Delta Alpha Integration In Discrete Fractional Calculus

Saraswathi D, Britto Antony Xavier G, Abisha M and Abirami V

Abstract—**The objective of this article is to develop the theory of discrete version of the fundamental theorem on $\alpha(\ell)$-delta integration, utilizing ∞-order $\alpha(\ell)$ integrable functions. This theory is subsequently applied to establish several fundamental theorems and illustrate examples concerning fractional order sums in the field of discrete fractional calculus.**

Index Terms—**Delta integrable function, Discrete Delta integration, Fractional sum, Closed form, Summation form, Numerical analysis.**

I. Introduction

OVER the past few decades, there has been substantial attention in the literature of discrete fractional calculus [1], [2], [3], [6], [9], [13], [18]. The authors concentrated on establishing precise definitions for discrete fractional differential equations and developing effective solution methodologies. To achieve this, various researchers rigorously investigated the commutative properties of fractional sum and difference operators. The study introduced a fractional difference equation of order ν-th order $(0 < \nu \leq 1)$, analyzed a nonlinear problem with an initial condition, and provided a linear constant-coefficient problem as an illustrative example [1].

In [2], the authors have paid attention in finite differences of fractional order, $\Delta^\alpha f$ which provides a description of differences of arbitrary order and computes them for numerous specific functions. The researchers developed a novel discrete transform method by extending the discrete Laplace transform. The authors established many properties, including a comprehensive exponential law and the critical Leibniz rule, which is used to solve 2^{nd}-order linear differential equations [4], [5].

In [7], the authours explores solutions to difference equations using elementary analysis and linear algebra. Readers may have encountered difference equations in various contexts, such as Newton's Method approximation, discretization, computation, combinatorics, and modeling economic or biological phenomena. In [8], the topics covered by differential equations, numerical analysis, discrete modeling, combinatorics and numerical methods. The wide range of uses to various mathematical subfields is a distinguishing feature of this iteration. In this study [10], [11], the researchers explained the generalized difference operator of the nth kind, represented by Δ_ℓ, where $\ell = \ell_1, \ell_2, \ldots, \ell_n$. They get some interesting findings. In number theory, formulas for the sum of universal partial sums of products of successive terms of an arithmetic progression are developed.

This article [12] discusses and provides solutions to the second order generalized difference problem. They demonstrate that there is no non-trivial solution for the given condition. They provide few formulas and examples respectively. In [14], the author delved into the theoretical and practical applications of computer technologies for modeling nonlinear systems. The research encompassed a variety of computational techniques, including high-precision operator approximation and innovative non-Lagrange interpolation methods.

The work [15] proposes fractional central differences and derivatives. The integer order derivatives and differences are extended to real orders, resulting in two new types of differences and derivatives. For each type, an appropriate integral formulation is obtained. This book [16] offers a comprehensive preliminaries to fractional differential equations and fractional derivatives, covering essential special functions, foundational theory, existence and uniqueness proofs, and analytical and numerical solution methods. Additionally, it presents a range of practical applications.

The goal of this study in [17] is to develop discrete fundamental theorems for delta integrable functions using a novel mechanism known as the delta integration method. The ν^{th} fractional sum of a function f has both forms, which is summation and closed form. Additionally, which is extended to h-delta sum and integration. Finally, they assess their findings using diagrams of falling factorial, geometric and polynomial functions. Also authors in [19] developed ℓ-nabla integration of f and discrete fractional integration for factorials and geometric functions.

In this article, we have extended this concept to $\alpha(\ell)$-delta integration and its sum. Here, we derive several fundamental theorems using $\alpha(\ell)$-delta operator. Also, we validated our findings with suitable numerical examples.

Notations I.1. For $a \in \mathbb{R} = (-\infty, \infty)$, we denote $a + \mathbb{Z}_\ell = \{a, a \pm \ell, a \pm 2\ell, ...\}$ throughout this paper, we take $\ell > 0$. Also, $a + \mathbb{Z}_1 = \{a, a \pm 1, a \pm 2, ...\}$, $0 + \mathbb{Z}_1 = \mathbb{Z}$.

Manuscript received October 22, 2024; revised January 26, 2025. This work was supported in part by the Sacred Heart College for the Research grants for Carreno Grant Fellowships (SHC/Fr.Carreno Research Grant/2023/04), Sacred Heart Fellowship (SHC/SHFellowship/2022/14), Don Bosco Research Grant (SHC/DB Grant/2024/03) and DST for the FIST Fund (SR/FST/College-2017/130(c)).

Saraswathi D is a research scholar at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: dswathisaranya@gmail.com)

Britto Antony Xavier G is an Associate Professor at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (Corresponding Author e-mail: brittoshc@gmail.com)

Abisha M is a research scholar at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: mabisha1996@gmail.com)

Abirami V is a PG student at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. (e-mail: abiramidvs2221@gmail.com)

II. Preliminaries Related To $\alpha(\ell)$ - Delta Integration

In this section, we present the definitions related to ℓ-falling factorials, the $\alpha(\ell)$-delta operator and its anti difference delta operator on real valued function, and a summation formula arrived via the anti difference $\alpha(\ell)$-delta operator.

Definition II.1. Let $f : a+\mathbb{Z}_\ell \to \mathbb{R}$, where $\ell > 0$ and $a+\mathbb{Z}_\ell = \{a, a\pm\ell, a\pm 2\ell, ...\}$. The $\alpha(\ell)$-delta operator on f is defined as

$$\Delta_{\alpha(\ell)} f(\zeta) = f(\zeta+\ell) - \alpha f(\zeta), \zeta \in a+\mathbb{Z}_\ell. \tag{1}$$

The inverse of $\alpha(\ell)$-delta operator $\Delta^{-1}_{\alpha(\ell)}$ on f is defined by

$$\Delta^{-1}_{\alpha(\ell)} f = f_1(\zeta) + c, \tag{2}$$

where $f_1 : a+\mathbb{Z}_\ell \to \mathbb{R}$ such that $\Delta_{\alpha(\ell)} f_1(\zeta) = f(\zeta)$ and c is an arbitrary constant.

Definition II.2. [17] For $n \in \mathbb{N} = \{1,2,3,\ldots\}$, the n^{th}-order ℓ-falling factorial of ζ, $\zeta_\ell^{(n)}$ is defined by

$$\zeta_\ell^{(n)} = \prod_{r=0}^{n-1}(\zeta - r\ell) \textit{ and } \zeta_\ell^{(0)} = 1, \zeta \in \mathbb{R}.$$

For ζ, $\nu \in \mathbb{R}$ and $\ell > 0$, the ν^{th}-order ℓ-falling factorial is defined by

$$\zeta_\ell^{(\nu)} = \frac{\Gamma\left(\frac{\zeta}{\ell}+1\right)}{\Gamma\left(\frac{\zeta}{\ell}-\nu+1\right)}\ell^\nu, \frac{\zeta}{\ell}-\nu+1 \notin \{0,-1,-2,-3,...\}. \tag{3}$$

Lemma II.3. *[17] The Gamma function satisfies the relation*

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1), \alpha \in \mathbb{R} - \{0,-1,-2,...\}. \tag{4}$$

Lemma II.4. *Let $\zeta \in (-\infty,\infty)$, $f(\zeta) = \zeta_\ell^{(n)}$ and $n > 0$. Then,*

$$\Delta_{\alpha(\ell)}\zeta_\ell^{(n)} = [(\zeta+\ell) - \alpha(\zeta-(n-1)\ell)]\zeta_\ell^{(n-1)}. \tag{5}$$

Proof: From the definition of $\alpha(\ell)$-delta operator,

$$\begin{aligned}\Delta_{\alpha(\ell)}\zeta_\ell^{(n)} &= (\zeta+\ell)_\ell^{(n)} - \alpha\zeta_\ell^{(n)}\\ &= (\zeta+\ell)\zeta(\zeta-\ell)\ldots(\zeta-(n-2)\ell)\\ &\quad -\alpha(\zeta)(\zeta-\ell)\ldots(\zeta-(n-2)\ell)(\zeta-(n-1)\ell)\\ &= \zeta_\ell^{(n-1)}[(\zeta+\ell)-\alpha(\zeta-(n-1)\ell)],\end{aligned}$$

which gives (5). ■

Theorem II.5. *If $\Delta_{\alpha(\ell)} f_1(\zeta) = f(\zeta)$ and $m = \frac{\zeta-a}{\ell} \in \mathbb{N}$ for $\zeta \in a+\mathbb{Z}_\ell$, then*

$$f_1(\zeta+\ell) - \alpha^m f_1(a+\ell) = \sum_{s=1}^{m}\alpha^{m-s} f(a+s\ell). \tag{6}$$

Proof: From the given hypothesis, $\Delta_{\alpha(\ell)} f_1(\zeta) = f(\zeta)$, and the equation (1), we get

$$f_1(\zeta+\ell) = f(\zeta) + \alpha f_1(\zeta). \tag{7}$$

By finding the values of $f_1(\zeta - r\ell)$, $r = 0,1,2,\ldots,m$ by replacing ζ by $\zeta - r\ell$, $r = 1,2,\ldots,m$ respectively in equation (7), and then substituting again in (7), we arrive following steps:

$$f_1(\zeta+\ell) = f(\zeta) + \alpha f(\zeta-\ell) + \alpha^2 f_1(\zeta-\ell)$$

$$f_1(\zeta+\ell) = f(\zeta) + \alpha f(\zeta-\ell) + \alpha^2 f(\zeta-2\ell) + \alpha^3 f_1(\zeta-2\ell),$$

$$f_1(\zeta+\ell) = f(\zeta) + \alpha f(\zeta-\ell) + \alpha^2 f(\zeta-2\ell) + \cdots + \alpha^4 f_1(\zeta-3\ell). \tag{8}$$

Proceeding this upto m-steps, equation (8) yields

$$f_1(\zeta+\ell) - \alpha^m f_1(\zeta-(m-1)\ell) = \sum_{k=1}^{m}\alpha^{k-1} f(\zeta-(k-1)\ell). \tag{9}$$

Now, equation (6) follows by taking $\zeta - m\ell = a$, in (9). ■

The following corollary motivates us to develop integer order $\alpha(\ell)$ delta integration of certain function.

Corollary II.6. *Let $\zeta \in a+\mathbb{Z}_\ell$, $\Delta^{-1}_{\alpha(\ell)} f(\zeta) = f_1(\zeta)$ and $m = \frac{\zeta-a}{\ell} \in N$. Then,*

$$\Delta^{-1}_{\alpha(\ell)} f(\zeta) - \alpha^m \Delta^{-1}_{\alpha(\ell)} f(a) = \sum_{s=0}^{m-1}\alpha^{m-1-s} f(a+s\ell). \tag{10}$$

Proof: This corollary is proved by taking $\Delta^{-1}_{\alpha(\ell)} f(\zeta) = f_1(\zeta)$ and replacing ζ by $\zeta - \ell$ in Theorem II.5. ■

III. Integer Order $\alpha(\ell)$ - Delta Integration

The relation (10) is a basic theorem of $\alpha(\ell)$-delta integration. The expressions (6) and (9) are said to be the 1^{st}-order discerete $\alpha(\ell)$-delta integration of f. Here, we obtain an important theorem for integer order $\alpha(\ell)$-delta integration. This is a generalization of the relation (10).

Definition III.1. Let $f : a+\mathbb{Z}_\ell \to \mathbb{R}$ be an m^{th}-order $\alpha(\ell)$-delta integrable function and if there exists a sequence of functions, $(f_1, f_2, ..., f_m)$ such that

$$\Delta^k_{\alpha(\ell)} f_k = f, k = 1,2,3,...,m. \tag{11}$$

Then $(f_1, f_2, ..., f_m)$ is said to be $\alpha(\ell)$-delta integrating sequence of f.

Example III.2. *(i) Let $f(\zeta) = 2^\zeta$, $\zeta \in \mathbb{R}$, is an m^{th}-order $\alpha(\ell)$-delta integrable function having integrating sequence $\left(\frac{2^\zeta}{2^\ell-\alpha}, \frac{2^\zeta}{(2^\ell-\alpha)^2}, ..., \frac{2^t}{(2^\ell-\alpha)^m}\right)$, Since*

$$f(\zeta) = 2^\zeta = \frac{\Delta_{\alpha(\ell)}2^\zeta}{2^\ell-\alpha} = \frac{\Delta^2_{\alpha(\ell)}2^\zeta}{(2^\ell-\alpha)^2} = ... = \frac{\Delta^m_{\alpha(\ell)}2^\zeta}{(2^\ell-\alpha)^m}, \tag{12}$$

$\forall\ m \in \mathbb{N}$.

(ii) Let $f(\zeta) = c^\zeta$, $c \neq 1$, $\zeta \in \mathbb{R} = J_1$ is m^{th}-order $\alpha(\ell)$-delta integrable function having integrating sequence $\left(\frac{c^\zeta}{c^\ell-\alpha}, \frac{c^\zeta}{(c^\ell-\alpha)^2}, ..., \frac{c^\zeta}{(c^\ell-\alpha)^m}\right)$, since

$$f(\zeta) = c^\zeta = \frac{\Delta_{\alpha(\ell)}c^\zeta}{c^\ell-\alpha} = \frac{\Delta^2_{\alpha(\ell)}c^\zeta}{(c^\ell-\alpha)^2} = ... = \frac{\Delta^m_{\alpha(\ell)}c^\zeta}{(c^\ell-\alpha)^m}, \tag{13}$$

$\forall\ m \in \mathbb{N}$.

For $\alpha = 1$, the delta integrable functions are clearly mentioned in [17]

Definition III.3. Let $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ be an $\alpha(\ell)$-delta integrable function having $\alpha(\ell)$-delta integrating sequence $(f_1, f_2, ..., f_n)$. If $\zeta \in a + \mathbb{Z}_\ell$, $m = \dfrac{\zeta - a}{\ell}$ and $n \in \mathbb{N}$, then n^{th}-order $\alpha(\ell)$-delta integration of f based at a is defined by

$$F_a^n(\zeta) := f_n(\zeta) - \sum_{r=0}^{n-1} \alpha^{m-r} m^{(r)} \frac{f_{n-r}}{r!}(a). \tag{14}$$

From this, we have the following example for n^{th}-order $\alpha(\ell)$-delta integration.

Example III.4. *(i) Let $f(\zeta) = 2^\zeta$, where $\zeta \in a+\mathbb{Z}_\ell$. Then the n^{th}-order $\alpha(\ell)$-delta integration of 2^ζ based at a is defined as,*

$$F_a^n(\zeta) = f_n(\zeta) - [\alpha^m f_n(a) + \frac{m}{1!}\alpha^{m-1} f_{n-1}(a) + \ldots + \frac{m^{(n-1)}}{(n-1)!}\alpha^{m-(n-1)} f_1(a)]$$

$$F_a^n(\zeta) = \frac{2^\zeta}{(2^\ell - \alpha)^n} - \left[\frac{\alpha^m 2^a}{(2^\ell-\alpha)^n} + \frac{m\alpha^{m-1}2^a}{(1!)(2^\ell-\alpha)^{n-1}} + \ldots + \frac{m^{(n-1)}\alpha^{m-(n-1)}2^a}{((n-1)!)(2^\ell-\alpha)}\right]$$

$$F_a^n(\zeta) = \frac{2^\zeta}{(2^\ell - \alpha)^n} - \sum_{r=0}^{n-1} \frac{\alpha^{m-r} m^{(r)}}{r!} \frac{2^a}{(2^\ell-\alpha)^{n-r}}.$$

(ii) Let $f(\zeta) = b^\zeta$, where $\zeta \in a + \mathbb{Z}_\ell$ and $b^\ell \neq \alpha$. Then the n^{th}-order $\alpha(\ell)$-delta integration of b^ζ based at a as,

$$F_a^n(\zeta) = f_n(\zeta) - [\alpha^m f_n(a) + \frac{m}{1!}\alpha^{m-1} f_{n-1}(a) + \ldots + \frac{m^{(n-1)}}{(n-1)!}\alpha^{m-(n-1)} f_1(a)]$$

$$F_a^n(\zeta) = \frac{b^\zeta}{(b^\ell - \alpha)^n} - \left[\frac{\alpha^m b^a}{(b^\ell-\alpha)^n} + \frac{m\alpha^{m-1}b^a}{(1!)(b^\ell-\alpha)^{n-1}} + \ldots + \frac{m^{(n-1)}\alpha^{m-(n-1)}b^a}{((n-1)!)(b^\ell-\alpha)}\right]$$

$$F_a^n(\zeta) = \frac{b^\zeta}{(b^\ell - \alpha)^n} - \sum_{r=0}^{n-1} \frac{\alpha^{m-r} (m)^{(r)}}{r!} \frac{b^a}{(b^\ell-\alpha)^{n-r}}.$$

Theorem III.5. *Assume $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ having $\alpha(\ell)$-delta integrating sequence $(f_1, f_2, ..., f_n)$. If $\zeta \in a + \mathbb{Z}_\ell$, $m = \dfrac{\zeta - a}{\ell}$ and $n \in \mathbb{N}$ such that $m - n$ be a positive integer and $\Delta_{\alpha(\ell)}^{-n} f(\zeta)$ be the n^{th}-order $\alpha(\ell)$-delta integration of f based at a, then we have*

$$_a\Delta_{\alpha(\ell)}^{-n} f(\zeta) = \frac{1}{(n-1)!} \sum_{s=0}^{m-n} \alpha^{m-n-s} \left(m - (s+1)\right)^{(n-1)} f(a + s\ell). \tag{15}$$

Proof: The 1^{st}-order $\alpha(\ell)$-delta integration is proved from Corollary II.6.
Taking inverse delta operator $\Delta_{\alpha(\ell)}^{-1}$ on equation (10),

$$\Delta_{\alpha(\ell)}^{-2} f(\zeta) - \alpha^m \Delta_{\alpha(\ell)}^{-2} f(a) = \sum_{s=0}^{m-1} \alpha^{m-1-s} \Delta_{\alpha(\ell)}^{-1} f(a + s\ell)$$
$$= \Delta_{\alpha(\ell)}^{-1} f(a + (m-1)\ell) + \alpha\Delta_{\alpha(\ell)}^{-1} f(a + (m-2)\ell) + \cdots + \alpha^{m-2}\Delta_{\alpha(\ell)}^{-1} f(a + \ell) + \alpha^{m-1}\Delta_{\alpha(\ell)}^{-1} f(a)$$

Since $a + m\ell = \zeta$, we get

$$\Delta_{\alpha(\ell)}^{-2} f(\zeta) - \alpha^m \Delta_{\alpha(\ell)}^{-2} f(a) = \Delta_{\alpha(\ell)}^{-1} f(\zeta - \ell) + \alpha\Delta_{\alpha(\ell)}^{-1} f(\zeta - 2\ell) + \alpha^2\Delta_{\alpha(\ell)}^{-1} f(\zeta - 3\ell) + \cdots + \alpha^{m-2} f_1(\zeta - (m-1)\ell) + \alpha^{m-1} f_1(\zeta - m\ell)$$

Substituting Equation (10) in every term of RHS of above equation, which yields

$$\Delta_{\alpha(\ell)}^{-2} f(\zeta) - \alpha^m \Delta_{\alpha(\ell)}^{-2} f(a) = f(\zeta - 2\ell) + 2\alpha f(\zeta - 3\ell) + 3\alpha^2 f(\zeta - 4\ell) + 4\alpha^3 f(\zeta - 5\ell) + \cdots + (m-1)\alpha^{m-2} f(\zeta - m\ell) + m\alpha^{m-1} f_1(\zeta - m\ell). \tag{16}$$

$$\Delta_{\alpha(\ell)}^{-2} f(\zeta) - \alpha^m \Delta_{\alpha(\ell)}^{-2} f(a) - m\alpha^{m-1}\Delta_{\alpha(\ell)}^{-1} f(a) = f(a + (m-2)\ell) + 2\alpha f(a + (m-3)\ell) + \cdots + (m-1)\alpha^{m-2} f(a)$$

$$\Delta_{\alpha(\ell)}^{-2} f(\zeta) - \alpha^m \Delta_{\alpha(\ell)}^{-2} f(a) - m\alpha^{m-1}\Delta_{\alpha(\ell)}^{-1} f(a) = \sum_{s=0}^{m-2} \alpha^{m-2-s} \left(m - (s+1)\right) f(a + s\ell). \tag{17}$$

Equation (17) is the 2^{nd}-order $\alpha(\ell)$-delta integration formula. Applying $\Delta_{\alpha(\ell)}^{-1}$ operator on the Equation (16) and then continuing the same way, we get

$$\Delta_{\alpha(\ell)}^{-3} f(\zeta) - \alpha^m \Delta_{\alpha(\ell)}^{-3} f(a) - m\alpha^{m-1}\Delta_{\alpha(\ell)}^{-2} f(a) - \frac{m^{(2)}}{2!}\alpha^{m-2}\Delta_{\alpha(\ell)}^{-1} f(a) = \frac{1}{2!}\sum_{s=0}^{m-3} \alpha^{m-3-s} \left(m - (s+1)\right)^{(2)} f(a + s\ell). \tag{18}$$

Similarly applying the $\Delta_{\alpha(\ell)}^{-1}$ operator repeatedly, we arrive the $(n-1)^{th}$-order $\alpha(\ell)$-delta integration as

$$\Delta_{\alpha(\ell)}^{-(n-1)} f(\zeta) - \sum_{r=0}^{n-2} \alpha^{m-r} \frac{m^{(r)}}{r!} \Delta_{\alpha(\ell)}^{r-(n-1)} f(a) = \frac{1}{(n-2)!} \sum_{s=0}^{m-(n-1)} \alpha^{m-(n-1)-s} \left(m - (s+1)\right)^{(n-2)} f(a+s\ell).$$

Hence, again taking the $\Delta_{\alpha(\ell)}^{-1}$ operator on both sides of above equation, we can easily obtain the n^{th}-order $\alpha(\ell)$-delta integration

$$\Delta_{\alpha(\ell)}^{-(n)} f(\zeta) - \sum_{r=0}^{n-1} \alpha^{m-r} \frac{m^{(r)}}{r!} \Delta_{\alpha(\ell)}^{r-n} f(a) = \frac{1}{(n-1)!} \sum_{s=0}^{m-n} \alpha^{m-n-s} \left(m - (s+1)\right)^{(n-1)} f(a + s\ell)$$

On taking

$$_a\Delta_{\alpha(\ell)}^{-n} f(\zeta) = \Delta_{\alpha(\ell)}^{-(n)} f(\zeta) - \sum_{r=0}^{n-1} \alpha^{m-r} \frac{m^{(r)}}{r!} \Delta_{\alpha(\ell)}^{r-n}\ f(a),$$

we get Equation (15). ■

Theorem III.6. *Let $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ be a function having $\alpha(\ell)$-delta inegrating sequence $(f_1, f_2, ..., f_n)$. If $\zeta \in a+\mathbb{Z}_\ell$, $m = \dfrac{\zeta - a}{\ell}$ such that $m-n$ be a positive integer and $F_a^n(\zeta)$ be the n^{th}-order $\alpha(\ell)$-delta integration of f based at a as* *defined in* (14)*, then*

$$F_a^n(\zeta) := \frac{\sum_{s=0}^{m-n} \alpha^{m-n-s} (m-(s+1))^{(n-1)} f(a+s\ell)}{(n-1)!}. \tag{19}$$

Proof: The proof follows by mathematical induction method.
From II.5, we have

$$F_a^1(\zeta) := f_1(\zeta) - \alpha^m f_1(a) = \sum_{s=0}^{m-1} \alpha^{m-1-s} f(a+s\ell)$$

and hence Equation (19) is true for $n = 1$
Let us assume Equation (19) is true for $n-1$ case,

$$F_a^{n-1}(\zeta) := f_{n-1}(\zeta) - \sum_{r=0}^{n-2} \alpha^{m-r} \frac{m^{(r)}}{r!} f_{n-1-r}(a)$$
$$= \frac{\sum_{s=0}^{m-(n-1)} \alpha^{m-n+1-s} (m-(s+1))^{(n-2)} f(a+s\ell)}{(n-2)!}. \tag{20}$$

Now, we are going to prove Equation (20) is true for n^{th} case.
On taking $\frac{\zeta - a}{\ell} = m$, $a = \zeta - m\ell$ and expanding the Equation (20), we arrive on

$$\begin{aligned} f_{n-1}(\zeta) &- \alpha^m f_{n-1}(a) - \alpha^{m-1}\frac{m^{(1)}}{1!} f_{n-2}(a) - \cdots \\ &- \alpha^{m-(n-2)} \frac{m^{(n-2)}}{(n-2)!} f_1(a) \\ &= \frac{1}{(n-2)!}\Big[\alpha^{m-n+1}(m-1)^{(n-2)} f(\zeta - m\ell) \\ &+ \alpha^{m-n}(m-2)^{(n-2)} f(\zeta-(m-1)\ell) \\ &+ \cdots + (n-2)^{(n-2)} f(\zeta-(n-1)\ell)\Big] \end{aligned}$$

Applying $\Delta^{-1}_{\alpha(\ell)}$ in the above equation, we get

$$\begin{aligned} f_n(\zeta) &- \alpha^m f_n(a) - \alpha^{m-1}\frac{m^{(1)}}{1!} f_{n-1}(a) - \cdots \\ &- \alpha^{m-(n-2)} \frac{m^{(n-2)}}{(n-2)!} f_2(a) \\ &= \frac{1}{(n-2)!}\Big[(n-2)^{(n-2)} f_1(\zeta-(n-1)l) \\ &+ \alpha(n-1)^{(n-2)} f_1(\zeta - nl) + \cdots \\ &+ \alpha^{m-n}(m-2)^{(n-2)} f_1(\zeta-(m-1)\ell) \\ &+ \alpha^{m-n+1}(m-1)^{(n-2)} f_1(\zeta - m\ell)\Big] \end{aligned}$$

Applying Theorem II.5 in every term of RHS of above equation, which arrive

$$\begin{aligned} f_n(\zeta) &- \alpha^m f_n(a) - \alpha^{m-1}\frac{m^{(1)}}{1!} f_{n-1}(a) - \cdots \\ &- \alpha^{m-(n-2)} \frac{m^{(n-2)}}{(n-2)!} f_2(a) \\ = \frac{(n-2)^{(n-2)}}{(n-2)!} & f(\zeta - nl) + \alpha \frac{n^{(n-1)}}{(n-2)!(n-1)} f(\zeta-(n+1)\ell) \\ &+ \alpha^2 \frac{(n+1)^{(n-1)}}{(n-2)!(n-1)} f(\zeta-(n+2)\ell) + \cdots \\ &+ \alpha^{m-n} \frac{(m-1)^{(n-1)}}{(n-2)!(n-1)} f(\zeta - m\ell) \\ &+ \alpha^{m-n+1} \frac{m^{(n-1)}}{(n-2)!(n-1)} f_1(\zeta - m\ell) \end{aligned}$$

Now multipling and dividing $(n-1)!$ in 1^{st} term only of the right side, we get

$$\begin{aligned} f_n(\zeta) &- \alpha^m f_n(a) - \alpha^{m-1}\frac{m^{(1)}}{1!} f_{n-1}(a) - \cdots \\ &- \alpha^{m-(n-2)} \frac{m^{(n-2)}}{(n-2)!} f_2(a) - \alpha^{m-(n-1)} \frac{m^{(n-1)}}{(n-1)!} f_1(a) \\ &= \frac{(n-1)^{(n-1)}}{(n-1)!} f(\zeta - n\ell) + \alpha \frac{n^{(n-1)}}{(n-1)!} f(\zeta-(n+1)\ell) \\ &+ \alpha^2 \frac{(n+1)^{(n-1)}}{(n-1)!} f(\zeta-(n+2)\ell) + \cdots \\ &+ \alpha^{m-n} \frac{(m-1)^{(n-1)}}{(n-1)!} f(\zeta - m\ell) \end{aligned}$$

which implies on

$$f_n(\zeta) - \sum_{r=0}^{n-1} \alpha^{m-r} \frac{m^{(r)}}{r!} f_{n-r}(a)$$
$$= \frac{1}{(n-1)!} \sum_{s=0}^{m-n} \alpha^{m-n-s} (m-(s+1))^{(n-1)} f(a+s\ell)$$

From Equation (14), we get equation (19) and hence by the induction method the theorem is true for n^{th} case. ■

Example III.7. *Taking* $f(\zeta) = 2^\zeta$ *and* $a = 2$, $n = 3$ *in Equation* (19)*, we get*

$$F_a^3(\zeta) := \frac{1}{2!} \sum_{s=0}^{m-3} \alpha^{m-3-s} (m-(s+1))^{(2)} f(a+s\ell) \tag{21}$$

By taking $\ell = 1, \zeta = 5, \alpha = 1$ *in example III.4 and inserting* $m = \frac{5-2}{1} = 3$ *in* (21)*, we get*

$$F_a^3(\zeta) = \frac{2^5}{(2^1-1)^3} - \sum_{r=0}^{3-1} \frac{1^{3-r} 3^{(r)}}{r!} \frac{2^2}{(2^1-1)^{3-r}} = 4.$$

$$\frac{1}{2!}\sum_{s=0}^{m-3} \alpha^{m-3-s} (m-(s+1))^{(2)} f(a+s\ell)$$
$$= \frac{1}{2!} \alpha^{3-3-0} (3-(0+1))^{(2)} f(a+s\ell) = 4.$$

Hence, the equation (21) *is verified.*

Corollary III.8. *Let* $\zeta \in a + \mathbb{Z}_\ell$ *and* $n \in \mathbb{N}$ *such that* $m - n$ *be a positive integer. If* f *be an* n^{th}*-order* $\alpha(\ell)$*-delta integrable function based at* a*, then*

$$F_a^n(\zeta) =_a \Delta^{-n}_{\alpha(\ell)} f(\zeta). \tag{22}$$

Proof: The proof completes by Equation (15) and (19) which leads to (22). ■

Corollary III.9. *Let* $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ *be* n^{th}*-order* $\alpha(\ell)$*-delta integrable function based at* a*. If* $a < b < \zeta$ *such that both,* $\frac{\zeta - a}{\ell} - n$ *and* $\frac{\zeta - b}{\ell} - n$ *be a positive integer, then*

$$_a\Delta^{-n}_{\alpha(\ell)} f(\zeta) = F_a^n(\zeta) - F_a^n(b). \tag{23}$$

Proof: From (22)*, we have* $\Delta^{-n}_{\alpha(\ell)} f(\zeta) = F_a^n(\zeta)$ *and* $\Delta^{-n}_{\alpha(\ell)} f(b) = F_a^n(b)$*.*

Now (23) *follows from* $_a\Delta^{-n}_{\alpha(\ell)} f(\zeta) = \Delta^{-n}_{\alpha(\ell)} f(\zeta) - \Delta^{-n}_{\alpha(\ell)} f(b)$ *arrived by* (24)*.* ■

Remark III.10. If f is n^{th}-order $\alpha(\ell)$-delta integrable function based at a, then from Corallary III.8 and Equation (3), we obtain

$$
\begin{aligned}
{}_a\Delta^{-n}_{\alpha(\ell)} f(\zeta) &= f_n(\zeta) - \sum_{r=0}^{n-1} \alpha^{m-r} \frac{f_{n-r}(a)}{\Gamma(r+1)} \frac{\Gamma(m+1)}{\Gamma(m-r+1)} \\
&= \frac{1}{\Gamma(n)} \sum_{s=0}^{m-n} \alpha^{m-n-s} \frac{\Gamma(m-s)}{\Gamma(m-s-(n-1))} f(a+s\ell) \qquad (24)
\end{aligned}
$$

Example III.11. *Putting $n = 1$ in the Equation (24), we get*

$$
\begin{aligned}
f_1(\zeta) - \sum_{r=0}^{0} \alpha^{m-r} \frac{f_{1-r}(a)}{\Gamma(r+1)} \frac{\Gamma(m+1)}{\Gamma(m-r+1)} \\
= \frac{1}{\Gamma(1)} \sum_{s=0}^{m-1} \alpha^{m-1-s} \frac{\Gamma(m-s)}{\Gamma(m-s-0)} f(a+s\ell).
\end{aligned}
$$

Now taking $f(\zeta) = 2^\zeta$, $\zeta = 5.5$, $a = 1.5$, $\ell = 2$, $\alpha = 1$ and inserting $m = 2$ in the above equation, we get

$$
f_1(\zeta) - \alpha^m \frac{f_1(a)}{\Gamma(1)} \frac{\Gamma(3)}{\Gamma(3)} = \sum_{s=0}^{1} 1^{2-1-s} \frac{\Gamma(2-s)}{\Gamma(2-s)} f(1.5+2s) \qquad (25)
$$

On expanding both the sides of the above equation, we get

$$
LHS = \frac{2^\zeta}{2^{\ell-\alpha}} - 1 \times \frac{2^a}{2^{\ell-\alpha}} = 14.14213562
$$

$$
\begin{aligned}
RHS = \sum_{s=0}^{1} 2^{1.5+2s} = \frac{2^{5.5}}{2^2-1} - \frac{2^{1.5}}{2^2-1} = 2^{1.5} + 2^{3.5} \\
= 14.14213562
\end{aligned}
$$

Hence, the equation (25) is verified.

IV. Fractional Order $\alpha(\ell)$ - Delta Integration

The relation (24) in Remark III.10 motivates us to form a conjecture in fractional order $\alpha(\ell)$-delta integration. Here, we extend this to fractional ν^{th}-order $\alpha(\ell)$-delta integration value equal to ν^{th}-order fractional sum of f based at a. In this section, the TheoremIII.5 and Theorem III.6 yield new definition when n becomes fractional, that is, for any real $\nu > 0$.

Definition IV.1. [17] Let $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ be an n^{th}-order delta integrable function based at a for every $n \in N$, then f is called the $\infty - order$ delta integrable function.

Remark IV.2. [17] All the functions mentioned in Example III.4 are ∞-order $\alpha(\ell)$-delta integrable functions.

Definition IV.3. Let $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ be a function $\nu \geq 0$, $\zeta \in a+\mathbb{Z}_\ell$, $m = \dfrac{\zeta - a}{\ell}$ such that $m-\nu$ be a positive integer. The fractional order (ν^{th}-order) $\alpha(\ell)$-delta sum of f based at a is defined by,

$$
\Delta^{-\nu}_{\alpha(\ell)} f(\zeta) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{m-\nu} \alpha^{m-n-s} \frac{\Gamma(m-s)}{\Gamma(m-s-(\nu-1))} f(a+s\ell). \qquad (26)
$$

Definition IV.4. Let $f : a+\mathbb{Z}_\ell \to \mathbb{R}$, $\nu \geq 0$, $\zeta \in a+\mathbb{Z}_\ell$ and $m = \dfrac{\zeta - a}{\ell}$ such that $m - \nu$ be a positive integer. If there exists a function $f^\nu_a : a + \nu + N \to \mathbb{R}$ such that

$$
f^\nu_a(\zeta) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{m-\nu} \alpha^{m-n-s} \frac{\Gamma(m-s)}{\Gamma(m-s-(\nu-1))} f(a+s\ell), \qquad (27)
$$

then the function f^ν_a is called the ν^{th}-order $\alpha(\ell)$-delta integration of f based at a.

Conjecture: Let $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ be an infinte order $\alpha(\ell)$-delta integrable function based at a having integrating sequence $(f_n)_{n=1}^\infty$. If $f_n(a) = 0$ when $n = 1, 2, \ldots$, then $f^\nu_a(\zeta)$ exists and satisfies (27) for $\nu > 0$.

Theorem IV.5. *If $f : a + \mathbb{Z}_\ell \to \mathbb{R}$ be a function having $\alpha(\ell)$-delta integrating sequence $(f_1, f_2, \cdots, f_n)$ and also f is in geometric progression, $\zeta \in a+\mathbb{Z}_\ell$, $a \in \mathbb{R}$ and ${}_a\Delta^{-n}_q f(t)$ is the n^{th}-order $\alpha(\ell)$-delta integration of f based at a, then*

$$
\begin{aligned}
&f_n(\zeta) - \\
&\frac{\left[\frac{(r+1)^{(n-1)}}{(n-1)!} \alpha^{r+2-n} f(\zeta-(r+2)\ell)\right]^2}{\frac{(r+1)^{(n-1)}}{(n-1)!} \alpha^{r+2-n} f(\zeta-(r+2)\ell) - \frac{(r+2)^{(n-1)}}{(n-1)!} \alpha^{r+3-n} f(\zeta-(r+3)\ell)} \\
&\qquad = \sum_{s=n-1}^{r} \frac{s^{(n-1)}}{(n-1)!} \alpha^{s-n+1} f(\zeta-(s+1)\ell) \qquad (28)
\end{aligned}
$$

Proof: Consider the expression (24) in Remark III.10,

$$
\begin{aligned}
f_n(\zeta) - \alpha^m f_n(a) - m\alpha^{m-1} f_{n-1}(a) - \frac{m^{(2)}}{2!} \alpha^{m-2} f_{n-2}(a) \\
- \cdots - \frac{m^{(n-1)}}{(n-1)!} \alpha^{m-(n-1)} f_1(a) \\
= \frac{(n-1)^{(n-1)}}{(n-1)!} f(\zeta-n\ell) + \frac{(n)^{(n-1)}}{(n-1)!} \alpha f(\zeta-(n+1)\ell) \\
+ \cdots + \frac{(m-1)^{(n-1)}}{(n-1)!} \alpha^{m-n} f(\zeta-m\ell)
\end{aligned}
$$

As $m \to \infty$, $f_s(a) \to 0$ when $s = 1, 2, \ldots, n$, then substituting this into the above equation and it becomes

$$
f_n(\zeta) = \frac{(n-1)^{(n-1)}}{(n-1)!} f(\zeta-n\ell) + \frac{(n)^{(n-1)}}{(n-1)!} \alpha f(\zeta-(n+1)\ell) + \cdots
$$

Now, we spilt the above infinite series into two series

$$
\begin{aligned}
f_n(\zeta) = \Bigg[\frac{(n-1)^{(n-1)}}{(n-1)!} f(\zeta-n\ell) + \frac{(n)^{(n-1)}}{(n-1)!} \alpha f(\zeta-(n+1)\ell) \\
+ \cdots + \frac{(r)^{(n-1)}}{(n-1)!} \alpha^{r+1-n} f(\zeta-(r+1)\ell)\Bigg] \\
+ \Bigg[\frac{(r+1)^{(n-1)}}{(n-1)!} \alpha^{r+2-n} f(\zeta-(r+2)\ell) \\
+ \frac{(r+2)^{(n-1)}}{(n-1)!} \alpha^{r+3-n} f(\zeta-(r+3)\ell) + \cdots\Bigg] \\
(29)
\end{aligned}
$$

Consider the second series of the equation (29)

$$
\begin{aligned}
&\frac{(r+1)^{(n-1)}}{(n-1)!} \alpha^{r+2-n} f(\zeta-(r+2)\ell) \\
&\qquad + \frac{(r+2)^{(n-1)}}{(n-1)!} \alpha^{r+3-n} f(\zeta-(r+3)\ell) + \cdots \\
&= \frac{\left[\frac{(r+1)^{(n-1)}}{(n-1)!} \alpha^{r+2-n} f(\zeta-(r+2)\ell)\right]^2}{\frac{(r+1)^{(n-1)}}{(n-1)!} \alpha^{r+2-n} f(\zeta-(r+2)\ell) - \frac{(r+2)^{(n-1)}}{(n-1)!} \alpha^{r+3-n} f(\zeta-(r+3)\ell)}.
\end{aligned}
$$

Substitute this into the equation (29), we get (28). ■

Corollary IV.6. *If f is ν^{th}-order $\alpha(\ell)$-delta integrable function based at a, then*

$$
\begin{aligned}
&f_\nu(\zeta) - \\
&\frac{\left[\frac{1}{\Gamma\nu} \frac{\Gamma(r+2)}{\Gamma(r-\nu+3)} \alpha^{r+2-\nu} f(\zeta-(r+2)\ell)\right]^2}{\frac{1}{\Gamma\nu} \frac{\Gamma(r+2)}{\Gamma(r-\nu+3)} \alpha^{r+2-\nu} f(\zeta-(r+2)\ell) - \frac{1}{\Gamma\nu} \frac{\Gamma(r+3)}{\Gamma(r-\nu+4)} \alpha^{r+3-\nu} f(\zeta-(r+3)\ell)} \\
&\qquad = \frac{1}{\Gamma\nu} \sum_{s=\nu-1}^{r} \frac{\Gamma(s+1)}{\Gamma(s-\nu+2)} f(\zeta-(s+1)\ell). \qquad (30)
\end{aligned}
$$

Proof: The proof follows by Theorem IV.5 and convert (28) this into gamma function. ■

Example IV.7. *Applying* $f(\zeta) = 2^{\zeta}$, $\zeta = 0.5$, $r = 4.5$, $\ell = 0.2$, $\alpha = 0.3$, $\nu = 2.5$ *in* (30)*, then it will becomes*

$$f_{2.5}(\zeta) - \frac{\left[\frac{1}{\Gamma 2.5}\frac{\Gamma 6.5}{\Gamma 5}(0.3)^4 f(0.5-1.3)\right]^2}{\frac{1}{\Gamma 2.5}\frac{\Gamma 6.5}{\Gamma 5}(0.3)^4 f(0.5-1.3) - \frac{1}{\Gamma 2.5}\frac{\Gamma 7.5}{\Gamma 6}(0.3)^5 f(0.5-1.5)}$$

$$= \frac{1}{\Gamma 2.5}\sum_{s=1.5}^{4.5}\frac{\Gamma(s+1)}{\Gamma(s-0.5)} f(0.5-(s+1)(0.2))$$

$$\begin{aligned} LHS &= \frac{2^{\zeta}}{(2^{\ell}-\alpha)^{2.5}} \\ &- \frac{\left[\frac{1}{\Gamma 2.5}\frac{\Gamma 6.5}{\Gamma 8}(0.3)^4 2^{0.5-1.3}\right]^2}{\frac{1}{\Gamma 2.5}\frac{\Gamma 6.5}{\Gamma 5}(0.3)^4 2^{0.5-1.3} - \frac{1}{\Gamma 2.5}\frac{\Gamma 7.5}{\Gamma 6}(0.3)^5 2^{0.5-1.5}} \\ &= 2.13124 - \frac{0.001762}{0.027726} = 2.13124 - 0.06355 \\ &= 2.06769 \end{aligned}$$

$$\begin{aligned} RHS &= \frac{1}{\Gamma 2.5}\sum_{s=1.5}^{4.5}(0.3)^{s-1.5}\frac{\Gamma(s+1)}{\Gamma(s-0.5)} 2^{0.5-(s+1)(0.2)} \\ &= \frac{1}{1.3296}[1.3296 + 0.868113 + 0.396762 + 0.15543] \\ &= \frac{1}{1.3296}[2.749905] \\ &= 2.06822 \end{aligned}$$

Hence, the equation (30) *is verified.*

V. Conclusion

While the fractional order $\alpha(\ell)$-delta sum of given function f based at a is available in the literature, no one has yet attempted to derive the fractional order $\alpha(\ell)$-delta integration of f. We have developed this discrete fractional integration for factorials and geometric functions. Here, the fractional sum of f is derived by using the Newton's formula. These results generate several identies and formulae.

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