Statistical Inference for Integrated Reflected Ornstein-Uhlenbeck Processes

Xuekang Zhang, Haofei Liu

Abstract—**In this paper, we consider the problem of statistical inference for integrated reflected Ornstein-Uhlenbeck processes based on continuous-time observations. We examine the consistency and asymptotic distribution of the trajectory fitting estimator by employing the Toeplitz lemma and the strong law of large numbers.**

Index Terms—**trajectory fitting estimator, integrated reflected OUP, consistency, asymptotic distribution.**

I. Introduction

THE Ornstein-Uhlenbeck process (OUP) has demonstrated a wide range of practical applications and considerable theoretical significance in physics, finance, and other fields since its inception. In practical applications, this process effectively combines the randomness of Brownian motion with the stability of mean regression, enabling accurate simulations of the fluctuation characteristics of financial variables such as stock prices, option pricing, and interest rates (see, e.g., Nicolato and Venardos [1]; Fouque et al. [2]). During the last decades, asymptotic properties of parameter estimator for OUP have been studied by many statisticians. (see, e.g., Bercu et al. [3]; Bercu and Rouault [4]; Dietz [5]; Florens-Landais and Pham [6]; Gao and Jiang [7]; Jiang [8]; Jiang and Dong [9]; Jiang and Xie [10]; Jiang and Zhang [11]; Kutoyants [12]).

In various applications, the modeled quantity often needs to meet additional constraints on its permissible values. For example, models for populations or interest rates must remain positive. It is widely recognized that reflected stochastic differential equations with white noises are better suited for capturing these types of phenomena. The reflected OUP modifies the standard OUP by introducing an additional regulator that ensures the reflected OUP remains nonnegative. It is known that the OUP $\{X_t,\, t \geq 0\}$ reflected at the boundary zero is defined as follows:

$$\begin{cases} dX_t = \theta X_t dt + dB_t + dL_t, \quad X_0 = x_0, \\ X_t \geq 0, \quad 0 \leq t \leq T, \end{cases} \tag{1}$$

where $\theta \in R$, $\{B_t, t \geq 0\}$ is a one dimensional standard Brownian process and $L = \{L_t, t \geq 0\}$ is the minimal nondecreasing and nonnegative process, which makes the process $X_t \geq 0$ for all $t \geq 0$, so that

$$\int_0^\infty I(X_t > 0) dL_t = 0,$$

where $I(\cdot)$ denotes the indicator function. It can be concluded that (see, e.g., Harrison [13]; Whitt [14]) the expression of process L is

$$\begin{aligned} L_t &= \max\left[0, \sup_{0\leq s\leq t}\left(-x_0 - \theta\int_0^s X_u du - B_s\right)\right] \\ &= \max\left[0, \sup_{0\leq s\leq t}(L_s - X_s)\right]. \end{aligned} \tag{2}$$

In recent decades, reflected OUP have been widely utilized in research on queueing systems, financial engineering, mathematical biology, and more (see, e.g., Bo et al. [15]; Ricciardi and Sacerdote [16]; Ward and Glynn [17]). Recently, statisticians have closely examined the asymptotic properties of parameter estimation for the reflected OUP and have made significant progress (see, e.g., Bo and Yang [18]; Bo et al. [19]; Jiang and Yang [20]; Hu et al. [21]; Zang and Zhang [22]; Zang and Zhu [23]).

However, there are instances when we do not directly observe the stochastic processes. Instead, we examine its integrals over non-overlapping time intervals. Furthermore, these observations are presumed to be affected by measurement errors. Integrated stochastic processes could provide a more effective explanation for various modern econometric phenomena, as current observations often represent the cumulative effects of all prior perturbations (see, e.g., Barndorff-Nielsen [24]; Barndorff-Nielsen and Shephard [25]; Nicolau [26]). Motivated by the aforementioned works, in this paper we consider the integrated reflected OUP

$$dY_t = X_t dt, \quad Y_0 = y_0, \tag{3}$$

where $X = \{X_t,\, t \geq 0\}$ is defined in equation (1). We assume that the process X is unobservable, whereas the process $Y = \{Y_t,\, t \geq 0\}$ is observable. This article aims to study consistency and asymptotic distribution of the trajectory fitting estimator (TFE) for the unknown drift parameter $\theta \in \mathbb{R}$ in equation (3) by utilizing continuous observations of the trajectory of the process $\{Y_t, t \geq 0\}$ over the interval $[0, T]$, where $T > 0$. TFE was first introduced by Kutoyants [27] as a numerically appealing alternative to the established maximum likelihood estimators for continuous diffusion processes (see, e.g., Kutoyants [12]; Dietz and Kutoyants [28]; Shu et al. [29]; Zhang and Shu [30]).

To obtain the TFE, we can rewrite equation (3) as follows:

$$dY_t = (x_0 + \theta Y_t + B_t + L_t)dt, \quad 0 \leq t \leq T. \tag{4}$$

It follows that

$$Y_t = y_0 + x_0 t + \theta C_t + \int_0^t B_s ds + \int_0^t L_s ds, \tag{5}$$

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Xuekang Zhang is an associate professor in the School of Mathematics-Physics and Finance, Anhui Polytechnic University, Wuhu 241000, China (Corresponding author, e-mail: xkzhang@ahpu.edu.cn).

Haofei Liu is a postgraduate student in the School of Mathematics-Physics and Finance, Anhui Polytechnic University, Wuhu 241000, China (e-mail: haofeiliu2023@163.com).

where $C_t = \int_0^t Y_s ds$. The TFE of θ is to minimize the following objective function:

$$\int_0^T |Y_t - y_0 - x_0 t - \theta C_t|^2 dt.$$

It is clear that the minimum is achieved when θ is given by

$$\widehat{\theta}_T = \frac{\int_0^T (Y_t - y_0 - x_0 t) C_t dt}{\int_0^T C_t^2 dt}. \tag{6}$$

Substituting equation (5) into equation (6) yields that

$$\widehat{\theta}_T = \theta + \frac{\int_0^T \left(\int_0^t B_s ds + \int_0^t L_s ds\right) C_t dt}{\int_0^T C_t^2 dt}. \tag{7}$$

II. Consistency of the TFE $\widehat{\theta}_T$

In the section, we investigate the consistency of the TFE $\widehat{\theta}_T$. Throughout the article, we shall use the notation " $\to_p$" to denote "convergence in probability" and the notation " $\Rightarrow$" to denote "convergence in distribution". We write " $\stackrel{d}{=}$" for equality in distribution.

We introduce an important lemmas as follows.

Lemma 2.1: (Dietz and Kutoyants [28]) If φ_T is a probability measure defined on $[0,\infty)$ such that $\varphi_T([0,T]) = 1$ and $\varphi_T([0,K]) \to 0$ as $T \to \infty$ for each $K > 0$, then

$$\lim_{T\to\infty} \int_0^T f_t \varphi_T(dt) = f_\infty$$

for every bounded and measure function $f : [0,\infty) \to \mathbb{R}$ for which the limit $f_\infty := \lim_{t\to\infty} f_t$ exists.

Theorem 2.1: (i) Under $\theta > 0$, we have

$$\lim_{T\to\infty} (\widehat{\theta}_T - \theta) = 0, \quad \text{a.s.} \tag{8}$$

(ii) Under $\theta = 0$, we have

$$\widehat{\theta}_T - \theta \to_p 0, \quad \text{as} \quad T \to \infty. \tag{9}$$

(iii) Under $\theta < 0$, we have

$$\lim_{T\to\infty} \widehat{\theta}_T = 0, \quad \text{a.s.} \tag{10}$$

Proof: (i) Under $\theta > 0$, it follows from Zang and Zhang [22] that

$$\lim_{t\to\infty} \frac{Y_t}{e^{\theta t}} = \frac{\eta_\infty + \beta_\infty}{\theta}, \quad \text{a.s.}, \tag{11}$$

and

$$\lim_{t\to\infty} \frac{L_t}{t^{\frac{1}{2}}} = 0, \quad \text{a.s.}, \tag{12}$$

where $\beta_\infty = \max\left[0, -x_0 + \max_{0\le s\le \frac{1}{2\theta}} \check{B}_s\right]$, $\{\check{B}_t, t \ge 0\}$ is another Brownian motion and $-\int_0^t e^{-\theta s} dB_s = \check{B}_{\frac{1-e^{-2\theta t}}{2\theta}}$, and $\eta_\infty = x_0 - \check{B}_{\frac{1}{2\theta}}$. Combining Lemma 2.1 and (11) gives that

$$\lim_{t\to\infty} \frac{C_t}{e^{\theta t}} = \lim_{t\to\infty} \frac{\int_0^t Y_s ds}{e^{\theta t}} = \lim_{t\to\infty} \frac{\int_0^t \frac{Y_s}{e^{\theta s}} \frac{e^{\theta s}}{\int_0^t e^{\theta s} ds} ds}{\frac{e^{\theta t}}{\int_0^t e^{\theta s} ds}} = \frac{\eta_\infty + \beta_\infty}{\theta^2}, \quad \text{a.s.} \tag{13}$$

By (12) and Lemma 2.1, one has

$$\lim_{t\to\infty} \frac{\int_0^t L_s ds}{e^{\theta t}} = \lim_{t\to\infty} \frac{\int_0^t \frac{L_s}{s^{\frac{1}{2}}} \frac{s^{\frac{1}{2}}}{e^{\theta s}} \frac{e^{\theta s}}{\int_0^t e^{\theta s} ds} ds}{\frac{e^{\theta t}}{\int_0^t e^{\theta s} ds}} = 0, \quad \text{a.s.} \tag{14}$$

According to the strong law of large numbers and Lemma 2.1, we have

$$\lim_{t\to\infty} \frac{\int_0^t B_s ds}{e^{\theta t}} = \lim_{t\to\infty} \frac{\int_0^t \frac{B_s}{s} \frac{s}{e^{\theta s}} \frac{e^{\theta s}}{\int_0^t e^{\theta s} ds} ds}{\frac{e^{\theta t}}{\int_0^t e^{\theta s} ds}} = 0, \quad \text{a.s.} \tag{15}$$

By (7), (13)-(15) and Lemma 2.1, one sees that

$$\begin{aligned} &\lim_{T\to\infty} (\widehat{\theta}_T - \theta) \\ &= \lim_{T\to\infty} \frac{\int_0^T \frac{C_t}{e^{\theta t}} \frac{\int_0^t B_s ds}{e^{\theta t}} \frac{e^{2\theta t}}{\int_0^T e^{2\theta t} dt} dt}{\int_0^T (\frac{C_t}{e^{\theta t}})^2 \frac{e^{2\theta t}}{\int_0^T e^{2\theta t} dt} dt} + \lim_{T\to\infty} \frac{\int_0^T \frac{C_t}{e^{\theta t}} \frac{\int_0^t L_s ds}{e^{\theta t}} \frac{e^{2\theta t}}{\int_0^T e^{2\theta t} dt} dt}{\int_0^T (\frac{C_t}{e^{\theta t}})^2 \frac{e^{2\theta t}}{\int_0^T e^{2\theta t} dt} dt} \\ &= 0, \quad \text{a.s.} \end{aligned} \tag{16}$$

This completes the desired proof.

(ii) Under $\theta = 0$, we have

$$Y_t = y_0 + x_0 t + \int_0^t B_s ds + \int_0^t L_s ds, \quad 0 \le t \le T. \tag{17}$$

By the scaling properties of Brownian motion and equation (2), there exists another standard Brownian motion $\{\widetilde{B}_t, t \ge 0\}$ on the enlarged probability space, such that

$$\{(\widetilde{B}_t, \widetilde{L}_t), t \ge 0\} \stackrel{d}{=} \Big\{T^{\frac{1}{2}} B_{\frac{t}{T}}, T^{\frac{1}{2}} \max\left[0, -\frac{x_0}{T^{\frac{1}{2}}} - \max_{0\le s\le t}(-B_{\frac{s}{T}})\right], t \ge 0\Big\}. \tag{18}$$

Then, we have

$$\begin{aligned} &\int_0^T C_t^2 dt \\ &= \int_0^T \left[\int_0^t \left(y_0 + x_0 s + \int_0^s (B_u + L_u) du\right) ds\right]^2 dt \\ &= T \int_0^1 \left[\int_0^{T\nu} \left(y_0 + x_0 s + \int_0^s (B_u + L_u) du\right) ds\right]^2 d\nu \\ &= T^3 \int_0^1 \left[\int_0^{\nu} \left(y_0 + x_0 Tr + \int_0^{Tr} (B_u + L_u) du\right) dr\right]^2 d\nu \\ &= T^6 \int_0^1 \left[\int_0^{\nu} \left(\frac{y_0 + x_0 Tr}{T^{\frac{3}{2}}} + \int_0^r \frac{B_{Tq} + L_{Tq}}{T^{\frac{1}{2}}} dq\right) dr\right]^2 d\nu \\ &\stackrel{d}{=} T^6 \int_0^1 \left[\int_0^{\nu} \left(\frac{y_0 + x_0 Tr}{T^{\frac{3}{2}}} + \int_0^r (\widetilde{B}_q + \widetilde{L}_q) dq\right) dr\right]^2 d\nu, \end{aligned}$$

and

$$\begin{aligned} &\int_0^T C_t \left(\int_0^t B_s ds + \int_0^t L_s ds\right) dt \\ &= T \int_0^1 \int_0^{T\nu} \left(y_0 + x_0 s + \int_0^s (B_u + L_u) du\right) ds \end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^{T\nu}(B_s+L_s)dsd\nu \\
&= T^5\int_0^1\int_0^\nu\left(\frac{y_0+x_0Tr}{T^{\frac{3}{2}}}+\int_0^r\frac{B_{Tq}+L_{Tq}}{T^{\frac{1}{2}}}dq\right)dr \\
&\cdot\int_0^\nu\frac{B_{Tr}+L_{Tr}}{T^{\frac{1}{2}}}drd\nu \\
&\stackrel{d}{=} T^5\int_0^1\int_0^\nu\left(\frac{y_0+x_0Tr}{T^{\frac{3}{2}}}+\int_0^r(\widetilde{B}_q+\widetilde{L}_q)dq\right)dr \\
&\cdot\int_0^\nu(\widetilde{B}_r+\widetilde{L}_r)drd\nu.
\end{aligned}
$$

Therefore, it follows from (7) and the continuous mapping theorem that

$$
\begin{aligned}
&\widehat{\theta}_T-\theta \\
&= \frac{\int_0^T\left(\int_0^t B_sds+\int_0^t L_sds\right)C_tdt}{\int_0^T C_t^2dt} \\
&\stackrel{d}{=}\frac{1}{T}\left(\int_0^1\int_0^\nu\left(\frac{y_0+x_0Tr}{T^{\frac{3}{2}}}+\int_0^r(\widetilde{B}_q+\widetilde{L}_q)dq\right)dr\right. \\
&\left.\cdot\int_0^\nu(\widetilde{B}_r+\widetilde{L}_r)drd\nu\right)\times\left(\int_0^1\left[\int_0^\nu\left(\frac{y_0+x_0Tr}{T^{\frac{3}{2}}}\right.\right.\right. \\
&\left.\left.\left.+\int_0^r(\widetilde{B}_q+\widetilde{L}_q)dq\right)dr\right]^2d\nu\right)^{-1} \\
&\to_p 0,\quad \text{as}\quad T\to\infty. \qquad (19)
\end{aligned}
$$

This completes the desired proof.

(iii) Under $\theta<0$, it follows from Zang and Zhang [22] that

$$\lim_{t\to\infty}\frac{Y_t}{t}=\frac{1}{\sqrt{-\pi\theta}},\quad \text{a.s.}, \qquad (20)$$

and

$$\lim_{t\to\infty}\frac{L_t}{t}=\sqrt{\frac{-\theta}{\pi}},\quad \text{a.s.} \qquad (21)$$

Combining (20) and Lemma 2.1 yields that

$$
\begin{aligned}
\lim_{t\to\infty}\frac{C_t}{t^2}&=\lim_{t\to\infty}\frac{\int_0^t Y_sds}{t^2} \\
&=\lim_{t\to\infty}\frac{\int_0^t\frac{Y_s}{s}\frac{s}{\int_0^t sds}ds}{\frac{t^2}{\int_0^t sds}} \\
&=\frac{1}{2\sqrt{-\pi\theta}},\quad \text{a.s.} \qquad (22)
\end{aligned}
$$

Applying Lemma 2.1 and the strong law of large numbers, we find that

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\int_0^t B_sds}{t^2}&=\lim_{t\to\infty}\frac{\int_0^t\frac{B_s}{s}\frac{s}{\int_0^t sds}ds}{\frac{t^2}{\int_0^t sds}} \\
&=0,\quad \text{a.s.} \qquad (23)
\end{aligned}
$$

By (21) and Lemma 2.1, one has

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\int_0^t L_sds}{t^2}&=\lim_{t\to\infty}\frac{\int_0^t\frac{L_s}{s}\frac{s}{\int_0^t sds}ds}{\frac{t^2}{\int_0^t sds}} \\
&=\frac{1}{2}\sqrt{\frac{-\theta}{\pi}},\quad \text{a.s.} \qquad (24)
\end{aligned}
$$

By (7), (22)-(24) and Lemma 2.1, it is not difficult to see that

$$
\begin{aligned}
&\lim_{T\to\infty}(\widehat{\theta}_T-\theta) \\
&=\lim_{T\to\infty}\frac{\int_0^T C_t\int_0^t B_sdsdt}{\int_0^T C_t^2dt}+\lim_{T\to\infty}\frac{\int_0^T C_t\int_0^t L_sdsdt}{\int_0^T C_t^2dt} \\
&=\lim_{T\to\infty}\frac{\int_0^T\frac{C_t}{t^2}\frac{\int_0^t B_sds}{t^2}\frac{t^4}{\int_0^T t^4dt}dt}{\int_0^T(\frac{C_t}{t^2})^2\frac{t^4}{\int_0^T t^4dt}dt} \\
&\quad+\lim_{T\to\infty}\frac{\int_0^T\frac{C_t}{t^2}\frac{\int_0^t L_sds}{t^2}\frac{t^4}{\int_0^T t^4dt}dt}{\int_0^T(\frac{C_t}{t^2})^2\frac{t^4}{\int_0^T t^4dt}dt} \\
&=-\theta,\quad \text{a.s.} \qquad (25)
\end{aligned}
$$

This implies that

$$\lim_{T\to\infty}\widehat{\theta}_T=0,\quad \text{a.s.} \qquad (26)$$

This completes the proof. ■

III. Asymptotic distribution of the TFE $\widehat{\theta}_T$

In the section, we study the asymptotic distribution of the TFE $\widehat{\theta}_T$.

Theorem 3.1: (i) Assume $\theta>0$, we have

$$\frac{e^{\theta T}}{T^{\frac{3}{2}}}(\widehat{\theta}_T-\theta)\Rightarrow\frac{2\sqrt{3}\theta^2}{3}\frac{N}{\eta_\infty+\beta_\infty}, \qquad (27)$$

as $T\to\infty$, where N is a standard normal random variable independent of η_∞ and β_∞.
(ii) Assume $\theta=0$, we have

$$
\begin{aligned}
&T(\widehat{\theta}_T-\theta) \\
&\Rightarrow\frac{\int_0^1\int_0^\nu\int_0^r(\widetilde{B}_q+\widehat{L}_q)dqdr\int_0^\nu(\widetilde{B}_r+\widehat{L}_r)drd\nu}{\int_0^1\left(\int_0^\nu\int_0^r(\widetilde{B}_q+\widehat{L}_q)dqdr\right)^2d\nu}, \qquad (28)
\end{aligned}
$$

as $T\to\infty$, where $\{\widetilde{B}_u,\,u\geq 0\}$ be another standard Brownian motion on the enlarged probability space, and $\widehat{L}_s=\max[0,\max_{0\leq u\leq s}(-\widetilde{B}_u)]$.

Proof: (i) Under $\theta>0$,

$$
\begin{aligned}
&\frac{e^{\theta T}}{T^{\frac{3}{2}}}(\widehat{\theta}_T-\theta) \\
&=\frac{e^{\theta T}\int_0^T C_t\int_0^t B_sdsdt}{T^{\frac{3}{2}}\int_0^T {C_t}^2dt}+\frac{e^{\theta T}\int_0^T C_t\int_0^t L_sdsdt}{T^{\frac{3}{2}}\int_0^T {C_t}^2dt} \\
&=: I_1+I_2. \qquad (29)
\end{aligned}
$$

For I_1, we can decompose it as follows:

$$
\begin{aligned}
I_1&=\frac{(\eta_T+\beta_T)^2}{e^{-2\theta T}\int_0^T {C_t}^2dt}\left(\frac{e^{-\theta T}T^{-\frac{3}{2}}\int_0^T C_tdt\int_0^T B_sds}{(\eta_T+\beta_T)^2}\right. \\
&\left.-\frac{e^{-\theta T}T^{-\frac{3}{2}}\int_0^T C_t\int_t^T B_sdsdt}{(\eta_T+\beta_T)^2}\right) \\
&=: I_{11}(I_{12}-I_{13}). \qquad (30)
\end{aligned}
$$

Combining Lemma 2.1 and (13) gives that

$$\lim_{T\to\infty}\frac{\int_0^T C_t^2dt}{e^{2\theta T}}=\lim_{T\to\infty}\frac{\int_0^T\left(\frac{C_t}{e^{\theta t}}\right)^2\frac{e^{2\theta t}}{\int_0^T e^{2\theta t}dt}dt}{\frac{e^{2\theta T}}{\int_0^T e^{2\theta t}dt}}$$

$$= \frac{(\eta_\infty + \beta_\infty)^2}{2\theta^5}, \quad \text{a.s.}$$

This implies that

$$\lim_{T\to\infty} I_{11} = 2\theta^5, \quad \text{a.s.} \tag{31}$$

For I_{12}, we have

$$I_{12} = \frac{e^{-\theta T}\int_0^T C_t dt}{\eta_T + \beta_T} \cdot \frac{T^{-\frac{3}{2}}\int_0^T B_s ds}{\eta_T + \beta_T}. \tag{32}$$

For the first half of I_{12}, it can be obtained using Lemma 2.1 and equation (13) as follows:

$$\lim_{T\to\infty} \frac{e^{-\theta T}\int_0^T C_t dt}{\eta_T + \beta_T} = \frac{1}{\theta^3}, \quad \text{a.s.} \tag{33}$$

For the second half of I_{12}, by integration by parts, it can be expressed as

$$\frac{T^{-\frac{3}{2}}\int_0^T B_s ds}{\eta_T + \beta_T} = \frac{T^{-\frac{3}{2}}\int_0^{\sqrt{T}}(T-s)dB_s + T^{-\frac{3}{2}}\int_{\sqrt{T}}^T (T-s)dB_s}{\eta_{\sqrt{T}} + (\eta_T - \eta_{\sqrt{T}}) + \beta_T}. \tag{34}$$

The following claims can be drawn from (34).

(i) Obviously, the random variable $T^{-\frac{3}{2}}\int_{\sqrt{T}}^T (T-s)dB_s$ follows a normal distribution $N(0, \frac{1}{3} + T^{-1} - T^{-\frac{1}{2}} - \frac{1}{3}T^{-\frac{3}{2}})$, which weakly converges to the normal random variable $N(0, \frac{1}{3})$ as $T \to \infty$.

(ii) By integration by parts, strong law of large numbers and Lemma 2.1, we have

$$\begin{aligned}
&\lim_{T\to\infty} T^{-\frac{3}{2}}\int_0^{\sqrt{T}}(T-s)dB_s \\
&= \lim_{T\to\infty} \frac{B_{\sqrt{T}}(T-\sqrt{T}) + \int_0^{\sqrt{T}} B_s ds}{T^{\frac{3}{2}}} \\
&= 0, \quad \text{a.s.}
\end{aligned} \tag{35}$$

(iii) It is easy to see that

$$\lim_{T\to\infty} \eta_{\sqrt{T}} = \eta_\infty, \quad \text{a.s.},$$

and

$$\lim_{T\to\infty} \beta_T = \beta_\infty, \quad \text{a.s.}$$

(iv) $T^{-\frac{3}{2}}\int_{\sqrt{T}}^T (T-s)dB_s$ is independent of $\eta_{\sqrt{T}}$ and β_T.

(v) By Zang and Zhang [22], it follows that $\eta_T - \eta_{\sqrt{T}} \to_p 0$ as $T \to \infty$.

Then, we can obtain that

$$I_{12} \Rightarrow \frac{\sqrt{3}}{3\theta^3}\frac{N}{\eta_\infty + \beta_\infty}, \tag{36}$$

as $T \to \infty$, where N is a standard normal random variable independent of η_∞ and β_∞. Next, let's consider I_{13}. Note that

$$\begin{aligned}
&e^{-\theta T}T^{-\frac{3}{2}}\int_0^T C_t \int_t^T B_s ds dt \\
&\le e^{-\theta T}T^{-\frac{3}{2}}\int_0^T \left|C_t e^{-\theta t}\right| \left|\int_t^T B_s ds\right| e^{\theta t} dt \\
&\le \sup_{t\ge 0}\left|C_t e^{-\theta t}\right| e^{-\theta T}T^{-\frac{3}{2}}\int_0^T \left|\int_t^T B_s ds\right| e^{\theta t} dt.
\end{aligned} \tag{37}$$

Since $\sup_{t\ge 0}|C_t e^{-\theta t}|$ is almost surely finite, we only need to show that

$$e^{-\theta T}T^{-\frac{3}{2}}\int_0^T \left|\int_t^T B_s ds\right| e^{\theta t} dt \to_p 0, \tag{38}$$

as $T \to \infty$. Applying Markov inequality and integration by parts, we have for T large enough and any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left(e^{-\theta T}T^{-\frac{3}{2}}\int_0^T \left|\int_t^T B_s ds\right| e^{\theta t} dt \ge \varepsilon\right) \\
&\le \varepsilon^{-1}\mathbb{E}\left[e^{-\theta T}T^{-\frac{3}{2}}\int_0^T \left|\int_t^T B_s ds\right| e^{\theta t} dt\right] \\
&\le \frac{\int_0^T e^{-\theta u} u du}{\varepsilon T} + \frac{\int_0^T e^{-\theta u} u^{\frac{1}{2}} du}{\varepsilon T^{\frac{1}{2}}} + \frac{\int_0^T e^{-\theta u} u^{\frac{1}{2}} du}{\varepsilon T^{\frac{1}{2}}} \\
&\to 0, \quad \text{as} \quad T \to \infty.
\end{aligned} \tag{39}$$

This implies that

$$I_{13} \to_p 0, \quad \text{as} \quad T \to \infty. \tag{40}$$

Hence, by (30), (31), (36) and (40), one can get that

$$I_1 \Rightarrow \frac{2\sqrt{3}\theta^2}{3}\frac{N}{\eta_\infty + \beta_\infty}, \quad \text{as} \quad T \to \infty. \tag{41}$$

By (13), (14) and Lemma 2.1, one sees that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{\int_0^T C_t \int_0^t L_s ds dt}{e^{\theta T}T^{\frac{3}{2}}} \\
&= \lim_{T\to\infty} \frac{\int_0^T \frac{C_t}{e^{\theta t}} \frac{\int_0^t L_s ds}{t^{\frac{3}{2}}} \frac{e^{\theta t}t^{\frac{3}{2}}}{\int_0^T e^{\theta t}t^{\frac{3}{2}} dt} dt}{\frac{e^{\theta T}T^{\frac{3}{2}}}{\int_0^T e^{\theta t}t^{\frac{3}{2}} dt}} \\
&= 0, \quad \text{a.s.}
\end{aligned} \tag{42}$$

It follows from (13) and (42) that

$$\begin{aligned}
\lim_{T\to\infty} I_2 &= \frac{\int_0^T C_t \int_0^t L_s ds dt}{e^{\theta T}T^{\frac{3}{2}}} \cdot \frac{e^{2\theta T}}{\int_0^T C_t^2 dt} \\
&= 0, \quad \text{a.s.}
\end{aligned} \tag{43}$$

Therefore, by (29), (41) and (43), we can conclude that (27) holds. This completes the desired proof.

(ii) Under $\theta = 0$, by (19), we can conclude that (28) holds. This completes the desired proof. ∎

IV. Conclusion

This paper presents new results on statistical inference for integrated reflected Ornstein-Uhlenbeck processes. The main findings are derived using integration by parts, Lemma 2.1, and the strong law of large numbers. These results contribute to the further development of asymptotic theory in statistical inference for integrated stochastic processes.

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