Analysis of Asymptotic Behavior and Growth Conditions for a Decreasing Function

Bahloul Tarek

***Abstract*—We investigate a positive, decreasing function $h(t)$ that satisfies the inequality $G(\int_0^t e^s h(s)ds) \leq e^{\xi t} h(t)$. leading to the condition $(h^{-\alpha})'' \geq 0$. This condition implies the inequality $E\left(\vartheta\sqrt{\psi} + \psi t\right) \leq \left(1 + \tau\sqrt{\psi} t\right)^{-\frac{1}{\psi}}$, our study explores the impact of $h(t)$ and another positive, decreasing function $E(t)$. The results contribute critical insights into these functions, with significant implications for differential equations, and dynamical systems. This work presents a framework for generating novel inequalities, enhancing the theoretical understanding and practical applications of these mathematical concepts.**

***Index Terms*—Non-increasing functions, asymptotic behavior, integral properties, power-law decay, inequalities.**

I. Introduction

The study of decreasing functions is crucial in various mathematical and scientific disciplines, particularly in analyzing differential equations and dynamical systems. This paper delves into the asymptotic behavior and growth conditions of such functions, focusing on $h(t)$, a positive, decreasing function that adheres to specific inequalities and constraints. Our investigation centers around the disparity:

$$G(\int_0^t e^s h(s)ds) \leq e^{\xi t} h(t),$$

which leads to a critical condition:

$$(h^{-\alpha})'' \geq 0.$$

This condition is instrumental in understanding the asymptotic behavior of $h(t)$ and its relationship with another positive, decreasing function $E(t)$. This analysis derives important inequalities and growth conditions for these functions.

The paper aims to elucidate how the exponential decay of $E(t)$ is transformed into corresponding bounds for $h(t)$ through the function G. Additionally, we explore how constraints on derivatives and integral properties of $E(t)$ impact $h(t)$. These insights are not only theoretical but also have practical implications for mathematical modeling and dynamical systems, offering a framework for generating new inequalities and enhancing the understanding of these functions' behavior.

By presenting a detailed exploration of these functions and their interactions, this research contributes to a deeper understanding of asymptotic analysis and growth conditions, with broad applications in differential equations and dynamical systems.

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II. Asymptotic Analysis and Inequalities for Bounded Functions of Class $\mathcal{C}^2$

This section provides scientific insights into the asymptotic behavior and inequalities of bounded, twice continuously differentiable functions ($\mathcal{C}^2$ class). We present key lemmas that describe the decay rates, integral relationships, and inequalities involving these functions and their derivatives. Understanding these properties is essential in fields such as differential equations and control theory, where the stability and long-term behavior of systems are analyzed.

In each lemma presented in this section, we assume that γ, γ', a, b, and t are strictly positive real numbers, satisfying the condition $\gamma > \gamma' + b$ and $\gamma' > b$.

Lemma 1: Let $h(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function of class $\mathcal{C}^2$ such that

$$h(t) \geq ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)} h(s)ds, \tag{1}$$

then there exists $\delta > 0$ verifying:

$$h''(t) \geq \delta h(t).$$

$h(t)$ decreasing, and

$$\lim_{t\to+\infty} h'(t) = 0, \quad \lim_{t\to+\infty} h(t) = 0$$

and for all $t \geq 0$.

$$h'(t) + \sqrt{\delta} h(t) \leq 0$$

Proof:

Let $h(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function of class $\mathcal{C}^2$ such that

$$h(t) \geq ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)} h(s)\, ds, \tag{2}$$

Differentiating both sides with respect to t, we get

$$h'(t) \geq -a\gamma' e^{-\gamma' t} - b\gamma \int_0^t e^{-\gamma(t-s)} h(s)\, ds + bh(t), \tag{3}$$

Differentiating again with respect to t, we obtain

$$h''(t) \geq a(\gamma')^2 e^{-\gamma' t} + b(\gamma)^2 \int_0^t e^{-\gamma(t-s)} h(s)\, ds \tag{4}$$
$$-b\gamma h(t) + bh'(t).$$

Substituting $h'(t)$ from (3) into (4), we get

$$h''(t) \geq a(\gamma')^2 e^{-\gamma' t} + b(\gamma)^2 \int_0^t e^{-\gamma(t-s)} h(s)\, ds \tag{5}$$
$$-b\gamma h(t)$$
$$+b\left(-a\gamma' e^{-\gamma' t} - b\gamma \int_0^t e^{-\gamma(t-s)} h(s)\, ds + bh(t)\right),$$

Simplifying the above equation, we have

$$h''(t) \geq a(\gamma')^2 e^{-\gamma' t} - ba\gamma' e^{-\gamma' t} + b^2 a e^{-\gamma' t} \quad (6)$$

$$+ b(\gamma)^2 \int_0^t e^{-\gamma(t-s)} h(s)\, ds - b\gamma h(t) \quad (7)$$

$$(8)$$

$$+b\left(-b\gamma \int_0^t e^{-\gamma(t-s)} h(s)\, ds + b^2 \int_0^t e^{-\gamma(t-s)} h(s)\, ds\right),$$

which simplifies to

$$h''(t) \geq a\left(\gamma'(\gamma' - b) + b^2\right) e^{-\gamma' t} \quad (9)$$

$$+ b\left(\gamma(\gamma - b) + b^2\right) \int_0^t e^{-\gamma(t-s)} h(s)\, ds \quad (10)$$

$$-b\gamma h(t).$$

Thus, we have

$$h''(t) \geq a\left(\gamma'(\gamma' - b) + b^2\right) e^{-\gamma' t} \quad (11)$$

$$+ b\left(\gamma'(\gamma' - b) + b^2\right) \int_0^t e^{-\gamma(t-s)} h(s)\, ds \quad (12)$$

$$-b\gamma h(t).$$

Therefore, we can write

$$h''(t) \geq \left(\gamma'(\gamma' - b) - b(\gamma - b)\right) h(t), \quad (13)$$

If we take $\delta = \gamma'(\gamma' - b) - b(\gamma - b) > 0$, then (Table III)

$$h''(t) \geq \delta h(t), \quad (14)$$

such that there exists δ verifying: $h(t)$ decreasing, and we have $h''(t) \geq \delta h(t) \geq 0$. Thus, $h'(t)$ increases. Hence, $h'(t)$ has a limit in $\overline{\mathbb{R}}$ at $+\infty$. Moreover, in the neighborhood of $+\infty$, $h'(t)$ has a fixed sign and $h(t)$ is monotone.
Since $h(t)$ is bounded, it therefore has a finite limit at $+\infty$. Consequently, $\int_0^{+\infty} h'(t)\, dt$ converges, and the limit of $h'(t)$ at $+\infty$ is necessarily 0.

$h'(t)$ increases and $\lim_{t\to+\infty} h'(t) = 0$. Thus, $h'(t) \leq 0$ and $h(t)$ decreases. Let $l = \lim_{t\to+\infty} h(t)$. We have $l \geq 0$, and $h''(t) \geq \delta h(t) \geq \delta l$. Hence,

$$\forall t \geq 0, \quad h'(t) - h'(0) \geq \delta l.$$

But $\lim_{t\to+\infty} h'(t) = 0$, so $l = 0$. Let $\theta(t) = (h'(t) + \sqrt{\delta} h(t)) e^{-t\sqrt{\delta}}$. We have:

$$\theta'(t) = (h''(t) - \delta h(t)) e^{-t\sqrt{\delta}} \geq 0.$$

Thus, θ is increasing. But $\lim_{t\to+\infty} \theta(t) = 0$. Hence, $\theta \leq 0$. Then, let $\varphi(t) = h(t) e^{\sqrt{\delta} t}$. We have $\varphi'(t) = \theta(t) e^{2\sqrt{\delta} t}.\varphi$ decreases, so

$$h(t) \leq h(0) e^{-\sqrt{\delta} t}$$

for all $t \geq 0$.

$$h'(t) \leq -h(0)\sqrt{\delta} e^{-\sqrt{\delta} t} \leq -\sqrt{\delta} h(t)$$

■

Lemma 2: Let $h(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function of class $\mathcal{C}^2$ such that

$$h(t) \geq a e^{-\gamma' t} + b \int_0^t e^{-\gamma(t-s)} h(s) ds, \quad (15)$$

TABLE I
EXAMPLES OF $\delta = \gamma'(\gamma' - b) - b(\gamma - b) > 0$

Example	γ	γ'	b	δ
1	0.9876	0.6543	0.3214	0.0036
2	1.2345	0.7654	0.3456	0.0141
3	0.5678	0.3214	0.1234	0.0087

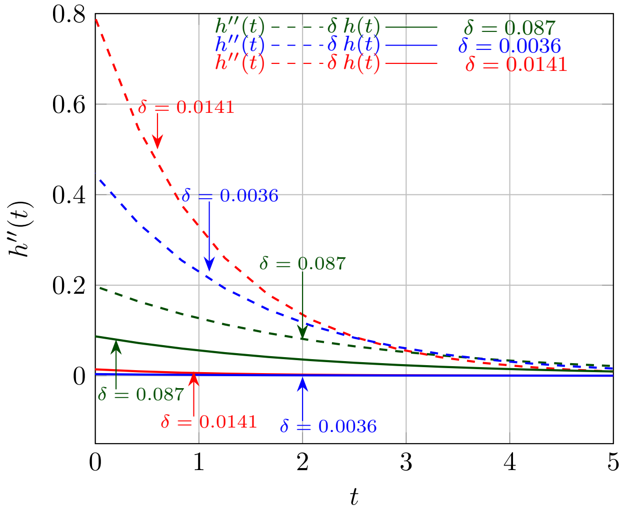

Fig. 1. Function $\delta h(t) \leq h''(t)$

then

$$\lim_{t\to+\infty} h(t) = \lim_{t\to+\infty} h'(t) = 0,$$

$$-\beta \leq \frac{h'(t)}{h(t)} \leq -\sqrt{\delta}$$

$$lim_{t\to+\infty} \frac{h'(t)}{h(t)} = -\sqrt{\delta},$$

and

$$lim_{t\to+\infty} \frac{h''(t)}{h(t)} = \delta.$$

Proof:
According to Lemma 1, we have $h'(t) \leq 0$, which implies that $h(t)$ is a non-increasing function. Since $h(t)$ is always positive, we conclude that $h(t)$ is decreasing and bounded below by 0. Thus, by the properties of monotone sequences, we can assert that:

$$\exists l \in \mathbb{R}_+, \qquad l = \lim_{t\to+\infty} h(t).$$

Applying the Monotone Convergence Theorem, we consider the auxiliary function:

$$g(t) = h'(t) + \beta h(t),$$

where $\beta = \gamma - b$. We then have:

$$e^{\beta t} g(t) = e^{\beta t} h'(t) + e^{\beta t} \beta h(t).$$

Rewriting the above equation, we get:

$$e^{\beta t} g(t) = \left(e^{\beta t} h(t)\right)'.$$

Integrating both sides from 0 to t:

$$\int_0^t e^{\beta s} g(s)\, ds = e^{\beta t} h(t) - h(0).$$

Rearranging, we obtain:

$$h(t) = h(0)e^{-\beta t} + e^{-\beta t}\int_0^t e^{\beta s} g(s)\, ds.$$

Or equivalently:

$$h(t) = h(0)e^{-\beta t} + \int_0^t e^{-\beta(t-s)} g(s)\, ds.$$

Next, we apply the Cauchy-Schwarz inequality:

$$\left(\int_0^t e^{\beta s} g(s)\, ds\right)^2 \leq \int_0^t g^2(s)\, ds \int_0^t e^{2\beta s}\, ds$$
$$\leq \frac{1}{2\beta} e^{2\beta t}\int_0^{+\infty} g^2(s)\, ds.$$

Therefore, we find that:

$$|h(t)| \leq |h(0)| + \frac{1}{\sqrt{2\beta}}\sqrt{\int_0^{+\infty} g^2(s)\, ds.}$$

This implies that $h(t)$ is bounded. To establish further properties of $h(t)$, consider:

$$|h(t+\xi)| \leq |h(\xi)|e^{-\beta t} + \frac{\epsilon}{2},$$

where $\xi > 0$ and $\sqrt{\int_\xi^{+\infty} g^2(s)\, ds} \leq \frac{\epsilon}{2\sqrt{2\beta}}$.

Thus,

$$\forall t \geq \frac{\xi}{\beta}, \quad |h(t)| \leq |h(\xi)|e^{\xi-\beta t} + \frac{\epsilon}{2}.$$

Moreover,

$$\lim_{t\to+\infty} |h(\xi)|e^{\xi-\beta t} = 0.$$

Therefore, $\exists \zeta > 0$ such that $\forall t \geq 0, \quad |h(\xi)|e^{\xi-\beta t} \leq \frac{\epsilon}{2}$.

Thus,

$$\forall t \geq \sup(\zeta, \xi), \quad |h(t)| \leq \epsilon.$$

This implies that ultimately,

$$\lim_{t\to+\infty} h(t) = 0.$$

Then, we have:

$$\lim_{t\to+\infty} h'(t) = 0, \quad \text{and} \quad -\beta \leq \frac{h'(t)}{h(t)} \leq -\sqrt{\delta}.$$

Thus,

$$\lim_{t\to+\infty} \frac{h'(t)}{h(t)} = -\sqrt{\delta}.$$

By L'Hopital's rule, we get:

$$\lim_{t\to+\infty} \frac{h'(t)}{h(t)} = \lim_{t\to+\infty} \frac{h''(t)}{h'(t)} = \lim_{t\to+\infty} \frac{\frac{h''(t)}{h(t)}}{\frac{h'(t)}{h(t)}}.$$

Therefore,

$$-\sqrt{\delta} = \lim_{t\to+\infty} \frac{\frac{h''(t)}{h(t)}}{-\sqrt{\delta}},$$

and hence:

$$\lim_{t\to+\infty} \frac{h''(t)}{h(t)} = \delta.$$

■

Lemma 3: If a C^2-function $h(t) > 0$ satisfies for $t \geq 0$,

$$h(t) \geq ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)} h(s) ds, \tag{16}$$

then

$$lim_{t\to+\infty} \frac{e^{-\beta t}\int_0^t e^{\beta s} h(s) ds}{h(t)} = \frac{1}{\beta - \sqrt{\delta}},$$

and

$$lim_{t\to+\infty} \frac{\int_0^t e^{\beta s} g(s) ds}{e^{\beta t} h(t)} = \frac{1}{(\beta - \sqrt{\delta})^2}.$$

Proof:

According to Lemma 2, we have

$$g(t) = e^{-\beta t}\int_0^t e^{\beta s} h(s)\, ds$$

which is the forcing term of the ordinary differential equation (7). As $t \to \infty$, it can be shown that $g(t) \to 0$.

For all $t \geq \tau \geq 0$, we have:

$$g(t) = e^{-\beta t}\int_0^t e^{\beta s} h(s)\, ds$$

$$\leq e^{-\beta t}\int_0^\tau e^{\beta s} h(s)\, ds + \sup_{s\in[\tau,t]} h(s) e^{-\beta t}\int_\tau^t e^{\beta s}\, ds.$$

Thus,

$$g(t) \leq e^{-\beta t}\int_0^\tau e^{\beta s} h(s)\, ds + \frac{1}{\beta}\sup_{s\in[\tau,t]} h(s)(1 - e^{-\beta(t-\tau)}).$$

As a result,

$$g(t) \leq e^{-\beta t}\int_0^\tau e^{\beta s} h(s)\, ds + \frac{1}{\beta}\sup_{s\in[\tau,t]} h(s).$$

Let $\epsilon > 0$. Choose $\tau \geq 0$ such that for all $t \geq \tau$, we have $h(t) \leq \frac{\epsilon\beta}{2}$. The function

$$t \mapsto e^{-\beta t}\int_0^\tau e^{\beta s} h(s)\, ds$$

tends to 0 as t tends to $+\infty$. Therefore, there exists $t_0 \geq 0$ such that

$$t \geq t_0 \quad implies \quad e^{-\beta t}\int_0^\tau e^{\beta s} h(s)\, ds \leq \frac{\epsilon}{2}.$$

For $t \geq \sup(t_0, \tau)$, we then have $0 \leq g(t) \leq \epsilon$. This implies that $g(t)$ tends to 0 as t tends to $+\infty$.

Applying L'Hopital's rule, we obtain:

$$\lim_{t\to\infty} \frac{g(t)}{h(t)} = \lim_{t\to\infty} \frac{e^{-\beta t}\int_0^t e^{\beta s} h(s)\, ds}{h(t)}$$
$$= \lim_{t\to\infty} \frac{\int_0^t e^{\beta s} h(s)\, ds}{e^{\beta t} h(t)}.$$

Evaluating the limit, we get:

$$\lim_{t\to\infty} \frac{\int_0^t e^{\beta s} h(s)\, ds}{e^{\beta t} h(t)} = \lim_{t\to\infty} \frac{e^{\beta t} h(t)}{\beta e^{\beta t} h(t) + e^{\beta t} h'(t)}.$$

Simplifying, we find:

$$\lim_{t\to\infty} \frac{h(t)}{\beta h(t) + h'(t)} = \lim_{t\to\infty} \frac{1}{\beta + \frac{h'(t)}{h(t)}} = \frac{1}{\beta - \sqrt{\delta}}.$$

Using L'Hopital's rule again, we have:

$$\lim_{t\to\infty} \frac{\int_0^t e^{\beta s} g(s)\, ds}{e^{\beta t} h(t)} = \lim_{t\to\infty} \frac{e^{\beta t} g(t)}{\beta e^{\beta t} h(t) + e^{\beta t} h'(t)}.$$

Therefore,

$$\lim_{t\to\infty} \frac{e^{\beta t}g(t)}{\beta e^{\beta t}h(t)+e^{\beta t}h'(t)} = \lim_{t\to\infty} \frac{\frac{g(t)}{h(t)}}{\beta+\frac{h'(t)}{h(t)}}$$

$$= \frac{\frac{1}{\beta-\sqrt{\delta}}}{\beta-\sqrt{\delta}} = \frac{1}{(\beta-\sqrt{\delta})^2}.$$

■

Lemma 4: If a C^2-function $h(t)>0$ satisfies for $t\geq 0$,

$$h(t) \geq ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)}h(s)ds, \tag{17}$$

then

$$h'(t)+\beta h(t) \geq 0 \quad with \quad \beta=\gamma-b, \tag{18}$$

and

$$1-\frac{h(0)}{\beta} \geq 1-\int_0^\infty h(s)ds \geq 1-\frac{h(0)}{\sqrt{\delta}}.$$

Proof:

Starting from the given inequality:

$$h(t) \geq ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)}h(s)\,ds, \tag{19}$$

we differentiate both sides concerning t:

$$h'(t) \geq -a\gamma' e^{-\gamma' t} - b\gamma\int_0^t e^{-\gamma(t-s)}h(s)\,ds + bh(t), \tag{20}$$

which can be rewritten as:

$$h'(t)-bh(t) \geq -a\gamma' e^{-\gamma' t} - b\gamma\int_0^t e^{-\gamma(t-s)}h(s)\,ds. \tag{21}$$

Since $\gamma > \gamma'+b$, we have:

$$h'(t)-bh(t) \geq -a\gamma e^{-\gamma' t} - b\gamma\int_0^t e^{-\gamma(t-s)}h(s)\,ds, \tag{22}$$

which simplifies to:

$$h'(t)-bh(t) \geq -\gamma h(t). \tag{23}$$

Thus,

$$h'(t)+\beta h(t) \geq 0 \quad \text{with} \quad \beta=\gamma-b. \tag{24}$$

Given that $\beta>\sqrt{\delta}$, we have (Figure 2):

$$h(0)e^{-\beta t} \leq h(t) \leq h(0)e^{-\sqrt{\delta}t}.$$

Therefore, we can integrate $h(t)$ over $[0,\infty)$ to obtain:

$$\frac{h(0)}{\beta} \leq \int_0^\infty h(s)\,ds \leq \frac{h(0)}{\sqrt{\delta}}.$$

Hence,

$$1-\frac{h(0)}{\beta} \geq 1-\int_0^\infty h(s)\,ds \geq 1-\frac{h(0)}{\sqrt{\delta}}.$$

■

Lemma 5: If a C^2-function $h(t)>0$ satisfies for $t\geq 0$,

$$h(t) \geq ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)}h(s)ds, \tag{25}$$

with $\beta>\sqrt{\delta}$, then

$$h(t)h''(t)-(1+\alpha)(h'(t))^2 \leq 0.$$

TABLE II

EXAMPLES OF $\beta>\sqrt{\delta}$ AND $\beta=\gamma-b$

Example	γ	β	b	$\sqrt{\delta}$
1	0.9876	0.6662	0.3214	0.06
2	1.2345	0.8889	0.3456	0.1187
3	0.5678	0.4444	0.1234	0.0932

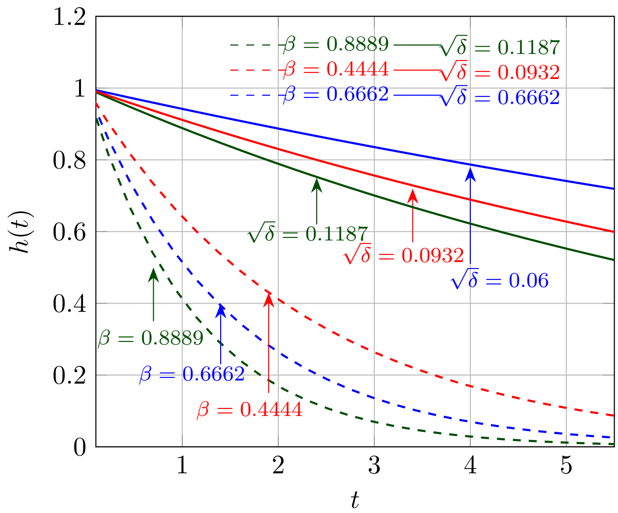

Fig. 2. Function $h(0)e^{-\beta t} \leq h(t) \leq h(0)e^{-\sqrt{\delta}t}$

TABLE III

EXAMPLES OF β AND $\sqrt{\delta}$

Example	β	$\sqrt{\delta}$
1	0.8889	0.1187
2	0.6662	0.06
3	0.4444	0.0932

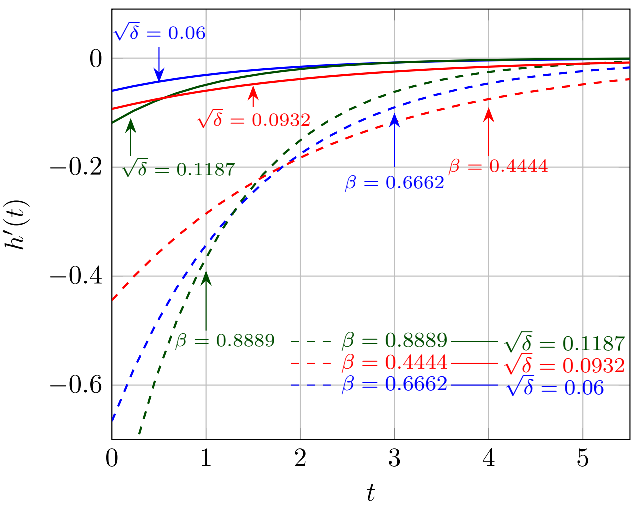

Fig. 3. Function $-\beta h(t) \leq h'(t) \leq -\sqrt{\delta}h(t)$

Proof:

Given: (Figure 3)

$$-\beta h(t) \leq h'(t) \leq -\sqrt{\delta}h(t),$$

we have:

$$-\beta h(t)h'(t) \geq (h'(t))^2 \geq -\sqrt{\delta}h(t)h'(t).$$

Thus,

$$h(t)h''(t)+\beta(1+\alpha)h(t)h'(t) \leq h(t)h''(t)$$

$$-(1+\alpha)(h'(t))^2 \le h(t)h''(t) + (1+\alpha)\sqrt{\delta}h(t)h'(t).$$

Next, we can write:

$$h(t)h''(t) + \beta(1+\alpha)h(t)h'(t) \ge \left(\delta - \beta^2(1+\alpha)\right)h^2(t)$$
$$\left[-\left((\beta-\sqrt{\delta})(\beta+\sqrt{\delta})\right) - \alpha\beta^2\right]h^2(t) \le 0$$

which leads to:

$$\left[-\left((\beta-\sqrt{\delta})(\beta+\sqrt{\delta})\right) - \alpha\beta^2\right]h^2(t) \le h(t)h''(t)$$
$$-(1+\alpha)(h'(t))^2 \le -\alpha h(0)h(t)e^{-\sqrt{\delta}t} < 0.$$

Therefore, we obtain:

$$h(t)h''(t) - (1+\alpha)(h'(t))^2 \le 0.$$

■

III. Exponential Bounds and Integral Relationships for Non-Increasing Functions

This section explores key properties of non-increasing functions, focusing on their exponential decay and integral relationships. We present theorems that provide bounds on these functions and their integrals, offering insights into their long-term behavior and decay rates. This analysis is essential for understanding the dynamics of systems described by non-increasing functions in various scientific and engineering applications.

Theorem 1: Let E: $R^+ \to R^+$ be a non-increasing function.
Then

$$E(t) \le \frac{E(0)}{e^{-2(1+\alpha)}}e^{-2(1+\alpha)e^{-\beta t}}. \tag{26}$$

By invoking Lemma 5, we derive the following expression:

$$\left[(h')^2 E\right]' = 2h''h'E + (h')^2E'. \tag{27}$$

Given the inequality

$$h(t)h''(t) \le (1+\alpha)(h'(t))^2,$$

we can further infer:

$$2h'h''(t)E \ge 2(1+\alpha)\frac{(h'(t))^3}{h(t)}E.$$

Combining the above, we obtain:

$$2h'h''(t)E + (h')^2E' \ge 2(1+\alpha)\frac{(h'(t))^3}{h(t)}E + (h')^2E'.$$

Therefore:

$$0 \ge 2(1+\alpha)\frac{(h'(t))^3}{h(t)}E + (h')^2E'.$$

Dividing through by $(h')^2E$, we get:

$$\frac{E'}{E} \le -2(1+\alpha)\frac{h'(t)}{h(t)}.$$

Considering the condition $h' + \beta h \ge 0$ and the fact that $\beta e^{-\beta t} < \beta$, it follows that:

$$0 \le h' + \beta e^{-\beta t}h \le h' + \beta h.$$

This implies:

$$0 \ge -(1+\alpha)h' - (1+\alpha)\beta e^{-\beta t}h \ge -(1+\alpha)h' - (1+\alpha)\beta h,$$

and consequently:

$$0 \ge -(1+\alpha)\frac{h'}{h} - (1+\alpha)\beta e^{-\beta t} \ge -(1+\alpha)\frac{h'}{h} - (1+\alpha)\beta.$$

Thus:

$$\frac{E'}{E} \le 2\beta(1+\alpha)e^{-\beta t}.$$

Integrating both sides, we obtain:

$$\frac{E}{E(0)} \le e^{2\beta(1+\alpha)\int_0^t e^{-\beta s}ds} = \frac{e^{-2(1+\alpha)e^{-\beta t}}}{e^{-2(1+\alpha)}},$$

and therefore:

$$E(t) \le \frac{E(0)}{e^{-2(1+\alpha)}}e^{-2(1+\alpha)e^{-\beta t}}.$$

Theorem 2: Let E: $R^+ \to R^+$ be a non-increasing function.
If

$$h(t) = e^{\frac{t}{T}}\int_t^\infty E(s)ds, \quad t \in R^+,\ T = \frac{\sqrt{\delta(1+\alpha)}}{\delta}. \tag{28}$$

then

$$\int_t^\infty E(s)ds \le \frac{1+\alpha}{4\delta}E(t), \tag{29}$$

and

$$E(t) \le \frac{1}{4}\frac{\sqrt{\delta(1+\alpha)}}{\delta}E(0)e^{1-t}, \quad t \in R^+. \tag{30}$$

Proof: We begin by noting that h is locally absolutely continuous and non-increasing, as established in Lemma 5:

$$h'(t) = \frac{1}{T}h(t) - e^{\frac{t}{T}}E(t), \quad t \in \mathbb{R}^+, \quad T > 0. \tag{31}$$

Considering the expression for the squared derivative, we have:

$$(1+\alpha)\left(h'(t)\right)^2 = \frac{(1+\alpha)}{T^2}h^2(t) \tag{32}$$
$$-\frac{2(1+\alpha)}{T}e^{\frac{t}{T}}h(t)E(t)$$
$$+(1+\alpha)e^{\frac{2t}{T}}E^2(t), \quad t \in \mathbb{R}^+, \quad T > 0.$$

Given that:

$$h''(t)h(t) \ge \delta h^2(t), \tag{33}$$

we obtain the inequality:

$$\frac{(1+\alpha)}{T^2}h^2(t) - \frac{2(1+\alpha)}{T}e^{\frac{t}{T}}h(t)E(t) \tag{34}$$
$$+(1+\alpha)e^{\frac{2t}{T}}E^2(t) \ge \delta h^2(t).$$

Assuming $\delta = \frac{(1+\alpha)}{T^2}$, we get:

$$e^{\frac{t}{T}}E(t) \ge \frac{2}{T}h(t). \tag{35}$$

By multiplying both sides by $\frac{T}{2}$, we have:

$$\frac{T}{2}e^{\frac{t}{T}}E(t) \ge h(t). \tag{36}$$

Moreover, considering the integral form, we obtain:

$$\frac{T}{2}e^{\frac{t}{T}}E(t) \ge \frac{2}{T}e^{\frac{t}{T}}\int_t^\infty E(s)ds. \tag{37}$$

Hence, it follows that:

$$\int_t^\infty E(s)ds \le \frac{1+\alpha}{4\delta}E(t). \tag{38}$$

Given that $h(t) = e^{\frac{t}{T}} \int_t^\infty E(s)ds$, we can state:

$$h(t) \leq h(0) = \int_0^\infty E(s)ds \leq \frac{1+\alpha}{4\delta}E(0), \qquad (39)$$
$$t \in \mathbb{R}^+.$$

Thus:

$$\int_t^\infty E(s)ds \leq \frac{1+\alpha}{4\delta}E(0)e^{-\frac{\sqrt{\delta(1+\alpha)}}{(1+\alpha)}t}, \qquad (40)$$
$$t \in \mathbb{R}^+.$$

Since E is nonnegative and non-increasing, we have:

$$\int_t^\infty E(s)ds \geq \int_t^{t+T} E(s)ds \geq TE(t+T). \qquad (41)$$

Therefore:

$$E(t+T) \leq \frac{1}{4}\frac{\sqrt{\delta(1+\alpha)}}{\delta}E(0)e^{-\frac{\sqrt{\delta(1+\alpha)}}{(1+\alpha)}t}, \quad t \in \mathbb{R}^+. \qquad (42)$$

By setting $t := t + \frac{\sqrt{\delta(1+\alpha)}}{\delta}$, we obtain:

$$E(t) \leq \frac{1}{4}\frac{\sqrt{\delta(1+\alpha)}}{\delta}E(0)e^{1-t}, \quad t \in \mathbb{R}^+. \qquad (43)$$

■

Theorem 3: Let $E : R^+ \to R^+$ be a non-increasing function and assume that there is a constant $\alpha > 0$.
If

$$h(t) = \int_t^\infty E^{\zeta+1}(s)ds, \quad t \in R^+, \qquad (44)$$

then

$$\int_t^\infty E^{\zeta+1}(s)ds \leq \frac{\sqrt{\delta(1+\alpha)}}{\delta}E^\zeta(0)E(t), \quad t \in R^+. \qquad (45)$$

and

$$E\left(\frac{\sqrt{\delta(1+\alpha)}}{\delta} + (\alpha+1)t\right)$$
$$\leq \left(1 + \frac{\delta\alpha\sqrt{\delta(1+\alpha)}t}{(1+\alpha)}\right)^{-\frac{1}{\alpha+1}}, \quad t \in R^+.$$

Proof: Given the inequality:

$$(1+\alpha)\left(h'(t)\right)^2 \geq h''(t)h(t) \geq \delta h^2(t) \quad t \in \mathbb{R}^+,$$

we can rewrite it as:

$$\delta h^2(t) - (1+\alpha)\left(h'(t)\right)^2 \leq 0.$$

Factoring the left-hand side, we obtain:

$$\left(\sqrt{(1+\alpha)}h'(t) + \sqrt{\delta}h(t)\right)\left(\sqrt{\delta}h(t) - \sqrt{(1+\alpha)}h'(t)\right) \leq 0.$$

From the product being non-positive, we conclude:

$$\left(\sqrt{(1+\alpha)}h'(t) + \sqrt{\delta}h(t)\right) \leq 0.$$

This implies:

$$h'(t) \leq \frac{-\sqrt{\delta}}{\sqrt{(1+\alpha)}}h(t).$$

Given that

$$-h'(t) = E^{\zeta+1}(t),$$

we have:

$$-E^{\zeta+1}(t) \leq \frac{-\sqrt{\delta}}{\sqrt{(1+\alpha)}}h(t).$$

Or equivalently:

$$E^{\zeta+1}(t) \geq \frac{\sqrt{\delta}}{\sqrt{(1+\alpha)}}h(t).$$

Then

$$h(t) \leq \frac{\sqrt{(1+\alpha)}}{\sqrt{\delta}}E^{\zeta+1}(t).$$

Because $E(t) \leq E(0)$ for $t \in \mathbb{R}^+$, we find:

$$h(t) \leq \frac{\sqrt{(1+\alpha)}}{\sqrt{\delta}}E^\zeta(0)E(t).$$

Finally,

$$\int_t^\infty E^{\zeta+1}(s)\,ds \leq \frac{\sqrt{\delta(1+\alpha)}}{\delta}E^\zeta(0)E(t). \qquad (46)$$

By differentiating the function once again, and utilizing equation (46), we may assume $E(0) = 1$, leading to the following relationships:

$$h(t) = \int_t^\infty E^{\zeta+1}(s)\,ds, \quad -h'(t) = E^{\zeta+1}(t)$$

where $h(t)$ represents the integral of $E(s)^{\zeta+1}$ over the range $[t, \infty)$, and $h'(t)$ is its first derivative. Proceeding from this, we derive:

$$h^{\alpha+1}(t) = \left(\int_t^\infty E^{\zeta+1}(s)\,ds\right)^{\alpha+1}$$
$$\leq \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{\alpha+1} E^{\alpha+1}(t)$$

This inequality provides an upper bound on $h^{\alpha+1}(t)$, involving the factor $\frac{\sqrt{\delta(1+\alpha)}}{\delta}$, where $\delta > 0$ represents a constant parameter of the system. Consequently, this implies:

$$\left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} h^{\alpha+1}(t) \leq E^{\alpha+1}(t)$$

By assuming $\zeta = \alpha$, we further deduce:

$$\left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} h^{\alpha+1}(t) \leq -h'(t)$$

We now employ the identity:

$$(h^{-\alpha}(t))' = -\alpha h'(t)h^{-\alpha-1}(t)$$

Thus, multiplying by $h^{\alpha+1}(t)$ and using the previous results, we establish:

$$\frac{1}{\alpha}(h^{-\alpha}(t))'h^{\alpha+1}(t) = -h'(t) \geq \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} h^{\alpha+1}(t)$$

Therefore:

$$(h^{-\alpha}(t))' \geq \alpha\left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1}$$

Upon differentiating again, we obtain:

$$(h^{-\alpha}(t))'' = -\alpha h^{-\alpha-2}(t)\left(h''(t)h(t) - (\alpha+1)(h'(t))^2\right)$$
$$\geq 0$$

This inequality suggests that $h^{-\alpha}(t)$ is concave upwards, ensuring the following relation:

$$(h^{-\alpha}(t)) - (h^{-\alpha}(s)) \geq (h^{-\alpha}(s))'(t-s)$$

$$\geq \alpha \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} (t-s)$$

which gives us:

$$h^{-\alpha}(t) \geq h^{-\alpha}(s) + \alpha \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} (t-s)$$

In particular, we obtain the bound:

$$h^{-\alpha}(t) \geq \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha} + \alpha \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} (t-s)$$

Furthermore, since $E(t)$ is nonnegative and non-increasing, we conclude that:

$$h(t) = \int_t^\infty E^{\zeta+1}(\tau)\, d\tau \geq \int_t^{\frac{\sqrt{\delta(1+\alpha)}}{\delta}+(\zeta+1)t} E^{\zeta+1}(\tau)\, d\tau$$

This integral inequality leads to:

$$h(t) \geq \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta} + \zeta t\right] E^{\zeta+1}\left(\frac{\sqrt{\delta(1+\alpha)}}{\delta} + (\zeta+1)t\right)$$

which holds for large t. We can rewrite this as:

$$\left(\left[\frac{\sqrt{\delta(1+\alpha)}}{\delta} + \zeta t\right] E^{\zeta+1}\left(\frac{\sqrt{\delta(1+\alpha)}}{\delta} + (\zeta+1)t\right)\right)^{-\alpha}$$

$$+\alpha \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} s$$

$$\geq \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha} + \alpha \left[\frac{\sqrt{\delta(1+\alpha)}}{\delta}\right]^{-\alpha-1} s$$

Finally, we can approximate $E(t)$ for large t as follows:

$$E\left(\frac{\sqrt{\delta(1+\alpha)}}{\delta} + (\alpha+1)t\right) \leq \left(1 + \frac{\alpha\sqrt{\delta(1+\alpha)}t}{(1+\alpha)}\right)^{-\frac{1}{\alpha+1}}$$

■

IV. Conclusion

This study provides a clear understanding of the function $h(t) = G(E(t))$. It demonstrates how the exponential decay of $E(t)$ is transformed by G into corresponding bounds for $h(t)$. Additionally, it shows how derivative constraints and integral properties of $E(t)$ influence the behavior of $h(t)$. These findings enhance the understanding of these functions in mathematical modeling and dynamical systems.

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