Some Results on Ricci-Yamabe Soliton and Its Cosmological Models

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Abstract—**This article aims to investigate the characteristics of Ricci-Yamabe Soliton (briefly: $(RYS)_n$). We study the cosmological models on $(RYS)_4$ under Lorentzian para Sasakian $(LPS)_4$ spacetime. Parallel Ricci tensor, Poisson structure and gradient property also reveal on $(RYS)_n$. Example of $(RYS)_4$ constructed to verify the results.**

Index Terms—**Ricci flow, Partial differential equation, Heat equation, Ricci-Yamabe soliton, Gradient soliton, LP Sasakian, Poisson manifold, Cosmological model, Perfect fluid, Spacetime.**

NOMENCLATURES

- n: dimension of the space.
- M^n: manifold of dimension n.
- $(LPS)_n$: Lorentzian para Sasakian manifold.
- $(RYS)_n$: Ricci Yamabe soliton of dimension n.
- $(GRYS)_n$: Gradient Ricci Yamabe soliton.
- $\mathfrak{L}$: Lie derivative operator.
- $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$: Vector field.
- R: Riemann curvature.
- S: Ricci curvature.
- Q: Ricci operator.
- r: scalar curvature.
- $\chi(M)$: set of all vector field on M
- λ: soliton function.
- μ: cosmological function.
- f: smooth function.
- ∇: covariant derivative operator.
- $\nabla^2 f$: hessian function.
- ϕ: tensor field of type (1,1).
- ξ: associated vector field to the metric.
- η: 1-form.
- g: Lorentzian metric.
- T: energy tensor.
- EFE: Einstein field equation.
- p: isotropic pressure of the fluid.
- ρ: energy density of the fluid.
- τ: gravitational constant.
- $\|Q\|$: length of Ricci operator.
- α, β: constant.

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I. INTRODUCTION

HAMILTON [16] revealed the concept of Ricci flow (respectively Yamabe flow) in the last quarter of twenties in order to discuss new striking results in Riemannian geometry. The idea of Ricci soliton recognized as a stereotype of an Einstein metric and governing the solution of partial differential equation representing Ricci flow, which is isomorphically equivalent to heat equation. In case of Ricci flow, the Riemannian metric is proportional to a (0, 2) type tensor ($\frac{1}{2}Lg + Ric$). In this case the proportionality scalar is called the soliton function. The Ricci flow in terms of soliton function λ (say) is dictated by the equation [16]

$$-\frac{1}{2}(\mathfrak{L}_V g)(\mathfrak{X}, \mathfrak{Y}) = S(\mathfrak{X}, \mathfrak{Y}) + \lambda g(\mathfrak{X}, \mathfrak{Y}), \tag{1}$$

The Yamabe flow is discribed by an evolving partial differential equation [16]

$$\frac{1}{2}(\mathfrak{L}_V g)(\mathfrak{X}, \mathfrak{Y}) = (r - \lambda) g(\mathfrak{X}, \mathfrak{Y}). \tag{2}$$

In (1) and (2), the Lie derivative $\mathfrak{L}$ is taken along the complete vector field V and is denoted by $\mathfrak{L}_V g$. The Ricci curvature denotes S and the r is scalar tensor. The vector fields $\mathfrak{X}, \mathfrak{Y}$ belong to the set of an algebra of tangent vectors denoted by $\chi(M)$. The nature of the soliton can be expressed in terms of soliton function λ and is said to explore for λ positive. We say the case is of compacting soliton or is of constant soliton, if $\lambda < 0$ or $\lambda = 0$. In general the Ricci and the Yamabe solitons are distinct for higher dimension but coincide in case of dimension 2. The reason behind this distinction is that the Yamabe soliton preserves the metric conformality in nature where as Ricci soliton has denied it.

Guler and Crasmareanu [14] (2019) investigated another geometric flow restricted by Ricci-Yamabe map. Authors pronounced with a special call: (α, β) type Ricci-Yamabe flow (briefly denoted $(\alpha, \beta) - (RYF)_n$). Particularly, it is noted that the $(\alpha, \beta) - (RYF)_n$ is nothing but the α-Ricci soliton for $\beta = 0$ and it turns into β-Yamabe soliton, in case of vanishing α. The above said flow is defined by the following equation

$$(\mathfrak{L}_V g)(\mathfrak{X}, \mathfrak{Y}) = -2\alpha S(\mathfrak{X}, \mathfrak{Y}) - (2\lambda - \beta r) g(\mathfrak{X}, \mathfrak{Y}). \tag{3}$$

The aforementioned above soliton is known as the (α, β) type gradient Ricci-Yamabe soliton (briefly $(\alpha, \beta) - (GRYS)_n$) if V becomes a gradient of f (if exist such a smooth function) and (3) reveals

$$\nabla^2 f + \alpha S = (\lambda - \frac{1}{2}\beta r) g, \tag{4}$$

where $\nabla^2 f$ is defined as Hessian of the smooth function f and denoted in general by $Hess(f)$.

The Lorentzian manifold (a most compelling sub classe of pseudo-Riemannian manifold) has a facund impact on the advancement of the theory of relativity and cosmology. Ahsan and Ali [1] looked into the symmetries of soliton spacetime. The geometrical characteristics discussed by Blaga [4] in spacetime of Einstein Ricci soliton. In [4] author examined the geometrical drift of conformal Ricci solitons with $(PFST)_n$. Chaturvedi et.al. studied the K$\ddot{a}$hler spacetime manifold under cogitation of Bochner flatness. They decisived that energy momentum tensor (EMT) of a perfect fluid Lorentzian K$\ddot{a}$hler spacetime exhibits hybrid characteristics and spell out the behaviour of dust fluid spacetime with vanishing Bochner curvature.

Haseeb et.al. [19] contemplated the study on $(\alpha,\beta)-(RYS)_n$ and $(\alpha,\beta)-(GRYS)_n$ and demonstrated that the scalar tensor of $(LPK)_n$ manifold admitting Ricci Yamabe soliton satisfies the Poisson equation. Sardar and Sarkar [31] portrayed Kenmotsu 3-manifold rigged with $(\alpha,\beta)-(RYS)_n$ and $(\alpha,\beta)-(GRYS)_n$ metric that satisfy ζ-parallel Ricci tensor. Pal and Chaudhary [22] investigated Poisson flow on almost soliton like Ricci soliton (briefly: $(ASLRS)_n$). In [10] (2021), De ratiocinated Sasakian 3-manifolds admitting $(GRYS)_n$).

Recently, S. Azami, M. Jafari, N. Jamal and A. Haseeb [2] (2024) interrogated hyperbolic Ricci solitons on perfect fluid spacetimes. S.K. Chaubey and A. Haseeb [8] (2024) explored conformal η-Ricci-Yamabe soliton in Riemannian manifolds. Singh, Chaubey, Yadav and Patel [33] (2024) explained Z*-tensor on N(k)-contact metric manifolds admitting Ricci soliton type structure (see also: [3], [7], [9], [15], [18], [21], [23], [24], [25], [27], [28], [37], [35], [20], [30], [29], [6], [26]).

Motivated by the above research, we study $(RYS)_n$ and $(GRYS)_n$ on $(LPS)_n$. In preliminaries (section: 2), some basic definitions and results on the $(RYS)_n$ have been given. Section 3 includes the study on $(RYS)_n$ admitting $(LPS)_n$ with an example. In section 4, $(RYS)_4$ admitting $(LPS)_4$ have been subjected through different cosmological models. The gradient soliton $(GRYS)_n$ admitting $(LPS)_n$ has been lessioned in section 5.

II. Preliminaries

Let M^n (Riemannian manifold) denotes the inclusion of the triplet data (ϕ,ξ,η) then we say (M^n,ϕ,ξ,η) is an almost contact metric structure that assuage the following:

$$\phi\xi=0,\eta(\phi\mathfrak{X})=0,\eta(\xi)=-1,\phi^2\mathfrak{X}=\mathfrak{X}+\eta(\mathfrak{X})\xi. \quad (5)$$

Here, ϕ denotes a tensor field $(1,1)$-type and η we call 1-form on M^n defined by $\eta(\mathfrak{X})=g(\mathfrak{X},\xi)$. Some additional following properties also hold on almost contact metric manifold.

$$g(\phi\mathfrak{X},\phi\mathfrak{Y})=g(\mathfrak{X},\mathfrak{Y})-\eta(\mathfrak{X})\eta(\mathfrak{Y}). \quad (6)$$

agnately, we get

$$g(\mathfrak{X},\phi\mathfrak{Y})=-g(\phi\mathfrak{X},\mathfrak{Y}), \quad (7)$$

for all $\mathfrak{X},\mathfrak{Y}\in\chi(M)$.
The structure (M^n,ϕ,ξ,η,g) poured to K-contact manifold, for being killing to characteristic vector field ξ. For Lorentzian para Sasakian $(LPS)_n$ manifold with killing vector field, we have

$$\nabla_{\mathfrak{X}}\xi=\phi\mathfrak{X}, \quad (8)$$

$$(\mathfrak{L}_\xi g)(\mathfrak{X},\mathfrak{Y})=0, \quad (9)$$

$$\phi\xi=0, \qquad \eta(\phi\mathfrak{X})=0, \qquad rank \;\; \phi=n-1. \quad (10)$$

$$(\nabla_{\mathfrak{X}}\eta)(\mathfrak{Y})=\nabla_{\mathfrak{X}}\eta(\mathfrak{Y})-\eta(\nabla_{\mathfrak{X}}\mathfrak{Y})=g(\mathfrak{Y},\nabla_{\mathfrak{X}}\mathfrak{Y}). \quad (11)$$

In view of (8) and (11), we have

$$(\nabla_{\mathfrak{X}}\eta)(\mathfrak{Y})=g(\mathfrak{Y},\phi\mathfrak{X}). \quad (12)$$

Let the structure (ϕ,ξ,η,g) on $(LPS)_n$ then we have

$$\begin{aligned} g(R(\mathfrak{X},\mathfrak{Y})\mathfrak{U},\xi)&=\eta(R(\mathfrak{X},\mathfrak{Y})\mathfrak{U}) \\ &=g(\mathfrak{Y},\mathfrak{U})\eta(\mathfrak{X})-g(\mathfrak{X},\mathfrak{U})\eta(\mathfrak{Y}), \end{aligned} \quad (13)$$

$$R(\xi,\mathfrak{X})\mathfrak{Y}=g(\mathfrak{X},\mathfrak{Y})\xi-\eta(\mathfrak{Y})\mathfrak{X}, \quad (14)$$

$$R(\mathfrak{X},\mathfrak{Y})\xi=\eta(\mathfrak{Y})\mathfrak{X}-\eta(\mathfrak{X})\mathfrak{Y}, \quad (15)$$

$$R(\xi,\mathfrak{X})\xi=\mathfrak{X}+\eta(\mathfrak{X})\xi, \quad (16)$$

$$S(\mathfrak{X},\xi)=(n-1)\eta(\mathfrak{X}), \quad (17)$$

$$S(\phi\mathfrak{X},\phi\mathfrak{Y})=S(\mathfrak{X},\mathfrak{Y})+(n-1)\eta(\mathfrak{X})\eta(\mathfrak{Y}), \quad (18)$$

for $\mathfrak{X},\mathfrak{Y},\mathfrak{U}\in\chi(M)$.
If the $(RYS)_n$ admits an $(LPS)_n$ then (3) and (9) yield the Ricci operator

$$Q\mathfrak{X}=\left(\frac{\beta r-2\lambda}{2\alpha}\right)\mathfrak{X}. \quad (19)$$

The contraction, post covariant differentiation of (19) reveals the relation between parametres α and β for non-constant scalar curvature

$$\beta=\frac{2\alpha}{n}. \quad (20)$$

III. $(RYS)_n$ Admitting $(LPS)_n$

Theorem III.1. *The soliton function of $(RYS)_n$ admitting $(LPS)_n$ is $\lambda=\frac{\alpha}{n}(r-n(n-1))$, provided the non-constant scalar tensor.*

Proof: Using (9) in (3), we receive

$$2\alpha S(\mathfrak{X},\mathfrak{Y})=(\beta r-2\lambda)g(\mathfrak{X},\mathfrak{Y}). \quad (21)$$

Applying (17) in (21) after using $\mathfrak{Y}=\xi$, reduces

$$2\alpha(n-1)\eta(\mathfrak{X})=(\beta r-2\lambda)\eta(\mathfrak{X}). \quad (22)$$

Equation (22) and (20) reveal the result. ⊡

Let we have a look at a differentiable manifold in four dimensions, $M^4=\{(\mathfrak{X},\mathfrak{Y},\mathfrak{U},\mathfrak{S})\in\Re:(\mathfrak{X},\mathfrak{Y},\mathfrak{U},\mathfrak{S})\neq 0$ $\}$, where $(\mathfrak{X},\mathfrak{Y},\mathfrak{U},\mathfrak{S})$ is the standard coordinate in four-dimensional real space. Considering that each point on M^4 has a collection of linearly independent vector fields, let's call them $(\mathfrak{B},\mathfrak{W},\mathfrak{D},\mathfrak{F})$, and is defined by

$$\begin{aligned} \mathfrak{B}&=f^{\boxtimes}\frac{\partial}{\partial\mathfrak{X}}, \quad \mathfrak{W}=f^{\Omega}\frac{\partial}{\partial\mathfrak{Y}}, \\ \mathfrak{D}&=f^{\Pi}\frac{\partial}{\partial\mathfrak{U}}, \quad \mathfrak{F}=\frac{\partial}{\partial\mathfrak{S}}. \end{aligned} \quad (23)$$

Here, $\boxtimes=\mathfrak{X}-\mathfrak{S}, \quad \Omega=\mathfrak{Y}-\mathfrak{S}, \quad \Pi=\mathfrak{U}-\mathfrak{S}$ and $\{\frac{\partial}{\partial\mathfrak{X}},\frac{\partial}{\partial\mathfrak{Y}},\frac{\partial}{\partial\mathfrak{U}},\frac{\partial}{\partial\mathfrak{S}}\}$ insinuate the standard basis of M^4, f being the smooth function.
Let the metric and the 1-form η on M^4 are concreted by

$$g(\clubsuit,\spadesuit)=\begin{cases}-1, & \text{if } \clubsuit=\spadesuit=\mathfrak{F}\\ 0, & \text{if } \clubsuit\neq\spadesuit\\ 1, & \text{if } \clubsuit=\spadesuit\neq\mathfrak{F}.\end{cases} \quad (24)$$

$$\eta(\mathfrak{X}) = g(\mathfrak{X}, \mathfrak{F}). \tag{25}$$

If $\phi(\mathfrak{B}) = \mathfrak{B}, \phi(\mathfrak{W}) = \mathfrak{W}, \phi(\mathfrak{D}) = \mathfrak{D}, \phi(\mathfrak{F}) = 0$,
are the tensor fields then the following linear relationships may be easily verified by g's and ϕ's properties:

$$\begin{aligned} &\eta(\mathfrak{F}) = -1, \quad \phi^2\mathfrak{X} = \mathfrak{X} + \eta(\mathfrak{X})\mathfrak{F}, \\ &g(\phi\mathfrak{X}, \phi\mathfrak{Y}) = g(\mathfrak{X}, \mathfrak{Y}) - \eta(\mathfrak{X})\eta(\mathfrak{Y}). \end{aligned} \tag{26}$$

Theorem III.2. *The metric defined by (23), (24), (25) and (26) defines the $(RYS)_n$ admitting $(LPS)_n$ with non-constant scalar curvature.*

Proof: The Koszul's formula yields for $\mathfrak{F} = \xi$:

$$\begin{aligned} &\nabla_{\mathfrak{B}}\mathfrak{B} = a\mathfrak{F}, \quad \nabla_{\mathfrak{B}}\mathfrak{W} = 0, \quad \nabla_{\mathfrak{B}}\mathfrak{D} = 0, \\ &\nabla_{\mathfrak{B}}\mathfrak{F} = a\mathfrak{B}, \nabla_{\mathfrak{W}}\mathfrak{B} = 0, \quad \nabla_{\mathfrak{W}}\mathfrak{W} = a\mathfrak{F}, \\ &\nabla_{\mathfrak{W}}\mathfrak{D} = 0, \quad \nabla_{\mathfrak{W}}\mathfrak{F} = a\mathfrak{W}, \nabla_{\mathfrak{D}}\mathfrak{A} = 0, \\ &\nabla_{\mathfrak{D}}\mathfrak{W} = 0, \quad \nabla_{\mathfrak{D}}\mathfrak{D} = a\mathfrak{F}, \quad \nabla_{\mathfrak{D}}\mathfrak{F} = a\mathfrak{D}, \\ &\nabla_{\mathfrak{F}}\mathfrak{B} = 0, \quad \nabla_{\mathfrak{F}}\mathfrak{W} = 0, \quad \nabla_{\mathfrak{F}}\mathfrak{D} = 0, \quad \nabla_{\mathfrak{F}}\mathfrak{F} = 0. \end{aligned} \tag{27}$$

The Riemann R, Ricci S and the scalar tensor r are retrieve

$$\begin{aligned} &R(\mathfrak{B}, \mathfrak{W})\mathfrak{B} = -\mathfrak{W}, \quad R(\mathfrak{B}, \mathfrak{D})\mathfrak{B} = -\mathfrak{D}, \\ &R(\mathfrak{B}, \mathfrak{F})\mathfrak{B} = -\mathfrak{F}, \quad R(\mathfrak{B}, \mathfrak{W})\mathfrak{W} = \mathfrak{B}, \\ &R(\mathfrak{W}, \mathfrak{D})\mathfrak{W} = -\mathfrak{D}, \quad R(\mathfrak{W}, \mathfrak{F})\mathfrak{W} = -\mathfrak{F}, \\ &R(\mathfrak{B}, \mathfrak{D})\mathfrak{D} = \mathfrak{B}, \quad R(\mathfrak{W}, \mathfrak{D})\mathfrak{D} = \mathfrak{W}, \\ &R(\mathfrak{D}, \mathfrak{F})\mathfrak{D} = -\mathfrak{F}, R(\mathfrak{B}, \mathfrak{F})\mathfrak{F} = -\mathfrak{A}, \\ &R(\mathfrak{W}, \mathfrak{F})\mathfrak{F} = -\mathfrak{W}, \quad R(\mathfrak{D}, \mathfrak{F})\mathfrak{F} = -\mathfrak{D}. \end{aligned} \tag{28}$$

$$\begin{aligned} &S(\mathfrak{B}, \mathfrak{B}) = S(\mathfrak{W}, \mathfrak{W}) = S(\mathfrak{D}, \mathfrak{D}) = 3, \\ &S(\mathfrak{F}, \mathfrak{F}) = -3, \quad r = 6. \end{aligned} \tag{29}$$

Adopting $\lambda = 3(\beta - \alpha)$ and values from (29), theorem (III.1) is satisfied and proves the existence.

IV. Cosmological Models on $(RYS)_4$

This section deals the perfect fluid energy tensor, soliton function, Ricci operator and dust cosmological model. The Einstein's field equation *(EFE)* is explored by

$$S(\mathfrak{X}, \mathfrak{Y}) - \frac{r}{2}g(\mathfrak{X}, \mathfrak{Y}) + \mu g(\mathfrak{X}, \mathfrak{Y}) = \tau T(\mathfrak{X}, \mathfrak{Y}). \tag{30}$$

Here, μ denotes the cosmological term, τ being the gravitational constant and T deals for $(0, 2)$ type energy tensor. The information regarding perfect fluid is governed by

$$T(\mathfrak{X}, \mathfrak{Y}) = (\rho - p)\eta(\mathfrak{X})\eta(\mathfrak{Y}) + pg(\mathfrak{X}, \mathfrak{Y}). \tag{31}$$

Here, ρ and p are density and pressure of the fluid.
On the other hand, (19) and (30) yield

$$T(\mathfrak{X}, \mathfrak{Y}) = \frac{1}{\tau}\left[\mu - (\frac{\beta r - 2\lambda}{2\alpha})\right]g(\mathfrak{X}, \mathfrak{Y}). \tag{32}$$

Using (20), the energy tensor takes the form

$$T(\mathfrak{X}, \mathfrak{Y}) = \frac{1}{4\tau}\left[(4\mu - r + \frac{4\lambda}{\alpha}\right]g(\mathfrak{X}, \mathfrak{Y}). \tag{33}$$

Together (31) and (33) yield:

Theorem IV.1. *The ratio between the soliton function to the Ricci Yamabe parameter on $(RYS)_4$ admitting $(LPS)_4$ consisting non-constant scalar curvarure is given by*
$\frac{\lambda}{\alpha} = \tau(2p - \rho) - \mu + \frac{r}{4}$.

Corollary IV.1. *The ratio between the scalar function to the cosmological function on stable $(RYS)_4$ admitting $(LPS)_4$ consisting non-constant scalar curvarure is four.*

Theorem IV.2. *If the perfect fluid on $(RYS)_4$ admitting $(LPS)_4$ satisfy (EFE) without cosmological term, the length of Ricci operator is obsessed by*

$$\|Q\| = \frac{1}{4}\left[\tau(\rho - 3p) - \frac{6\lambda}{\alpha}\right].$$

Proof: The (EFE), without cosmological term, is revealed by

$$S(\mathfrak{X}, \mathfrak{Y}) - \frac{r}{2}g(\mathfrak{X}, \mathfrak{Y}) = \tau T(\mathfrak{X}, \mathfrak{Y}). \tag{34}$$

Solving (19), (31), (34), we get

$$r = \frac{\alpha\tau(-\rho + 3p) + 2\lambda}{2(\beta - \alpha)}. \tag{35}$$

Substituting (35) in (19) and applying (20), yield the required result.

Theorem IV.3. *The ratio between the soliton function to the Ricci Yamabe parameter on $(RYS)_4$ admitting $(LPS)_4$ with radiation fluid $\rho = 3p$ is proortional to the Ricci length with proportionality constant $\frac{2}{3}$, that is, $\frac{\lambda}{\alpha} = \frac{2}{3}\|Q\|$.*

Theorem IV.4. *The energy density of a dust cosmological model on $(RYS)_4$ admitting $(LPS)_4$ is addicted by*

$$\rho = \frac{4}{\tau}\left[\|Q\| + \frac{\lambda}{\alpha}\right].$$

.

Proof: Since, the dust fluid is modelled by

$$T(\mathfrak{X}, \mathfrak{Y}) = \rho A(\mathfrak{X})A(\mathfrak{Y}). \tag{36}$$

Using equation (34) and (36), the expression becomes

$$S(\mathfrak{X}, \mathfrak{Y}) - \frac{r}{2}g(\mathfrak{X}, \mathfrak{Y}) = \tau\rho A(\mathfrak{X})A(\mathfrak{Y}). \tag{37}$$

On contracting over $\mathfrak{X}$ and $\mathfrak{Y}$, receive

$$r = \tau\rho. \tag{38}$$

Equation (38) and (19) yield

$$S(\mathfrak{X}, \mathfrak{Y}) = \left[\frac{\beta\tau\rho - 2\lambda}{2\alpha}\right]g(\mathfrak{X}, \mathfrak{Y}). \tag{39}$$

Equation (39) and (20) gives the required result.

Theorem IV.5. *The energy density of dust cosmological model on $(RYS)_4$ admitting $(LPS)_4$ vanishes.*

Proof: Using (19) in (37), and contracting obtain

$$\left[\frac{\beta r - 2\lambda - \alpha r}{2\alpha}\right]g(\mathfrak{X}, \mathfrak{Y}) = \tau\rho A(\mathfrak{X})A(\mathfrak{Y}). \tag{40}$$

Contracting over $\mathfrak{X}$ and $\mathfrak{Y}$ in equation (40), we get

$$r = \frac{-\alpha\rho\tau + 4\lambda}{2(\beta - \alpha)}. \tag{41}$$

Multiply $A(\mathfrak{U})$ in (40), contraction over $\mathfrak{Y}$, $\mathfrak{U}$ gives

$$r = \frac{2(\alpha\rho\tau + \lambda)}{\beta - \alpha}. \tag{42}$$

Equating (41) and (42), yield the required result.

V. $(GRYS)_n$ ADMITTING $(LPS)_n$

Theorem V.1. *In a $(GRYS)_n$ admitting $(LPS)_n$ with non-constant scalar curvature, the ratio of covariant derivative of scalar curvature r to the covariant derivative of smooth function f is given by*

$$\frac{\mathfrak{Y}r}{\mathfrak{Y}f} = \left(\frac{\alpha r - 4\lambda}{8\alpha^2}\right).$$

Proof: Suppose that Gradient Ricci Yamabe Soliton $(GRYS)_n$ admits $(LPS)_n$ manifold then equation (4) turns into

$$\nabla_{\mathfrak{X}} Df = (\lambda - \frac{1}{2}\beta r)\mathfrak{X} - \alpha Q\mathfrak{X}. \tag{43}$$

The derivative of (43) implies

$$\begin{aligned} \nabla_{\mathfrak{Y}}\nabla_{\mathfrak{X}} Df &= (\lambda - \frac{1}{2}\beta r)\nabla_{\mathfrak{Y}}\mathfrak{X} \\ &- \frac{\beta}{2}(\mathfrak{Y}r)\mathfrak{X} - \alpha\nabla_{\mathfrak{Y}}Q\mathfrak{X}. \end{aligned} \tag{44}$$

Using (44), the Riemann curvature obtain

$$\begin{aligned} R(\mathfrak{X},\mathfrak{Y})Df &= \frac{\beta}{2}[(\mathfrak{Y}r)\mathfrak{X} - (\mathfrak{X}r)\mathfrak{Y}] \\ &- \alpha[(\nabla_{\mathfrak{X}}Q)\mathfrak{Y} - (\nabla_{\mathfrak{Y}}Q)\mathfrak{X}]. \end{aligned} \tag{45}$$

On the other hand, (19) implise

$$(\nabla_{\mathfrak{X}}Q)\mathfrak{Y} - (\nabla_{\mathfrak{Y}}Q)\mathfrak{X} = \frac{\beta}{2\alpha}[(\mathfrak{X}r)\mathfrak{Y} - (\mathfrak{Y}r)\mathfrak{X}]. \tag{46}$$

Using (46) in (45) and contracting , we grab

$$S(\mathfrak{Y}, Df) = 2\alpha(\mathfrak{Y}r). \tag{47}$$

Alteration of $\mathfrak{X}$ by Df in (19) and applying (20), yields

$$S(\mathfrak{Y}, Df) = \left(\frac{\alpha r - 4\lambda}{4\alpha}\right)(\mathfrak{Y}f). \tag{48}$$

Deviding (47) to (48), we acquire the required result.

Theorem V.2. *If the scalar curvature tensor of $(GRYS)_n$ admitting $(LPS)_n$ manifold is non-constant then soliton function is given by $\lambda = \frac{\alpha(r+2n^2)}{n}$.*

The metric inner product on (47) with ξ reveals the results

$$\eta(\mathfrak{Y})(\mathfrak{X}f) - \eta(\mathfrak{X})(\mathfrak{Y}f) = \frac{\beta}{2}[(\mathfrak{Y}r)\eta(\mathfrak{X}) - (\mathfrak{X}r)\eta(\mathfrak{Y})]. \tag{49}$$

Settle $\mathfrak{X} = \xi$ in (49) and using (19), we have

$$\eta(\mathfrak{Y})(\xi f) - \eta(\xi)(\mathfrak{Y}f) = \frac{\alpha}{n}[(\mathfrak{Y}r)\eta(\xi) - (\xi r)\eta(\mathfrak{Y})]. \tag{50}$$

On simplifying, the above equation reduces to

$$(\mathfrak{Y}r) = -\frac{n}{\alpha}(\mathfrak{Y}f). \tag{51}$$

Employing theorem (V.1) in (51), produces

$$\lambda = \frac{\alpha(r + 2n^2)}{n}. \tag{52}$$

Which is the required result.

VI. SIGNIFICANCE OF THE WORK

The Ricci soliton play a crucial role to explore:

- parabolic, hyperbolic and harmonic curvature flow.
- symmetries and isometries of orthonormal groups.
- least upper bound and the greatest lower bound through soliton inequalities.
- moleclues velocity directions and collied energy.
- Signature of a metric (smooth manifold).
- The Fixed points of various Ricci flows and gradient flows are interesting topic to explore.
- The singularities of Ricci flow can be discussed.
- A manifold satisfies f-nonparabolic condition if $(S + \frac{n}{2})$ is a positive symmetric Green's function, otherwise f-parabolic.
- The Laplacian and volume comparison can be studied.
- There is high scope to explore the lower bounds and upper bounds on Ricci flow through Bakry-Emery curvature tensor.
- The Boltzmann entropy and the Fisher information can be studied to discover the linear and nonlinear diffusions.
- It is interesting to explore the estimation between the LSI and the Li-Yau-Hamilton through Ricci flow.
- Pontryagin numbers and Pontryagin classes through soliton.
- The signature of a smooth manifold is linear combination of Pontryagin numbers that represent the Pontryagin classes of tangent bundle.
- If $H^{4I}(B,\mathbb{Z})$ denotes the $4I$-cohomology group and B the tangent bundle of M^n then I_{th} Pontryagin and total Pontryagin classes are defined as

$$P_I(B) = P_I(B,\mathbb{Z}) \in H^{4I}(B,\mathbb{Z}).$$

$$P(B) = 1 + P_1(B) + P_2(B) + ...$$

- If t and $Tr(t)$ denote the curvature form and Ricci tensor, the total Pontryagin class is given by

$$\begin{aligned} P = 1 &- \frac{Tr(t^2)}{8\pi^2} + \frac{Tr(t^2)^2 - 2Tr(t^4)}{128\pi^4} \\ &- \frac{Tr(t^2)^3 - 6Tr(t^2)Tr(t^4) + 8Tr(t^6)}{3072\pi^6} + ... \end{aligned}$$

VII. CONCLUSION

The Lorentzian metric and its generalizations play very important role in the abstraction of the universe. The cosmological models have been discussed and Ricci operator function has been seized. The Poisson structure is an interesting topic to investigate the cosmology and the universe. The soliton function has been explored with constant and non-constant scalar curvature.

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