On Stability Analysis of Nonlinear Systems

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***Abstract*—In this study, an extensive review of nonlinear systems and their stability analysis is given. In particular, this paper attempts to explore the differences between closed-loop and open-loop systems and demonstrates how each plays a different function in control theory. The idea of hyperbolic equilibria is investigated, along with the stable and unstable manifolds that go along with it. This exploration sheds light on the behavior of dynamical systems both locally and globally. A solid foundation for assessing the stability of nonlinear systems is provided by discussing several forms of stability, such as exponential, asymptotic, and input-to-state stability (ISS).**

***Index Terms*—Nonlinear systems, Stability analysis, Control theory, Hyperbolic equilibria, Exponential stability, Asymptotic stability, Input-to-state stability.**

I. Introduction

Nonlinear systems encompass a range of equations—algebraic, differential, integral, functional, or operator-based—that describe physical processes or devices not adequately represented by linear equations. When such systems involve equations depicting the evolution of solutions over time, often with varying parameters or control inputs, they are termed dynamical systems [1]–[6].

Since the 19th century, there has been a major evolution in the study of nonlinear systems (dynamical in this context), sometimes known as nonlinear control systems when the control inputs are taken into account. This theoretical framework is now crucial in understanding and modeling a vast array of phenomena across multiple disciplines, including life sciences, social sciences, and engineering [7]–[11]. Applications are found in fields as diverse as physics, chemistry, biology, medicine, economics, and various engineering sectors. In order to see the application of the stability on the fractional-order systems, the reader may refer to the references [12]–[22].

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A core aspect of system engineering, especially within control systems and automation, is stability theory. This theory is vital for ensuring that a system's outputs and internal signals remain within acceptable bounds (bounded-input/bounded-output stability) or, more rigorously, that they tend to a specific equilibrium state (asymptotic stability). Stability in nonlinear dynamics and control systems can be categorized into three main types: stability relative to equilibrium points, orbital stability of trajectories, and structural stability of the system.

The origins of stability theory trace back to 1644 with E. Torricelli's investigation of a rigid body's equilibrium under gravity. The classical stability theorem formulated by G. Lagrange in 1788 remains a pivotal result, stating that a conservative system's equilibrium is stable if its potential energy is at a minimum. The concepts of system and trajectory stability have been refined over centuries, leading to A. M. Lyapunov's landmark 1892 dissertation, "The General Problem of Motion Stability." Lyapunov's work laid the foundation for modern stability analysis and controller design, influencing both theoretical research and practical applications in dynamical systems. This article aims to provide a concise overview of Lyapunov's stability theory, its criteria, methodologies, and associated stability concepts relevant to nonlinear dynamical systems.

II. Preliminaries of Nonlinear System

A. Control System as Nonlinear

To understand nonlinear control systems, consider a system described by the continuous-time differential equation:

$$\dot{x} = f(x, t, u), \quad t \in [t_0, \infty).$$

The system's state vector, $x(t)$ in this case, is limited to a normally bounded area $\Omega_x \subset \mathbb{R}^n$. Within yet another restricted area $\Omega_u \subset \mathbb{R}^m$—typically $m \leq n$—lies the control input vector u. With respect to any valid control input and initial condition $x(t_0) = x_0 \in \Omega_x$, there exists a unique solution for the nonlinear function f, which is either Lipschitz continuous or continuously differentiable. The temporal evolution and dependence on the starting state x_0 are displayed by the trajectory of the state $x(t)$, often written as $\phi(t, x_0)$.

Systems are referred to as autonomous when the time variable t does not exist independently of the state vector in f. Using a state feedback control, for instance, $u(t) = h(x(t))$, we can write this as:

$$\dot{x} = f(x), \quad x(t_0) = x_0 \in \mathbb{R}^n.$$

If the time variable t appears independently, the system is nonautonomous. This terminology also applies to discrete-time systems, although their characteristics might differ.

A nonlinear control system in a discrete-time setting is expressed by the difference equation:

$$\begin{cases} x_{k+1} = f(x_k, k, u_k), \\ y_k = g(x_k, k, u_k) \end{cases}.$$

Here, the notations are defined similarly to the continuous-time case. This article primarily focuses on the control system defined by $\dot{x} = f(x, t, u)$, or the first equation of the discrete-time system. For simplicity, the state x is also considered the system output. An equilibrium or fixed point x^* of the autonomous system $\dot{x} = f(x)$ (or $x_{k+1} = f(x_k)$ in discrete time) satisfies:

$$f(x^*) = 0.$$

This implies that at equilibrium, the state vector x remains constant. In the discrete-time case, the equilibrium is a solution of:

$$x^* = f(x^*).$$

A control system is termed deterministic if each change in system parameters or initial states results in a unique outcome. Conversely, it is stochastic if changes lead to multiple possible outcomes according to some probability distribution. This discussion is confined to deterministic systems. In control systems, an observation or measurement equation often accompanies the state evolution equation:

$$y = f(x, t, u),$$

where the system output is represented by $y(t) \in \mathbb{R}^\ell$ with $1 \le \ell \le n$. A continuous or smooth nonlinear function is represented by the function f. Single-input/single-output (SISO) systems have $n = \ell = 1$, whereas systems with multiple inputs and outputs (MIMO) have both $n, \ell > 1$. Accordingly, systems that are multi-input/single-output (MISO) or single-input/multi-output (SIMO) are also specified. The initial time is usually taken as $t_0 = 0$ unless specified otherwise. The state space $\mathbb{R}^n$ encompasses all possible states of the system.

B. Study of Open-Loop/Closed-Loop Systems

Let us consider a Multiple-Input Multiple-Output (MIMO) control system S. This system might have continuous time or discrete time, be deterministic or stochastic, and be linear or nonlinear. Over the temporal domain $\mathcal{D} = [a, b]$, the permissible input and matching output signals are determined, where $-\infty < a < b < \infty$. In general, $a = t_0 = 0$ and $b = \infty$ apply to control systems. An open-loop map may be used to illustrate this relationship:

$$S : u \to y \quad \text{or} \quad y(t) = S(u(t)).$$

This map is illustrated in Figure 1's block diagram. This kind of map may basically be used to any control system that can be explained by differential or difference equations. In these instances, the equation and starting conditions implicitly determine the map S.

Control systems denoted by equations (1) or (3) can be implemented in a closed-loop configuration if the control inputs are functions of the state vectors, $u = h(x, t)$. A typical closed-loop system is shown in Figure 3, where S_1 represents the plant that is described by f and S_2 represents

Fig. 1: The block diagram of an open-loop system.

the controller that is described by h. They can, however, play different roles.

Examining the example of a discrete-time system in Figure 2 can help you comprehend the distinctions between open-loop and closed-loop systems better. Let $u_1(k) = 1$ for all $k = 0, 1, \ldots$ and assume that S_1^{-1} and S_2^{-1} exist. Even if S_1 and S_2 could each be BIBO-stable (Bounded Input Bounded Output), it can be shown that when k rises, $y_1(k)$ will diverge to infinity. This highlights the need for a more rigorous criterion that takes into consideration the interaction between S_1 and S_2, demonstrating that the stability of individual components does not ensure the stability of the closed-loop system.

In summary, both open-loop and closed-loop configurations have their distinct roles and implications in control systems. Open-loop systems are simpler and rely on a direct mapping from input to output, whereas closed-loop systems use feedback to dynamically adjust the control inputs, often leading to more robust performance in the face of disturbances and uncertainties.

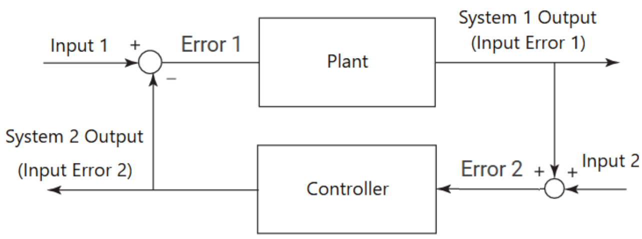

Fig. 2: Typical closed-loop control system.

C. Study of Hyperbolic Equilibria and Their Manifolds

Consider the autonomous differential system:

$$\dot{z} = g(z), \quad z(t_0) = z_0 \in \mathbb{R}^n. \tag{1}$$

The Jacobian matrix $J(z)$ of this system is defined by:

$$J(z) = \frac{\partial g}{\partial z}. \tag{2}$$

This matrix depends on time and becomes a constant when evaluated at specific states like z^* or z_0, determined by g and these states. An equilibrium z^* is classified as hyperbolic if the Jacobian matrix's eigenvalues at z^* have nonzero real parts. For a periodic solution $\tilde{z}(t)$ with period $T > 0$, the Jacobian $J(\tilde{z}(t))$ is T-periodic:

$$J(\tilde{z}(t + T)) = J(\tilde{z}(t)) \quad \text{for all } t \in [t_0, \infty). \tag{3}$$

In this scenario, there exists a T-periodic nonsingular matrix $N(t)$ and a constant matrix P such that the fundamental solution matrix associated with $J(\tilde{z}(t))$ is:

$$\Phi(t) = N(t) e^{tP}. \tag{4}$$

Here, $\Phi(t)$ consists of n linearly independent solution vectors of the linear differential equation:

$$\dot{z} = J(\tilde{z}(t))z, \quad z(t_0) = z_0. \tag{5}$$

The Floquet multipliers of the Jacobian are the eigenvalues of the matrix e^{TP}. If every Floquet multiplier associated with the periodic solution $\tilde{z}(t)$ has nonzero real portions, then the orbit is called a hyperbolic periodic orbit. Now examine an equilibrium z^* and its neighborhood D. z^*'s local stable and unstable manifolds are described as follows:

$$W^s_{\text{loc}}(z^*) = \{z \in D \mid \varphi_t(z) \in D : \varphi_t(z) \to z^* \text{ as } t \to \infty\} \tag{6}$$

for all $t \geq t_0$, and

$$W^u_{\text{loc}}(z^*) = \{z \in D \mid \varphi_t(z) \in D : \varphi_t(z) \to z^* \text{ as } t \to -\infty\} \tag{7}$$

for all $t \leq t_0$. An equilibrium is stable if nearby trajectories approach it and unstable if they move away. The stable and unstable manifolds of z^* are further defined as:

$$W^s(z^*) = \{z \in D \mid \varphi_t(z) \cap W^s_{\text{loc}}(z^*) \neq \emptyset\}, \tag{8}$$

and

$$W^u(z^*) = \{z \in D \mid \varphi_t(z) \cap W^u_{\text{loc}}(z^*) \neq \emptyset\}. \tag{9}$$

For instance, the system:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_1(1 - z_1^2) \end{cases} \tag{10}$$

has a hyperbolic equilibrium at $(z_1^*, z_2^*) = (0, 0)$. Figures 3a and 3b illustrate the local and global stable and unstable manifolds of this equilibrium, respectively.

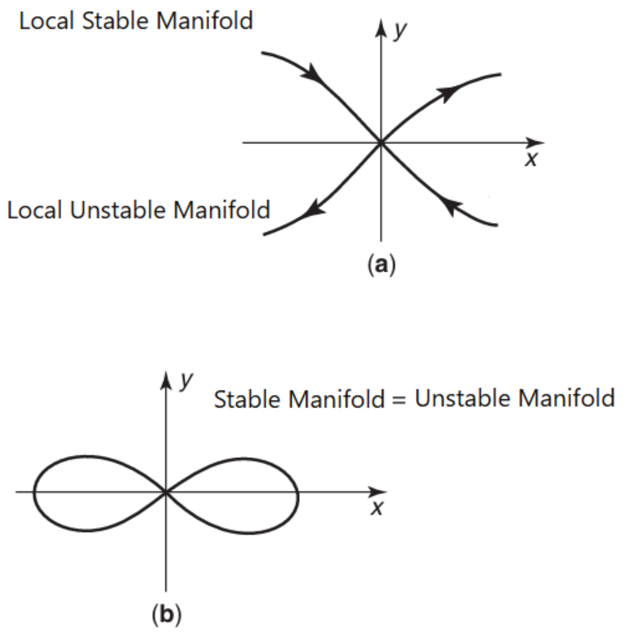

Fig. 3: Difference between stable and unstable manifolds

A hyperbolic equilibrium features only stable and/or unstable manifolds due to its Jacobian having exclusively stable and/or unstable eigenvalues. The behavior near a hyperbolic equilibrium is typically straightforward, either converging (stable) or diverging (unstable), and thus, complex dynamics like chaos are seldom linked with isolated hyperbolic equilibria.

III. Different Types of Stabilities

A. Stability in the Sense of Input-to-State Stability (ISS)

Consider this non-autonomous system:

$$x(t) = f(x, t); \quad x(t_0) = x_0 \in \mathbb{R}^n.$$

If, for any initial state $x(t_0)$ and any bounded input $u(t)$, the system state $x(t)$ remains bounded and converges to a bounded neighborhood of the equilibrium as $t \to \infty$, then this system is stable in the sense of Input-to-State Stability (ISS) with respect to the equilibrium $x^* = 0$. In particular, γ and β are class $\mathcal{K}$ functions such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\|u\|_\infty),$$

for all $t \geq t_0$, where $\|u\|_\infty$ denotes the essential supremum of the input $u(t)$.

Unlike Lyapunov stability, ISS explicitly considers the effect of external inputs on the system's state. This is particularly important for practical systems where disturbances and control inputs cannot be ignored. For instance, consider the linear time-varying system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

where $A(t)$ and $B(t)$ are time-varying matrices. The system is ISS if there exist positive constants c and λ such that the following holds:

$$V(x(t)) \leq e^{-\lambda(t-t_0)}V(x(t_0)) + \frac{c}{\lambda}\|u\|_\infty,$$

for some Lyapunov function $V(x)$. This definition emphasizes that the system's response to initial conditions and external inputs is crucial for determining its stability. As such, ISS provides a more comprehensive framework for analyzing the stability of nonlinear systems subjected to external disturbances.

The concept of ISS extends naturally to systems with time-varying inputs and non-autonomous systems, offering a robust method for stability analysis in practical applications. This approach ensures that the system not only remains stable in the presence of bounded inputs but also recovers its equilibrium state once the disturbances diminish.

B. Study of Asymptotic and Exponential Stabilities

Assume that the system in question is asymptotically stable with respect to its equilibrium point $x^* = 0$. This indicates that it is stable in the sense of Lyapunov. Moreover, a constant $\delta = \delta(t_0) > 0$ exists such that:

$$\|x(t_0)\| < \delta \implies \|x(t)\| \to 0 \text{ as } t \to \infty.$$

Uniform asymptotic stability occurs when the constant δ is independent of t_0 over the interval $[0, \infty)$. Furthermore, the stability is considered global if the convergence $\|x(t)\| \to 0$ holds irrespective of the initial state $x(t_0)$ across the entire spatial domain where the system is defined, such as when $\delta = \infty$. Additionally, if the system satisfies:

$$\|x(t_0)\| < \delta \implies \|x(t)\| \leq ce^{-st}.$$

If c and s are positive constants, the equilibrium is considered exponentially stable.

It is evident that asymptotic stability follows from exponential stability, and that stability in the Lyapunov sense follows from this. The opposite isn't always true, though. As an illustration, a system with the output trajectory $x_1(t) = x_0 \sin(t)$ is not asymptotically stable, although it is Lyapunov stable around 0. Asymptotically stable (and thus Lyapunov stable) if $t_0 < 1$, a system with the output $x_2(t) = x_0(1 + t - t_0)^{-1}$ is not exponentially stable around 0. On the other hand, a system with $x_3(t) = x_0 e^{-t}$ is Lyapunov stable and asymptotically stable due to its exponential stability.

C. Orbital Stability

Orbital stability focuses on the stability of a system's trajectory under small disturbances, differing from Lyapunov stability. Consider a periodic solution $\phi(t)$ with period $T > 0$ for the autonomous system:

$$\dot{x}(t) = f(x), \quad x(t_0) = x_0 \in \mathbb{R}^n.$$

Let Γ denote the closed orbit of $\phi(t)$ in the state space:

$$\Gamma = \{y \mid y = \phi(t),\ 0 \le t \le T\}.$$

If for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for any x_0 satisfying

$$\text{dist}(x_0, \Gamma) = \inf_{y \in \Gamma} \|x_0 - y\| < \delta.$$

so that the system's solution $\phi(t)$ satisfies

$$\text{dist}(\phi(t, x_0), \Gamma) < \epsilon \quad \text{for all } t \ge t_0,$$

for which $\phi(t)$ is considered orbitally stable at that point. Figure 7 provides an illustration of this idea. For example, a stable periodic solution is orbitally stable because, similar to a stable equilibrium, it converges to adjacent trajectories, preserving proximity even in the face of tiny disturbances.

A more sophisticated definition is provided by the Zhukovskij stability. If, for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any y_0 inside a ball of radius δ centered at x_0, there exist homeomorphisms $t_1(t)$ and $t_2(t)$ with $t_1(0) = t_2(0) = 0$, guaranteeing Zhukovskij stability for $\phi(t)$ in which

$$\|\phi(t_1(t), x_0) - \phi(t_2(t), x_0)\| < \epsilon \quad \text{for all } t \ge t_0.$$

Furthermore, a Zhukovskij stable solution $\phi(t)$ is asymptotically stable in the sense of Zhukovskij if, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if y_0 is anywhere within a ball of radius δ centered at x_0, there exist homeomorphisms $t_1(t)$ and $t_2(t)$ guaranteeing

$$\|\phi(t_1(t), x_0) - \phi(t_2(t), x_0)\| \to 0 \quad \text{as } t \to \infty,$$

where t_1 and t_2 are continuous mappings with continuous inverses. Asymptotic Zhukovskij stability implies asymptotic Lyapunov stability, although the converse is not necessarily true. If $\phi(t)$ is an equilibrium, both types of stability are equivalent. This comprehensive approach provides a nuanced understanding of orbital stability and its implications for system behavior under perturbations.

D. Structural Stability

The concept of topological orbital equivalence defines two systems as equivalent if a homeomorphism exists that maps the trajectory families of one system onto those of another while preserving the direction of motion. This essentially means that the geometric configuration of the trajectory families in both systems should be similar, without any additional features such as knots, sharp corners, or branching. For example, the systems described by $\dot{x} = x$ and $\dot{x} = 2x$ are topologically orbitally equivalent, whereas the systems $\dot{x} = x$ and $\dot{x} = \sqrt{x}$ are not. Figure 8 illustrates the trajectories of these three systems.

Returning to the autonomous system described by $\dot{x} = f(x)$, if the system's behavior in state space is significantly altered by minor external disturbances, such as the emergence of a new equilibrium or a new periodic orbit, the system is deemed structurally unstable. To be more specific, consider a set of functions defined as:

$$S = \left\{ g(x) \mid \|g(x)\| < 1 \text{ and } \left\| \frac{\partial g(x)}{\partial x} \right\| < 1 \right\},$$

for all $x \in \mathbb{R}^n$.

An autonomous system $\dot{x} = f(x)$ is structurally stable if, for any $g \in S$, there exists an $\epsilon > 0$ such that the systems $\dot{x} = f(x)$ and $\dot{x} = f(x) + \epsilon g(x)$ are topologically orbitally equivalent. For instance, the system $\dot{x} = x$ is structurally stable, whereas $\dot{x} = x^2$ is not near the origin. This instability arises because a small perturbation, like $\dot{x} = x^2 + \epsilon$ where $\epsilon > 0$, results in a system with two equilibria, $x = \sqrt{\epsilon}$ and $x = -\sqrt{\epsilon}$, contrasting the original system which has only one equilibrium at $x = 0$. In summary, if minor external perturbations lead to a drastic change in the system's dynamics in the state space, such as the creation of new equilibria or periodic orbits, the system is considered structurally unstable. Systems like $\dot{x} = x$ maintain their structural stability, whereas those like $\dot{x} = x^2$ do not, due to their sensitivity to such perturbations.

IV. BIBO Stability

This section examines bounded-input/bounded-output (BIBO) stability, which is a basic but less rigorous kind of stability. Any bounded input to the system will always result in a bounded output thanks to BIBO stability. Consider a linear system represented by:

$$\dot{z} = Cz + Du.$$

If the matrix C is asymptotically stable, then this system is BIBO stable. In this case, we look at the input-output map (17), which is shown in Figure 2. When there are non-negative constants b_i and b_o for each admissible input $u \in U$ and the corresponding output $z \in Z$, then a system Q is said to be BIBO stable from an input set U to an output set Z. This means that

$$\|u\|_U \le b_i \implies \|z\|_Z \le b_o.$$

Given that all norms are equivalent for finite-dimensional vectors, the specific norms used for input and output signals to define and achieve BIBO stability are generally inconsequential. However, it's crucial to recognize that a system remains BIBO stable even if b_i is minimal and b_o is substantial, which may limit the practicality of this stability criterion in certain applications.

A. Study of Small-Gain Theorem

A helpful criteria for confirming the BIBO stability of a closed-loop control system is provided by the small-gain theorem. Given that the system is formally constructed to satisfy the theory's requirements, this theorem may be applied to nearly any kind of system, including nonlinear and linear, discrete and continuous-time, with delays, and of any dimension. But this standard has a tendency to be too cautious. Examine the standard closed-loop system depicted in Figure 3, where internal signals, outputs, and inputs are connected by:

$$Q_1(e_1) = e_2 - u_2,$$

and

$$Q_2(e_2) = u_1 - e_1.$$

It is important to remember that the BIBO stability of the closed-loop system as a whole is not guaranteed by the BIBO stability of Q_1 and Q_2. For instance, both Q_1 and Q_2 are BIBO stable independently in a discrete-time setting with $Q_1 = 1$ and $Q_2 = -1$, and $u_1(k) = 1$ for all $k = 0, 1, \ldots$. However, as k increases, $z_1(k) = k \to \infty$ throughout the discrete-time setting. Consequently, a more robust condition characterizing the interplay between Q_1 and Q_2 is required.

Theorem 1: **(Small-Gain Theorem)** If there exist constants L_1, L_2, M_1, M_2 with $L_1 L_2 < 1$ such that:

$$\|Q_1(e_1)\| \leq M_1 + L_1\|e_1\|$$

and

$$\|Q_2(e_2)\| \leq M_2 + L_2\|e_2\|.$$

Then:

$$\|e_1\| \leq (1 - L_1 L_2)^{-1}(\|u_1\| + L_2\|u_2\| + M_2 + L_2 M_1)$$

and

$$\|e_2\| \leq (1 - L_1 L_2)^{-1}(\|u_2\| + L_1\|u_1\| + M_1 + L_1 M_2).$$

The signal spaces are where the norms $\|\cdot\|$ are defined. Thus, if u_1 and u_2 are bounded inputs, then $Q_1(e_1)$ and $Q_2(e_2)$ are correspondingly bounded outputs.

B. Contraction Mapping Theorem

Contraction mapping theorems are basically what the small-gain theorem is. With the right system formulation, this theorem may determine the BIBO stability of a system defined by a map. A global contraction mapping theorem is as follows: The contraction mapping theorem: In the event that $\|Q\| < 1$ is satisfied by the operator norm of the input-output map Q, which is specified on $\mathbb{R}^n$, then the system mapping:

$$z(t) = Q(z(t)) + c$$

has a unique solution (not trivial) for any constant vector $c \in \mathbb{R}^n$. This solution satisfies:

$$\|z\| \leq (1 - \|Q\|)^{-1}\|c\|.$$

Particularly, the solution of:

$$z_{k+1} = Q(z_k), z_0 \in \mathbb{R}^n, k = 0, 1, \ldots$$

satisfies:

$$\|z_k\| \to 0 \text{ as } k \to \infty.$$

V. Conclusion

In this research, a comprehensive overview of nonlinear systems and their stability analysis has been provided. The exploration began with an introduction to nonlinear dynamical systems, highlighting their significance across various fields such as physics, biology, and engineering. A detailed examination of control systems, both in continuous and discrete time, was presented, emphasizing the importance of equilibrium points and their classification. The study delved into the distinction between open-loop and closed-loop systems, illustrating their respective roles in control theory. The concept of hyperbolic equilibria and their associated stable and unstable manifolds was explored, providing insight into the local and global behavior of dynamical systems. Different types of stability, including Input-to-State Stability (ISS), asymptotic stability, and exponential stability, were discussed, offering a robust framework for analyzing the stability of nonlinear systems. The research underscored the importance of considering external inputs and disturbances in practical applications, highlighting the relevance of ISS in real-world scenarios. By integrating classical stability concepts with modern approaches, this research contributes to a deeper understanding of nonlinear systems and their control. This knowledge is crucial for developing effective control strategies and ensuring the reliable operation of various systems in engineering and applied sciences.

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