Topp-Leone Exponential Distribution for Symmetric Loss Functions with Different Priors

Saridha D, Radha R.K, Venkatesan P

Abstract – **This study estimates the parameters of the Topp-Leone exponential distribution through the Bayesian method by applying Lindley's and Tierney-Kadane's (T-K) approximation techniques. The shape and scale parameters are derived using symmetric loss functions such as Squared Error Loss Function (SELF) and Quadratic Loss Function (QLF). The posterior distribution for Exponential, Gamma, Log-normal, and Weibull priors are analyzed by comparing the estimators based on Bayes risk for simulated and real data. It is seen that Bayes estimators using the T-K approximation method produce lower Bayes risk than Lindley's approximation for both the shape and scale parameters under QLF for simulated data whereas, Lindley's approximation outperforms the T-K method for both parameters in the case of real data.**

Index Terms- **Bayesian estimators, Lindley's Approximation, Tierney and Kadane, Bayes risk, Prior, SELF, QLF.**

I. Introduction

Bayesian estimation is a non-classical approach to statistical inference, and it has a wide range of applications worldwide. Topp and Leone[17] formulated the distribution of Topp-Leone, as a univariate continuous distribution with a J-shaped density function. Al-Shomrani. et al. [2] studied by describing their characteristics and applications of the distribution. Tabassum Naz Sindhu .et.al [15] investigated the distribution with different loss functions for informative and non-informative priors. Maryan Khalid [11] studied the analysis using non-informative prior. Prem Lata Gautam [13] focused on parameter estimation for the generalized Rayleigh distribution and identified Bayes risk in type II censoring using SELF, QLF and GELF. Kamaran Abbas [6] applied Lindley's approximation method for estimating the parameters of the Frechet distribution, particularly focusing on LINEX and GELF using medical data. Afaq Ahamed [1] estimated both the parameters of the Lomex distribution by applying Lindley's approximation technique using gamma, exponential and Levy priors with different loss functions. Manoj Kumar Rastogi [10] adopted the Weibull Rayleigh distribution to estimate the parameters and reliability characteristics using symmetric and asymmetric loss functions. Kahkashan Ateeq .et.al [5] applied a Bayesian technique to analyze the Lomex-Gumbel {Frechet} distribution by comparing Bayes estimators under various loss functions. For the comparison, they used Lindley's and Tierney-Kadane's approximation techniques. Uzma Jan .et.al.[18] studied the Inverse Lomax distribution, using Normal and Tierney-Kadane's approximation methods to study the shape parameter. Their study used informative and non-informative priors to estimate the parameters. The proposed distribution is fairly new to statisticians. Bayesian paradigm has not been examined by many statisticians earlier. Saridha. et. al [14] discussed the Topp-Leone exponential distribution for estimating the parameters with symmetric loss functions with identical priors. This paper estimates parameters using the SELF and QLF loss functions for Lindley's and T-K approximation methods. The posterior distribution for the unknown shape parameter and scale parameters for η and δ assumed to follow different priors for the Topp-Leone exponential distribution. The parameters are derived using the following six priors shown in Table I.

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TABLE I
NON-IDENTICAL PRIORS SELECTION

Priors	Non-Identical priors	
	Shape Parameter η	Scale Parameter δ
Gamma- Exponential	Gamma	Exponential
Lognormal-Exponential	Lognormal	Exponential
Weibull-Exponential	Weibull	Exponential
Lognormal-Gamma	Lognormal	Gamma
Weibull-Gamma	Weibull	Gamma
Weibull-Lognormal	Weibull	Lognormal

II. Maximum Likelihood Estimation

Al-Shomrani. et.al [2] introduced the p.d.f of the Topp-Leone exponential distribution given by :

$$f(x;\eta,\delta) = 2\eta\delta\, e^{-2\delta x}\left(1-e^{-2\delta x}\right)^{\eta-1} ; x,\eta,\delta \geq 0 \quad (1)$$

with η and δ as shape and scale parameters, then likelihood function:

$$L(x;\eta,\delta) = (2\eta\delta)^n\, e^{-2\delta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1} \quad (2)$$

Taking the log of likelihood function given in equation (2).

$$LogL = nlog(2\eta\delta) - 2\delta\sum_{i=1}^{n} x_i + (\eta-1)\sum_{i=1}^{n} log\left(1-e^{-2\delta x_i}\right) \quad (3)$$

The differentiating the log-likelihood function partially $w.r.to.\eta$ and δ are as follows:

$$\frac{\partial LogL}{\partial \eta}=\frac{n}{\eta}+\sum_{i=1}^{n} log\left(1-e^{-2\delta x_i}\right)=0 \quad (4)$$

$$\frac{\partial LogL}{\partial \delta}=\frac{n}{\delta}-2\sum_{i=1}^{n} x_i+(\eta-1)\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}=0 \quad (5)$$

The maximum likelihood estimation (MLEs) of η and δ, say $\hat{\eta}$ and $\hat{\delta}$, respectively, are the solution of the equations (4) and (5). Unfortunately, analytic solutions for η and δ are not in the closed form. To estimate the parameters η and δ Newton Raphson's method is used.

III. Priors And Posterior Distributions

The Bayesian estimates of the Topp-Leone exponential distribution for the parameters η and δ being independent random variables with different combinations of non-identical priors as follows:

(i) $\eta\sim\mathrm{Gamma}(a_1,b_1)$ and $\delta\sim\mathrm{Exponential}\,(a_2)$ i.e., Gamma-Exponential.
(ii) $\eta\sim\mathrm{Lognormal}\,(a_3,b_2)$ and $\delta\sim\mathrm{Exponential}\,(a_2)$ i.e., Lognormal-Exponential.
(iii) $\eta\sim\mathrm{Weibull}\,(a_4,b_3)$ and $\delta\sim\mathrm{Exponential}\,(a_2)$ i.e., Weibull-Exponential.
(iv) $\eta\sim\mathrm{Lognormal}\,(a_3,b_2)$ and $\delta\sim\mathrm{Gamma}\,(a_5,b_4)$ i.e., i.e., Lognormal-Gamma.
(v) $\eta\sim\mathrm{Weibull}\,(a_3,b_2)$ and $\delta\sim\mathrm{Gamma}\,(a_5,b_4)$ i.e. Weibull-Gamma.
(vi) $\eta\sim\mathrm{Weibull}\,(a_4,b_3)$ and $\delta\sim\mathrm{Lognormal}\,(a_6,b_5)$ i.e. Weibull- Lognormal.

The posterior distribution parameters for the combination of the above priors are discussed below.

A. Posterior Distribution for Topp- Leone exponential distribution using Non-Identical Priors

1) Gamma- Exponential Prior:

The combined prior density function for the parameters η and δ $i.e.,\eta\sim G(a_1,b1)$ and $\delta\sim Exp(a_2)$ is given by:

$$p_1(\eta,\delta)=\frac{b_1^{a_1}}{\Gamma a_1}a_2 e^{-(b_1\eta+a_2\delta)}\eta^{a_1-1}\,;\eta,\delta,a_1,a_2,b_1>0 \quad (6)$$

The joint posterior density function of η and δ is given by:

$$\pi_1(\eta,\delta|x)=\frac{1}{C_1}\,e^{-(b_1\eta+a_2\delta)}\eta^{a_1-1}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1} \quad (7)$$

where $C_1=\int_0^\infty\int_0^\infty e^{-(b_1\eta+a_2\delta)}\eta^{a_1-1}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta$

2) LogNormal - Exponential Prior:

The combined prior density function for the parameters η and δ $i.e.,\eta\sim LN(a_3,b_2)$ and $\delta\sim Exp(a_2)$ is given by:

$$p_2(\eta,\delta)=\frac{a_2}{\eta b_2\sqrt{2\pi}}e^{-\left(a_2\delta+\frac{(log\eta-a_3)^2}{2b_2^2}\right)};\quad \eta,\delta,a_1,a_3,b_2>0 \quad (8)$$

The joint posterior density function of η and δ is given by:

$$\pi_2(\eta,\delta|x)=\frac{1}{C_2\eta}e^{-\left(a_2\delta+\frac{(log\eta-a_3)^2}{2b_2^2}\right)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1} \quad (9)$$

where

$$C_2=\int_0^\infty\int_0^\infty\frac{1}{\eta}\;e^{-\left(a_2\delta+\frac{(log\eta-a_3)^2}{2b_2^2}\right)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta$$

3) Weibull - Exponential Prior:

The combined prior density function for the parameters η and δ $i.e.,\eta\sim W(a_4,b_3)$ and $\delta\sim Exp(a_2)$ is given by:

$$p_3(\eta,\delta)=\frac{a_2a_4}{b_3^{a_4}}\eta^{a_4-1}e^{-\left(a_2\delta+\left(\frac{\eta}{b^3}\right)^{a_4}\right)}\;;\;a_1,a_4,b_3>0 \quad (10)$$

The joint posterior density function of η and δ is given by:

$$\pi_3(\eta,\delta|x)=\frac{1}{C_3}\frac{a_1a_4}{b_3^{a_4}}\eta^{a_4-1}e^{-\left(a_2\delta+\left(\frac{\eta}{b^3}\right)^{a_4}\right)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta \quad (11)$$

where $C_3=\int_0^\infty\int_0^\infty\eta^{a_4-1}e^{-\left(a_1\delta+\left(\frac{\eta}{b^3}\right)^{a_4}\right)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1}\,d\eta d\delta$

4) LogNormal- Gamma Prior:

The combined prior density function for the parameters η and δ $i.e.,\eta\sim LN(a_3,b_2)$ and $\delta\sim G(a_5,b_4)$ is given by:

$$p_4(\eta,\delta)=\frac{b_4^{a_5}}{\eta b_2\sqrt{2\pi\Gamma a_5}}\delta^{a_5-1}e^{-\left(b_4\delta+\frac{(log\eta-a_3)^2}{2b_2^2}\right)};\quad a_3,a_5,b_2,b_4>0 \quad (12)$$

The joint posterior density function of η and δ is given by:

$$\pi_4(\eta,\delta|x)=\frac{1}{C_4}\frac{1}{\eta}\delta^{a_5-1}e^{-\left(b_4\delta+\frac{(log\eta-a_3)^2}{2b_2^2}\right)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta \quad (13)$$

where

$$C_4=\int_0^\infty\int_0^\infty\frac{1}{\eta}\delta^{a_5-1}e^{-\left(b_4\delta+\frac{(log\eta-a_3)^2}{2b_2^2}\right)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1}\,d\eta d\delta$$

5) Weibull-Gamma Prior:

The combined prior density function for the parameters η and δ $i.e.$, $\eta\sim W(a_4, b_3)$ and $\delta\sim G(a_5, b_4)$ is given by:

$$p_5(\eta,\delta) = \frac{a_4 b_4^{a_5}}{b_3^{a_4}\Gamma a_5}\eta^{a_4-1}\delta^{a_5-1}\, e^{-\left(b_4\delta+\left(\frac{\eta}{b_3}\right)^{a_4}\right)}; \quad a_4, a_5, b_3, b_4 > 0 \tag{14}$$

The joint posterior density function of η and δ is given by:

$$\pi_5(\eta,\delta|x) = \frac{1}{C_5}\,\eta^{a_4-1}\delta^{a_5-1}\, e^{-\left(b_4\delta+\left(\frac{\eta}{b_3}\right)^{a_4}\right)}(2\eta\delta)^n \, e^{-2\delta\sum_{i=1}^{n} x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta \tag{15}$$

where

$$C_5 = \int_0^{\infty}\int_0^{\infty} \eta^{a_4-1}\delta^{a_5-1}\, e^{-\left(b_4\delta+\left(\frac{\eta}{b_3}\right)^{a_4}\right)}(2\eta\delta)^n\, e^{-2\delta\sum_{i=1}^{n} x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta$$

6) Weibull-LogNormal Prior:

The combined prior density function of the parameters η and δ $i.e.$, $\eta\sim W(a_4, b_3)$ and $\delta\sim LN(a_6, b_5)$ is given by:

$$p_6(\eta,\delta) = \frac{a_4}{b_3^{a_4}\mathrm{b}_5\delta\sqrt{2\pi}}\, e^{-\left(\frac{(log\delta-a_6)^2}{2b_5^2}+\left(\frac{\eta}{b_3}\right)^{a_4}\right)}; \quad a_4, a_5, b_3, b_4 > 0 \tag{16}$$

The joint posterior density function of η and δ is given by:

$$\pi_6(\eta,\delta|x) = \frac{1}{C_6}\frac{1}{\delta}\, e^{-\left(\frac{(log\delta-a_6)^2}{2b_5^2}+\left(\frac{\eta}{b_3}\right)^{a_4}\right)}(2\eta\delta)^n \, e^{-2\delta\sum_{i=1}^{n} x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta \tag{17}$$

where

$$C_6 = \int_0^{\infty}\int_0^{\infty}\frac{1}{\delta}\, e^{-\left(\frac{(log\delta-a_6)^2}{2b_5^2}+\left(\frac{\eta}{b_3}\right)^{a_4}\right)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n} x_i}\prod_{i=1}^{n}\left(1-e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta$$

IV. Bayes Estimates for Different Loss Functions

The Bayes estimates η and δ and with Bayes risk of Topp-Leone exponential distribution is estimated for symmetric loss functions such as the Squared Error Loss Function (SELF) and the Quadratic Loss Function (QLF) in TABLE II. The parameters η and δ, as well as their associated Bayes risk of the Topp-Leone exponential distribution, which involves computing the expectation of a function of parameters under a posterior distribution from equations (7), (9), (11),(13),(15) and (17). Since solving these equations analytically is impractical, the Lindley approximation method [3] and the T-K approximation method [16] for numerical estimation were used.

V. Bayes Estimates for Lindley's Approximation

The posterior expectation is given by:

$$E[u(\eta,\delta)|x] = \frac{\int u(\eta,\delta)\exp[L(\eta,\delta)+\rho(\eta,\delta)]\, d(\eta,\delta)}{\int \exp[L(\eta,\delta)+\rho(\eta,\delta)]\, d(\eta,\delta)} \tag{18}$$

where $u(\eta,\delta)$ which is a function of η and δ only, $L(\eta,\delta)$ refers to the log-likelihood and $\rho(\eta,\delta)$ represents the log of the joint prior to η and δ. Based on Lindley [9], suggests that for a sufficiently large sample size n, the Equation (18) can be approximately computed through:

$$\begin{aligned} I(x) = {} & u(\hat{\eta},\hat{\delta}) + \frac{1}{2}[(u_{\eta\eta}+2u_\eta\rho_\eta)\sigma_{\eta\eta} + (u_{\delta\eta}+2u_\delta\rho_\eta)\sigma_{\delta\eta} \\ & +(u_{\eta\delta}+2u_\eta\rho_\delta)\sigma_{\eta\delta} + (u_{\delta\delta}+2u_\delta\rho_\delta)\sigma_{\delta\delta} + \\ & \frac{1}{2}[(u_\eta\sigma_{\eta\eta}+u_\delta\sigma_{\eta\delta}) (L_{\eta\eta\eta}\sigma_{\eta\eta}+L_{\eta\delta\eta}\sigma_{\eta\delta}+L_{\delta\eta\eta}\sigma_{\delta\eta}+L_{\delta\delta\eta}\sigma_{\delta\delta})] \\ & +\frac{1}{2}[(u_\eta\sigma_{\delta\eta}+u_\delta\sigma_{\delta\delta}) (L_{\eta\eta\delta}\sigma_{\eta\eta}+L_{\eta\delta\delta}\sigma_{\eta\delta}+L_{\delta\eta\delta}\sigma_{\delta\eta}+L_{\delta\delta\delta}\sigma_{\delta\delta})] \end{aligned} \tag{19}$$

where $\hat{\eta}$ and $\hat{\delta}$ are the MLEs of η and δ.

$$u_\eta = \frac{\partial u(\eta,\delta)}{\partial\eta};\; u_\delta = \frac{\partial u(\eta,\delta)}{\partial\delta};\; u_{\eta\eta} = \frac{\partial^2 u(\eta,\delta)}{\partial\eta^2};$$
$$u_{\delta\delta} = \frac{\partial^2 u(\eta,\delta)}{\partial\delta^2};\; u_{\eta\delta} = \frac{\partial^2 u(\eta,\delta)}{\partial\eta\partial\delta};\; \frac{\partial^2 LogL}{\partial\eta^2} = L_{\eta\eta}$$

and so on.

For equation (19), the estimated values of the Topp-Leone exponential distribution are given by:

$$\begin{aligned} E[u(\hat{\eta},\hat{\delta})|x] = {} & u(\hat{\eta},\hat{\delta}) + \frac{1}{2}[(u_{\eta\eta}+2u_\eta\rho_\eta)\sigma_{\eta\eta} \\ & +(u_{\delta\eta}+2u_\delta\rho_\eta)\sigma_{\delta\eta} \\ & +(u_{\eta\delta}+2u_\eta\rho_\delta)\sigma_{\eta\delta} \\ & +(u_{\delta\delta}+2u_\delta\rho_\delta)\sigma_{\delta\delta} \\ & +\frac{1}{2}[(u_\eta\sigma_{\eta\eta}+u_\delta\sigma_{\eta\delta})(S_1)] \\ & +\frac{1}{2}[(u_\eta\sigma_{\delta\eta}+u_\delta\sigma_{\delta\delta})(S_2)] \end{aligned} \tag{20}$$

where $S_1 = L_{\eta\eta\eta}\sigma_{\eta\eta} + L_{\delta\delta\eta}\sigma_{\delta\delta}$

$S_2 = L_{\eta\delta\delta}\sigma_{\eta\delta} + L_{\delta\eta\delta}\sigma_{\delta\eta} + L_{\delta\delta\delta}\sigma_{\delta\delta}$

(i)The log of the joint prior density for Gamma-Exponential prior is:

$$\rho(\eta,\delta) = (a_1-1)log\eta - b_1\eta - a_2\delta$$

$$\rho(\eta) = \frac{\partial\rho(\eta,\delta)}{\partial\eta} = \frac{(a_1-1)}{\eta} - b_1 \tag{21}$$

$$\rho(\delta) = \frac{\partial\rho(\eta,\delta)}{\partial\delta} = -a_2 \tag{22}$$

(ii) The log of the joint prior density for LogNormal-Exponential prior is:

$$\rho(\eta,\delta) = -a_2\delta + \log\left(\frac{1}{\eta}\right) - \frac{(log\eta-a_3)^2}{2b_2^2}$$

TABLE II
BAYES ESTIMATORS AND BAYES RISK FOR VARIOUS LOSS FUNCTIONS

Loss Function Expression	Bayes Estimator		Bayes risk	
	Parameter η	Parameter δ	Parameter η	Parameter δ
$SELF = L((\hat{\eta}-\eta)=(\hat{\eta}-\eta)^2$	$\hat{\eta}_{SELF}=E(\eta\|x)$	$\hat{\delta}_{SELF}=E(\delta\|x)$	$E(\eta^2\|x)-[E(\eta\|x)]^2$	$E(\delta^2\|x)-[E(\delta\|x)]^2$
$QLF = L(\hat{\eta}-\eta)=\left(\frac{\hat{\eta}-\eta}{\eta}\right)^2$	$\hat{\eta}_{QLF}=\left(\frac{E(\eta^{-1}\|x)}{E(\eta^{-2}\|x)}\right)$	$\hat{\delta}_{QLF}=\left(\frac{E(\delta^{-1}\|x)}{E(\delta^{-2}\|x)}\right)$	$1-\frac{[E(\eta^{-1}\|x)]^2}{E(\eta^{-2}\|x)}$	$1-\frac{[E(\delta^{-1}\|x)]^2}{E(\delta^{-2}\|x)}$

$$\rho(\eta)=\frac{\partial\rho(\eta,\delta)}{\partial\eta}=-\frac{1}{\eta}-\frac{(log\eta-a_3)}{\eta b_2^2} \quad (23)$$

$$\rho(\delta)=\frac{\partial\rho(\eta,\delta)}{\partial\delta}=-a_2 \quad (24)$$

(iii) The log of the joint prior density for Weibull-Exponential prior is:

$$\rho(\eta,\delta)=-a_2\delta+(a_4-1)log\eta-\left(\frac{\eta}{b^3}\right)^{a_4}$$

$$\rho(\eta)=\frac{\partial\rho(\eta,\delta)}{\partial\eta}=\frac{a_4-1}{\eta}-\left(\frac{a_4}{b_3}\right)\left(\frac{\eta}{b^3}\right)^{a_4-1} \quad (25)$$

$$\rho(\delta)=\frac{\partial\rho(\eta,\delta)}{\partial\delta}=-a_2 \quad (26)$$

(iv) The log of the joint prior density for LogNormal-Gamma prior is:

$$\rho(\eta,\delta)=(a_5-1)log\delta-b_4\delta+\log\left(\frac{1}{\eta}\right)-\frac{(log\eta-a_3)^2}{2b_2^2}$$

$$\rho(\eta)=\frac{\partial\rho(\eta,\delta)}{\partial\eta}=-\frac{1}{\eta}-\frac{(log\eta-a_3)}{\eta b_2^2} \quad (27)$$

$$\rho(\delta)=\frac{\partial\rho(\eta,\delta)}{\partial\delta}=\frac{a_5-1}{\delta}-b_4 \quad (28)$$

(v) The log of the joint prior density for Weibull-Gamma prior is:

$$\rho(\eta,\delta)=(a_4-1)log\eta-b_4\delta+(a_5-1)log\delta-\left(\frac{\eta}{b_3}\right)^{a_4}$$

$$\rho(\eta)=\frac{\partial\rho(\eta,\delta)}{\partial\eta}=\frac{a_4-1}{\eta}-\frac{a_4}{b_3}\left(\frac{\eta}{b_3}\right)^{a_4-1} \quad (29)$$

$$\rho(\delta)=\frac{\partial\rho(\eta,\delta)}{\partial\delta}=\frac{a_5-1}{\delta}-b_4 \quad (30)$$

(vi) The log of the joint prior density for Weibull-LogNormal prior is:

$$\rho(\eta,\delta)=\log\left(\frac{1}{\delta}\right)-\frac{(log\delta-a_6)^2}{2b_5^2}+(a_4-1)log\eta-\left(\frac{\eta}{b_3}\right)^{a_4}$$

$$\rho(\eta)=\frac{\partial\rho(\eta,\delta)}{\partial\eta}=\frac{a_4-1}{\eta}-\frac{a_4}{b_3}\left(\frac{\eta}{b_3}\right)^{a_4-1} \quad (31)$$

$$\rho(\delta)=\frac{\partial\rho(\eta,\delta)}{\partial\delta}=-\frac{1}{\delta}-\frac{(log\delta-a_6)}{\delta b_5^2} \quad (32)$$

A. Lindley's Approximation for the parameters η and δ using SELF

Lindley's approximation for Bayes estimate of the parameter η using Gamma-Exponential, LogNormal-Exponential, Weibull-Exponential, LogNormal-Gamma, Weibull-Gamma, and Weibull -LogNormal priors are derived respectively as follows:

$$\hat{\eta}_{G-E}=\hat{\eta}+\left(\frac{(a_1-1)}{\hat{\eta}}-b_1\right)\sigma_{\eta\eta}+(-a_2)\sigma_{\eta\delta}+\frac{1}{2}\sigma_{\eta\eta}S_1+\frac{1}{2}\sigma_{\delta\eta}S_2 \quad (33)$$

$$\hat{\eta}_{LN-E}=\hat{\eta}+\left(-\frac{1}{\hat{\eta}}-\frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\eta\eta}+(-a_2)\sigma_{\eta\delta}+\frac{1}{2}\sigma_{\eta\eta}S_1+\frac{1}{2}\sigma_{\delta\eta}S_2 \quad (34)$$

$$\hat{\eta}_{W-E}=\hat{\eta}+\left(\frac{a_4-1}{\hat{\eta}}-\left(\frac{a_4}{b_3}\right)\left(\frac{\hat{\eta}}{b^3}\right)^{a_4-1}\right)\sigma_{\eta\eta}+(-a_2)\sigma_{\eta\delta}+\frac{1}{2}\sigma_{\eta\eta}S_1+\frac{1}{2}\sigma_{\delta\eta}S_2 \quad (35)$$

$$\hat{\eta}_{LN-G}=\hat{\eta}+\left(-\frac{1}{\hat{\eta}}-\frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\eta\eta}+\left(\frac{a_5-1}{\hat{\delta}}-b_4\right)\sigma_{\eta\delta}+\frac{1}{2}\sigma_{\eta\eta}S_1+\frac{1}{2}\sigma_{\delta\eta}S_2 \quad (36)$$

$$\hat{\eta}_{W-G}=\hat{\eta}+\left(\frac{a_4-1}{\hat{\eta}}-\frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\eta\eta}+\left(\frac{a_5-1}{\hat{\delta}}-b_4\right)\sigma_{\eta\delta}+\frac{1}{2}\sigma_{\eta\eta}S_1+\frac{1}{2}\sigma_{\delta\eta}S_2 \quad (37)$$

$$\hat{\eta}_{W-LN}=\hat{\eta}+\left(\frac{a_4-1}{\hat{\eta}}-\frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\eta\eta}+\left(-\frac{1}{\hat{\delta}}-\frac{(log\hat{\delta}-a_6)}{\hat{\delta}b_5^2}\right)\sigma_{\eta\delta}+\frac{1}{2}\sigma_{\eta\eta}S_1+\frac{1}{2}\sigma_{\delta\eta}S_2 \quad (38)$$

Lindley's approximation for Bayes estimate of the parameter δ using Gamma-Exponential, LogNormal-Exponential, Weibull-Exponential, LogNormal-Gamma, Weibull-Gamma and Weibull-LogNormal priors are derived respectively as follows:

$$\hat{\delta}_{G-E}=\hat{\delta}+\left(\frac{(a_1-1)}{\hat{\eta}}-b_1\right)\sigma_{\delta\eta}+(-a_2)\sigma_{\delta\delta}+\frac{1}{2}\sigma_{\eta\delta}S_1+\frac{1}{2}\sigma_{\delta\delta}S_2 \quad (39)$$

$$\hat{\delta}_{LN-E}=\hat{\delta}+\left(-\frac{1}{\hat{\eta}}-\frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\delta\eta}+(-a_2)\sigma_{\delta\delta}+\frac{1}{2}\sigma_{\eta\delta}S_1+\frac{1}{2}\sigma_{\delta\delta}S_2 \quad (40)$$

$$\hat{\delta}_{W-E} = \hat{\delta} + \left(\frac{a_4 - 1}{\hat{\eta}} - \left(\frac{a_4}{b_3}\right)\left(\frac{\hat{\eta}}{b^3}\right)^{a_4-1}\right)\sigma_{\delta\eta} + (-a_2)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (41)$$

$$\hat{\delta}_{LN-G} = \hat{\delta} + \left(-\frac{1}{\hat{\eta}} - \frac{(log\hat{\eta} - a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\delta\eta} + \left(\frac{a_5 - 1}{\hat{\delta}} - b_4\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (42)$$

$$\hat{\delta}_{W-G} = \hat{\delta} + \left(\frac{a_4 - 1}{\hat{\eta}} - \frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\delta\eta} + \left(\frac{a_5 - 1}{\hat{\delta}} - b_4\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (43)$$

$$\hat{\delta}_{W-LN} = \hat{\delta} + \left(\frac{a_4 - 1}{\hat{\eta}} - \frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\delta\eta} + \left(-\frac{1}{\hat{\delta}} - \frac{(log\hat{\delta} - a_6)}{\hat{\delta}b_5^2}\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (44)$$

B. Lindley's Approximation for the parameters η and δ using QLF

Lindley's approximation for Bayes estimate of the parameter η using Gamma-Exponential, LogNormal-Exponential, Weibull-Exponential, LogNormal-Gamma, Weibull-Gamma, and Weibull-LogNormal priors are derived respectively as follows:

$$\hat{\eta}_{G-E} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{(a_1-1)}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}(-a_2)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{(a_1-1)}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}(-a_2)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (45)$$

$$\hat{\eta}_{LN-E} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(-\frac{1}{\hat{\eta}} - \frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}(-a_2)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(-\frac{1}{\hat{\eta}} - \frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}(-a_2)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (46)$$

$$\hat{\eta}_{W-E} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_4-1}{\hat{\eta}} - \left(\frac{a_4}{b_3}\right)\left(\frac{\hat{\eta}}{b^3}\right)^{a_4-1}\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}(-a_2)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_4-1}{\hat{\eta}} - \left(\frac{a_4}{b_3}\right)\left(\frac{\hat{\eta}}{b^3}\right)^{a_4-1}\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}(-a_2)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (47)$$

$$\hat{\eta}_{LN-G} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(-\frac{1}{\hat{\eta}} - \frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_5-1}{\hat{\delta}} - b_4\right)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(-\frac{1}{\hat{\eta}} - \frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_5-1}{\hat{\delta}} - b_4\right)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (48)$$

$$\hat{\eta}_{W-G} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_4-1}{\hat{\eta}} - \frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_5-1}{\hat{\delta}} - b_4\right)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_4-1}{\hat{\eta}} - \frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_5-1}{\hat{\delta}} - b_4\right)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (49)$$

$$\hat{\eta}_{W-LN} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_4-1}{\hat{\eta}} - \frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(-\frac{1}{\hat{\delta}} - \frac{(log\hat{\delta}-a_6)}{\hat{\delta}b_5^2}\right)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_4-1}{\hat{\eta}} - \frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(-\frac{1}{\hat{\delta}} - \frac{(log\hat{\delta}-a_6)}{\hat{\delta}b_5^2}\right)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (50)$$

Lindley's approximation for Bayes estimate of the parameter δ using Gamma-Exponential, LogNormal-Exponential, Weibull-Exponential, LogNormal-Gamma, Weibull-Gamma, and Weibull -LogNormal priors are derived respectively as follows:

$$\hat{\delta}_{G-E} = \frac{\frac{1}{\hat{\delta}} - \frac{1}{\hat{\delta}^2}\left(\frac{(a_1-1)}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} + \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta} - \frac{1}{\hat{\delta}^2}(-a_2)\sigma_{\delta\delta} - \frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1 - \frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2} + \frac{3}{\hat{\delta}^4}\sigma_{\delta\delta} - \frac{2}{\hat{\delta}^3}\left(\frac{(a_1-1)}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} - \frac{2}{\hat{\delta}^3}(-a_2)\sigma_{\eta\delta} - \frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1 - \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \quad (51)$$

$$\hat{\delta}_{LN-E}=\frac{\frac{1}{\hat{\delta}}-\frac{1}{\hat{\delta}^2}\left(-\frac{1}{\hat{\eta}}-\frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\delta\eta}+\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}-\frac{1}{\hat{\delta}^2}(-a_2)\sigma_{\delta\delta}-\frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1-\frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2}+\frac{3}{\hat{\delta}^4}\sigma_{\delta\delta}-\frac{2}{\hat{\delta}^3}\left(-\frac{1}{\hat{\eta}}-\frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\delta\eta}-\frac{2}{\hat{\delta}^3}(-a_2)\sigma_{\eta\delta}-\frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1-\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \quad (52)$$

$$\hat{\delta}_{W-E}=\frac{\frac{1}{\hat{\delta}}-\frac{1}{\hat{\delta}^2}\left(\frac{a_4-1}{\hat{\eta}}-\left(\frac{a_4}{b_3}\right)\left(\frac{\hat{\eta}}{b^3}\right)^{a_4-1}\right)\sigma_{\delta\eta}+\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}-\frac{1}{\hat{\delta}^2}(-a_2)\sigma_{\delta\delta}-\frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1-\frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2}+\frac{3}{\hat{\delta}^4}\sigma_{\delta\delta}-\frac{2}{\hat{\delta}^3}\left(\frac{a_4-1}{\hat{\eta}}-\left(\frac{a_4}{b_3}\right)\left(\frac{\hat{\eta}}{b^3}\right)^{a_4-1}\right)\sigma_{\delta\eta}-\frac{2}{\hat{\delta}^3}(-a_2)\sigma_{\eta\delta}-\frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1-\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \quad (53)$$

$$\hat{\delta}_{LN-G}=\frac{\frac{1}{\hat{\delta}}-\frac{1}{\hat{\delta}^2}\left(-\frac{1}{\hat{\eta}}-\frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\delta\eta}+\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}-\frac{1}{\hat{\delta}^2}\left(\frac{a_5-1}{\hat{\delta}}-b_4\right)\sigma_{\delta\delta}-\frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1-\frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2}+\frac{3}{\hat{\delta}^4}\sigma_{\delta\delta}-\frac{2}{\hat{\delta}^3}\left(-\frac{1}{\hat{\eta}}-\frac{(log\hat{\eta}-a_3)}{\hat{\eta}b_2^2}\right)\sigma_{\delta\eta}-\frac{2}{\hat{\delta}^3}\left(\frac{a_5-1}{\hat{\delta}}-b_4\right)\sigma_{\eta\delta}-\frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1-\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \quad (54)$$

$$\hat{\delta}_{W-G}=\frac{\frac{1}{\hat{\delta}}-\frac{1}{\hat{\delta}^2}\left(\frac{a_4-1}{\hat{\eta}}-\frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\delta\eta}+\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}-\frac{1}{\hat{\delta}^2}\left(\frac{a_5-1}{\hat{\delta}}-b_4\right)\sigma_{\delta\delta}-\frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1-\frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2}+\frac{3}{\hat{\delta}^4}\sigma_{\delta\delta}-\frac{2}{\hat{\delta}^3}\left(\frac{a_4-1}{\hat{\eta}}-\frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\delta\eta}-\frac{2}{\hat{\delta}^3}\left(\frac{a_5-1}{\hat{\delta}}-b_4\right)\sigma_{\eta\delta}-\frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1-\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \quad (55)$$

$$\hat{\delta}_{W-LN}=\frac{\frac{1}{\hat{\delta}}-\frac{1}{\hat{\delta}^2}\left(\frac{a_4-1}{\hat{\eta}}-\frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\delta\eta}+\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}-\frac{1}{\hat{\delta}^2}\left(-\frac{1}{\hat{\delta}}-\frac{(log\hat{\delta}-a_6)}{\hat{\delta}b_5^2}\right)\sigma_{\delta\delta}-\frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1-\frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2}+\frac{3}{\hat{\delta}^4}\sigma_{\delta\delta}-\frac{2}{\hat{\delta}^3}\left(\frac{a_4-1}{\hat{\eta}}-\frac{a_4}{b_3}\left(\frac{\hat{\eta}}{b_3}\right)^{a_4-1}\right)\sigma_{\delta\eta}-\frac{2}{\hat{\delta}^3}\left(-\frac{1}{\hat{\delta}}-\frac{(log\hat{\delta}-a_6)}{\hat{\delta}b_5^2}\right)\sigma_{\eta\delta}-\frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1-\frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \quad (56)$$

VI. Bayes Estimates for Tierney-Kadane (T-K) Approximation

The T-K method is utilized to find Bayes estimates of the Topp-Leone exponential distribution. This technique approximates the solution through a ratio of two integrals. The posterior expectation is evaluated according to the specified equation (18).

$$E[u(\eta,\delta)|x]=\frac{\int_\eta\int_\delta e^{n\left[\frac{1}{n}u(\eta,\delta;x)+\frac{1}{n}L(\eta,\delta;x)+\frac{1}{n}\rho(\eta,\delta)\right]}d(\eta,\delta)}{\int_\eta\int_\delta e^{n\left[\frac{1}{n}L(\eta,\delta;x)+\frac{1}{n}\rho(\eta,\delta)\right]}d(\eta,\delta)} \quad (57)$$

$$E[u(\eta,\delta)|x]=\frac{\int_\eta\int_\delta e^{n\Delta^*(\eta,\delta)}d(\eta,\delta)}{\int_\eta\int_\delta e^{n\Delta(\eta,\delta)}d(\eta,\delta)}$$

We consider the following functions

$$\Delta^*(\eta,\delta)=\frac{1}{n}lnu(\eta,\delta)+\frac{1}{n}lnL(\eta,\delta;x)+\frac{1}{n}ln\rho(\eta,\delta) \quad (58)$$

$$\Delta(\eta,\delta)=\frac{1}{n}lnL(\eta,\delta;x)+\frac{1}{n}ln\rho(\eta,\delta) \quad (59)$$

Tierney and Kadane [16], equation (57) can be approximated in the following form:

$$E[u(\eta,\delta)]=\sqrt{\frac{|\Sigma_\eta^*|}{|\Sigma|}}\exp[n\Delta^*(\hat{\eta}^*,\hat{\delta}^*)-n\Delta\left((\hat{\eta},\hat{\delta}\right)] \quad (60)$$

where $L(\eta,\delta)$ refers to the log-likelihood function given in (3) and $\rho(\eta,\delta)=ln\rho(\eta,\delta)$ denotes the log of the joint prior density and also assumes that $\Delta^*(\hat{\eta}^*,\hat{\delta}^*)$ and $\Delta(\hat{\eta},\hat{\delta})$ maximize functions $\Delta^*(\eta,\delta)$and $\Delta(\eta,\delta)$ respectively. Σ^* and Σ are the negative inverse of the Hessian matrix of the $\Delta^*(\eta,\delta)$ and $\Delta(\eta,\delta)$ respectively computed at $(\hat{\eta}^*,\hat{\delta}^*)$ and $(\hat{\eta},\hat{\delta})$, where

$$|\Sigma|=\left[\frac{\partial^2\Delta(\eta,\delta)}{\partial\eta^2}\frac{\partial^2\Delta(\eta,\delta)}{\partial\delta^2}-\frac{\partial^2\Delta}{\partial\eta\partial\delta}\frac{\partial^2\Delta}{\partial\delta\partial\eta}\right]^{-1}$$

$$|\Sigma_\eta^*|=\left[\frac{\partial^2\Delta^*(\eta,\delta)}{\partial\eta^2}\frac{\partial^2\Delta^*(\eta,\delta)}{\partial\delta^2}-\frac{\partial^2\Delta^*}{\partial\eta\partial\delta}\frac{\partial^2\Delta^*}{\partial\delta\partial\eta}\right]^{-1}$$

A. *Tierney and Kadane approximation for η under SELF*

Legendra [8] and Gauss [4] proposed SELF as:

$$L(\eta,\hat{\eta}_{SELF})=(\eta-\hat{\eta}_{SELF})^2$$

To obtain a Bayesian estimator for η_{SELF} of a function $u=u(\eta,\delta)$ and incorporating Gamma-Exponential prior is given below :
If $u(\eta,\delta)=\eta$ then

$$E(\eta|X)=\frac{\int_0^\infty\int_0^\infty\frac{\eta(2\eta\delta)^n\,e^{-2\delta\sum_{i=1}^n x_i}}{\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}}\frac{b_1^{a_1}}{\Gamma a_1}a_2e^{-(b_1\eta+a_2\delta)}\eta^{a_1-1}d\eta d\delta}{\int_0^\infty\int_0^\infty\frac{(2\eta\delta)^n\,e^{-2\delta\sum_{i=1}^n x_i}}{\prod_{i=1}^n(1-e^{-2\delta x_i})^{\eta-1}}\frac{b_1^{a_1}}{\Gamma a_1}a_2e^{-(b_1\eta+a_2\delta)}\eta^{a_1-1}d\eta d\delta} \quad (61)$$

therefore, the T-K approximation will be used as follows:

$$ln\rho(\eta,\delta)=lna_2-b_1\eta-a_2\delta+(a_1-1)ln\,\eta+a_1\ln b_1-ln\Gamma(a_1).$$

substituting (3) and (6) in (59) we get

$$\Delta_\eta(\eta,\delta)=\frac{1}{n}\begin{bmatrix} nlog(2\eta\delta)-2\delta\sum_{i=1}^{n}x_i \\ +(\eta-1)\sum_{i=1}^{n}log\left(1-e^{-2\delta x_i}\right)+lna_2 \\ -b_1\eta-a_2\delta+(a_1-1)\ln\eta \\ +a_1\ln b_1-ln\Gamma(a_1)\end{bmatrix} \quad (62)$$

Now, based on (58), $\Delta_\eta^*(\eta,\delta)$ can be obtained as follows:

$$\Delta_\eta^*(\eta,\delta)=\frac{1}{n}\begin{bmatrix} nlog(2\eta\delta)-2\delta\sum_{i=1}^{n}x_i \\ +(\eta-1)\sum_{i=1}^{n}log\left(1-e^{-2\delta x_i}\right) \\ +lna_2-b_1\eta-a_2\delta \\ +(a_1-1)\ln\eta \\ +a_1\ln b_1-ln\Gamma(a_1)\end{bmatrix}+\frac{ln\,\eta}{n} \quad (63)$$

then $(\hat{\eta},\hat{\delta})$ are solved using the equations given below:

$$\frac{\partial\Delta(\eta,\delta)}{\partial\eta}=\frac{1}{n}\left[\frac{n}{\eta}+\sum_{i=1}^{n}\begin{matrix} log\left(1-e^{-2\delta x_i}\right)-b_1 \\ +\frac{a_1-1}{\eta}\end{matrix}\right]=0 \quad (64)$$

$$\frac{\partial\Delta(\eta,\delta)}{\partial\delta}=\frac{1}{n}\begin{bmatrix} \frac{n}{\delta}-2\sum_{i=1}^{n}x_i+ \\ (\eta-1)\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}-a_2\end{bmatrix}=0 \quad (65)$$

The second order derivatives of $\Delta(\eta,\delta)$ w.r.to η and δ are derived as follows:

$$\frac{\partial^2\Delta(\eta,\delta)}{\partial\eta^2}=\frac{1}{n}\left[-\frac{n}{\eta^2}-\frac{a_1-1}{\eta^2}\right]$$

$$\frac{\partial^2\Delta(\eta,\delta)}{\partial\delta^2}=\frac{1}{n}\left[-\frac{n}{\delta^2}+(\eta-1)\sum_{i=1}^{n}\frac{(-4x_i^2)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})^2}\right]$$

$$\frac{\partial^2\Delta(\eta,\delta)}{\partial\eta\partial\delta}=\frac{\partial^2\Delta(\eta,\delta)}{\partial\delta\partial\eta}=\frac{1}{n}\left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]$$

Using the above equations, we compute the inverse of the negative Hessian of a matrix of $\Delta(\eta,\delta)$ at $(\hat{\eta},\hat{\delta})$ as follows:

$$|\Sigma|=\left[\frac{1}{n}\begin{bmatrix}\left[-\frac{n}{\eta^2}-\frac{a_1-1}{\eta^2}\right]\begin{bmatrix}-\frac{n}{\delta^2}+(\eta-1) \\ \sum_{i=1}^{n}\frac{(-4x_i^2)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})^2}\end{bmatrix} \\ -\left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]\left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]\end{bmatrix}\right]^{-1} \quad (66)$$

then $(\hat{\eta}_\Delta^*,\hat{\delta}_\Delta^*)$ are obtained by solving the following equations:

$$\frac{\partial\Delta_\eta^*(\eta,\delta)}{\partial\eta}=\frac{1}{n}\left[\frac{n}{\eta}+\sum_{i=1}^{n}\begin{matrix} log\left(1-e^{-2\delta x_i}\right)-b_1 \\ +\frac{a_1-1}{\eta}+\frac{1}{\eta}\end{matrix}\right]=0$$

$$\frac{\partial\Delta_\eta^*(\eta,\delta)}{\partial\delta}=\frac{1}{n}\begin{bmatrix} \frac{n}{\delta}-2\sum_{i=1}^{n}x_i+ \\ (\eta-1)\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}-a_2\end{bmatrix}=0$$

We obtain the second order derivatives of $\Delta_\eta^*(\eta,\delta)$ w.r.to η and δ as

$$\frac{\partial^2\Delta_\eta^*}{\partial\eta^2}=\frac{1}{n}\left[-\frac{n}{\eta^2}-\frac{a_1-1}{\eta^2}-\frac{1}{\eta^2}\right]$$

$$\frac{\partial^2\Delta_\eta^*}{\partial\delta^2}=\frac{1}{n}\left[-\frac{n}{\delta^2}+(\eta-1)\sum_{i=1}^{n}\frac{(-4x_i^2)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})^2}\right]$$

$$\frac{\partial^2\Delta_\eta^*}{\partial\eta\partial\delta}=\frac{1}{n}\left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]$$

$$\frac{\partial^2\Delta_\eta^*}{\partial\delta\partial\eta}=\frac{1}{n}\left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]$$

To compute $\Sigma_{\eta SELF}^*$ we make use of the expression given below.

$$\left|\Sigma_{\eta SELF}^*\right|=\left[\frac{1}{n}\begin{bmatrix}\left[\left[-\frac{n}{\eta^2}-\frac{a_1-1}{\eta^2}-\frac{1}{\eta^2}\right]\right] \\ \begin{bmatrix}-\frac{n}{\delta^2}+(\eta-1) \\ \sum_{i=1}^{n}\frac{(-4x_i^2)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})^2}\end{bmatrix}- \\ \left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right] \\ \left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]\end{bmatrix}\right]^{-1} \quad (67)$$

The approximate Bayes estimator of η under SELF using Gamma-Exponential prior is given by:

$$\hat{\eta}_{SELF}=\sqrt{\frac{\left|\Sigma_{\eta SELF}^*\right|}{|\Sigma|}}\exp\left[n\Delta_\eta^*(\hat{\eta}^*,\hat{\delta}^*)-n\Delta\left((\hat{\eta},\hat{\delta})\right]\right. \quad (68)$$

B. Tierney and Kadane approximation for δ under SELF

Legendra [8] and Gauss [4] proposed SELF as:

$$L(\delta,\hat{\delta}_{SELF})=\left(\delta-\hat{\delta}_{SELF}\right)^2$$

To obtain a Bayesian estimator for δ_{SELF} of a function $u=u(\eta,\delta)$ under Gamma-Exponential prior is given below

If $u(\eta,\delta)=\delta$ then, $\Delta_\delta^*(\eta,\delta)$ is :

$$\Delta_\delta^*(\eta,\delta)=\Delta(\eta,\delta;x)+\frac{1}{n}\ln(\delta)$$

We obtain the second order derivatives of $\Delta_\delta^*(\eta,\delta)$ w.r.to η and δ as

$$\frac{\partial^2\Delta_\eta^*}{\partial\eta^2}=\frac{1}{n}\left[-\frac{n}{\eta^2}-\frac{a_1-1}{\eta^2}\right]$$

$$\frac{\partial^2\Delta_\delta^*}{\partial\delta^2}=\frac{1}{n}\left[-\frac{n}{\delta^2}+(\eta-1)\sum_{i=1}^{n}\frac{(-4x_i^2)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})^2}-\frac{1}{\delta^2}\right]$$

$$\frac{\partial^2\Delta_\delta^*}{\partial\eta\partial\delta}=\frac{1}{n}\left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]=\frac{\partial^2\Delta_\delta^*}{\partial\delta\partial\eta}$$

To compute $\Sigma^*_{\delta SELF}$, we make use of the expression given below.

$$|\Sigma^*_{\delta SELF}| = \left[\frac{1}{n}\begin{bmatrix} \left[-\frac{n}{\eta^2}-\frac{a_1-1}{\eta^2}\right] \\ \left[-\frac{n}{\delta^2}+(\eta-1)\sum_{i=1}^{n}\frac{(-4x_i^2)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})^2}-\frac{1}{\delta^2}\right] \\ -\left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right] \\ \left[\sum_{i=1}^{n}\frac{(-2x_i)e^{-2\delta x_i}}{(1-e^{-2\delta x_i})}\right]\end{bmatrix}\right]^{-1} \quad (69)$$

The approximate Bayes estimator of δ under SELF using Gamma-Exponential prior is given by:

$$\hat{\delta}_{SELF} = \sqrt{\frac{|\Sigma^*_{\delta SELF}|}{|\Sigma|}}\exp[n\Delta^*_{\delta}(\hat{\eta}^*,\hat{\delta}^*) - n\Delta((\hat{\eta},\hat{\delta})] \quad (70)$$

C. *Tierney and Kadane's approximation for η under QLF*

The Quadratic Loss Function(QLF) is given by [4]:

$$L(\eta,\hat{\eta}_{QLF}) = \left(\frac{\eta-\hat{\eta}_{QLF}}{\eta}\right)^2$$

under QLF the Bayes estimate of η is:

$$\hat{\eta}_{QLF} = \frac{E(\eta^{-1}|X)}{E(\eta^{-2}|X)}$$

To obtain a Bayesian estimator for η_{QLF} of a function $u = u(\eta,\delta)$ under Gamma-Exponential prior is given below.

Let $u(\eta,\delta) = \eta^{-1}$ then $\Delta^*_{\eta QLF_1}$ function is defined as:

$$\Delta^*_{\eta QLF_1} = \Delta(\eta,\delta;x) - \frac{1}{n}\ln(\eta)$$

We obtain the second order derivatives of $\Delta^*_{\eta\text{QLF1}}(\eta,\delta)$ w.r.to η and δ as.

$$\frac{\partial^2\Delta^*_{\eta QLF1}}{\partial\eta^2} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\eta^2} + \frac{1}{n\eta^2}$$

$$\frac{\partial^2\Delta^*_{\eta QLF1}}{\partial\delta^2} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\delta^2};\ \frac{\partial\Delta^*_{\eta QLF1}}{\partial\eta\partial\delta} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\eta\partial\delta}$$

$$\frac{\partial^2\Delta^*_{\eta QLF1}}{\partial\delta\partial\eta} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\delta\partial\eta}$$

To compute $\Sigma^*_{\eta QLF1}$, we make use of the expression given below.

$$|\Sigma^*_{\eta QLF1}| = \begin{bmatrix} \left[\frac{\partial^2\Delta^*_{\eta QLF1}}{\partial\eta^2}\right]\left[\frac{\partial^2\Delta^*_{\eta QLF1}}{\partial\delta^2}\right] \\ -\left[\frac{\partial\Delta^*_{\eta QLF1}}{\partial\eta\partial\delta}\right]\left[\frac{\partial^2\Delta^*_{\eta QLF1}}{\partial\delta\partial\eta}\right]\end{bmatrix}^{-1} \quad (71)$$

$$\eta_{QLF1} = \sqrt{\frac{|\Sigma^*_{\eta QLF1}|}{|\Sigma|}}\exp[n\Delta^*_{\eta\text{QLF1}}(\hat{\eta}^*,\hat{\delta}^*) - n\Delta((\hat{\eta},\hat{\delta})]$$

Let $u(\eta,\delta) = \eta^{-2}$ then $\Delta^*_{\eta QLF_2}$ is defined as :

$$\Delta^*_{\eta QLF_2} = \Delta(\eta,\delta;x) - \frac{2}{n}ln(\eta)$$

We obtain the second order derivatives of $\Delta^*_{\eta\text{QLF2}}(\eta,\delta)$ w.r.to η and δ as.

$$\frac{\partial^2\Delta^*_{\eta QLF2}}{\partial\eta^2} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\eta^2} + \frac{2}{n\eta^2}$$

$$\frac{\partial^2\Delta^*_{\eta QLF2}}{\partial\delta^2} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\delta^2}$$

$$\frac{\partial^2\Delta^*_{\eta QLF2}}{\partial\eta\partial\delta} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\eta\partial\delta};\frac{\partial^2\Delta^*_{\eta QLF2}}{\partial\delta\partial\eta} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\delta\partial\eta}$$

To compute $|\Sigma^*_{\eta QLF2}|$, we make use of the expression given below.

$$|\Sigma^*_{\eta QLF2}| = \begin{bmatrix} \left[\frac{\partial^2\Delta^*_{\eta QLF2}}{\partial\eta^2}\right]\left[\frac{\partial^2\Delta^*_{\eta QLF2}}{\partial\delta^2}\right] \\ -\left[\frac{\partial\Delta^*_{\eta QLF2}}{\partial\eta\partial\delta}\right]\left[\frac{\partial^2\Delta^*_{\eta QLF2}}{\partial\delta\partial\eta}\right]\end{bmatrix}^{-1} \quad (72)$$

$$\eta_{QLF2} = \sqrt{\frac{|\Sigma^*_{\eta QLF2}|}{|\Sigma|}}\exp[n\Delta^*_{\eta\text{QLF2}}(\hat{\eta}^*,\hat{\delta}^*) - n\Delta((\hat{\eta},\hat{\delta})]$$

The approximate Bayes estimator of η under QLF using Gamma-Exponential prior is given by:

$$\hat{\eta}_{QLF} = \frac{\eta_{QLF1}}{\eta_{QLF2}} \quad (73)$$

D. *Tierney and Kadane approximation for δ under QLF*

The Quadratic Loss Function (QLF) is given by:

$$L(\delta,\hat{\delta}_{QLF}) = \left(\frac{\delta-\hat{\delta}_{QLF}}{\delta}\right)^2$$

The Bayes estimate of δ under QLF is :

$$\hat{\delta}_{QLF} = \frac{E(\delta^{-1}|X)}{E(\delta^{-2}|X)}$$

To obtain a Bayesian estimator for δ_{QLF} of a function $u = u(\eta,\delta)$ under Gamma-Exponential prior is given below:

Let $u(\eta,\delta) = \delta^{-1}$, then $\Delta^*_{\delta_{QLF1}}$ is :

$$\Delta^*_{\delta_{QLF1}} = \Delta(\eta,\delta;x) - \frac{1}{n}\ln(\delta)$$

We obtain the second order derivatives of $\Delta^*_{\delta\text{QLF2}}(\eta,\delta)$ w.r.to η and δ as.

$$\frac{\partial^2\Delta^*_{\delta QLF1}}{\partial\eta^2} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\eta^2}$$

$$\frac{\partial^2\Delta^*_{\delta QLF1}}{\partial\delta^2} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\delta^2} + \frac{1}{n\delta^2}$$

$$\frac{\partial\Delta^*_{\delta QLF1}}{\partial\eta\partial\delta} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\eta\partial\delta};\frac{\partial^2\Delta^*_{\delta QLF1}}{\partial\delta\partial\eta} = \frac{\partial^2\Delta(\eta,\delta)}{\partial\delta\partial\eta}$$

To compute $|\Sigma^*_{\delta QLF1}|$, we make use of the expression given below.

$$|\Sigma^*_{\delta QLF1}| = \begin{bmatrix} \left[\frac{\partial^2\Delta^*_{\delta QLF1}}{\partial\eta^2}\right]\left[\frac{\partial^2\Delta^*_{\delta QLF1}}{\partial\delta^2}\right] \\ -\left[\frac{\partial\Delta^*_{\delta QLF1}}{\partial\eta\partial\delta}\right]\left[\frac{\partial^2\Delta^*_{\delta QLF1}}{\partial\delta\partial\eta}\right]\end{bmatrix}^{-1} \quad (74)$$

$$\delta_{QLF1} = \sqrt{\frac{|\Sigma^*_{\delta QLF1}|}{|\Sigma|}}\exp[n\Delta^*_{\delta\text{QLF1}} - n\Delta((\hat{\eta},\hat{\delta})]$$

Let $u(\eta,\delta)=\delta^{-2}$, then $\Delta^*_{\delta_{QLF2}}$ function is defined as :

$$\Delta^*_{\delta_{QLF2}}=\Delta(\eta,\delta;x)-\frac{2}{n}\ln(\delta)$$

We obtain the second order derivatives of $\Delta^*_{\delta QLF2}\,(\eta,\delta)$ w.r.to η and δ as.

$$\frac{\partial^2\Delta^*_{\delta QLF2}}{\partial\eta^2}=\frac{\partial^2\Delta(\eta,\delta)}{\partial\eta^2}$$

$$\frac{\partial\Delta^*_{\delta QLF2}}{\partial\eta\partial\delta}=\frac{\partial^2\Delta(\eta,\delta)}{\partial\eta\partial\delta};\frac{\partial^2\Delta^*_{\delta QLF2}}{\partial\delta\partial\eta}=\frac{\partial^2\Delta(\eta,\delta)}{\partial\delta\partial\eta}$$

$$\frac{\partial^2\Delta^*_{\delta QLF2}}{\partial\delta^2}=\frac{\partial^2\Delta(\eta,\delta)}{\partial\delta^2}+\frac{2}{n\delta^2}$$

To compute $|\Sigma^*_{\delta QLF2}|$, we make use of the expression given below.

$$|\Sigma^*_{\delta QLF2}|=\begin{bmatrix}\left[\frac{\partial^2\Delta^*_{\delta QLF2}}{\partial\eta^2}\right]\left[\frac{\partial^2\Delta^*_{\delta QLF2}}{\partial\delta^2}\right]\\-\left[\frac{\partial\Delta^*_{\delta QLF2}}{\partial\eta\partial\delta}\right]\left[\frac{\partial^2\Delta^*_{\delta QLF2}}{\partial\delta\partial\eta}\right]\end{bmatrix}^{-1} \quad (75)$$

$$\delta_{QLF2}=\sqrt{\frac{|\Sigma^*_{\delta QLF2}|}{|\Sigma|}}\exp[n\Delta^*_{\delta QLF2}(\hat{\eta}^*,\hat{\delta}^*)-n\Delta\left((\hat{\eta},\hat{\delta})\right]$$

The approximate Bayes estimator of δ under QLF is given by:

$$\hat{\delta}_{QLF}=\frac{\delta_{QLF1}}{\delta_{QLF2}} \quad (76)$$

The derivation of Bayes Estimators and their Bayes risks for different priors, such as LogNormal-Exponential, Weibull-Exponential, LogNormal-Gamma, Weibull-Gamma, and Weibull-LogNormal using the above method are similarly obtained.

VII. Simulation Study

Bayes parameters and their corresponding risks of η and δ are estimated by applying Lindley's and T-K approximation methods under simulation study. This study undergoes the following steps:

- Random samples of size n were generated from a uniform distribution over the interval (0,1), labeled $x_1,x_2,\ldots.x_n$.Using the inverse transformation method, with $\eta=1,1.3$ and $\delta=1.5$ the samples are generated from the Topp-Leone exponential distribution.
- A sample of size n = 25, 50,100 is being considered and the number of replications is fixed as N=1000.
- The hyperparameters are chosen as $a_i=1\,,i=1,\ldots6\ and\ b_1=b_4=1.5, b_2=b_5=1, b_3=2$ for both methods.
- The parameters are estimated through loss functions SELF and QLF based on non-identical priors.
- Bayes risk for both parameters is also found in this study for non-identical priors under different loss functions.
- The results of this study are tabulated and shown in Table: III-IV.
- All the calculations were performed using R packages.

It is evident that from Table III-IV the Bayes risk for the shape and scale parameters of the Topp-Leone exponential distribution consistently decreases as the sample size increases. Further analysis reveals that, under these loss functions when the true values of the parameters η and δ are smaller, the Bayes risk tends to decrease. Also, it is seen that the T-K approximation method produces a lower Bayes risk than Lindley's method for shape and scale parameters. Among the six non-identical priors assessed, the Weibull-lognormal prior combination under the QLF yields a lower Bayes risk than the SELF.

VIII. Real Data Application

A real-life example is considered to evaluate the effectiveness of the estimates.

Data Set I: Data was extracted from "Statistical Methods for Survival Data Analysis" by Lee and Wang [7]. The data set consists of remission times of 128 bladder cancer patients.

Data Set II: This data set refers to the time between failures for 30 repairable items provided by Murthy et al. [12].

sTable V-VI shows the estimated parameters along with Bayes risk. Table V shows that the parameters of SELF with prior Weibull-LogNormal produce minimum risk. In contrast, for QLF, both the parameters with prior Gamma-Exponential perform better under Lindley's approximation method. In Table VI the data set reveals that for the Gamma-Exponential prior, the loss functions are SELF and QLF for scale and shape parameters. In comparison to both methods, Lindley's approximation method has minimum risk.

IX. Conclusion

The Bayes estimator of the Topp-Leone exponential distribution parameters is studied with different priors under symmetric loss functions. As the posterior distribution does not possess a closed form structure, parameter estimates and their risk are obtained through Lindley's and Tierney and Kadane's approximation methods. On comparison of loss functions for simulated data with Bayes risk, QLF with Weibull-LogNormal prior performs better under the T-K approximation method. For the real data sets I and II, QLF for the shape parameter and SELF for the scale parameter perform better for Lindley's approximation.

The real data sets do not yield conclusive results regarding the optimal choice of prior distribution further studies are needed to confirm these findings.

TABLE III
PARAMETER ESTIMATES AND THEIR RISK AT $\eta = 1$AND $\delta = 1.5$ WITH NON-IDENTICAL PRIORS FOR VARIOUS LOSS FUNCTION

SAMPLE SIZES	PRIOR	Lindley's Approximation				T-K Approximation			
		SELF		QLF		SELF		QLF	
		η	δ	η	δ	η	δ	η	δ
25	Gamma-Exponential	1.00072 (0.06412)	1.50413 (0.14005)	0.93467 (0.02782)	1.40336 (0.02872)	1.1946 (0.06269)	1.80894 (0.11947)	1.13021 (0.02360)	1.71015 (0.02534)
	LogNormal-Exponential	1.14681 (0.10656)	1.63222 (0.18449)	1.00915 (0.05679)	1.47048 (0.04690)	1.13432 (0.04471)	1.75285 (0.10018)	1.08160 (0.02082)	1.66445 (0.02347)
	Weibull- Exponential	1.10991 (0.10497)	1.59743 (0.17862)	0.98607 (0.05019)	1.45044 (0.04205)	1.17234 (0.04874)	1.78980 (0.10634)	1.11785 (0.0206)	1.69783 (0.02395)
	LogNormal-Gamma	1.100160 (0.10114)	1.53534 (0.15484)	0.98215 (0.04853)	1.41870 (0.03355)	1.19351 (0.06774)	1.87304 (0.15929)	1.12101 (0.02702)	1.74691 (0.03127)
	Weibull-Gamma	1.06326 (0.09557)	1.50055 (0.14157)	0.96167 (0.04084)	1.40175 (0.02783)	1.23228 (0.07458)	1.91033 (0.16657)	1.15737) (0.02681)	1.78046 (0.03159)
	Weibull-LogNormal	1.17476 (0.10722)	1.73286 (0.19021)	1.02637 (0.06099)	1.53568 (0.05794)	1.09998 (0.02746)	1.64170 (0.05694)	1.06472 (0.01458)	1.58297 (0.01703)
50	Gamma-Exponential	1.01208 (0.03867)	1.51935 (0.07991)	0.95818 (0.02405)	1.44191 (0.02352)	1.07553 (0.01111)	1.63493 (0.02201)	1.06078 (0.00610)	1.61004 (0.00736)
	Log- Normal-Exponential	1.07191 (0.04203)	1.57531 (0.08590)	1.00219 (0.03166)	1.48237 (0.02866)	1.0502 (0.00960)	1.61033 (0.01975)	1.03723 (0.00549)	1.58754 (0.00684)
	Weibull- Exponential	1.05436 (0.04208)	1.55835 (0.08499)	0.98814 (0.02981)	1.46955 (0.02721)	1.06807 (0.00993)	1.62787 (0.02070)	1.05479 (0.00553)	1.60421 (0.00704)
	LogNormal-Gamma	1.05241 (0.04184)	1.53197 (0.08177)	0.98690 (0.02958)	1.45070 (0.02481)	1.07167 (0.01195)	1.65791 (0.02716)	1.05552 (0.00675)	1.62804 (0.00869)
	Weibull-Gamma	1.03486 (0.04118)	1.51501 (0.07933)	0.97371 (0.02733)	1.43902 (0.02302)	1.08951 (0.01233)	1.67545 (0.02818)	1.07308 (0.00676)	1.64474 (0.00886)
	Weibull-LogNormal	1.08203 (0.04180)	1.62005 (0.08577)	1.01043 (0.03256)	1.51853 (0.03164)	1.03940 (0.00752)	1.56402 (0.01381)	1.02936 (0.00423)	1.54700 (0.00530)
100	Gamma-Exponential	1.01104 (0.01850)	1.51045 (0.03946)	0.98000 (0.01458)	1.46547 (0.01430)	1.03704 (0.00215)	1.56039 (0.00245)	1.03274 (0.00207)	1.55691 (0.00115)
	Log- Normal-Exponential	1.03872 (0.01896)	1.53674 (0.04055)	1.00392 (0.01654)	1.48782 (0.01568)	1.02506 (0.00193)	1.54879 (0.00205)	1.02111 (0.00193)	1.54576 (0.00101)
	Weibull- Exponential	1.03006 (0.01900)	1.52839 (0.04038)	0.99615 (0.01605)	1.48057 (0.01528)	1.03380 (0.00198)	1.55729 (0.00224)	1.02979 (0.00194)	1.55405 (0.00107
	LogNormal-Gamma	1.02975 (0.01899)	1.51642 (0.03982)	0.99591 (0.01603)	1.47044 (0.01465)	1.03443 (0.00228)	1.57000 (0.00324)	1.02986 (0.00220)	1.56559 (0.00143)
	Weibull-Gamma	1.02109 (0.01887)	1.50808 (0.03930)	0.98839 (0.01541)	1.46353 (0.01415)	1.04316 (0.00233)	1.57850 (0.00342)	1.03854 (0.0022)	1.57389 (0.00149)
	Weibull-LogNormal	1.04297 (0.01889)	1.55767 (0.04039)	1.007806 (0.01674)	1.50668 (0.01645)	1.02069 (0.00160)	1.52756 (0.00108)	1.01734 (0.00165)	1.52568 (0.00066)

TABLE IV
PARAMETER ESTIMATES AND THEIR RISK AT $\eta = 1.3$AND $\delta = 1.5$WITH NON-IDENTICAL PRIORS FOR VARIOUS LOSS FUNCTIONS.

SAMPLE SIZES	PRIOR	Lindley's Approximation				T-K Approximation			
		SELF		QLF		SELF		QLF	
		η	δ	η	δ	η	δ	η	δ
25	Gamma-Exponential	1.22389 (0.01782)	1.45300 (0.09044)	1.18726 (0.01442)	1.38702 (0.01943)	1.62864 (0.23452)	1.80159 (0.11393)	1.49549 (0.03260)	1.70628 (0.02341)
	Log- Normal-Exponential	1.48766 (0.20965)	1.61135 (0.15663)	1.29934 (0.05831)	1.46986 (0.04154)	1.52213 (0.11226)	1.73784 (0.08712)	1.43219 (0.02606)	1.66151 (0.02064)
	Weibull- Exponential	1.44110 (0.20087)	1.58129 (0.15109)	1.27356 (0.05138)	1.45184 (0.03765)	1.57435 (0.14404)	1.77024 (0.09781)	1.47244 (0.02720)	1.68873 (0.02141)
	LogNormal-Gamma	1.42351 (0.19429)	1.52882 (0.13386)	1.26579 (0.04871)	1.42332 (0.03068)	1.60637 (0.16700)	1.83930 (0.12607)	1.48496 (0.03264)	1.73431 (0.02648)
	Weibull-Gamma	1.37696 (0.17716)	1.49876 (0.12277)	1.24296 (0.04087)	1.40781 (0.02608)	1.66247 (0.22514)	1.87151 (0.14723)	1.52548 (0.03408)	1.76159 (0.02754)
	Weibull-LogNormal	1.53011 (0.21234)	1.69702 (0.16221)	1.32403 (0.06393)	1.52713 (0.05058)	1.46878 (0.08268)	1.64275 (0.05548)	1.40057 (0.02001)	1.58702 (0.01596)
50	Gamma-Exponential	1.29205 (0.06667)	1.49730 (0.06542)	1.22663 (0.02218)	1.43441 (0.01921)	1.41886 (0.02577)	1.62901 (0.01966)	1.39349 (0.00793)	1.60693 (0.00655)
	Log- Normal-Exponential	1.39328 (0.08057)	1.56446 (0.07404)	1.29527 (0.03375)	1.48320 (0.02523)	1.38383 (0.02076)	1.60508 (0.01703)	1.36265 (0.00677)	1.58534 (0.00595)
	Weibull- Exponential	1.37298 (0.08022)	1.55045 (0.07325)	1.27976 (0.03186)	1.47236 (0.02412)	1.40429 (0.02199)	1.61932 (0.01784)	1.38219 (0.00696)	1.59884 (0.00611)
	LogNormal-Gamma	1.36670 (0.07960)	1.52704 (0.07063)	1.27545 (0.03124)	1.45510 (0.02213)	1.41313 (0.02526)	1.64584 (0.02252)	1.38735 (0.00815)	1.62086 (0.00734)
	Weibull-Gamma	1.34641 (0.07809)	1.51303 (0.0687)	1.26093 (0.02901)	1.44512 (0.02076)	1.43368 (0.02693)	1.66006 (0.02365)	1.40697 (0.00831)	1.63438 (0.00751)
	Weibull-LogNormal	1.41068 (0.08032)	1.60383 (0.07429)	1.30893 (0.03515)	1.51552 (0.02769)	1.36476 (0.01706)	1.56396 (0.01270)	1.34762 (0.00550)	1.54859 (0.00477)

SAMPLE SIZES	PRIOR	Lindley's Approximation				T-K Approximation			
		SELF		QLF		SELF		QLF	
		η	δ	η	δ	η	δ	η	δ
100	Gamma-Exponential	1.30463 (0.03382)	1.50049 (0.03368)	1.26292 (0.01493)	1.46228 (0.01217)	1.35681 (0.00434)	1.55816 (0.00214)	1.35015 (0.00245)	1.55517 (0.00098)
	Log- Normal-Exponential	1.35021 (0.03576)	1.53167 (0.03524)	1.30083 (0.01793)	1.48888 (0.01376)	1.34087 (0.00374)	1.54707 (0.00174)	1.33495 (0.00221)	1.54456 (0.00084)
	Weibull- Exponential	1.34045 (0.03575)	1.52487 (0.03509)	1.29226 (0.01743)	1.48290 (0.01346)	1.35066 (0.00387)	1.55393 (0.00187)	1.34460 (0.00224)	1.55126 (0.00088)
	LogNormal-Gamma	1.33802 (0.03569)	1.51399 (0.03461)	1.29021 (0.01729)	1.47356 (0.01293)	1.35360 (0.00440)	1.56547 (0.00262)	1.34681 (0.00251)	1.56191 (0.00116)
	Weibull-Gamma	1.32826 (0.03544)	1.50719 (0.03422)	1.28193 (0.01669)	1.46783 (0.01256)	1.36340 (0.00454)	1.57233 (0.00276)	1.35648 (0.00254)	1.56862 (0.00120)
	Weibull-LogNormal	1.35798 (0.03565)	1.55036 (0.03520)	1.30778 (0.01826)	1.50584 (0.01439)	1.33279 (0.00314)	1.52802 (0.0010)	1.32769 (0.00192)	1.52639 (0.00057)

TABLE V
PARAMETER ESTIMATES AND THEIR RISK WITH NON-IDENTICAL PRIORS FOR VARIOUS LOSS FUNCTIONS

PRIOR	Lindley's Approximation				T-K Approximation			
	SELF		QLF		SELF		QLF	
	η	δ	η	δ	η	δ	η	δ
Gamma-Exponential	1.210838 (0.022102)	0.060596 (0.000046)	1.176787 (0.013726)	0.059130 (0.011952)	1.229183 (0.022275)	0.061176 (0.000046)	1.193375 (0.014722)	0.059658 (0.012614)
Log- Normal-Exponential	1.240480 (0.021645)	0.061530 (0.000045)	1.204211 (0.014805)	0.060013 (0.012459)	1.275974 (0.023697)	0.063714 (0.000047)	1.239283 (0.014529)	0.062238 (0.011756)
Weibull-Exponential	1.232990 (0.021926)	0.061294 (0.000046)	1.197035 (0.014632)	0.059785 (0.012372)	1.241214 (0.022872)	0.061550 (0.000046)	1.204812 (0.014822)	0.060026 (0.012584)
Log-Normal-Gamma	1.240131 (0.021660)	0.061507 (0.000045)	1.203873 (0.014799)	0.059990 (0.012452)	1.226255 (0.021877)	0.061111 (0.000046)	1.191003 (0.014525)	0.059602 (0.012549)
Weibull-Gamma	1.232641 (0.021936)	0.061271 (0.000046)	1.196705 (0.014623)	0.059763 (0.012362)	1.226893 (0.022197)	0.061129 (0.000046)	1.191147 (0.014724)	0.059610 (0.012629)
Weibull-Log-Normal	1.265967 (0.019847)	0.063471 (0.000038)	1.230001 (0.014839)	0.062035 (0.012008)	1.238064 (0.022455)	0.061478 (0.000046)	1.202234 (0.014623)	0.059964 (0.012518)

TABLE VI
PARAMETER ESTIMATES AND THEIR RISK WITH NON-IDENTICAL PRIORS FOR VARIOUS LOSS FUNCTIONS

PRIOR	Lindley's Approximation				T-K Approximation			
	SELF		QLF		SELF		QLF	
	η	δ	η	δ	η	δ	η	δ
Gamma-Exponential	1.799276 (0.240054)	0.458607 (0.008294)	1.700452 (0.015648)	0.437046 (0.018897)	2.088803 (0.306312)	0.496867 (0.009304)	1.812799 (0.069375)	0.457661 (0.041715)
Log- Normal-Exponential	2.194542 (0.340035)	0.509526 (0.010082)	1.905003 (0.064243)	0.471750 (0.036453)	2.193678 (0.343185)	0.509999 (0.009619)	1.902069 (0.069611)	0.470652 (0.040701)
Weibull-Exponential	2.144375 (0.344658)	0.503063 (0.010143)	1.872066 (0.059592)	0.466645 (0.034895)	2.141523 (0.326937)	0.503465 (0.009504)	1.854849 (0.070298)	0.463955 (0.041472)
Log-Normal-Gamma	2.172315 (0.342705)	0.504453 (0.010137)	1.890085 (0.062253)	0.467723 (0.035249)	2.207723 (0.347758)	0.513221 (0.009700)	1.914124 (0.069643)	0.473799 (0.040514)
Weibull-Gamma	2.122147 (0.345097)	0.497991 (0.010131)	1.858281 (0.057355)	0.462800 (0.033516)	2.155118 (0.331082)	0.506647 (0.009583)	1.866624 (0.070299)	0.467074 (0.041268)
Weibull-Log-Normal	2.249972 (0.329074)	0.527161 (0.009492)	1.944655 (0.068673)	0.486992 (0.039451)	2.237223 (0.345254)	0.525929 (0.009461)	1.947270 (0.067913)	0.488624 (0.037222)

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