Spectral Properties and Energy of Generalized Hypercubes

Jinxia Yang, Jier Liu, Zhipeng Liang*

Abstract—**The generalized hypercube $Q(d_1,d_2,\cdots,d_n)$ is a topological structure renowned for its exceptional symmetry and performance. Based on its structure and properties, this paper derives explicit expressions for its adjacency and Laplacian spectra apply these results to compute its energy. MATLAB is utilized for verification and to develop program commands that calculate the adjacency spectrum, Laplacian spectrum,and energy $Q(d_1,d_2,\cdots,d_n)$ for specified parameters $d_1,d_2,\cdots,d_n$.**

Index Terms—**Generalized hypercube, Characteristic polyno mial, Adjacency spectrum, Laplacian spectrum, Graph energy**

I. Introduction

THE topology of an interconnection network is usually typically modeled as a mathematical graph, where ver tices correspond to servers and edges represent connections. The Generalized Hypercube, proposed in [1] as a variant of the hypercube [2], is known for its excellent interconnection properties[2].It features several advantageous properties, suc h as regularity, symmetry, embeddability, and a short diamet er. Its connection mode and recursive structure make it easy to construct, and its topology is versatile.

Over the past five decades, spectral graph theory has become an important field in graph theory and attracted wide attention. Its research holds theoretical significance and practical application in network optimization design. The research covers many aspects, with the investigation of the standard adjacency spectrum of hypercubes and variants being common [3-12]. This paper examines the adjacency and Laplacian spectral characterization of generalized hypercubes constructed using Cartesian product operations. These results are applied to consider the energy problem of the generalized hypercubes. We use MATLAB to thoroughly verify the conclusion and design program commands for deriving the adjacency spectrum, Laplacian spectrum and energy of the generalized hypercube with given parameters $d_1,d_2,\cdots,d_n$.

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II. Basic concepts

For convenience, let $u \sim v$ represent vertex u, and v be adjacent if the vertex is connected to vertex $u,v \in V(G)$.

Definition2.1:[1] Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs. The Cartesian product of these graphs $G_1 \times G_2$ is defined as follows:

The vertex set is $V(G_1 \times G_2)=V_1 \times V_2$;

The Edge set is

$$E(G_1 \times G_2)=\{\{(u_1v_1)(u_2v_2)\mid u_1=u_2, and\ v_1 \sim v_2 \\ or\ v_1=v_2, and\ u_1 \sim u_2\}.$$

Definition 2.2: [1-2] The topology of a hypercube network refers to n dimensional cube, with a graph-theoretic model as a simple undirected graph. This is typically denoted as Q_n. Harary provides many equivalent definitions, of which two are the most common:

1. The vertex set of Q_n is the ordered n element array from $\{0,1\}$, even if $V=\{x_1x_2\cdots x_n : x_i \in \{0,1\}, i=1,2,\cdots,n\}$, two vertices $x=x_1x_2\cdots x_n$ and $y=y_1y_2\cdots y_n$ being adjacent if and only if x and y differ in exactly one coordinate, even if $\sum_{i=1}^{n}|x_i-y_i|=1$.

2. Q_n can be recursively defined as a Cartesian product:

$$Q_1=K_2,\ \ldots,\ Q_n=Q_{n-1}\times Q_1=\underbrace{K_2\times K_2\times\cdots\times K_2}_{n},\ n\geq 2.$$

Hypercubes possess several advantageous properties. In this study, we focus on the regularity n of the hypercube Q_n, characterized by 2^n vertices and $n2^{n-1}$ edges.

Definition2.3:[1] Bhuyan and Agrawal generalize Q_n to the n dimensional generalized hypercube network, denoted as follows:

$Q(d_1,d_2,\cdots,d_n)$, $d_i \geq 2$is integer,$i=1,2,\cdots n$, and are defined as follows:

The set of vertices of the generalized hypercube is $V=\{x_1x_2\cdots x_n : x_i \in \{0,1,\cdots d_i-1\}, i=1,2,\cdots,n\}$, where two vertices $x=x_1x_2\cdots x_n$ and $y=y_1y_2\cdots y_n$ are connected if and only if they differ in exactly one coordinate.

Alternatively, the generalized hypercube can be defined as

$$Q(d_1,d_2,\cdots,d_n)=K_{d_1}\times K_{d_2}\times\cdots\times K_{d_n}.$$

Specifically, when $d_1=d_2=\cdots=d_n=d\geq 2$, generalized hypercube $Q(d,d,\cdots,d)$ is called the d element n-dimension cube $Q_n(d)$. Clearly, when $d=2$, $Q_n(2)$ corresponds to the famous hypercube network Q_n.

Generalized hypercube exhibits several desirable properties.This paper uses the following important properties:

$Q(d_1,d_2,\cdots,d_n)$ is $\sum_{i=1}^{n} d_i - n$ regular, with $\prod_{i=1}^{n} d_i$ vertices and $\frac{1}{2}(\prod_{i=1}^{n} d_i)(\sum_{i=1}^{n} d_i - n)$ edges.

Definition 2.4:[3-4] Let G be a simple graph whose adjacency matrix is represented as $A(G)$, the Laplace matrix is expressed as $L(G)=\Delta(G)-A(G)$, where $\Delta(G)$ is the degree-diagonal matrix of the graph G.

i.e. $\Delta(G) = diag\{\underbrace{d_G(v_1), d_G(v_2), \cdots, d_G(v_n)}_{n}\}$.

The matrix $\lambda I - A(G)$ with unknown quantities λ is called the characteristic matrix of $A(G)$ ($\lambda I - L(G)$ called the characteristic matrix), $\det(\lambda I - A(G))$ is called the adjacency characteristic polynomial of the graph ($\det(\lambda I - L(G))$ called the Laplace characteristic polynomial). The zeros of $\det(\lambda I - A(G))$ are referred to as the adjacency eigenvalues of graph G, while the zeros of $\det(\lambda I - L(G))$ are known as the Laplace eigenvalues of $L(G)$. When all the adjacency eigenvalues (Laplace eigenvalues) along with their respective multiplicities are combined, they form the adjacency spectrum (Laplacian spectrum) of graph G. These spectra can be simply denoted as $Sp\ A(G)$ ($Sp\ L(G)$).

Definition 2.5:[8-9] If G is an (n,m) -graph, and its Laplacian eigenvalues are $\mu_1, \mu_2, \cdots, \mu_n$, then the Laplacian energy of G, denoted by $LE(G)$, is equal to $LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$.

In addition, " $\otimes$ " represents the tensor product of the matrix (Kronecker product). The three important matrices of the generalized hypercube $Q(d_1,d_2,\cdots,d_n)$ are in order:

(1) Adjacency matrix $A(G)$

$A(Q(d_1)) = A(K_{d_1})$,

$A(Q(d_1,d_2)) = I_{d_1} \otimes A(K_{d_2}) + A(K_{d_1}) \otimes I_{d_2}$,

$A(Q(d_1,d_2,d_3)) = I_{d_3} \otimes A(Q(d_1,d_2)) + A(K_{d_3}) \otimes I_{d_1 d_2}$,

……

$$A(Q(d_1,d_2,\cdots,d_n)) = I_{d_n} \otimes A(Q(d_1,d_2,\cdots,d_{n-1})) + A(K_{d_n}) \otimes I_{d_1 d_2 \cdots d_{n-1}}.$$

(2) Degree matrix $\Delta(G)$

According to the properties of $Q(d_1,d_2,\cdots,d_n)$, let $\sum_{i=1}^{n} d_i - n$ be the regularity degree and $\prod_{i=1}^{n} d_i$ be the number of vertices. Thus, the degree sequence

$$\pi(Q(d_1,d_2,\cdots,d_n)) = diag\{\underbrace{\sum_{i=1}^{n} d_i - n, \cdots, \sum_{i=1}^{n} d_i - n}_{\prod_{i=1}^{n} d_i}\}$$

of $Q(d_1,d_2,\cdots,d_n)$. Therefore, the degree matrix is:

$$\Delta(Q(d_1,d_2,\cdots,d_n)) = diag\{\underbrace{\sum_{i=1}^{n} d_i - n, \cdots, \sum_{i=1}^{n} d_i - n}_{\prod_{i=1}^{n} d_i}\}.$$

(3) The Laplace matrix $L(G)$

$$L(Q(d_1,d_2,\cdots,d_n)) = diag\{\underbrace{\sum_{i=1}^{n} d_i - n, \cdots, \sum_{i=1}^{n} d_i - n}_{\prod_{i=1}^{n} d_i}\} - A(Q(d_1,d_2,\cdots,d_n))$$

The generalized hypercube $Q(d_1,d_2)$ has $d_1 d_2$ vertices. According to the decimal and binary conversion relationship, we use the decimal number to represent the vertex. Thus, the adjacency matrix of $Q(2,3)$ and $Q(3,5)$ can be expressed as (see Figure 1 and Figure 2)

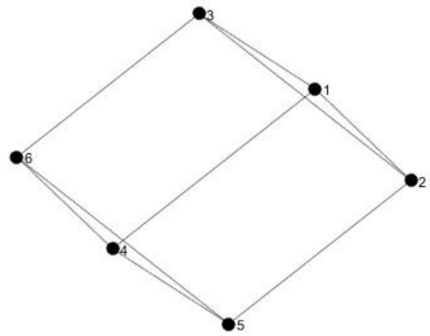

Fig.1. The generalized hypercube $Q(2,3)$

$$A(Q(2,3)) = \left(\begin{array}{ccc:ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) = \begin{pmatrix} A(K_3) & I_3 \\ I_3 & A(K_3) \end{pmatrix}$$

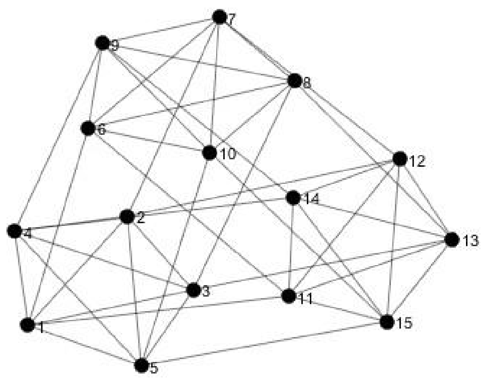

Fig.2. The generalized hypercube $Q(3,5)$

$$A(Q(3,5)) = \begin{pmatrix} A(K_5) & I_5 & I_5 \\ I_5 & A(K_5) & I_5 \\ I_5 & I_5 & A(K_5) \end{pmatrix}$$

It is not difficult to verify that this fact is easily extended to the general case. If $d_1 \le d_2$, the following lemma holds.

Lemma 1: The adjacency matrix A of $Q(d_1,d_2)$ is a block matrix of $d_1 \times d_1$ order, where each subblock on the main diagonal is the adjacency matrix of the complete graph K_{d_2}, and the rest of the subblocks are the identity matrix of the same order, expressed as

$$A(Q(d_1,d_2)) = \begin{pmatrix} A(K_{d_2}) & \cdots & I_{d_2} \\ \vdots & \ddots & \vdots \\ I_{d_2} & \cdots & A(K_{d_2}) \end{pmatrix}_{d_1 \times d_1}.$$

Lemma 2: " ab "type determinant

$$D_n = \begin{vmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{vmatrix} = [a+(n-1)b](a-b)^{n-1}.$$

Lemma 3: Let the adjacency matrix A of the generalized hypercube, $Q(d_1,d_2,\cdots,d_n)$ and the Laplace matrix be L, the eigenvalue $\mu = \Delta - \lambda$, where Δ is the regularity degree of $Q(d_1,d_2,\cdots,d_n)$ and λ is the eigenvalue of A.

Proof: The characteristic polynomial of the matrix L is:

$$|\lambda I - L| = |\lambda I - (\Delta - A)| = |(\lambda - \Delta)I + A|$$

$$= (-1)^{\prod_{i=1}^{n} d_i} |-(\lambda - \Delta)I - A|$$

$$= (-1)^{\prod_{i=1}^{n} d_i} |(\Delta - \lambda)I - A| \qquad \text{(I)}$$

Case 1: When $d_1,d_2,\cdots,d_n$ has at least one even number, equation (I) simplifies to $|\lambda I - L| = |(\Delta - \lambda)I - A|$, that is, the eigenvalue of the matrix L is $\mu = \Delta - \lambda$.

Case 2: When $d_1,d_2,\cdots,d_n$ are odd, equation (I) is simplified to $|\lambda I - L| = (-1)|(\Delta - \lambda)I - A|$. Let it be equal to zero, both sides simultaneously eliminate -1, and we obtain the eigenvalue of the matrix L is $\mu = \Delta - \lambda$.

Considering both cases, the eigenvalue of the Laplace matrix L of $Q(d_1,d_2,\cdots,d_n)$ is $\mu = \Delta - \lambda$.

III. Spectral Characterization of $Q(d_1,d_2)$

3.1 The adjacency spectrum of $Q(d_1,d_2)$

Theorem 3.1.1: For $d_1 < d_2$, the adjacency spectrum $Q(d_1,d_2)$ has four different eigenvalues:

$$Sp\ A(Q(d_1,d_2)) = \begin{pmatrix} d_2+d_1-2 & d_2-2 & d_1-2 & -2 \\ 1 & d_1-1 & d_2-1 & (d_1-1)(d_2-1) \end{pmatrix}.$$

Proof: For $d_1 \le d_2$, according to Lemma 1 and 2, the characteristic polynomial of the adjacency matrix A of $Q(d_1,d_2)$ of the generalized hypercube can be derived as follows:

$$|\lambda I - A| = \begin{vmatrix} \lambda I_{d_2} - A(K_{d_2}) & \cdots & -I_{d_2} \\ \vdots & \ddots & \vdots \\ -I_{d_2} & \cdots & \lambda I_{d_2} - A(K_{d_2}) \end{vmatrix}_{d_1 \times d_1}$$

$$= \left| [(\lambda I_{d_2} - A(K_{d_2})) + (d_1-1)\cdot(-I_{d_2})]\cdot[(\lambda I_{d_2} - A(K_{d_2})) - (-I_{d_2})]^{d_1-1} \right|$$

$$= \{[(\lambda - (d_1-1) + (d_2-1)(-1)]\cdot[(\lambda - (d_1-1) - (-1)]^{(d_2-1)}\}\cdot$$
$$\{(\lambda+1) + (d_2-1)(-1)]\cdot[(\lambda+1) - (-1)]^{d_2-1}\}^{d_1-1}$$

$$= [\lambda - (d_1+d_2-2)][\lambda - (d_1-2)]^{d_2-1}[\lambda - (d_2-2)]^{d_1-1} (\lambda+2)^{(d_1-1)(d_2-1)} \qquad (1)$$

Therefore, if the above equation (1) is zero, there are four different eigenvalues of the adjacency matrix of $Q(d_1,d_2)$, and the spectrum is:

$$Sp\ A(Q(d_1,d_2)) = \begin{pmatrix} \sum_{i=1}^{2} d_i - 2 & d_1-2 & d_2-2 & -2 \\ 1 & d_2-1 & d_1-1 & \prod_{i=1}^{2}(d_i-1) \end{pmatrix}.$$

According to the conclusion proved in the above theorem, specifically, for $d_1 = d_2 \ge 2$, adjacency spectrum of the generalized hypercube $Q(d_1,d_2)$ is concluded as:

Corollary 3.1.2: If $d_1 = d_2 \ge 2$,then $Q(d_1,d_2)$ has three different eigenvalues, and adjacency spectrum is:

$$Sp\ A(Q(d_1,d_2)) = \begin{pmatrix} 2(d_1-1) & d_1-2 & -2 \\ 1 & 2(d_1-1) & (d_1-1)^2 \end{pmatrix}.$$

3.2 The Laplacian spectrum of $Q(d_1,d_2)$

From Lemma 1, the adjacency matrix A of $Q(d_1,d_2)$ is a block matrix of order $d_1 \times d_1$. Therefore, the Laplace matrix $L = \Delta - A$ of $Q(d_1,d_2)$ is expressed as follows:

Lemma 3.2.1: The Laplace matrix L of the generalized hypercube $Q(d_1,d_2)$ is expressed as：

$$L(Q(d_1,d_2)) = \begin{pmatrix} \Delta - A(K_{d_2}) & \cdots & -I_{d_2} \\ \vdots & \ddots & \vdots \\ -I_{d_2} & \cdots & \Delta - A(K_{d_2}) \end{pmatrix}_{d_1 \times d_1}.$$

Theorem 3.2.2: There are four different eigenvalues of $Q(d_1,d_2)$. The spectrum of the Laplace matrix is as follows:

$$Sp\ L(Q(d_1,d_2)) = \begin{pmatrix} 0 & d_2 & d_1 & d_2+d_1 \\ 1 & d_2-1 & d_1-1 & (d_1-1)(d_2-1) \end{pmatrix}.$$

Proof: From Lemma 3, it is only necessary to determine the regularity degree of the generalized hypercube $Q(d_1,d_2)$, $\Delta = d_1+d_2-2$, and to find the eigenvalue λ of the adjacency matrix A, so as to obtain the eigenvalue of the Laplace matrix L. Therefore, the Laplacian spectrum of the generalized hypercube $Q(d_1,d_2)$ is:

$$Sp\ L(Q(d_1,d_2)) = \begin{pmatrix} 0 & d_2 & d_1 & d_2+d_1 \\ 1 & d_2-1 & d_1-1 & (d_1-1)(d_2-1) \end{pmatrix}.$$

According to the conclusion proved in the above theorem, specifically, if $d_1 = d_2 \ge 2$, the spectrum of the Laplace matrix of $Q(d_1,d_2)$ is concluded as follows:

Corollary 3.2.3: If $d_1 = d_2 \ge 2$, there are three different eigenvalues of $Q(d_1,d_2)$, and the spectrum of the Laplace matrix is as follows:

$$Sp\ L(Q(d_1,d_2))=\begin{pmatrix}0 & d_1 & 2d_1\\ 1 & 2(d_1-1) & (d_1-1)^2\end{pmatrix}.$$

Example 1: MATLAB verification confirms that the adjacency spectrum of $Q(3,5)$ agrees with Theorem 3.1.1 and the Laplacian spectra comply with Theorem 3.2.3.

Solution: The generalized hypercube $Q(3,5)$, as shown in Figure 2 above, has an adjacency spectrum

$$Sp\ A(Q(d_1,d_2))=\begin{pmatrix}6 & 1 & 3 & -2\\ 1 & 4 & 2 & 8\end{pmatrix}.$$

This result is obtained using MATLAB commands. Meanwhile, according to the conclusion of Theorem 3.1.1, the result is consistent.

According to the MATLAB programming command, its Laplacian spectrum is

$$Sp\ L(Q(d_1,d_2))=\begin{pmatrix}0 & 5 & 3 & 8\\ 1 & 4 & 2 & 8\end{pmatrix}.$$

According to the conclusion of Theorem 3.2.2, the conclusion is consistent.

IV. SPECTRAL CHARACTERIZATION OF $Q(d_1,d_2,d_3)$

4.1 The adjacency spectrum of $Q(d_1,d_2,d_3)$

Let $d_1\le d_2\le d_3$, and based on Lemma 1, the following results are established:

Lemma 4.1.1: The adjacency matrix A of $Q(d_1,d_2,d_3)$ is a partition matrix of $d_3\times d_3$ order, where each subblock on the main diagonal is an adjacency matrix of $Q(d_1,d_2)$, and the rest of the subblocks are identity matrices of equal order as $Q(d_1,d_2)$, expressed as

$$A(Q(d_1,d_2,d_3))=\begin{pmatrix}A(Q(d_1,d_2)) & \cdots & I_{d_1d_2}\\ \vdots & \ddots & \vdots\\ I_{d_1d_2} & \cdots & A(Q(d_1,d_2))\end{pmatrix}_{d_3\times d_3}.$$

Theorem 4.1.2: The characteristic polynomial of the adjacency matrix of $Q(d_1,d_2,d_3)$ is given by:

$$\begin{aligned}&[\lambda-(d_1+d_2+d_3-3)]\cdot[\lambda-(d_1+d_3-3)]^{d_2-1}\\ &\cdot[\lambda-(d_2+d_3-3)]^{d_1-1}\cdot[\lambda-(d_3-3)]^{(d_1-1)(d_2-1)}\\ &\cdot[\lambda-(d_1+d_2-3)]^{d_3-1}\cdot[\lambda-(d_1-3)]^{(d_2-1)(d_3-1)}\\ &\cdot[\lambda-(d_2-3)]^{(d_1-1)(d_3-1)}\cdot[\lambda+3]^{(d_1-1)(d_2-1)(d_3-1)}\end{aligned}$$

Proof: From the above Lemma 2 and lemma 4.1.1, the characteristic polynomial of the adjacency matrix of $Q(d_1,d_2,d_3)$ is:

$$\begin{aligned}&\left|\lambda I-A(Q(d_1,d_2,d_3))\right|\\
&=\begin{vmatrix}\lambda I_{d_1d_2}-A(Q(d_1,d_2)) & \cdots & -I_{d_1d_2}\\ \vdots & \ddots & \vdots\\ -I_{d_1d_2} & \cdots & \lambda I_{d_1d_2}-A(Q(d_1,d_2))\end{vmatrix}_{d_3\times d_3}\\
&=\begin{vmatrix}[(\lambda I_{d_1d_2}-A(Q(d_1,d_2)))+(d_3-1)\cdot(-I_{d_1d_2})]\cdot\\ [(\lambda I_{d_1d_2}-A(Q(d_1,d_2)))-(-I_{d_1d_2})]^{d_3-1}\end{vmatrix}\\
&=\begin{vmatrix}[(\lambda-(d_3-1))I_{d_1d_2}-A(Q(d_1,d_2))]\cdot\\ [(\lambda+1)I_{d_1d_2}-A(Q(d_1,d_2))]^{d_3-1}\end{vmatrix}\\
&=\begin{vmatrix}[((\lambda-(d_3-1))I_{d_2}-A(Q(d_2)))+(d_1-1)(-I_{d_2})]\cdot\\ [((\lambda-(d_3-1))I_{d_2}-A(Q(d_2)))-(-I_{d_2})]^{d_1-1}\end{vmatrix}\\
&\quad\cdot\begin{vmatrix}[((\lambda+1)I_{d_2}-A(Q(d_2)))+(d_1-1)(-I_{d_2})]\cdot\\ [((\lambda+1)I_{d_2}-A(Q(d_2)))-(-I_{d_2})]^{d_1-1}\end{vmatrix}^{d_3-1}\\
&=\left|(\lambda-(d_1+d_3-2))I_{d_2}-A(Q(d_2))\right|\cdot\\
&\quad\left|[(\lambda-(d_3-2))I_{d_2}-A(Q(d_2))]^{d_1-1}\right|\\
&\quad\cdot\left|(\lambda-(d_1-2))I_{d_2}-A(Q(d_2))\right|^{d_3-1}\cdot\\
&\quad\left|[(\lambda+2)I_{d_2}-A(Q(d_2))]^{(d_1-1)(d_3-1)}\right|\\
&=[(\lambda-(d_1+d_3-2))+(d_2-1)(-1)]\cdot\\
&\quad[(\lambda-(d_1+d_3-2))-(-1)]^{d_2-1}\\
&\quad\cdot[(\lambda-(d_3-2))+(d_2-1)(-1)]^{d_1-1}\cdot\\
&\quad[(\lambda-(d_3-2))-(-1)]^{(d_1-1)(d_2-1)}\\
&\quad\cdot[(\lambda-(d_1-2))+(d_2-1)(-1)]^{d_3-1}\cdot\\
&\quad[(\lambda-(d_1-2))-(-1)]^{(d_2-1)(d_3-1)}\\
&\quad\cdot[(\lambda+2)+(d_2-1)(-1)]^{(d_1-1)(d_3-1)}\cdot\\
&\quad[(\lambda+2)-(-1)]^{(d_1-1)(d_2-1)(d_3-1)}\\
&=[\lambda-(d_1+d_2+d_3-3)]\cdot[\lambda-(d_1+d_3-3)]^{d_2-1}\\
&\quad\cdot[\lambda-(d_2+d_3-3)]^{d_1-1}\cdot[\lambda-(d_3-3)]^{(d_1-1)(d_2-1)}\\
&\quad\cdot[\lambda-(d_1+d_2-3)]^{d_3-1}\cdot[\lambda-(d_1-3)]^{(d_2-1)(d_3-1)}\\
&\quad\cdot[\lambda-(d_2-3)]^{(d_1-1)(d_3-1)}\cdot[\lambda+3]^{(d_1-1)(d_2-1)(d_3-1)}\qquad(3)\end{aligned}$$

If the above equation (3) is zero, the adjacency spectrum of $Q(d_1,d_2,d_3)$ is obtained as:

$$Sp\ A(Q(d_1,d_2,d_3))=\begin{pmatrix}\sum_{i=1}^{3}d_i-3 & \sum_{i\ne1}^{3}d_i-3 & \sum_{i\ne2}^{3}d_i-3 & \sum_{i\ne3}^{3}d_i-3\\ 1 & d_1-1 & d_2-1 & d_3-1\end{pmatrix.$$
$$\begin{matrix}d_1-3 & d_2-3 & d_3-3 & -3\\ \prod_{i\ne1}^{3}(d_i-1) & \prod_{i\ne2}^{3}(d_i-1) & \prod_{i\ne3}^{3}(d_i-1) & \prod_{i=1}^{3}(d_i-1)\end{matrix}\Bigg).$$

According to the conclusion proved in the above theorem, specifically, if $d_1=d_2=d_3\ge2$, the adjacency spectrum of $Q(d_1,d_2,d_3)$ is as follows:

Corollary 4.1.3: Let $d_1=d_2=d_3\ge2$. There are four different eigenvalues of $Q(d_1,d_2,d_3)$, and the adjacency spectrum is as follows:

$$Sp\ A(Q(d_1,d_2,d_3))=\begin{pmatrix}3(d_1-1) & 2d_1-3 & d_1-3 & -3\\ 1 & 3(d_1-1) & 3(d_1-1)^2 & (d_1-1)^3\end{pmatrix}.$$

4.2 The Laplacian spectrum of $Q(d_1,d_2,d_3)$

According to the definition of Laplace matrix $L=\Delta-A$ of $Q(d_1,d_2,d_3)$ and Lemma 4.1.1, the following lemma can be obtained:

Lemma 4.2.1: The Laplace matrix L of $Q(d_1,d_2,d_3)$ is expressed as:

$$L(Q(d_1,d_2,d_3)) = \begin{pmatrix} \Delta - A(Q(d_1,d_2)) & \cdots & -I_{d_1d_2} \\ \vdots & \ddots & \vdots \\ -I_{d_1d_2} & \cdots & \Delta - A(Q(d_1,d_2)) \end{pmatrix}_{d_3\times d_3}.$$

Theorem 4.2.2 There are eight different eigenvalues of $Q(d_1,d_2,d_3)$, and the spectrum of the Laplace matrix is as follows:

$$Sp\ L(Q(d_1,d_2,d_3)) = \begin{pmatrix} 0 & d_1 & d_2 & d_3 & d_2+d_3 & d_1+d_3 & d_1+d_2 & \sum_{i=1}^{3} d_i \\ 1 & d_1-1 & d_2-1 & d_3-1 & \prod_{i\neq 1}^{3}(d_i-1) & \prod_{i\neq 2}^{3}(d_i-1) & \prod_{i\neq 3}^{3}(d_i-1) & \prod_{i=1,2,3}(d_i-1) \end{pmatrix}.$$

Proof: From lemma 3, we only need to determine the regularity degree of $Q(d_1,d_2,d_3)$ as $\Delta = d_1+d_2+d_3-3$ and find the eigenvalue λ of the adjacency matrix A, thus obtaining the eigenvalue of the Laplace matrix L. Therefore, the Laplacian spectrum of $Q(d_1,d_2,d_3)$ is

$$Sp\ L(Q(d_1,d_2,d_3)) = \begin{pmatrix} 0 & d_1 & d_2 & d_3 & d_2+d_3 & d_1+d_3 & d_1+d_2 & \sum_{i=1}^{3} d_i \\ 1 & d_1-1 & d_2-1 & d_3-1 & \prod_{i\neq 1}^{3}(d_i-1) & \prod_{i\neq 2}^{3}(d_i-1) & \prod_{i\neq 3}^{3}(d_i-1) & \prod_{i=1,2,3}(d_i-1) \end{pmatrix}.$$

According to the proof conclusion of the above theorem, specifically, if $d_1=d_2=d_3\geq 2$, the spectrum of the Laplace matrix of $Q(d_1,d_2,d_3)$ is as follows:

Corollary 4.2.3: Let $d_1=d_2=d_3\geq 2$. The generalized hypercube $Q(d_1,d_2,d_3)$ has four different eigenvalues, and the Laplacian spectrum is as follows:

$$Sp\ L(Q(d_1,d_2,d_3)) = \begin{pmatrix} 0 & d_1 & 2d_1 & 3d_1 \\ 1 & 3(d_1-1) & 3(d_1-1)^2 & (d_1-1)^3 \end{pmatrix}.$$

Example 2: Experimentally verify that the adjacency spectrum of $Q(2,3,4)$ complies with Theorem 4.1.2 and that its Laplacian spectrum complies with Theorem 4.2.2.

Solution: The MATLAB programming command shows that the characteristic polynomial of $Q(2,3,4)$ is:

$$(-6+\lambda)(-4+\lambda)(-3+\lambda)^2(-2+\lambda)^3$$
$$(-1+\lambda)^2\lambda^3(1+\lambda)^6(3+\lambda)^6$$

The spectrum is,

$$Sp\ A(Q(2,3,4)) = \begin{pmatrix} 6 & 4 & 3 & 2 & 1 & 0 & -1 & -3 \\ 1 & 1 & 2 & 3 & 2 & 3 & 6 & 6 \end{pmatrix}.$$

According to the conclusion of Theorem 4.1.2, the adjacency spectrum is consistent with the theoretical result.

After the command designed by the MATLAB program, the Laplacian spectrum of $Q(2,3,4)$ is calculated as follows:

$$Sp\ L(Q(2,3,4)) = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 6 & 7 & 9 \\ 1 & 1 & 2 & 3 & 2 & 3 & 6 & 6 \end{pmatrix}.$$

According to the conclusion of Theorem 4.2.2, the Mathematica is consistent.

V. Spectral characterization of $Q(d_1,d_2,\cdots,d_n)$

5.1 The adjacency spectrum of $Q(d_1,d_2,\cdots,d_n)$

From Lemma 4.1.1, the following lemma can be obtained:

Lemma 5.1.1: The adjacency matrix A of the generalized hypercube $Q(d_1,d_2,\cdots,d_n)$ is a partition matrix of order $d_n\times d_n$, where each subblock on the main diagonal is an adjacency matrix of $Q(d_1,d_2,\cdots,d_{n-1})$, and the rest of the subblocks are identity matrices of equal order as $Q(d_1,d_2,\cdots,d_{n-1})$, expressed as

$$A(Q(d_1,\cdots,d_n)) = \begin{pmatrix} A(Q(d_1,\cdots,d_{n-1})) & \cdots & I_{d_1\cdots d_{n-1}} \\ \vdots & \ddots & \vdots \\ I_{d_1\cdots d_{n-1}} & \cdots & A(Q(d_1,\cdots,d_{n-1})) \end{pmatrix}_{d_n\times d_n}.$$

Theorem 5.1.2: The characteristic polynomial of the adjacency matrix A of $Q(d_1,d_2,\cdots,d_n)$ is given by:

$$[\lambda-(\sum_{i=1}^{n}d_i-n)]\cdot\prod_{i=1}^{n}[(\lambda-(d_i-n))]^{\prod_{j\neq i}^{n}(d_j-1)}\cdot\prod_{j=1}^{n}[(\lambda-(\sum_{i\neq j}^{n}d_i-n))]^{(d_j-1)}$$
$$\cdot\prod_{k\neq l}^{n}[(\lambda-(d_i+d_j-n))]^{\prod_{i,j\neq k,l}^{n}(d_k-1)(d_l-1)}\cdot[\lambda+n]^{\prod_{i=1}^{n}(d_i-1)}.$$

Proof: From the Lemma 2 and lemma 5.1.1 above, the characteristic polynomial of the adjacency matrix is as follows:

$$\begin{aligned}
&|\lambda I-A| \\
&=\begin{vmatrix} \lambda I_{d_1\cdots d_{n-1}}-A(Q(d_1,\cdots,d_{n-1})) & \cdots & -I_{d_1\cdots d_{n-1}} \\ \vdots & \ddots & \vdots \\ -I_{d_1\cdots d_{n-1}} & \cdots & \lambda I_{d_1\cdots d_{n-1}}-A(Q(d_1,\cdots,d_{n-1})) \end{vmatrix}_{d_n\times d_n} \\
&=\begin{vmatrix} [(\lambda I_{d_1d_2\cdots d_{n-1}}-A(Q(d_1d_2\cdots d_{n-1})))+(d_n-1)\cdot(-I_{d_1d_2\cdots d_{n-1}})] \\ \cdot[(\lambda I_{d_1d_2\cdots d_{n-1}}-A(Q(d_1d_2\cdots d_{n-1})))-(-I_{d_1d_2\cdots d_{n-1}})]^{d_n-1} \end{vmatrix} \\
&=\begin{vmatrix} [(\lambda-(d_n-1))I_{d_1d_2\cdots d_{n-1}}-A(Q(d_1d_2\cdots d_{n-1}))]\cdot \\ [(\lambda+1)I_{d_1d_2\cdots d_{n-1}}-A(Q(d_1d_2\cdots d_{n-1}))]^{d_n-1} \end{vmatrix} \\
&=\begin{vmatrix} [((\lambda-(d_n-1))I_{d_1\cdots d_{n-2}}-A(Q(d_1\cdots d_{n-2})))+(d_{n-1}-1)(-I_{d_1\cdots d_{n-2}})] \\ \cdot[((\lambda-(d_n-1))I_{d_1\cdots d_{n-2}}-A(Q(d_1\cdots d_{n-2})))-(-I_{d_1\cdots d_{n-2}})]^{d_{n-1}-1} \end{vmatrix} \\
&\quad\cdot\begin{vmatrix} [((\lambda+1)I_{d_1\cdots d_{n-2}}-A(Q(d_1\cdots d_{n-2})))+(d_{n-1}-1)(-I_{d_1\cdots d_{n-2}})] \\ \cdot[((\lambda+1)I_{d_1\cdots d_{n-2}}-A(Q(d_1\cdots d_{n-2})))-(-I_{d_1\cdots d_{n-2}})]^{d_{n-1}-1} \end{vmatrix}^{d_n-1} \\
&=\left|[(\lambda-(d_n+d_{n-1}-2))I_{d_1d_2\cdots d_{n-2}}-A(Q(d_1d_2\cdots d_{n-2}))]\right| \\
&\quad\cdot\left|[(\lambda-(d_n-2))I_{d_1d_2\cdots d_{n-2}}-A(Q(d_1d_2\cdots d_{n-2}))]^{d_{n-1}-1}\right| \\
&\quad\cdot\left|[(\lambda-(d_{n-2}-1))I_{d_1d_2\cdots d_{n-2}}-A(Q(d_1d_2\cdots d_{n-2}))]\right|^{d_n-1} \\
&\quad\cdot\left|[(\lambda+2)I_{d_1d_2\cdots d_{n-2}}-A(Q(d_1d_2\cdots d_{n-2}))]\right|^{(d_n-1)(d_{n-1}-1)} \\
&=\cdots \\
&=[\lambda-(\sum_{i=1}^{n}d_i-n)]\cdot[(\lambda-(\sum_{k\neq i}^{n}d_k-n))]^{d_i-1}\cdot[(\lambda-(\sum_{k\neq i,j}^{n}d_k-n))]^{\prod_{k=i,j}^{n}(d_k-1)}\cdots\cdots \\
&\quad\cdot[\lambda-(d_i-n)]^{\prod_{k\neq i}^{n}(d_k-1)}\cdot[\lambda+n]^{\prod_{i=1}^{n}(d_i-1)} \qquad (5)
\end{aligned}$$

Thus, setting the above characteristic equation (5) to zero, $Q(d_1,d_2,\cdots,d_n)$ have $C_n^0+C_n^1+C_n^2+\cdots+C_n^{n-1}+C_n^n=2^n$ different eigenvalues, with the following adjacency spectrum:

$$SpA(Q(d_1,\cdots,d_n)) = \begin{array}{c} \begin{matrix} C_n^0 & C_n^1 & C_n^2 & & C_n^{n-1} & C_n^n \end{matrix} \\ \begin{pmatrix} \sum_{i=1}^{n} d_i - n & \sum_{k\neq i}^{n} d_k - n & \sum_{k\neq i,j}^{n} d_k - n & \cdots & d_i - n & -n \\ 1 & d_i - 1 & \prod_{k=i,j}^{n}(d_k - 1) & \cdots & \prod_{k\neq i}^{n}(d_k - 1) & \prod_{i=1}^{n}(d_1 - 1) \end{pmatrix} \end{array}.$$

Specifically, if $d_1 = d_2 = \cdots = d_n \geq 2$,the adjacency spectrum of $Q(d_1,d_2,\cdots,d_n)$ is as follows:

Corollary 5.1.3: Let $d_1 = d_2 = \cdots = d_n \geq 2$, the adjacency spectrum of $Q(d_1,d_2,\cdots,d_n)$ is as follows:

$$Sp\ A(Q(d_1,d_2,\cdots,d_n)) = \begin{array}{c} \begin{matrix} C_n^0 & C_n^1 & & C_n^{n-1} & C_n^n \end{matrix} \\ \begin{pmatrix} n(d_1 - 1) & (n-1)d_1 - n & \cdots & d_1 - n & -n \\ 1 & C_n^1(d_1 - 1) & \cdots & C_n^{n-1}(d_1 - 1) & \prod_{i=1}^{n}(d_1 - 1) \end{pmatrix} \end{array}.$$

Specifically, if $d_1 = d_2 = \cdots = d_n = 2$, $Q(d_1,d_2,\cdots,d_n)$ is hypercube $Q(n)$ given adjacency spectrum:

$$Sp\ A(Q(n)) = \begin{array}{c} \begin{matrix} C_n^0 & C_n^1 & C_n^2 & & C_n^{n-1} & C_n^n \end{matrix} \\ \begin{pmatrix} n & n-2 & n-4 & \cdots & 2-n & -n \\ 1 & n & C_n^2 & \cdots & C_n^{n-1} & 1 \end{pmatrix} \end{array}.$$

This is consistent with the conclusion found in the literature [10].

5.2 The Laplacian spectrum of $Q(d_1,d_2,\cdots,d_n)$

Theorem 5.2.1: There are 2^n different eigenvalues of $Q(d_1,d_2,\cdots,d_n)$, and the spectrum of the Laplace matrix is determined as follows:

$$Sp\ L(Q(d_1,d_2,\cdots,d_n)) = \begin{array}{c} \begin{matrix} C_n^0 & C_n^1 & C_n^2 & & C_n^{n-1} & C_n^n \end{matrix} \\ \begin{pmatrix} 0 & d_i & d_i + d_j & \cdots & \sum_{k\neq i}^{n} d_k & \sum_{i=1}^{n} d_i \\ 1 & d_i - 1 & \prod_{k=i,j}^{n}(d_k - 1) & \cdots & \prod_{k\neq i}^{n}(d_k - 1) & \prod_{i=1}^{n}(d_i - 1) \end{pmatrix} \end{array}.$$

Proof: From Lemma 3, it suffices to determine the regularity degree $\Delta = \sum_{i=1}^{n} d_i - n$ of $Q(d_1,d_2,\cdots,d_n)$. By finding the eigenvalue λ of the adjacency matrix A , we can obtain the eigenvalue of the Laplace matrix L . Therefore, the Laplacian spectrum of $Q(d_1,d_2,\cdots,d_n)$ is

$$Sp\ L(Q(d_1,d_2,\cdots,d_n)) = \begin{array}{c} \begin{matrix} C_n^0 & C_n^1 & C_n^2 & & C_n^{n-1} & C_n^n \end{matrix} \\ \begin{pmatrix} 0 & d_i & d_i + d_j & \cdots & \sum_{k\neq i}^{n} d_k & \sum_{i=1}^{n} d_i \\ 1 & d_i - 1 & \prod_{k=i,j}^{n}(d_k - 1) & \cdots & \prod_{k\neq i}^{n}(d_k - 1) & \prod_{i=1}^{n}(d_i - 1) \end{pmatrix} \end{array}.$$

According to the proof conclusion of the above theorem, in particular, if $d_1 = d_2 = \cdots = d_n \geq 2$,the spectrum of $Q(d_1,d_2,\cdots,d_n)$ Laplace matrix is as follows:

Corollary5.2.2: Let $d_1 = d_2 = \cdots = d_n \geq 2$,the generalized hypercube $Q(d_1,d_2,\cdots,d_n)$ have $n+1$ different eigenvalues, and its Laplacian spectrum is given by

$$Sp\ L(Q(d_1,d_2,\cdots,d_n)) = \begin{pmatrix} 0 & d_1 & \cdots & (n-1)d_1 & nd_1 \\ C_n^0(d_1 - 1)^0 & C_n^1(d_1 - 1)^1 & \cdots & C_n^{n-1}(d_1 - 1)^{n-1} & C_n^n(d_1 - 1)^n \end{pmatrix}.$$

Specifically, if $d_1 = d_2 = \cdots = d_n = 2$, $Q(d_1,d_2,\cdots,d_n)$ reduces to hypercube $Q(n)$, and its Laplacian spectrum is

$$Sp\ L(Q(n)) = \begin{pmatrix} 0 & 2 & \cdots & 2(n-1) & 2n \\ C_n^0 & C_n^1 & \cdots & C_n^{n-1} & C_n^n \end{pmatrix}.$$

This result is consistent with the conclusion in the literature [11].

VI. THE ENERGY OF $Q(d_1,d_2,\cdots,d_n)$

Lemma 6.1:[13] For any regular graph G , $LE(G) = E(G)$.

Based on the conclusion of the adjacency spectrum and Laplacian spectrum of $Q(d_1,d_2,\cdots,d_n)$, and its regularity degree, $Q(d_1,d_2,\cdots,d_n)$ is $\Delta = \sum_{i=1}^{n} d_i - n$ and the programming approach for computing the energy and Laplace energy of $Q(d_1,d_2,\cdots,d_n)$ can be obtained as follows:

Algorithm 6.2:

Input: Input n and n variables of $d_1,d_2,\cdots,d_n$;

Step1: Design the adjacent matrix of $Q(d_1,d_2,\cdots,d_n)$ and the program command for commuting its characteristic polynomial;

Step2: Solve the characteristic polynomial equal to zero to obtain the eigenvalue and design the program command for computing them;

Step 3: According to the energy definition, the absolute value of the eigenvalue is found and summed up.

Output: Adjacent matrix, factorization of characteristic polynomial, eigenvalue, energy of $Q(d_1,d_2,\cdots,d_n)$.

Example 3: To compute the energy of $Q(2,3,4)$.

Solution: After the MATLAB programming command, the energy of $Q(2,3,4)$ is 48 .

VII. SUMMARY

Based on the structures and properties of the generalized hypercube $Q(d_1,d_2,\cdots,d_n)$, we have derived expressions for its adjacency and Laplace spectra. As an application of these results, we consider its energy problem. This is of great relevance in studying the structural properties of $Q(d_1,d_2,\cdots,d_n)$ in a new perspective. Meanwhile, MATLAB is used to verify the conclusion and design program commands for deriving the adjacency spectrum, Laplacian spectrum and energy of $Q(d_1,d_2,\cdots,d_n)$ with given parameters $d_1,d_2,\cdots,d_n$. In future work, we plan to extend our study to the spectral theory of other important interconnection networks.

APPENDIX

The analysis and visualization procedure for the adjacency matrices of generalized hypercube $Q(d_1, d_2, \cdots, d_n)$ is as follows:

```
n = input('Enter the number of variables n: ');
d = zeros(1, n);
for i = 1:n
   d(i) = input(['Enter d' num2str(i) ' = ']);
end
matrix_size = prod(d);
A_total = zeros(matrix_size);
for i = 1:n
   I_di = eye(d(i));
   K_di = ones(d(i)) - eye(d(i));
   A_i = 1;
   for j = 1:n
      if j == i
         A_i = kron(A_i, K_di);
      else
         A_i = kron(A_i, eye(d(j)));
      end
   end
   A_total = A_total + A_i;
end
disp('Final adjacency matrix A_total:');
disp(A_total);
syms lambda;
result = det(lambda * eye(matrix_size) - A_total);
factor_result = factor(result);
disp('Factorized determinant:');
disp(factor_result);
eigenvalues_A_total = eig(A_total);
disp('Eigenvalues of A_total:');
disp(eigenvalues_A_total);
sum_abs_eigenvalues = sum(abs(eigenvalues_A_total));
disp('Sum of absolute values of eigenvalues:');
disp(sum_abs_eigenvalues);
G = graph(A_total);
node_labels = arrayfun(@num2str, 1:matrix_size,
'UniformOutput', false);
figure;
p = plot(G, 'Layout', 'force', 'NodeLabel', node_labels,
'MarkerSize', 7);
p.EdgeColor = 'k';
p.NodeColor = 'k';
title('Graph of adjacency matrix A_total');
```

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