,$$

where θ and β are an unknown parameters. Given the observed data (t_i, x_{t_i}), $i = 1, 2, \cdots, n$ in which $t_{i+1} - t_i = 0.5$. By solving the optimization problem

$$\min \sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} - \theta(t_{i+1} - t_i))^2,$$

we obtain the estimator of θ

$$\begin{aligned}
\widehat{\theta} &= \frac{\sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i})(t_{i+1} - t_i)}{\sum_{i=1}^{n-1}(t_{i+1} - t_i)^2} \\
&= \frac{2}{n-1}\sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i}) = \frac{2}{n-1}(x_{t_n} - x_{t_1}).
\end{aligned}$$

Then, according to Eq. (7), we can get the estimator of β

$$\widehat{\beta} = \sqrt{\frac{4\sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} - \frac{1}{n-1}(x_{t_n} - x_{t_1}))^2}{\sum_{i=1}^{n-1} x_{t_i}^2}}.$$

Assume that we have 20 groups of observed data as shown in Table 1. Then, we derive the least squares estimators

$$\widehat{\theta} = 2.0926, \quad \widehat{\beta} = 0.0354.$$

Thus, the uncertain delay differential equation could be written as

$$dX_t = 2.0926dt + 0.0354X_{t-0.5}dC_t.$$

Hence, the γ-path X_t^γ $(0 < \gamma < 1)$ is the solution of following ordinary differential equation

$$dX_t^\gamma = 2.0926dt + 0.0354X_{t-0.5}^\gamma \frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}dt.$$

TABLE I
Observations of uncertain delay differential equation

n	1	2	3	4	5	6	7	8	9	10
t_i	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50
X_{t_i}	0.53	1.20	2.45	3.91	4.32	5.41	6.58	7.86	8.37	9.25
n	11	12	13	14	15	16	17	18	19	20
t_i	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.00	10.50
X_{t_i}	10.43	11.76	12.59	13.92	14.18	15.06	17.39	18.71	19.63	20.41

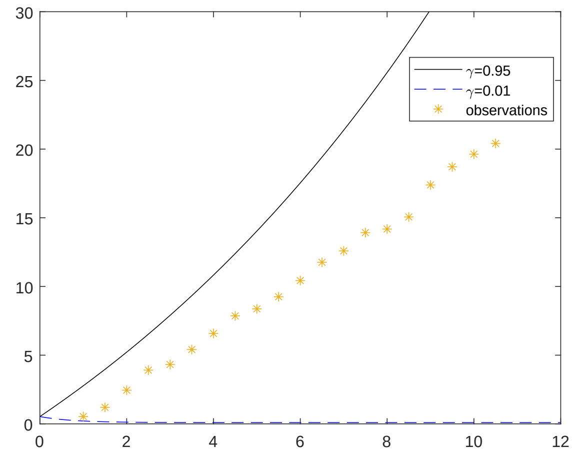

Fig. 1. Observations and γ-path of X_t

According to Figure 1, all observations fall into the area between 0.01-path $X_t^{0.01}$ and 0.95-path $X_t^{0.95}$. Therefore, the methods used in this paper are reasonable.

Example 2: Consider the following uncertain delay differential equation:

$$dX_t = \theta X_{t-0.2}dt + \beta X_t dC_t,$$

where θ and β are an unknown parameters. Given the observed data (t_i, x_{t_i}), $i = 1, 2, \cdots, n$ in which $t_{i+1} - t_i = 0.2$. By solving the optimization problem

$$\min \sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} - \theta x_{t_i}(t_{i+1} - t_i))^2,$$

we obtain the estimator of θ

$$\begin{aligned}
\widehat{\theta} &= \frac{\sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i})x_{t_i}(t_{i+1} - t_i)}{\sum_{i=1}^{n-1}(x_{t_i}(t_{i+1} - t_i))^2} \\
&= \frac{\sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i})x_{t_i}}{5\sum_{i=1}^{n-1}(x_{t_i})^2}.
\end{aligned}$$

Assume that we have 20 groups of observed data as shown in Table 2. Then, we derive the least squares estimators

$$\widehat{\theta} = 0.0154, \quad \widehat{\beta} = 0.3944.$$

Thus, the uncertain delay differential equation could be written as

$$dX_t = 0.0154X_{t-0.2}dt + 0.3944X_t dC_t.$$

Hence, the γ-path X_t^γ $(0 < \gamma < 1)$ is the solution of following ordinary differential equation

$$dX_t^\gamma = 0.0154X_{t-0.2}^\gamma dt + 0.3944X_t^\gamma \frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}dt.$$

TABLE II
OBSERVATIONS OF UNCERTAIN DELAY DIFFERENTIAL EQUATION

n	1	2	3	4	5	6	7	8	9	10
t_i	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80	2.00
X_{t_i}	0.17	0.69	1.53	1.98	2.76	3.25	3.81	4.32	5.07	5.83
n	11	12	13	14	15	16	17	18	19	20
t_i	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
X_{t_i}	6.34	7.18	7.92	8.51	9.13	9.64	10.35	11.17	12.36	13.18

According to Figure 2, all observations fall into the area between 0.15-path $X_t^{0.15}$ and 0.83-path $X_t^{0.83}$. Therefore, the methods used in this paper are reasonable.

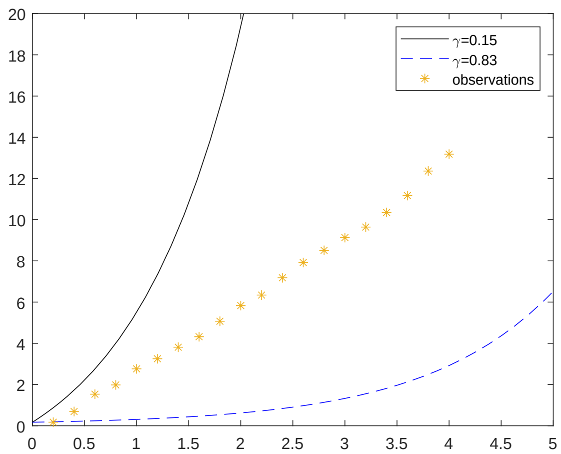

Fig. 2. Observations and γ-path of X_t

Example 3: Consider the following uncertain delay differential equation:

$$dX_t = (\theta_1 X_t + \theta_2 X_{t-0.3})dt + \beta X_t dC_t,$$

where θ_1, θ_2 and β are an unknown parameters. Given the observed data (t_i, x_{t_i}), $i = 1, 2, \cdots, n$ in which $t_{i+1} - t_i = 0.3$.

Assume that we have 20 groups of observed data as shown in Table 3. Then, we derive the least squares estimators

$$\widehat{\theta}_1 = 3.3333, \quad \widehat{\theta}_2 = -10.8436, \quad \widehat{\beta} = 0.9437.$$

Thus, the uncertain delay differential equation could be written as

$$dX_t = (3.3333X_t - 10.844X_{t-0.3})dt + 0.9437X_t dC_t,$$

Hence, the γ-path X_t^γ $(0 < \gamma < 1)$ is the solution of following ordinary differential equation

$$\begin{aligned} dX_t^\gamma &= (3.3333X_t^\gamma - 10.8436X_{t-0.3}^\gamma)dt \\ &\quad +0.9437X_t^\gamma \frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}dt. \end{aligned}$$

TABLE III
OBSERVATIONS OF UNCERTAIN DELAY DIFFERENTIAL EQUATION

n	1	2	3	4	5	6	7	8	9	10
t_i	0.30	0.60	0.90	1.20	1.50	1.80	2.10	2.40	2.70	3.00
X_{t_i}	0.19	1.58	3.69	6.12	4.35	8.01	6.52	9.72	3.83	7.54
n	11	12	13	14	15	16	17	18	19	20
t_i	3.30	3.60	3.90	4.20	4.50	4.80	5.10	5.40	5.70	6.00
X_{t_i}	10.23	15.18	11.29	6.27	12.39	9.37	17.53	11.89	15.61	18.36

According to Figure 3, all observations fall into the area between 0.05-path $X_t^{0.05}$ and 0.96-path $X_t^{0.96}$. Therefore, the methods used in this paper are reasonable.

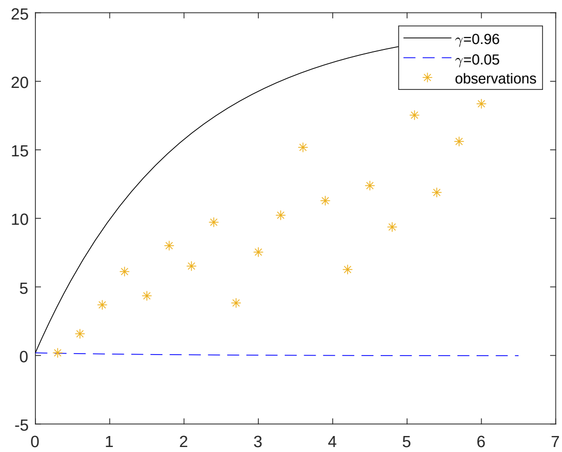

Fig. 3. Observations and γ-path of X_t

Remark 2: If the time delay in equation (2) is unknown, equation (3) can be rewritten as the following approximation equation by using Taylor expansion

$$\begin{aligned} X_{t_{i+1}} - X_{t_i} &= h(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}} - \tau\frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \theta) \\ &\quad (t_{i+1} - t_i) \\ &\quad +w(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}} - \tau\frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \beta) \\ &\quad (C_{t_{i+1}} - C_{t_i}). \end{aligned} \tag{6}$$

We can derive the estimator of θ by solving the optimization problem as follows:

$$\min \sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} - h(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}} \\ -\tau\frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \theta)(t_{i+1} - t_i))^2.$$

Then, the estimator of β can be obtained by solving the equation

$$\sum_{i=1}^{n-1} w^2(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}} - \tau\frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \beta) \\ (t_{i+1} - t_i)^2 \\ = \sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} \\ -h(t_{i+1}, X_{t_{i+1}}, X_{t_{i+1}} - \tau\frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \theta)(t_{i+1} - t_i))^2.$$

Example 4: Consider the following uncertain delay differential equation:

$$dX_t = \theta dt + X_{t-\tau}dC_t,$$

where θ and τ are an unknown parameters. Given the observed data (t_i, x_{t_i}), $i = 1, 2, \cdots, n$ in which $t_{i+1} - t_i = 1$. By solving the optimization problem

$$\min \sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} - \theta(t_{i+1} - t_i))^2,$$

we obtain the estimator of θ

$$\widehat{\theta} = \frac{\sum_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i})(t_{i+1} - t_i)}{\sum_{i=1}^{n-1}(t_{i+1} - t_i)^2}.$$

Assume that we have 20 groups of observed data as shown in Table 4. Then, we derive the least squares estimators

$$\widehat{\theta} = 0.7258, \quad \widehat{\tau} = 0.5472.$$

Thus, the uncertain delay differential equation could be written as

$$dX_t = 0.7258dt + X_{t-0.5472}dC_t.$$

Hence, the γ-path X_t^γ $(0 < \gamma < 1)$ is the solution of following ordinary differential equation

$$dX_t^\gamma = 0.7258dt + X_{t-0.5472}^\gamma \frac{\sqrt{3}}{\pi} \ln \frac{\gamma}{1-\gamma} dt.$$

According to Figure 4, all observations fall into the area between 0.15-path $X_t^{0.15}$ and 0.85-path $X_t^{0.85}$. Therefore, the methods used in this paper are reasonable.

V. Conclusions

In this paper, we have studied the problem of parameter estimation for UDDEs based on implicit Euler scheme from discrete observations. We have obtained the difference equation of UDDEs by using the implicit Euler scheme. Then, we have derived the equations which can obtain the parameter estimators both for drift item and diffusion item. Moreover, we have provided some examples to illustrate the methods used in this paper. We will consider the parameter estimation for partially observed UDDEs in future works.

TABLE IV
Observations of Uncertain Delay Differential Equation

n	1	2	3	4	5	6	7	8	9	10
t_i	1	2	3	4	5	6	7	8	9	10
X_{t_i}	1.28	3.65	5.17	8.75	3.39	7.11	9.27	10.94	12.81	6.38
n	11	12	13	14	15	16	17	18	19	20
t_i	11	12	13	14	15	16	17	18	19	20
X_{t_i}	10.39	14.28	11.81	8.45	12.97	7.12	15.68	11.25	18.61	15.07

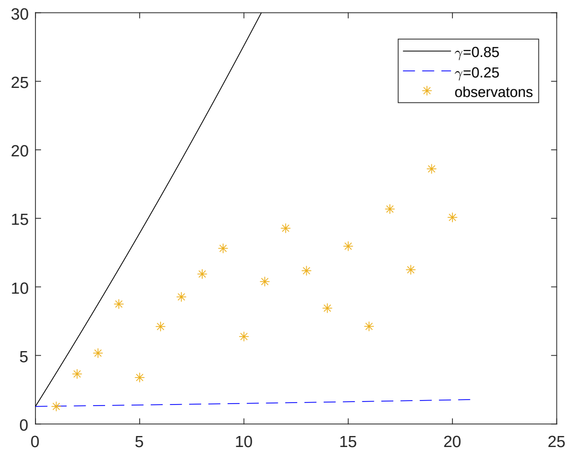

Fig. 4. Observations and γ-path of X_t

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