Multi-modal Multi-objective Algorithm with p-Norm and Adaptive Weight based Crowding Measurement

Hejun Xuan, Libo Kou, Liang Lei, Yan Ding, Junyi Hu, Yuhang Niu, and Yan Feng

***Abstract*—Multi-modal multi-objective optimization problem (MMOP) is a multi-objective optimization problem which has multiple Pareto Sets (PSs) corresponding to the same Pareto Front (PF). Since multiple PS can provide decision-makers with more diverse decision options, the researches for solving multi-modal multi-objective optimization problems has attracted much attention. At present, how to define a more effective crowding degree measurement, and how to make the distribution of solutions on the PF more uniform are still urgent issues. In order to obtain a uniformity, diversity, and convergence PF, and diversity PS, a L_p based crowding degree measurement for solution space and objective space was proposed. In addition, an adaptive weighting coefficient for the crowding degree measurement was designed. To obtain a more uniform PF and diversified PS, a solution merging strategy based on the clustering of solutions after the K-Means algorithm was designed. To demonstrate the efficiency of the proposed algorithm (denoted as p-ACDCM), p-ACDCM has been compared with eight benchmark algorithms in the CEC'2019 standard test function, and the experimental results show that the proposed algorithm in this paper can obtain better rPSP, rHV, IGDX, and IGDF than compared algorithms.**

***Index Terms*—Multi-modal; Multi-objective; Crowding measurement; L_p; Merging strategy**

I. Introduction

MULTI-OBJECTIVE Problem (MOPs) plays an important role in optimization science. They are widely used in many fields, such as base station location, network optimization, resource scheduling[1], [2], [3], [4], [5]. According to the number of objective functions, optimization problems can divided into two categories: global-objective optimization and multi-objective optimization[6], [7], [8], [9], [10]. Global-objective optimization focuses on achieving a single optimization goal, while multi-objective optimization is more complex and can handle multiple objective functions simultaneously[11]. A notable feature of multi-objective optimization is that it can generate a series of solutions, namely Pareto sets (PSs). These pareto sets can provide a variety of potential solutions. Therefore, multi-objective optimization provides a more flexible and effective tool for solving multi-objective and constraint challenges in real world. The mathematical formulation of multi-objective optimization problem is shown as follows:

$$\begin{cases} \min F(X) = \{f_1(x), f_2(x), \cdots, f_m(x)\}^T \in R^m \\ s.t. \\ \quad (a) g_j(x) \leq 0, j = 1, \cdots, p; \\ \quad (b) h_j(x) = 0, j = p+1, \cdots, q; \\ \quad x = (x_1, x_2, \cdots, x_n)^T \in R^n; \end{cases} \tag{1}$$

where X is a n-dimensional decision vector in the decision space R^n. R^m and R^n are the objective space and decision space, respectively. $g_j(x)$ is the inequality constraint, $h_j(x)$ is the equality constraint. $F(X) = \{f_1(x), f_2(x), \cdots, f_m(x)\}^T$ is an objective vector.

Multi-objective optimization can produce a series of solutions, namely PSs, which can provide diversity potential solutions. Thus, It allows decision-makers to independently weigh different objectives. Multi-objective optimization is much more suit to solve real word problems with complexity and diversity. It can provide a more flexible solutions for dealing with multi-objective and multi-constraint challenges in the real world.

Multi-modal multi-objective optimization problems, as a special form of multi-objective optimization problems, are characterized by having two or more different PSs corresponding to the same PF, as shown in Fig.1. Compared with single-modal optimization, multi-modal optimization increases the diversity of potential solution sets. At the same time, multimodality significantly improves the robustness of understanding by integrating information from different modalities. Not only complexity and diversity are increased, but also multimodals are richer.

In this paper, the multi-modal multi-objective optimization problem was the focused. A multi-objective optimization algorithm based on L_p and adaptive weights was designed. what's more, an improved crowding measurement, and a adaptive classification of cluster results are designed. In summary, the main contributions are summarized as follows:

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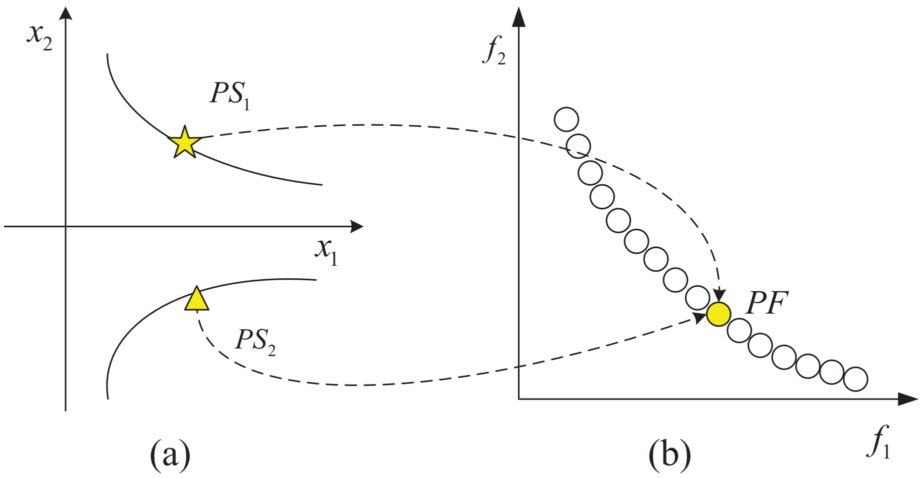

Fig. 1. MMOPs. (a)Pareto Set; (b)Pareto Front

- The crowding measurement of the solution was proposed, and a crowding degree evaluation method was constructed by defining a norm-based distance function.
- An adaptive weighting method based on the L_p crowding measurement is proposed to adjust the relative importance of different objectives. By assigning weights to each objective, the influence of each objective on the final optimization solution can be specified.
- An adaptive classification method for solutions is designed, which decomposed and merged classification results based on clustering results to obtain an adaptive classification scheme.

II. Related Works

In recent years, much more research has focused on multi-objective multi-modal optimization [11], [12], [13], [14]. Researches on multi-modal multi-objective evolutionary algorithm MMEA can be categorized into three main areas:

A. Pareto-based MMEA

The basic idea of Pareto-based multi-modal multi-objective evolutionary algorithms are to use evolutionary algorithms to explore the solution space, which employs the PF to maintain and identify solutions across different modalities. This class of algorithms is mainly based on the theory of PF and seeks for non-dominated solution sets in multi-objective problems. Some mechanisms were sued, such as effective Pareto ordering and crowding distance. In order to obtain the widest possible range of PF, this approach focused on maintaining diversity in the solution space. Deb et al [15] proposed the Omni-optimizer, which uses the NSGA-II framework and introduced a non-dominated sorting method to increase diversity in the decision space. In addition, selecting outstanding individuals for the next generation based on environment selection strategy. Zhang et al. [16] proposed the MOEA/D algorithm, which decomposes a multi-objective problem into multiple scalar optimization problems and optimizes these sub-problems simultaneously. Euclidean distance is used to define neighborhood relationships and manage objectives of different scales. Liang et al. [17] proposed a clustering-based differential evolutionary algorithm to solve MMOPs, which employs a specialized crowding distance ranking and elite selection. Zhao et al. [18] proposed a prediction-differential evolution strategy based on reinforced evolution for solving multi-modal multi-objective problems. A prediction strategy and reinforced evolution were used to accelerate population convergence and approach the global optimal solution. Yue [19] proposed a particle swarm optimization algorithm for solving MMOPs based on a ring topology. This algorithm used a metric-based ring topology to create stable ecological niches. Thus, it can enable the identification of more Pareto-optimal solutions. Yue et al. [20] proposed a multi-modal multi-objective differential evolutionary algorithm that crowding measurement is used to enhance solution diversity.

B. Decomposition-based MMEA

This type of algorithm employed decomposition techniques to break a multi-objective problem into a series of single-objective subproblems. By progressively optimizing the subproblems, this class of algorithms can more efficiently explore and exploit the multi-modal structure in the decision space. Liu [21] used convergence and diversity profiles to synergistically obtain multiple Pareto-optimal solutions, and enhanced the diversity of the decision space. Literature [22] proposed a reinforcement learning-based differential evolutionary algorithm (DE-RLFR) that designd a reward function to guide the population toward global convergence on the PF. Qi et al. [23] adjusted the MOEA/D-AWA by using weight vectors that adaptively redistribute the weights of the subproblems through periodic adjustments. Hu et al. [24] used the decomposition-based Multi-Objective Evolutionary Algorithm (MOEA/D) to achieve effective results.

C. Metrics-based MMEA

Such algorithms use various performance metrics to guide the search process, aiming to maintain both the diversity and balance of the solution set. Commonly, some metrics include convergence, distribution, which can help to discover multiple PF solutions in the decision space and better address multi-modal optimization problems. Zhang et al. [25] proposed the MMO-EvoKnee algorithm to identify the boundary between the global knee solution and the objective space and to retain solutions with good convergence. Literature [26] employs a multi-modal multi-objective evolutionary algorithm with a dual-archive recombination strategy that ensures diversity in the objective space. Bader et al. [27] proposed a hypervolume-based MOEA that measures the superiority of an individual by calculating its hypervolume. Subsequently, Moffaert [28] analyzed the effectiveness of a multi-objective evolutionary algorithm based on hypervolume metrics.

In the evolution of multi-objective multi-modal optimization algorithms, there are still several challenges, including inaccurate crowding measures and unbalanced classification of solutions. This problem and corresponding solutions were deeply analyzed. Crowding measurement is a crucial problem for optimization algorithms. In order to improve the accuracy of crowding evaluation, this paper introduced the norm based crowding measurement. The norm not only provides a flexible and mathematically rigorous method to measure crowding, but also guides multi-objective optimization algorithms to search efficiently the solution space. In addition, clustering methods are often used to deal with multi-modal problems to identify and distinguish solutions of different modes. However, traditional clustering techniques always lead to uneven or misclassified clustering results, as shown in Figure3. In order to obtain better classification results, this study proposed a clustering merging strategy to make the classification of solutions more uniformly. By introducing the norm and cluster merging method, this study aims to solve the key problems in multi-objective and multi-modal optimization algorithms.

III. Proposed Algorithm

A. Crowding degree measurement in decision space

In the decision space, the crowding distance (CD) mechanism is used to maintain diversity of the population. In general, higher diversity in the decision space indicates that the solutions has a better distribution. However, a good distribution of solutions in the decision space does not guarantee that the population will also have a corresponding level of diversity in the objective space. Therefore, further research is needed to address this issue rationally. To address this problem, a new solution crowding degree measurement is proposed. This involves designing a crowding degree measurement method by defining a distance function, which provides a basis for evaluating the similarity of solutions. The main objective of this method is to maintain diversity within the population and improve the uniformity of its distribution. The distribution in the decision space is shown in Fig.2(a), where PS_1, PS_2 belong to different PS. The formulation of crowding degree measurement in the decision space is given by Eq. (2)

$$CD_{j,x} = \sum_{i=1}^{n} \left(\alpha_i \frac{\sqrt[p]{|x_{j_1,i} - x_{j_2,i}|^p}}{\sqrt[p]{|x_{Max,i} - x_{Min,i}|^p}} \right) \tag{2}$$

where the

$$\alpha_i = \frac{\sum\limits_{j=1,j\neq i}^{n} \left(x^j_{max} - x^j_{min} \right)}{\sum\limits_{j=1}^{n} \left(x^j_{max} - x^j_{min} \right)} \tag{3}$$

where n is the decision space dimension, as expressed in Eq. (3). $x_{j_1,i}$, $x_{j_2,i}$ denote the two neighboring solutions in the same dimension. $x_{Max,i}$ and $x_{Min,i}$ denote the bounded minimum and bounded maximum of the solution in different dimensions, respectively. As an example, the distribution of solutions in the decision space is shown in Fig. 2(a), the 3^{rd} solution in the two-dimensional decision space has the solutions (1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}) on PS_1 and (6^{th}, 7^{th}, 8^{th}, 9^{th}, 10^{th}) on PS_2. The x_1 solution 3^{rd} has two adjacent solutions on PS_1 which are 2^{nd} and 4^{th}, combined with two adjacent solutions on PS_2 which are 8^{th} and 9^{th}, and two adjacent solutions on x_2 which are 2^{nd} and 4^{th}, so the crowding of the two-dimensional decision space is calculated as:

$$CD_{3,x} = \alpha_1 \frac{\sqrt[p]{|x_{9,1} - x_{8,1}|^p}}{\sqrt[p]{|x_{10,1} - x_{1,1}|^p}} + \alpha_2 \frac{\sqrt[p]{|x_{4,2} - x_{2,2}|^p}}{\sqrt[p]{|x_{8,2} - x_{1,2}|^p}} \tag{4}$$

B. Crowding degree measurement in the objective space

The main purpose of crowding degree measurement in the objective space is to maintain diversity. In addition, it can ensure a good distribution, and avoid over-concentration of solutions in localized regions. The crowding degree measurement can help to achieve an even distribution of solutions in the objective space, resulting in a more diversity and comprehensive set of solutions for decision-makers. This uniform distribution allows decision-makers to have a more complete view of the range of feasible solutions. Fig. 2(b) gives an example of distribution of the objective in the objective space. The crowding degree is calculated by Eq. (5).

$$CD_{j,f} = \sum_{i=1}^{m} \left(\beta_i \frac{\sqrt[p]{|y_{j_1,i} - y_{j_2,i}|^p}}{\sqrt[p]{|y_{Max,i} - y_{Min,i}|^p}} \right) \tag{5}$$

Similar to α_i, β_i is defined as:

$$\beta_i = \frac{\sum\limits_{j=1,j\neq i}^{m} \left(f^j_{max} - f^j_{min} \right)}{\sum\limits_{j=1}^{m} \left(f^j_{max} - f^j_{min} \right)} \tag{6}$$

where m represents the dimension of the objective space, $x_{j_1,i}$ and $x_{j_2,i}$ denote the two solutions adjacent to each other in the same dimension, respectively. x_{Min} and x_{Max} denote the bounded minimum and bounded maximum of the solutions in different dimensions, respectively. Objective space of two-dimensional is used as an example to compute the crowding degree, the two neighboring solutions on f_1 are 4^{th} and 6^{th}, and the two neighboring solutions on f_2 are 4^{th} and 6^{th}.

$$CD_{5,f} = \beta_1 \frac{\sqrt[p]{|y_{6,1} - y_{4,1}|^p}}{\sqrt[p]{|y_{9,1} - y_{1,1}|^p}} + \beta_2 \frac{\sqrt[p]{|y_{4,2} - y_{6,2}|^p}}{\sqrt[p]{|y_{1,2} - y_{9,2}|^p}} \tag{7}$$

An example of how the crowding degree measurement is calculated for the 5^{th} objective function. By introducing this novel measurement of crowding degree, it is possible to simultaneously maintain the diversity of the population and enhance the uniform distribution of the population in both the decision space and the objective space. This approach offers an effective way to address multi-modal multi-objective optimization problems.

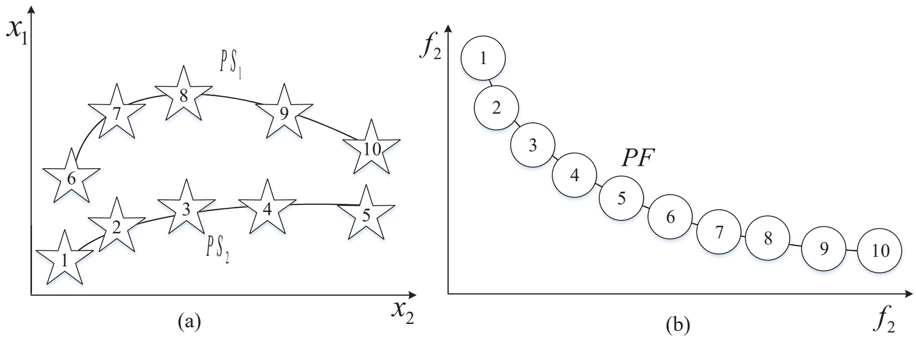

Fig. 2. Crowding Distance. (a)Decision Space; (b)Objective Space

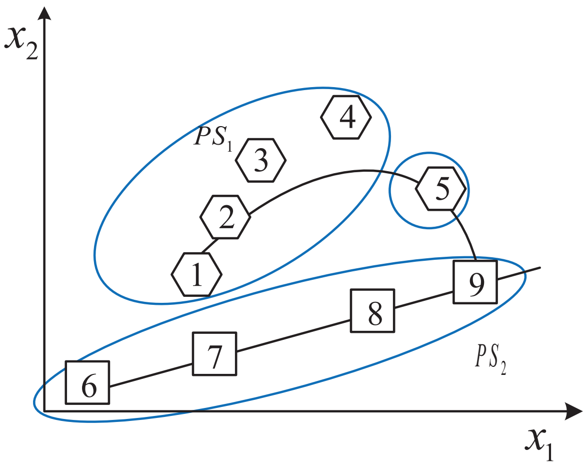

Fig. 3. Classification of solutions

C. Merging strategies for solutions

It is difficult to determine which individuals belong to the same PF. However, based on the analysis, it can be inferred that the neighborhood of individuals should be determined based on their surroundings. Therefore, clustering algorithms can be used to divide the solutions in the same non-dominated layer into multiple classes. The K-means algorithm was used for this purpose in literature [17]. However, the K-means algorithm may lead to an uneven distribution of solutions and produce isolated solutions. To handle this problem, a merging strategy has been designed. Fig. 3 gives an example in the decision space. The K-means algorithm was divided the solutions into three classes: (1^{st}, 2^{nd}, 3^{rd}, 4^{th}) (5^{th}) (6^{th}, 7^{th}, 8^{th}, 9^{th}). After this classification, a classification judgment of automated is made on solutions, and calculate the distance of isolated individual 5^{th} from the class of individual 4^{th}. If its distance is smaller than the average distance between several other classes, 5^{th} is merged into the class of individual 4^{th} to obtain a new classification result.

Algorithm 1: Merging strategies for solutions

Input: Total number of front $Total_{PF}$; average_distance of each front average_distances; individuals of each front POP;
Output: Merged front individuals POP;

Calculate the Euclidean distance between each pair of individuals, and storing the results in a distance matrix;
Compute the average distance for each individual, which is the average minimum distance to other individuals;
Sort individuals based on their average distance;
for $i = 1$ *to Total_PF* **do**
  if *cluster_sizes(i)* > 2 **then**
    Calculate the minimum average distance between the current front and other front;
    if *min_distance* $>$ *average_distances(i)* **then**
      Find the index of the nearest front nearest_cluster to current front;
      if *nearest_cluster* $\geq i$ **then**
        Increment the index of nearest_cluster by 1;
      end
      Merge the individuals of current front i into the nearest front nearest_cluster;
      Update the cluster size of nearest front nearest_cluster;
      Set the cluster size of current front i to 0;
      Set the average distance of current front i to infinity to prevent repeated merging;
    end
  end
end
Return the merged front POP;

IV. Experimental Setup

A. Benchmark and Compared Algorithms

In the experiments, comparisons were made with eight algorithms to demonstrate the efficiency of the proposed algorithm. The compared algorithm of MMODE_CSCD [17] designed a special crowding measurement method to calculate the degree of crowding in both the decision space and the objective space, and utilized a distance-based elite selection mechanism to generate new individuals. MO_Ring_PSO_SCD [19] used an index-based ring topology to create stable ecological niches, thereby facilitating the identification of more Pareto-optimal solutions. It also employed a specialized notion of crowding distances as a parametric measurement in both the decision

Algorithm 2: The MMO-pACDCM algorithmic framework

Input: population size: NP; maximum number of generations: G_{max};
Output: PF and PSs.

$G = 1$;
Generate the initial population P with NP individuals and evaluate all individuals on each objective;
while $G \leq G_{max}$ **do**
    Population after ordering based on decision space crowding to Eq.(2);
    Population after ordering based on crowded in the object space crowding to Eq.(5);
    Calculating crowding according to the formula;
    if *Classification results are not homogeneous* **then**
        Clustering the populations crowding to algorithm crowding to algorithm 1;
    end
    Selecting an exemplar for each individual;
    Performing mutation and crossover to obtain an offspring population OP;
    Combining offspring populations with parental populationsPerform the environmental selection on POP to obtain a new population P;
    G = G + 1;
end

and objective spaces. TriMOEA_TAR [21] is a multi-modal multi-objective evolutionary algorithm that utilized pairwise archiving and recombination strategies. MO_PSO_MM [18] was a novel multi-objective particle swarm optimizer with a self-organizing mechanism. DE_RLFR [22] is a differential evolutionary algorithm based on fitness-ranked reinforcement learning (using the original parameters). To obtain average results, each function was run 30 times independently for the 22 test functions.

B. Parameter Settings

The parameters setting for all 8 comparison algorithms were adopted as the references. The parameters of the proposed algorithms in this paper are set as follows: the values of α and β are as introduced. p is seted as 1/3, 1/2, 1,2, and 3. The best value is selected based on the experimental results. To ensure a fair comparison, each of the 22 test functions is iterated 30 times, independently. Both the average and standard deviation are calculated. In addition, a Wilcoxon test with a significance level of 0.05 is used to estimate the significance of the difference between two algorithms. The symbols "+" and "-" indicated that the comparative algorithm performed better or worse than the algorithm proposed in this paper, respectively. The symbol "≈" indicates that the performance of proposed algorithm and the compared algorithm are similarly.

C. Performance Metrics

In this paper, four performance indicators are used: (1) Inverse of Pareto Sets Proximity (1/PSP, rPSP) [22]. (2) Inverse of Hypervolume (1/HV, rHV) [29]. (3) Inverse of Generational Distance (IGD) in the decision space (IGDX) [30]. (4) IGD in the objective space (IGDF). Therefore, for all four indicators, smaller values are better. The rPSP reflects the overlap rate between the obtained PS and the true PS, as well as the diversity and convergence of the obtained solutions. rHV reflects the convergence and diversity of the resulting PF. IGD is a commonly used indicator for evaluating the convergence of the proposed PF to the true PF, and the coverage of the proposed PF. IGDX measures the convergence between the obtained PS and the true PS. IGDF reflects the convergence between the obtained PF and the true PF. The mathematical formulas for the four performance indicators are as follows:

$$IGDX = \frac{\sum\limits_{x^* \in X^*} min\left\{ED\left(x^*, X\right)\right\}}{|X^*|} \tag{8}$$

where min ED(x, X^*) represents the minimum Euclidean distance between the solutions obtained in x and X^*.

$$CR\left(X\right) = \left(\prod_{i=1}^{d} \eta_i\right)^{\frac{1}{2D}} \tag{9}$$

$$\eta_i = \left\{\frac{min\left(x_i^{*,max}, x_i^{max}\right) - max\left(x_i^{*,min}, x_i^{min}\right)}{x_i^{*,max} - x_i^{*,min}}\right\}^2 \tag{10}$$

where x_i and x_i^* represent the i^{th} solution of the true PSs and the reference point, respectively. When $x_i^{max} \leq x_i^{*,min}$, or $x_i^{min} \geq x_i^{*,max}$, $\eta_i = 0$. When $x_i^{*,man} = x_i^{*,min}$,

$$RPSP\left(X\right) = \frac{IGDX}{CR} \tag{11}$$

$$H = \sum_{i=1}^{N} h_i \tag{12}$$

where N is the number of solutions in the PF, and h_i is the volume contribution between each solution i and the reference point.

$$IGDF = \frac{\sum\limits_{f^* \in F^*} min\left\{ED\left(f^*, F\right)\right\}}{|F^*|} \tag{13}$$

where $minED(f, F^*)$ represents the minimum Euclidean distance between the solutions obtained in f and F^*.

V. Results and Discussion

A. Experimental results and analysis of crowding strategy based on different norms

For different scenes, the test results are shown in Table I to Table IV. In p-ACDCM algorithm, when the p-value is set to 1/2, the minimum mean values (rPSP, rHV, IGDX, and IGDF) are better than other p-values. Therefore, the a p-value of 1/2 in the other experiments. The L_p distance metric helps improve the convergence of the PF by computing distances between points. This metric facilitates operations such as clustering, selection, and merging, ensuring that the points on the PF are more evenly distributed. L_p metric can efficiently cluster or merge points, contributing to the convergence of optimization process. Selecting an appropriate value for p

TABLE I

rPSP results with different norms (mean and standard deviation)

	p=1/3	p=1/2	p=1	p=2	p=3
MMF1	0.0409±0.0012	0.0400±0.0413+	0.0413±0.0014−	0.0416±0.0015−	0.0412±0.0015−
MMF2	0.0258±0.0215	0.0091±0.0105+	0.0156±0.0092+	0.0303±0.0297−	0.0251±0.0046+
MMF3	0.0163±0.0083	0.0083±0.0077+	0.0117±0.0047+	0.0226±0.0201−	0.0182±0.0038−
MMF4	0.0218±0.0012	0.0217±0.0209+	0.0221±0.0010−	0.0217±0.0008+	0.0216±0.0008+
MMF5	0.0719±0.0029	0.0720±0.0680−	0.0723±0.0036−	0.0722±0.0030−	0.0731±0.0028−
MMF6	0.0628±0.0035	0.0639±0.0621−	0.0632±0.0025−	0.0625±0.0028+	0.0638±0.0021−
MMF7	0.0248±0.0026	0.0225±0.0224+	0.0231±0.0020+	0.0251±0.0037−	0.0237±0.0012+
MMF8	0.0516±0.0039	0.0471±0.0506+	0.0507±0.0069+	0.0524±0.0050−	0.0522±0.0025−
MMF9	0.0055±0.0002	0.0057±0.0058−	0.0056±0.0002−	0.0057±0.0002−	0.0057±0.0003−
MMF10	0.1165±0.1528	0.0017±0.0017+	0.0645±0.1196+	0.0717±0.1288+	0.0605±0.1077+
MMF11	0.0036±0.0002	0.0037±0.0034−	0.0035±0.0001+	0.0036±0.0002≈	0.0036±0.0003≈
MMF12	0.0015±0.0000	0.0015±0.0014≈	0.0015±0.0000≈	0.0015±0.0001≈	0.0015±0.0000≈
MMF13	0.0257±0.0007	0.0251±0.0254+	0.0262±0.0010−	0.0263±0.0008−	0.0264±0.0014−
MMF14	0.0624±0.0024	0.0646±0.0625−	0.0628±0.0022−	0.0634±0.0024−	0.0636±0.0022−
MMF15	0.0486±0.0018	0.0517±0.0470−	0.0507±0.0021−	0.0514±0.0023−	0.0519±0.0024−
MMF16	0.0296±0.0023	0.0284±0.0310+	0.0290±0.0011+	0.0294±0.0013+	0.0307±0.0011−
MMF17	1.4441±3.7947	0.3975±0.2143+	0.3738±0.3314+	0.4741±0.3352+	0.5555±0.0735+
MMF18	0.0738±0.0021	0.0754±0.0753−	0.0737±0.0021+	0.0751±0.0026−	0.0757±0.0024−
MMF19	0.0569±0.0019	0.0553±0.0545+	0.0599±0.0027−	0.0626±0.0035−	0.0635±0.0021−
MMF20	0.9557±1.2766	0.0641±0.0565+	0.2046±0.4516+	0.1318±0.2525−	0.1317±0.0036+
MMF21	0.3803±0.6594	0.0616±0.0504+	0.0985±0.2118+	0.3334±0.6054+	0.2707±0.0048+
MMF22	0.7211±0.2354	0.3722±0.5169+	0.6332±0.1745+	0.5495±0.1420+	0.5584±0.1313+
+/ − / ≈		14/7/1	12/9/1	7/13/2	8/10/2

TABLE II

rHV results with different norms (mean and standard deviation)

	p=1/3	p=1/2	p=1	p=2	p=3
MMF1	1.1457±0.0003	1.1453±0.0001+	1.1455±0.0003+	1.1456±0.0002≈	1.1457±0.0004≈
MMF2	1.1500±0.0006	1.1499±0.0009+	1.1503±0.0008−	1.1508±0.0012−	1.1510±0.0013−
MMF3	1.1492±0.0007	1.1488±0.0005+	1.1493±0.0006−	1.1498±0.0007−	1.1497±0.0009−
MMF4	1.8535±0.0012	1.8525±0.0008+	1.8529±0.0009−	1.8531±0.0009−	1.8536±0.0010−
MMF5	1.1457±0.0003	1.1454±0.0002+	1.1455±0.0002+	1.1456±0.0002+	1.1456±0.0003+
MMF6	1.1459±0.0005	1.1456±0.0005+	1.1457±0.0007−	1.1460±0.0007−	1.1459±0.0007−
MMF7	1.1454±0.0001	1.1453±0.0001+	1.1453±0.0001≈	1.1454±0.0001≈	1.1454±0.0002≈
MMF8	2.3749±0.0014	2.3746±0.0013+	2.3775±0.0119−	2.3759±0.0020−	2.3757±0.0017−
MMF9	0.1033±0.0000	0.1032±0.0000+	0.1033±0.0000≈	0.1033±0.0000≈	0.1033±0.0000≈
MMF10	0.0809±0.0038	0.0800±0.0034+	0.0795±0.0033+	0.0799±0.0034+	0.0794±0.0032+
MMF11	0.0689±0.0000	0.0688±4.8910+	0.0689±0.0000≈	0.0689±0.0000≈	0.0689±0.0000≈
MMF12	0.6354±0.0000	0.6354±0.0000≈	0.6354±0.0000≈	0.6355±0.0000−	0.6355±0.0000−
MMF13	0.0542±0.0000	0.0542±4.9817≈	0.0542±0.0000≈	0.0543±0.0000−	0.0542±0.0000≈
MMF14	0.3507±0.0158	0.3495±0.0182−	0.3549±0.0149−	0.3531±0.0253−	0.3504±0.0247+
MMF15	0.2547±0.0086	0.2519±0.0087+	0.2513±0.0092+	0.2516±0.0097−	0.2561±0.0107−
MMF16	1.1456±0.0003	1.1454±0.0002+	1.1455±0.0002+	1.1455±0.0002+	1.1457±0.0004−
MMF17	1.1643±0.0114	1.1736±0.0225−	1.1661±0.0077−	1.1712±0.0202−	1.1740±0.0188−
MMF18	0.3665±0.0537	0.3498±0.0229+	0.3476±0.0174+	0.3592±0.0221+	0.3456±0.0236+
MMF19	0.2504±0.0113	0.2495±0.0112+	0.2535±0.0075−	0.2548±0.0090−	0.2562±0.0130−
MMF20	0.0600±0.0005	0.0600±3.2389≈	0.0600±0.0000≈	0.0600±0.0000≈	0.0600±4.2533≈
MMF21	0.0600±0.0000	0.0600±3.5595≈	0.0600±0.0000≈	0.0600±0.0000≈	0.0600±0.0000≈
MMF22	0.0600±0.0000	0.0600±3.5595≈	0.0600±0.0000≈	0.0600±0.0000≈	0.0600±0.0000≈
+/ − / ≈		15/2/5	6/8/8	4/11/7	3/11/8

TABLE III
IGDX RESULTS WITH DIFFERENT NORMS (MEAN AND STANDARD DEVIATION)

	p=1/3	p=1/2	p=1	p=2	p=3
MMF1	0.0407±0.0012	0.0409±0.0010−	0.0411±0.0014−	0.0413±0.0015−	0.0409±0.0014−
MMF2	0.0258±0.0215	0.0127±0.0089+	0.0156±0.0092+	0.0303±0.0297−	0.0251±0.0235−
MMF3	0.0163±0.0083	0.0100±0.0049+	0.0117±0.0047+	0.0226±0.0201−	0.0182±0.0082−
MMF4	0.0217±0.0011	0.0213±0.0009+	0.0220±0.0010−	0.0216±0.0008−	0.0215±0.0010−
MMF5	0.0715±0.0028	0.0707±0.0017+	0.0719±0.0035−	0.0718±0.0029−	0.0727±0.0037−
MMF6	0.0625±0.0034	0.0634±0.0033−	0.0629±0.0024−	0.0622±0.0028+	0.0636±0.0026−
MMF7	0.0245±0.0027	0.0232±0.0026+	0.0229±0.0020+	0.0249±0.0036−	0.0235±0.0025−
MMF8	0.0508±0.0037	0.0476±0.0030+	0.0500±0.0065−	0.0516±0.0048−	0.0514±0.0047−
MMF9	0.0055±0.0002	0.0056±0.0002≈	0.0056±0.0002−	0.0057±0.0002−	0.0057±0.0002−
MMF10	0.1152±0.1526	0.0771±0.1295+	0.0628±0.1169+	0.0712±0.1287+	0.0588±0.1200+
MMF11	0.0036±0.0002	0.0036±0.0001−	0.0035±0.0001+	0.0036±0.0002−	0.0036±0.0001−
MMF12	0.0015±0.0000	0.0014±0.0000+	0.0015±0.0000−	0.0015±0.0001−	0.0015±0.0001−
MMF13	0.0254±0.0006	0.0259±0.0006−	0.0259±0.0009−	0.0260±0.0008−	0.0261±0.0007−
MMF14	0.0624±0.0024	0.0634±0.0018−	0.0628±0.0022−	0.0634±0.0024−	0.0636±0.0025−
MMF15	0.0486±0.0018	0.0492±0.0023≈	0.0507±0.0021−	0.0514±0.0023−	0.0519±0.0023−
MMF16	0.0294±0.0022	0.0271±0.0016+	0.0288±0.0011−	0.0292±0.0012−	0.0304±0.0041−
MMF17	0.6462±0.7721	0.3332±0.1469+	0.3257±0.2179+	0.3999±0.2314+	0.4441±0.4200−
MMF18	0.0735±0.0021	0.0746±0.0033−	0.0736±0.0021−	0.0750±0.0026−	0.0756±0.0036−
MMF19	0.0568±0.0019	0.0573±0.0031+	0.0597±0.0027−	0.0624±0.0035−	0.0633±0.0044−
MMF20	0.8705±1.0584	0.1651±0.3101+	0.2042±0.4509+	0.1316±0.2522+	0.1312±0.2508+
MMF21	0.3212±0.5185	0.1920±0.3488+	0.0925±0.1827+	0.2939±0.5212−	0.2272±0.4618−
MMF22	0.7083±0.2214	0.5905±0.1614+	0.6247±0.1708+	0.5415±0.1388+	0.5514±0.1265+
+/ − / ≈		14/6/2	9/13/0	5/17/0	4/18/0

TABLE IV
IGDF RESULTS WITH DIFFERENT NORMS (MEAN AND STANDARD DEVIATION)

	p=1/3	p=1/2	p=1	p=2	p=3
MMF1	0.0023±0.0000	0.0022±0.0000+	0.0023±0.0001−	0.0023±0.0001−	0.0023±0.0000−
MMF2	0.0042±0.0003	0.0040±0.0003+	0.0044±0.0003+	0.0047±0.0005−	0.0049±0.0006−
MMF3	0.0039±0.0002	0.0037±0.0002+	0.0039±0.0002−	0.0043±0.0004−	0.0043±0.0043−
MMF4	0.0023±0.0001	0.0023±0.0001≈	0.0023±0.0001≈	0.0023±0.0001≈	0.0023±0.0001≈
MMF5	0.0023±0.0000	0.0023±0.0000≈	0.0023±0.0000≈	0.0023±0.0000≈	0.0023±0.0000≈
MMF6	0.0023±0.0000	0.0022±0.0000+	0.0023±0.0000−	0.0023±0.0000+	0.0023±0.0001+
MMF7	0.0024±0.0000	0.0023±0.0000+	0.0023±0.0000+	0.0023±0.0000−	0.0024±0.0000−
MMF8	0.0028±0.0001	0.0027±0.0001+	0.0028±0.0002≈	0.0028±0.0001≈	0.0028±0.0001≈
MMF9	0.0144±0.0015	0.0108±0.0008+	0.0141±0.0013+	0.0144±0.0013≈	0.0152±0.0015−
MMF10	0.1220±0.1347	0.1534±0.1375−	0.0715±0.1134+	0.0801±0.1173−	0.0670±0.1077+
MMF11	0.0184±0.0039	0.0112±0.0008+	0.0144±0.0018+	0.0170±0.0021−	0.0175±0.0024+
MMF12	0.0020±0.0001	0.0020±0.0000≈	0.0020±0.0001≈	0.0020±0.0001≈	0.0020±0.0001≈
MMF13	0.0207±0.0078	0.0136±0.0028+	0.0204±0.0076+	0.0228±0.0111−	0.0184±0.0079+
MMF14	0.0907±0.0014	0.0913±0.0028−	0.0911±0.0018−	0.0917±0.0026−	0.0925±0.0023−
MMF15	0.0986±0.0037	0.0968±0.0033+	0.1009±0.0033−	0.1036±0.0039−	0.1048±0.0040−
MMF16	0.0023±0.0000	0.0021±0.0000+	0.0023±0.0000−	0.0022±0.0000+	0.0023±0.0011≈
MMF17	0.0078±0.0018	0.0083±0.0010−	0.0086±0.0018−	0.0092±0.0015−	0.0080±0.0019−
MMF18	0.0890±0.0021	0.0903±0.0027−	0.0896±0.0026−	0.0906±0.0022−	0.0906±0.0024−
MMF19	0.0999±0.0039	0.1004±0.0047−	0.1041±0.0042−	0.1104±0.0065−	0.1122±0.0021−
MMF20	0.0100±0.0009	0.0104±0.0010−	0.0100±0.0010−	0.0106±0.0010−	0.0101±0.0011−
MMF21	0.0101±0.0007	0.0100±0.0009+	0.0100±0.0013≈	0.0098±0.0009+	0.0101±0.0011≈
MMF22	0.0171±0.0027	0.0200±0.0033−	0.0201±0.0038−	0.0225±0.0042−	0.0202±0.0037−
+/ − / ≈		12/7/3	7/8/7	5/11/6	3/12/7

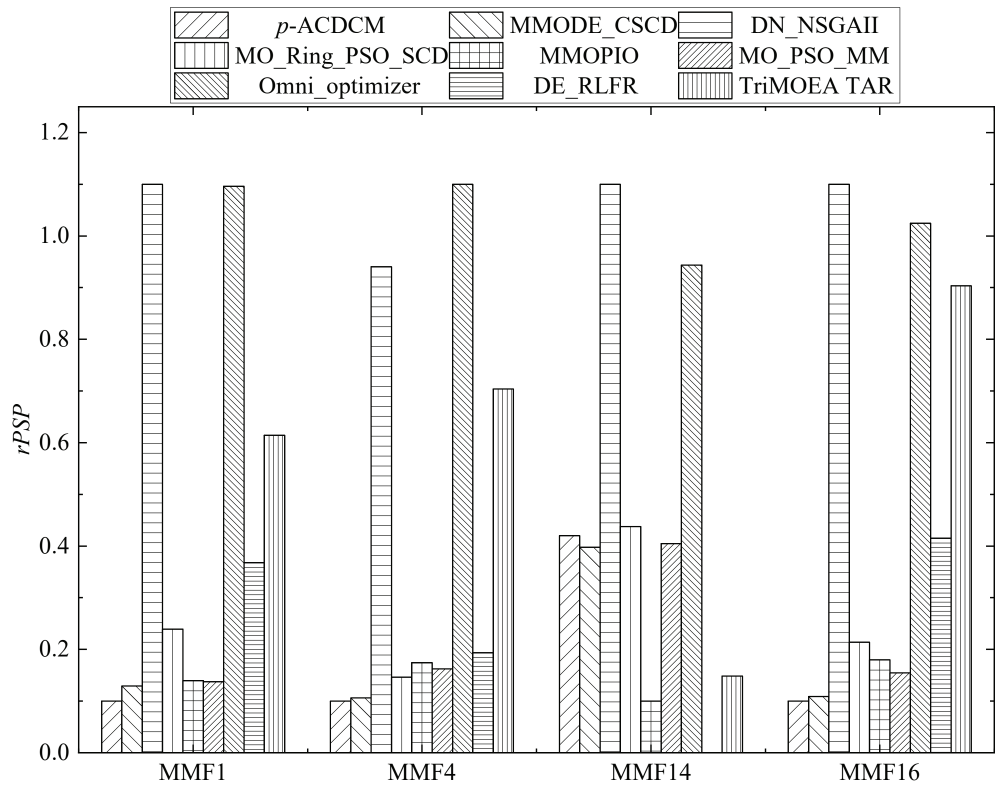

Fig. 4. Average rPSP obtained by nine algorithms on four categories functions.

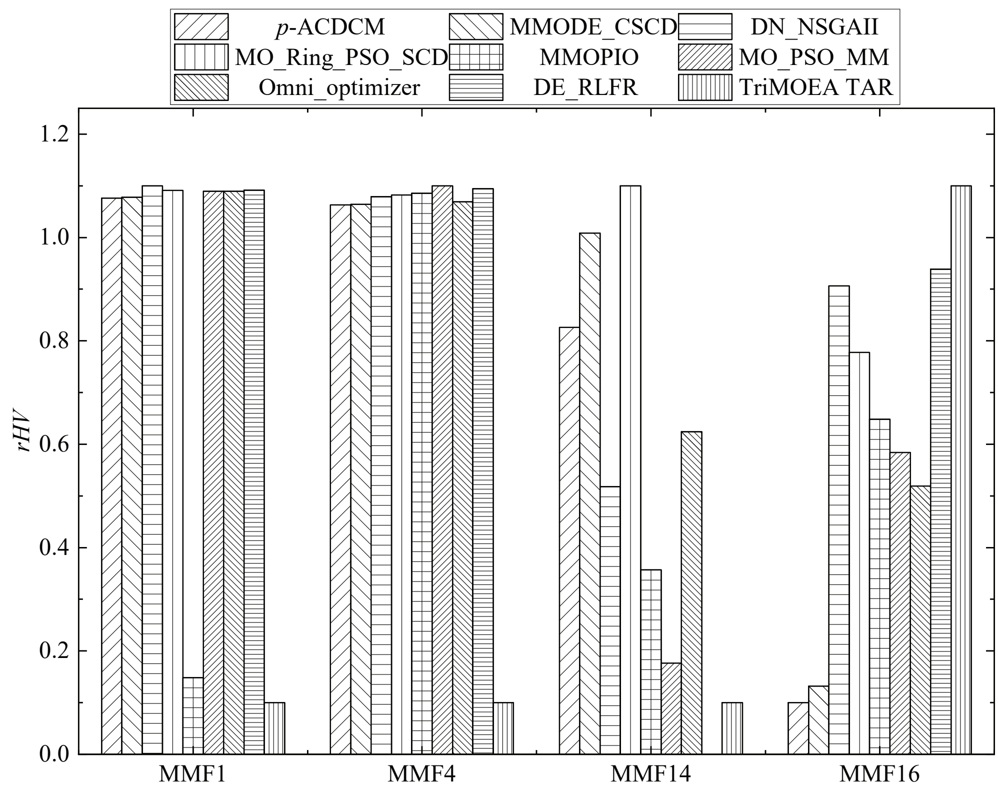

Fig. 5. Average rHV obtained by nine algorithms on four categories functions.

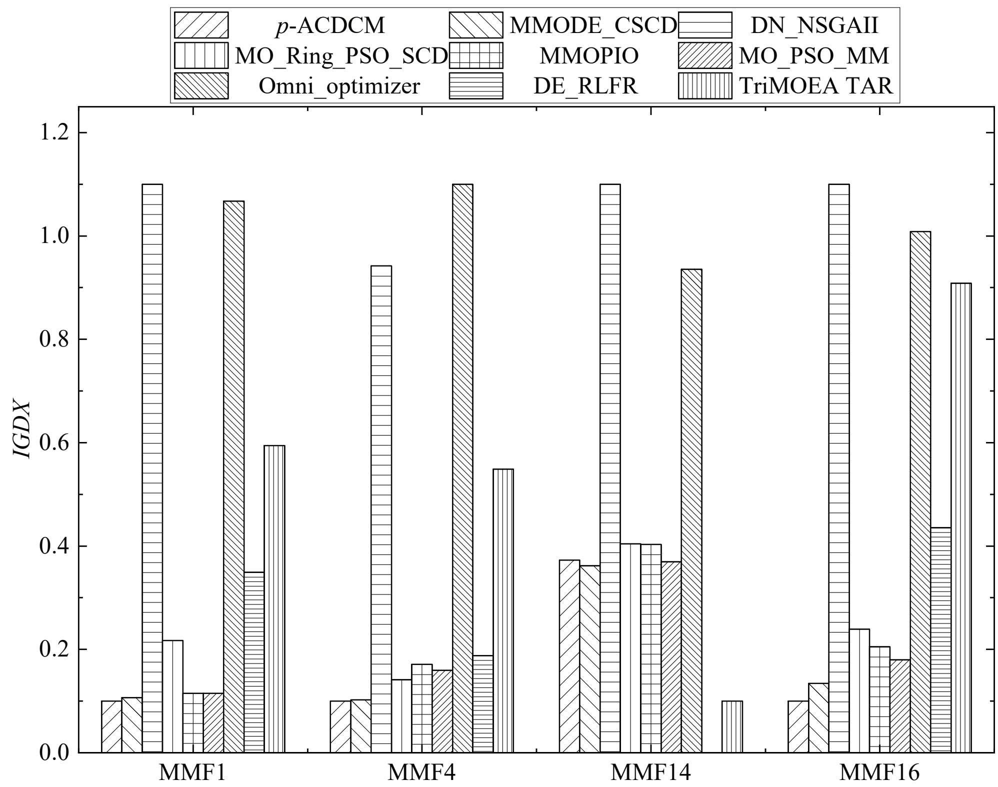

Fig. 6. Average IGDX obtained by nine algorithms on four categories functions.

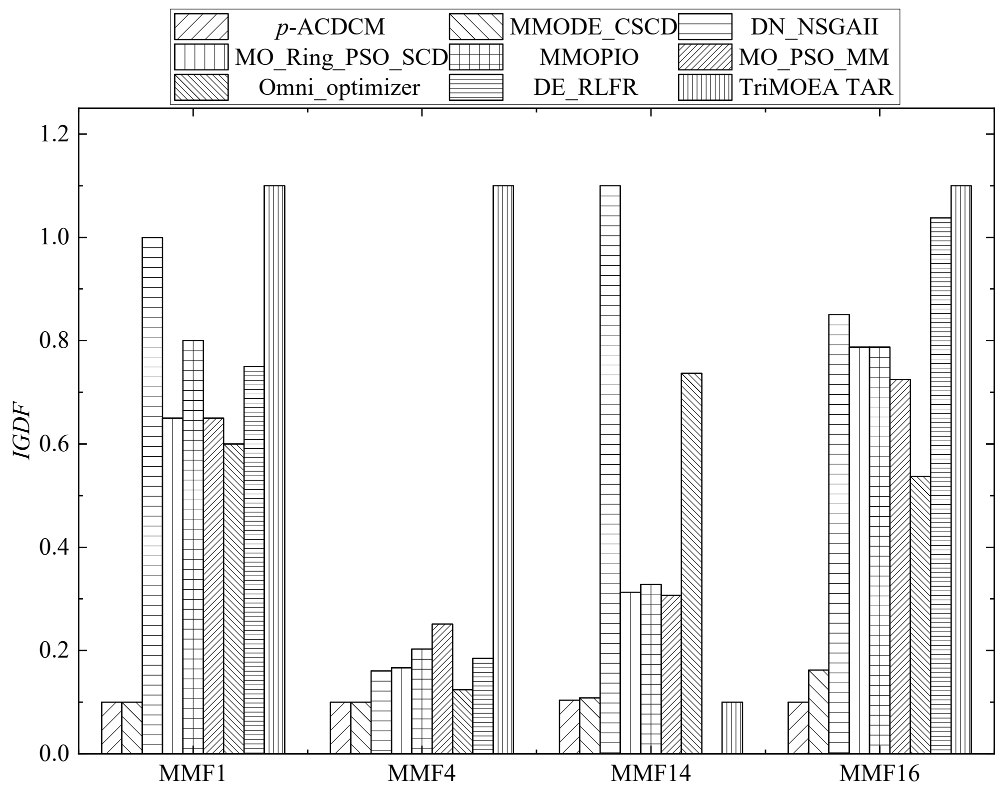

Fig. 7. Average IGDF obtained by nine algorithms on four categories functions.

TABLE V
STATISTICAL RESULTS (MEAN AND STANDARD DEVIATION) OF rPSP

	p-ACDCM	MMODE_CSCD	DN_NSGAII	MO_Ring_PSO_SCD	MMOPIO
MMF1	0.0400±0.0413	0.0414±0.0014 −	0.0882±0.0119 −	0.0467±0.0022 −	0.0419±0.0025 −
MMF2	0.0091±0.0105	0.0102±0.0018 −	0.1251±0.0776 −	0.0272±0.0108 −	0.0124±0.0036 −
MMF3	0.0083±0.0077	0.0085±0.0018 −	0.0827±0.0378 −	0.0188±0.0031 −	0.0118±0.0038 −
MMF4	0.0217±0.0209	0.0223±0.0011 −	0.1017±0.0365 −	0.0261±0.0019 −	0.0288±0.0042 −
MMF5	0.0720±0.0680	0.0721±0.0034 −	0.1604±0.0157 −	0.0800±0.0039 −	0.0847±0.0070 −
MMF6	0.0639±0.062	0.0625±0.0022 +	0.1388±0.0180 −	0.0683±0.0040 −	0.0722±0.0045 −
MMF7	0.0225±0.0224	0.0221±0.0014+	0.0461±0.0090 −	0.0257±0.0010 −	0.0346±0.0047 −
MMF8	0.0471±0.0506	0.0489±0.0031 −	0.3057±0.0970 −	0.0587±0.0032 −	0.0634±0.0132 −
MMF9	0.0057±0.0058	0.0057±0.0003 ≈	0.0228±0.0095 −	0.0072±0.0004 −	0.0122±0.0025 −
MMF10	0.0017±0.0017	0.0370±0.1063 −	0.1390±0.1457 −	0.0158±0.0047 −	0.0051±0.0022 −
MMF11	0.0037±0.0034	0.0041±0.0003 −	0.0046±0.0002 −	0.0047±0.0001 −	0.0072±0.0017 −
MMF12	0.0015±0.0014	0.0016±0.0001 −	0.0044±0.0094 −	0.0030±0.0003 −	0.0022±0.0005 −
MMF13	0.0251±0.0254	0.0277±0.0008 −	0.0720±0.0130 −	0.0316±0.0011 −	0.0335±0.0022 −
MMF14	0.0646±0.0625	0.0624±0.0016 +	0.1306±0.0141 −	0.0663±0.0023 −	0.0335±0.0022 +
MMF15	0.0517±0.0470	0.0504±0.0019 +	0.0848±0.0092 −	0.0505±0.0018 +	0.0489±0.0026 +
MMF16	0.0284±0.0310	0.0288±0.0009 −	0.0722±0.0162 −	0.0334±0.0016 −	0.0319±0.0025 −
MMF17	0.3975±0.2143	0.3617±0.1759 +	1.6845±1.0459 −	0.4114±0.0978 −	0.6117±0.3614 −
MMF18	0.0754±0.0753	0.0741±0.0026 +	0.1452±0.0114 −	0.0747±0.0019 +	0.0736±0.0025 +
MMF19	0.0553±0.0545	0.0591±0.0029 −	0.1095±0.0162 −	0.0550±0.0023 +	0.0548±0.0023 +
MMF20	0.0641±0.0565	0.0539±0.0039 +	5.0409±2.8973 −	0.1107±0.0156 −	0.0708±0.0105 −
MMF21	0.0616±0.0504	0.1075±0.2512 −	7.0084±9.0814 −	0.1309±0.0131 −	0.1215±0.2058 −
MMF22	0.3722±0.5169	0.5636±0.1177 −	1.7879±0.4169 −	0.3675±0.0820 ≈	0.6689±0.1403 −
+/ − / ≈		7/14/1	0/22/0	3/18/1	4/18/0

	p-ACDCM	MO_PSO_MM	Omni_optimizer	DE_RLFR	TriMOEA TAR
MMF1	0.0400±0.0413	0.0418±0.0025 −	0.0880±0.0170 −	0.0529±0.0059 −	0.0648±0.0080 −
MMF2	0.0091±0.0105	0.0191±0.0059 −	0.1154±0.0670 −	0.0846±0.0505 −	0.0724±0.0457 −
MMF3	0.0083±0.0077	0.0137±0.0018 −	0.0925±0.0692 −	0.0566±0.0390 −	0.1100±0.0736 −
MMF4	0.0217±0.0209	0.0276±0.0022 −	0.1169±0.0341 −	0.0306±0.0036 −	0.0792±0.1544 −
MMF5	0.0720±0.0680	0.0751±0.0036 −	0.1632±0.0163 −	0.0868±0.0091 −	0.1028±0.0100 −
MMF6	0.0639±0.062	0.0679±0.0030 −	0.1442±0.0293 −	0.0754±0.0045 −	0.0862±0.0140 −
MMF7	0.0225±0.0224	0.0321±0.0037 −	0.0387±0.0084 −	0.0392±0.0078 −	0.0546±0.0507 −
MMF8	0.0471±0.0506	0.0480±0.0033 −	0.3230±0.1714 +	0.0743±0.0164 −	0.4676±0.1072 −
MMF9	0.0057±0.0058	0.0098±0.0017 −	0.0280±0.0160 −	0.0066±0.0012 −	0.0032±0.0001 +
MMF10	0.0017±0.0017	0.0019±0.0003 −	0.0607±0.0925 −	0.0656±0.1397 −	0.0029±0.0001 −
MMF11	0.0037±0.0034	0.0069±0.0018 −	0.0044±0.0002 −	0.0048±0.0034 −	0.0037±0.0001 ≈
MMF12	0.0015±0.0014	0.0017±0.0002 −	0.0020±0.0001 −	0.0017±0.0004 −	0.0023±0.0001 −
MMF13	0.0251±0.0254	0.0282±0.0013 −	0.0710±0.0240 −	0.0317±0.0030 −	0.0533±0.0133 −
MMF14	0.0646±0.0625	0.0631±0.0023 ≈	0.1154±0.0118 −	−	0.0382±0.0006 +
MMF15	0.0517±0.0470	0.0475±0.0018 +	0.0710±0.0059 −	−	0.0385±0.0006 +
MMF16	0.0284±0.0310	0.0308±0.0018 −	0.0689±0.0196 −	0.0422±0.0049 −	0.0636±0.0146 −
MMF17	0.3975±0.2143	0.3043±0.1008 +	2.3871±1.8144 −	6.4334±4.6594 −	4.8973±2.5963 −
MMF18	0.0754±0.0753	0.0726±0.0024 +	0.1442±0.0161 −	−	0.0718±0.0016 +
MMF19	0.0553±0.0545	0.0535±0.0023 −	0.0915±0.0115 −	−	0.0476±0.0013 +
MMF20	0.0641±0.0565	0.0727±0.0093 −	5.4645±2.2086 −	0.0707±0.0155	0.0292±0.0165+
MMF21	0.0616±0.0504	0.1257±0.0104 −	6.9498±4.3783 −	0.1684±0.2523 −	1.9208±1.3825 −
MMF22	0.3722±0.5169	0.3133±0.0788 +	1.9829±0.5164 −	0.1502±0.0597 +	0.6523±0.3016 +
+/ − / ≈		4/17/1	1/21/0	1/17/0	7/14/1

TABLE VI
STATISTICAL RESULTS (MEAN AND STANDARD DEVIATION) OF rHV

	p-ACDCM	MMODE_CSCD	DN_NSGAII	MO_Ring_PSO_SCD	MMOPIO
MMF1	1.1453±0.0001	1.1455±0.0003−	1.1490±0.0016−	1.1476±0.0005−	1.1481±0.0009−
MMF2	1.1499±0.0009	1.1497±0.0009+	1.1655±0.0209−	1.1694±0.0045−	1.1504±0.0013−
MMF3	1.1488±0.0005	1.1488±0.0006≈	1.1574±0.0140−	1.1617±0.0030−	1.1495±0.0017−
MMF4	1.8525±0.0008	1.8529±0.0009−	1.8582±0.0017−	1.8595±0.0022−	1.8607±0.0039−
MMF5	1.1454±0.0002	1.1454±0.0003≈	1.1481±0.0011−	1.1473±0.0003−	1.1481±0.0016−
MMF6	1.1456±0.0005	1.1457±0.0006−	1.1482±0.0010−	1.1477±0.0007−	1.1474±0.0006−
MMF7	1.1453±0.0001	1.1454±0.0002−	1.1501±0.0024−	1.1481±0.0005−	1.1515±0.0016−
MMF8	2.3746±0.0013	2.3747±0.0018−	2.3805±0.0032−	2.3917±0.0098−	2.3799±0.0030−
MMF9	0.1032±0.0000	0.1032±0.0000≈	0.1033±0.0000−	0.1034±0.0000−	0.1034±0.0001−
MMF10	0.0800±0.0034	0.0781±0.0018+	0.0810±0.0034−	0.0788±0.0003+	0.0779±0.0004+
MMF11	0.0688±4.8910	0.0689±0.0000−	0.0689±0.0000−	0.0690±0.0000−	0.0690±0.0000−
MMF12	0.6354±0.0000	0.6355±0.0000−	0.6468±0.0402−	0.6375±0.0009−	0.6358±0.0002−
MMF13	0.0542±4.9817	0.0542±0.0000≈	0.0543±0.0000−	0.0544±0.0000−	0.0543±0.0000−
MMF14	0.3495±0.0182	0.3581±0.0151−	0.3350±0.0206−	0.3624±0.0345−	0.3274±0.0179+
MMF15	0.2519±0.0087	0.2443±0.0103+	0.2399±0.0138−	0.2448±0.0150−	0.2257±0.0103+
MMF16	1.1454±0.0002	1.1455±0.0002−	1.1479±0.0007−	1.1475±0.0004−	1.1471±0.0005−
MMF17	1.1736±0.0225	1.1711±0.0109+	1.1206±0.3944+	1.1647±0.0135+	1.1488±0.0018+
MMF18	0.3498±0.0229	0.3600±0.0199−	0.3175±0.0135+	0.3296±0.0318+	0.3079±0.0134+
MMF19	0.2495±0.0112	0.2498±0.0096−	0.2358±0.0157+	0.2396±0.0106+	0.2228±0.0068+
MMF20	0.0600±3.2389	0.0600±0.0000≈	0.0601±0.0000−	0.0602±0.0000−	0.0601±0.0000−
MMF21	0.0600±3.5595	0.0600±0.0000≈	0.0601±0.0000−	0.0603±0.0000	0.0601±0.0000−
MMF22	0.0188±7.4243	0.0190±0.0000−	0.0189±0.0000−	0.0190±0.0000−	0.0190±0.0000−
$+/-/\approx$		4/12/6	4/18/0	5/17/0	6/16/0

	p-ACDCM	MO_PSO_MM	Omni_optimizer	DE_RLFR	TriMOEA TAR
MMF1	1.1453±0.0001	1.1473±0.0007−	1.1473±0.0007−	1.1477±0.0007−	0.9924±0.4970+
MMF2	1.1499±0.0009	1.1602±0.0022−	1.1596±0.0190−	1.2002±0.0229−	1.1731±0.0098−
MMF3	1.1488±0.0005	1.1560±0.0011−	1.1640±0.0301−	1.1793±0.0183−	1.0572±0.3842−
MMF4	1.8525±0.0008	1.8659±0.0049−	1.8548±0.0005−	1.8638±0.0094−	1.5042±1.1165+
MMF5	1.1454±0.0002	1.1470±0.0007−	1.1467±0.0010−	1.1484±0.0030−	0.4526±2.2123−
MMF6	1.1456±0.0005	1.1473±0.0007−	1.1465±0.0005−	1.1489±0.0041−	1.0296±0.3743+
MMF7	1.1453±0.0001	1.1515±0.0011−	1.1470±0.0004−	1.1482±0.0008−	1.1743±0.0639−
MMF8	2.3746±0.0013	2.3792±0.0023−	2.3744±0.0008+	2.4169±0.1205−	2.0725±0.9758+
MMF9	0.1032±0.0000	0.1035±0.0001−	0.1033±0.0000−	0.1034±0.0001−	0.1047±0.0002−
MMF10	0.0800±0.0034	0.0778±0.0001+	0.0792±0.0024+	0.0790±0.0030+	0.0787±0.0000+
MMF11	0.0688±4.8910	0.0690±0.0000−	0.0689±0.0000−	0.0691±0.0002−	0.0696±0.0001−
MMF12	0.6354±0.0000	0.6385±0.0002−	0.6356±0.0000−	0.6395±0.0052−	0.6361±0.0000−
MMF13	0.0542±4.9817	0.0543±0.0000−	0.0543±0.0001−	0.0543±0.0000−	0.0550±0.0000−
MMF14	0.3495±0.0182	0.3189±0.0280x	0.3400±0.0109−	−	0.3153±0.0156+
MMF15	0.2519±0.0087	0.2286±0.0079−	0.2416±0.0130−	−	0.2217±0.0078+
MMF16	1.1454±0.0002	1.1469±0.0003−	1.1467±0.0010−	1.1480±0.0007−	1.1485±0.0023−
MMF17	1.1736±0.0225	1.1526±0.0017+	1.1571±0.0085+	1.1519±0.0074+	1.1520±0.0024+
MMF18	0.3498±0.0229	0.3148±0.0144+	0.3331±0.0104+	0.3432±0.0209−	−
MMF19	0.2495±0.0112	0.2273±0.0092+	0.2367±0.0124+	0.2296±0.0173−	−
MMF20	0.0600±3.2389	0.0601±0.0000−	0.0601±0.0000−	0.0601±0.0000−	0.0602±0.0000−
MMF21	0.0600±3.5595	0.0603±0.0000−	0.0601±0.0000−	0.0601±0.0000−	0.0601±0.0000−
MMF22	0.0188±7.4243	0.0190±0.0000−	0.0189±0.0000−	0.0190±0.0000−	0.0190±0.0000−
$+/-/\approx$		6/16/0	7/15/0	6/16/0	6/12/0

TABLE VII
STATISTICAL RESULTS (MEAN AND STANDARD DEVIATION) OF IGDX

	p-ACDCM	MMODE_CSCD	DN_NSGAII	MO_Ring_PSO_SCD	MMOPIO
MMF1	0.0409±0.0010	0.0412±0.0013−	0.0870±0.0114−	0.0463±0.0022−	0.0416±0.0025−
MMF2	0.0127±0.0089	0.0102±0.0018+	0.1030±0.0526+	0.0261±0.0106−	0.0120±0.0034≈
MMF3	0.0100±0.0049	0.0085±0.0018+	0.0738±0.0318−	0.0180±0.0030−	0.0115±0.0036−
MMF4	0.0219±0.0009	0.0221±0.0011−	0.1013±0.0359−	0.0258±0.0019−	0.0286±0.0041−
MMF5	0.0707±0.0017	0.0718±0.0034−	0.1594±0.0155−	0.0795±0.0037−	0.0841±0.0070−
MMF6	0.0634±0.0033	0.0622±0.0022+	0.1372±0.0169−	0.0678±0.0039−	0.0717±0.0043−
MMF7	0.0232±0.0026	0.0220±0.0013+	0.0454±0.0085−	0.0254±0.0010−	0.0344±0.0046−
MMF8	0.0476±0.0030	0.0484±0.0029−	0.3006±0.0967−	0.0582±0.0031−	0.0626±0.0121−
MMF9	0.0056±0.0002	0.0057±0.0003−	0.0228±0.0095−	0.0072±0.0004−	0.0122±0.0025−
MMF10	0.0771±0.1295	0.0368±0.1063+	0.1383±0.1461−	0.0154±0.0042+	0.0051±0.0022+
MMF11	0.0036±0.0001	0.0041±0.0003−	0.0046±0.0002−	0.0047±0.0001−	0.0072±0.0017−
MMF12	0.0015±0.0000	0.0016±0.0001−	0.0044±0.0094−	0.0030±0.0003−	0.0022±0.0005−
MMF13	0.0259±0.0006	0.0276±0.0008−	0.0719±0.0129−	0.0314±0.0011−	0.0333±0.0022−
MMF14	0.0634±0.0018	0.0624±0.0016+	0.1306±0.0141−	0.0663±0.0023−	0.0662±0.0027−
MMF15	0.0492±0.0023	0.0504±0.0019−	0.0848±0.0092−	0.0505±0.0018−	0.0488±0.0026≈
MMF16	0.0271±0.0016	0.0286±0.0009−	0.0709±0.0155−	0.0332±0.0015−	0.0317±0.0024−
MMF17	0.3332±0.1469	0.3198±0.1320≈	1.1126±0.5107−	0.3732±0.0725−	0.5041±0.2247−
MMF18	0.0746±0.0033	0.0740±0.0026−	0.1452±0.0114−	0.0745±0.0019−	0.0734±0.0025+
MMF19	0.0573±0.0031	0.0590±0.0029−	0.1095±0.0162−	0.0549±0.0022+	0.0547±0.0023+
MMF20	0.1651±0.3101	0.0538±0.0038+	4.3945±1.4891−	0.1105±0.0155+	0.0707±0.0105+
MMF21	0.1920±0.3488	0.0997±0.2171+	4.2530±1.7326−	0.1297±0.0130+	0.1205±0.2037+
MMF22	0.5905±0.1614	0.5560±0.1159+	1.6491±0.2145−	0.3633±0.0808+	0.6623±0.1386−
+/ − / ≈		9/12/1	1/21/0	5/17/0	5/15/2
	p-ACDCM	MO_PSO_MM	Omni_optimizer	DE_RLFR	TriMOEA TAR
MMF1	0.0409±0.0010	0.0416±0.0024−	0.0855±0.0156−	0.0524±0.0057−	0.0637±0.0071−
MMF2	0.0127±0.0089	0.0185±0.0057−	0.1025±0.0530−	0.0717±0.0367−	0.0656±0.0380−
MMF3	0.0100±0.0049	0.0133±0.0018−	0.0790±0.0459−	0.0500±0.0318−	0.0900±0.0398−
MMF4	0.0219±0.0009	0.0275±0.0022−	0.1162±0.0339−	0.0302±0.0035−	0.0642±0.1068−
MMF5	0.0707±0.0017	0.0749±0.0036−	0.1602±0.0153−	0.0860±0.0087−	0.1016±0.0098−
MMF6	0.0634±0.0033	0.0676±0.0030−	0.1415±0.0268−	0.0746±0.0042−	0.0847±0.0126−
MMF7	0.0232±0.0026	0.0320±0.0036−	0.0378±0.0079−	0.0360±0.0056−	0.0398±0.0223−
MMF8	0.0476±0.0030	0.0478±0.0033−	0.3105±0.1594−	0.0732±0.0157−	0.4127±0.0819−
MMF9	0.0056±0.0002	0.0098±0.0017−	0.0280±0.0160−	0.0066±0.0012−	0.0032±0.0001−
MMF10	0.0771±0.1295	0.0019±0.0003+	0.0602±0.0925+	0.0644±0.1388+	0.0029±0.0001+
MMF11	0.0036±0.0001	0.0069±0.0018−	0.0044±0.0002−	0.0047±0.0032−	0.0037±0.0001−
MMF12	0.0015±0.0000	0.0017±0.0002−	0.0020±0.0001−	0.0017±0.0004−	0.0023±0.0001−
MMF13	0.0259±0.0006	0.0279±0.0012−	0.0690±0.0173−	0.0310±0.0027−	0.0469±0.0096−
MMF14	0.0634±0.0018	0.0631±0.0023≈	0.1154±0.0118−	−	0.0382±0.0006+
MMF15	0.0492±0.0023	0.0475±0.0018+	0.0710±0.0059−	−	0.0385±0.0006+
MMF16	0.0271±0.0016	0.0306±0.0018−	0.0669±0.0179−	0.0418±0.0047−	0.0625±0.0138−
MMF17	0.3332±0.1469	0.2845±0.0683+	1.3087±0.6787−	2.3082±0.7334−	2.1337±0.6565−
MMF18	0.0746±0.0033	0.0724±0.0024−	0.1441±0.0161−	−	0.0718±0.0016+
MMF19	0.0573±0.0031	0.0534±0.0023+	0.0915±0.0115−	−	0.0475±0.0013+
MMF20	0.1651±0.3101	0.0726±0.0093+	5.0544±1.3630−	0.0705±0.0154+	0.0292±0.0165−
MMF21	0.1920±0.3488	0.1246±0.0102+	4.8056±1.8665−	0.1581±0.2130−	1.6371±1.1647−
MMF22	0.5905±0.1614	0.3104±0.0776+	1.7329±0.2275−	0.1484±0.0588+	0.6085±0.2512−
+/ − / ≈		7/14/1	1/21/0	4/14/0	5/17/0

TABLE VIII
STATISTICAL RESULTS (MEAN AND STANDARD DEVIATION) OF IGDF

	p-ACDCM	MMODE_CSCD	DN_NSGAII	MO_Ring_PSO_SCD	MMOPIO
MMF1	0.0023±0.0000	0.0023±0.0001≈	0.0041±0.0010−	0.0034±0.0002−	0.0037±0.0004−
MMF2	0.0040±0.0003	0.0043±0.0004−	0.0158±0.0176−	0.0135±0.0024−	0.0041±0.0004−
MMF3	0.0037±0.0002	0.0039±0.0003−	0.0087±0.0085−	0.0098±0.0014−	0.0037±0.0009≈
MMF4	0.0023±0.0001	0.0023±0.0001≈	0.0033±0.0003−	0.0034±0.0003−	0.0040±0.0007−
MMF5	0.0023±0.0000	0.0023±0.0000≈	0.0034±0.0002−	0.0032±0.0001−	0.0037±0.0007−
MMF6	0.0022±0.0000	0.0023±0.0001−	0.0034±0.0002−	0.0033±0.0001−	0.0034±0.0003−
MMF7	0.0023±0.0000	0.0024±0.0001−	0.0041±0.0005−	0.0037±0.0002−	0.0063±0.0009−
MMF8	0.0027±0.0001	0.0028±0.0001−	0.0037±0.0004−	0.0039±0.0002−	0.0034±0.0003−
MMF9	0.0108±0.0008	0.0104±0.0006+	0.0141±0.0018−	0.0149±0.0018−	0.0207±0.0056−
MMF10	0.1534±0.1375	0.0399±0.0883+	0.1432±0.1181+	0.0630±0.0101+	0.0223±0.0235+
MMF11	0.0112±0.0008	0.0114±0.0010−	0.0131±0.0013−	0.0159±0.0016−	0.0221±0.0048−
MMF12	0.0020±0.0000	0.0021±0.0001−	0.0057±0.0113−	0.0047±0.0003−	0.0038±0.0007−
MMF13	0.0136±0.0028	0.0149±0.0035−	0.0174±0.0039−	0.0242±0.0061−	0.0231±0.0067−
MMF14	0.0913±0.0028	0.0914±0.0029≈	0.1390±0.0098−	0.1013±0.0039−	0.1020±0.0036−
MMF15	0.0968±0.0033	0.1004±0.0036−	0.1744±0.0198−	0.1053±0.0033−	0.1093±0.0071−
MMF16	0.0022±0.0000	0.0023±0.0001−	0.0034±0.0003−	0.0033±0.0001−	0.0033±0.0002−
MMF17	0.0083±0.0010	0.0089±0.0023−	0.0150±0.0107−	0.0074±0.0007+	0.0035±0.0003+
MMF18	0.0903±0.0027	0.0894±0.0021+	0.1514±0.0164−	0.0971±0.0032−	0.0954±0.0034−
MMF19	0.1004±0.0047	0.1028±0.0048−	0.1825±0.0214−	0.1026±0.0043−	0.1088±0.0058−
MMF20	0.0104±0.0010	0.0105±0.0012−	0.0129±0.0014−	0.0245±0.0030−	0.0141±0.0019−
MMF21	0.0100±0.0009	0.0098±0.0011+	0.0145±0.0013−	0.0273±0.0037−	0.0139±0.0018−
MMF22	0.0200±0.0033	0.0207±0.0024−	0.0116±0.0005+	0.0361±0.0036−	0.0113±0.0014+
+/ − / ≈		4/14/4	2/20/0	2/20/0	3/18/1
	p-ACDCM	MO_PSO_MM	Omni_optimizer	DE_RLFR	TriMOEA TAR
MMF1	0.0023±0.0000	0.0034±0.0003−	0.0033±0.0003−	0.0036±0.0004−	0.0043±0.0011−
MMF2	0.0040±0.0003	0.0092±0.0012−	0.0115±0.0140−	0.0385±0.0203−	0.0174±0.0079−
MMF3	0.0037±0.0002	0.0071±0.0006−	0.0165±0.0256−	0.0207±0.0107−	0.0453±0.0824−
MMF4	0.0023±0.0001	0.0048±0.0010−	0.0027±0.0002−	0.0037±0.0003−	0.0188±0.0498−
MMF5	0.0023±0.0000	0.0033±0.0003−	0.0030±0.0002−	0.0036±0.0002−	0.0041±0.0013−
MMF6	0.0022±0.0000	0.0033±0.0003−	0.0029±0.0002−	0.0036±0.0004−	0.0035±0.0009−
MMF7	0.0023±0.0000	0.0061±0.0008−	0.0031±0.0002−	0.0041±0.0005−	0.0040±0.0014−
MMF8	0.0027±0.0001	0.0031±0.0001−	0.0030±0.0002−	0.0040±0.0003−	0.0058±0.0072−
MMF9	0.0108±0.0008	0.0254±0.0061−	0.0133±0.0013−	0.0192±0.0035−	0.0698±0.0070−
MMF10	0.1534±0.1375	0.0132±0.0032+	0.0707±0.0918+	0.0606±0.1018+	0.0378±0.0025+
MMF11	0.0112±0.0008	0.0198±0.0034−	0.0120±0.0009−	0.0275±0.0119−	0.0740±0.0067−
MMF12	0.0020±0.0000	0.0025±0.0001−	0.0023±0.0001−	0.0075±0.0045−	0.0057±0.0002−
MMF13	0.0136±0.0028	0.0177±0.0041−	0.0164±0.0068−	0.0075±0.0045+	0.1043±0.0082−
MMF14	0.0913±0.0028	0.1010±0.0050−	0.1216±0.0051−	−	0.0911±0.0011−
MMF15	0.0968±0.0033	0.1060±0.0047−	0.1400±0.0077−	−	0.0918±0.0013+
MMF16	0.0022±0.0000	0.0032±0.0001−	0.0029±0.0003−	0.0037±0.0004−	0.0038±0.0005−
MMF17	0.0083±0.0010	0.0053±0.0005+	0.0087±0.0072−	0.0060±0.0049+	0.0053±0.0007+
MMF18	0.0903±0.0027	0.0966±0.0033−	0.1308±0.0085−	−	0.0934±0.0013−
MMF19	0.1004±0.0047	0.1055±0.0051−	0.1487±0.0137−	−	0.0913±0.0026+
MMF20	0.0104±0.0010	0.0172±0.0024−	0.0119±0.0012−	0.0205±0.0035−	0.0400±0.0042−
MMF21	0.0100±0.0009	0.0295±0.0035−	0.0131±0.0016−	0.0238±0.0057−	0.0278±0.0033−
MMF22	0.0200±0.0033	0.0340±0.0036−	0.0102±0.0005+	0.0199±0.0051≈	0.0216±0.0052−
+/ − / ≈		2/20/0	2/20/0	3/14/1	3/19/0

ensures better coverage of the search space by the pareto front.

B. Experimental results and analysis of weight-based crowding strategy

The comparison results with different comparison algorithms on the four performance indicators (rPSP, rHV, IGDX and IGDF) of the test problems is given in Fig. 4-7 by using histograms.

The experimental results of the p-ACDCM algorithm and the eight comparison algorithms are shown in Table V to Table VI. Table V gives the statistical results (mean and variance) of rPSP, Table VI gives the statistical results (mean and variance) of rHV, Table VII gives the statistical results (mean and variance) for the statistical results (mean and variance) of IGDX. Table VIII gives the statistical results (mean and variance) of IGDF.

The p-ACDCM had more minimum averages than the other compared algorithms in four metrics, namely rPSP, rHV, IGDX, and IGDF. p-ACDCM used L_p for convergence, and adaptive weighting to maintain the diversity of the populations, and improve the population uniformity of distribution. The p-ACDCM algorithm used L_p and weights to increase the diversity of solutions. In decision space, lower decision space crowding usually means that the solutions are more widely distributed in the decision space, which can increase the diversity of PF so that the final solution set contains more different types of solution and helps to find diverse solutions in the search space. Second, the p-ACDCM algorithm can reduce the effect of local optimal solutions. Lower decision space crowding reduces the risk of falling into local optimal solutions. When the solutions are more widely distributed, the optimization algorithm is more likely to find the global optimal solution rather than being restricted to the vicinity of some local optimal solution. p-ACDCM algorithm reduces the decision space crowding. On the one hand, it helps to generate uniform PF. So, the solutions are uniformly distributed in the decision space. It improved uniformity in multi-objective problems and ensures good coverage in different problem domains. Smaller decision space usually reduces the size of the search space, which reduces the computational complexity of the optimization algorithm. In objective space, the p-ACDCM algorithm reduces the object space crowding, and reduces the excessive aggregation between the solutions on the PF. In genneral, lower objective space crowding usually increases the performance of the algorithm, including convergence and search ability, which helps the algorithm find better PF faster.

Clustering algorithms in the decision space may (e.g., K-means) lead to an uneven distribution of solutions because a value of K may lead to overfitting. With the merging strategy in p-ACDCM algorithm, reasonable clusters can be generated, which can better capture the details and internal structure of the data.

VI. Conclusion

In order to improve the diversity of solutions and provide more potential solutions for decision makers, this paper proposed p-ACDCM to solve multi-modal multi-objective problem. The algorithm can improve the performance of the algorithm by defining an adaptive crowding measurement formula and designing an automatic classification method for solutions. Experimental results show that the improved multi-modal multi-objective algorithm can effectively improve the performance of the algorithm. In future research, it is recommended to focus on finer and more accurate crowding measurement metrics and more efficient solution classification techniques.

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