Model Order Reduction of Continuous-time Linear Systems Using Linear Matrix Inequality: A Case Study of Heat Conduction Problem

Inna Kuswandari, Fatmawati, Chidozie Wiliam Chukwu, Julianti Lemantara, and Tania Fatma Wati

***Abstract*—This paper aims to demonstrate the necessary and sufficient conditions for the existence of a model order reduction of continuous-time linear systems using the Linear Matrix Inequality (LMI) method. Generally, systems derived from partial differential equations are of high order, making them inefficient and ineffective in terms of time and computational resources, requiring an observer for practical use. The solution involves using order reduction to produce a system that is similar and represents the properties of the original system using the LMI method. Through algebraic manipulations, it is concluded that the necessary and sufficient conditions for a system to have a minimal realization and be reducible involve the existence of certain positive definite matrices and a real matrix satisfying specific conditions obtained in this research. Furthermore, the lower bound and the infimum error of model order reduction problem are also stated in the subsequent discussion. With these findings, an algorithm is derived for obtaining an optimally reduced system using the LMI method. As a case study, the model order reduction with LMI method is applied to a heat conduction problem in continuous-time linear systems. The heat conduction problem is one of the cases on heat network and accordance with one of the themes in SDG 7.**

***Index Terms*—linear system, model order reduction, linear matrix inequality, heat conduction problem, heat network, scientific research.**

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I. INTRODUCTION

In many practical scenarios, systems derived from mathematical models often possess high orders [1]. The order of a system refers to the dimension of the state space formed as an alternative realization of the system. An example of a high-order system arises from the discretization of partial differential equations, such as heat conduction problems [1]. In heat conduction problem, the input supplied to the system comprises a heat source while the observed output corresponds to the temperature distribution along a rod. One of the objectives of temperature observation along the rod is to investigate temperature stability across its entire surface. However, this task becomes challenging due to the continuous nature of the points on the rod surface. Consequently, observations are made by measuring temperatures at selected locations on the rod. When the more partitions are taken from the rod, then the closer of the representation gets to the original system. The number of partitions indicates the order of the resulting system, thus yielding in a high-order system [1].

High-order systems lead to high-order controllers, which, from a numerical computation and implementation perspective are often impractical [2]. This is due to the high costs and extended computational burden associated with such controllers, as well as increased likelihood of numerical errors. Moreover, high-order systems are not efficient as certain states might have negligible influence on the input and output characteristics. The impact of states on these characteristics is represented by the Hankel singular values [3]. Hence, high-order systems require reduction to lower orders, resulting in lower-order controllers. The reduction must still adequately represent the original system, which means the model order reduction error should be minimal using the measured H_∞ norm.

Several methods are commonly used to reduce the order of systems, including Balanced Truncation (BT) and Linear Matrix Inequality (LMI) methods. The BT method reduces the system order based on the lower-ranked Hankel singular values [3], whereas the LMI method focuses on model order reduction to achieve a new system with minimal deviation from the original system [4]. The principle of LMI-based model order reduction aims to obtain a reduced system with an order lower than the original system while maintaining minimal error introduced by the reduction process. The error associated with the LMI approach is measured using the H_∞ norm and denoted as $\|E(s)\|_\infty$. The LMI-based model order reduction is performed by formulating linear matrix inequalities that transform the non-linear norm $\|E(s)\|_\infty$ into a linear matrix inequality. The study regarding the application of the LMI method in optimal control of satellite system models was conducted by [5]. Another research on model order reduction was carried out by [6], focusing on the dual reduction strategy for model reduction of periodic control systems.

Let $G(s)$ represent the transfer function of the system with order n, and $G_r(s)$ represent the transfer function of the reduced system with order r where $r < n$. The H_∞ norm of the error introduced by model order reduction is expressed as $\|G(s) - G_r(s)\|_\infty$. Utilizing the Bounded Real Lemma, $\|G(s) - G_r(s)\|_\infty$ can be transformed into a matrix inequality by finding a positive real value γ such that $\|G(s) - G_r(s)\|_\infty < \gamma$. However, this matrix inequality is nonlinear, requiring algebraic manipulations to linearize it.

The process of transforming non-linear matrix inequalities into linear matrix inequalities has been extensively studied by researchers. For instance, [7] derived the necessary and sufficient conditions for the existence of model order reduction through LMI-based approach. In [3], the authors provided an alternative derivation of necessary and sufficient conditions for the existence of model order reduction through LMI manipulation. The work in [3] also established lower bound and infimum for the error resulting from order reduction, along with an algorithm to obtain a suboptimal reduced system using the LMI method. Various alternatives for the necessary and sufficient conditions in the form of linear matrix inequalities, have been derived to facilitate understanding and implementation, particularly in the context of model order reduction.

Based on the explanation above, the purpose of this research is to overcome the inefficiency and ineffectiveness of high-order systems in terms of time and calculation process. To solve this problem, the research establishes the necessary and sufficient conditions for the existence of model order reduction in continuous-time linear systems using the LMI method. Additionally, the research also determines the lower bound and infimum of the error resulting from model order reduction. An algorithm is designed to facilitate the attainment of lower-order reduced systems through a computational approach. As a case study, the LMI method is applied to a heat conduction problem, which demonstrates the practical application of the solution in the form of algorithm.

II. Matrix and Linear System

In this section, the theoretical concepts of matrices and linear systems are discussed. These concepts are essential to support the research requirements and provide a basis for the subsequent analysis.

Definition 1 [8] (Definite) An $n \times n$ symmetric matrix A is said to be:

a. positive definite if $x^T Ax > 0$ for every $x \neq 0$;
b. positive semidefinite if $x^T Ax \geq 0$ for every x;
c. negative definite if $x^T Ax < 0$ for every $x \neq 0$;
d. negative semidefinite if $x^T Ax \leq 0$ for every x;

where $x \in R^n$.

Furthermore, a matrix A that is positive definite, positive semidefinite, negative definite, and negative semidefinite is denoted by $A > 0, A \geq 0, A < 0$, and $A \leq 0$, respectively. Additionally, $A > B$ equivalent to $A - B > 0$ or $A - B$ is a positive definite matrix.

Definition 2 [8] (Schur Complement) Consider a square matrix A partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11} and A_{22} are also square matrices.

a. If A_{11} is nonsingular, then

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

and

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix},$$

thus

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}.$$

Furthermore, the matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is referred as the Schur Complement of A_{11}, denoted by Δ.

b. If A_{22} is nonsingular, then

$$\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ C & A_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & B \\ 0 & A_{22} \end{bmatrix},$$

thus

$$\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

Furthermore, the matrix $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is referred as the Schur Complement of A_{22}, denoted by $\overline{\Delta}$.

Theorem 3 [8] (Matrix Inversion Formula) Given a square matrix A partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} and A_{22} also being square matrices.

a. If A_{11} and $\Delta = A_{22} - A_{21}A_{11}^{-1}A_{12}$ are both nonsingular, then

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}\Delta^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}\Delta^{-1} \\ -\Delta^{-1}A_{21}A_{11}^{-1} & \Delta^{-1} \end{bmatrix}.$$

b. If A_{22} and $\overline{\Delta} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ are both nonsingular, then

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \overline{\Delta}^{-1} & -\overline{\Delta}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}\overline{\Delta}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}\overline{\Delta}^{-1}A_{12}A_{22}^{-1} \end{bmatrix}.$$

Theorem 4 [9] (Schur Complement Theorem) Let A be a symmetric matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$, where A_{11} and A_{22} are also symmetric matrices.

The following statements are equivalent:

a. $A > 0$,
b. $A_{11} > 0$ and $A_{22} - A_{12}^T A_{11}^{-1} A_{12} > 0$,
c. $A_{22} > 0$ and $A_{11} - A_{12} A_{22}^{-1} A_{12}^T > 0$.

Definition 5 [10] A linear time-invariant (LTI) system is defined as a linear differential equation with constant coefficients as follows:

$$\dot{x}(t) = \frac{dx}{dt} = A \cdot x(t) + B \cdot u(t), \qquad (1)$$
$$y(t) = C \cdot x(t) + D \cdot u(t),$$

where $x(t) \in R^n$ is called the state vector, $u(t) \in R^p$ is called the input vector, $y(t) \in R^q$ is called the output vector, and A, B, C, D are constant real matrices of appropriate size.

Furthermore, equation (1) is denoted by the system (A, B, C, D) and the solution of the system (A, B, C, D) is

$$x(t, x_0, u) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

and the output vector $y(t)$ is

$$y(t, x_0, u) = C\left(e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\right) + Du(t),$$

where $x(t_0) = x_0$ is called the initial value of the system. The vector space whose elements are state vectors is called the state space.

Definition 6 [10] The order of the system (A, B, C, D) is the dimension of the state space.

Definition 7 [10] Let $Y(s)$ and $U(s)$ be the Laplace transforms of $y(t)$ and $u(t)$ respectively, with $x(0) = 0$. The transfer function of the system (A, B, C, D), denoted by $G(s)$, that is $G(s) = C(sI - A)^{-1}B + D$ which satisfies $Y(s) = G(s)U(s)$ and has the corresponding transfer function $G(s)$ in state-space form:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Definition 8 [10] A linear system $\dot{x} = Ax$ is considered asymptotically stable if all eigenvalues of A, denoted as $\lambda_i(A)$ have negative real parts that is

$$\mathrm{Re}(\lambda_i(A)) < 0.$$

Definition 9 [10] A system (A, B, C, D) is said to be controllable if for every pair of states x_0 and x_1 with $t_1 > 0$, there exist an input u such that $x(t_1, x_0, u) = x_1$.

Theorem 10 [10] A system (A, B, C, D) is controllable if and only if the controllability Gramian matrix

$$P = \int_0^t e^{A\tau}BB^T e^{A^T\tau} d\tau$$

is positive definite for all $t > 0$.

Definition 11 [10] A system (A, B, C, D) is considered observable if for every input u, there exist $t_1 > 0$ such that $y(t, x_0, u) = y(t, x_1, u)$ implies $x_0 = x_1$ for $t \in [0, t_1]$.

Theorem 12 [10] A system (A, B, C, D) is observable if and only if the observability Gramian matrix

$$Q = \int_0^t e^{A^T\tau}C^T C e^{A\tau} d\tau$$

is positive definite for all $t > 0$.

Theorem 13 [10] The state space realization of the system (A, B, C, D) is said to be the minimal realization of $G(s)$ if the state space of system (A, B, C, D) has the smallest dimensions.

Theorem 14 [10] The realization of the state space (A, B, C, D) of $G(s)$ is said to be minimal if and only if the system (A, B, C, D) is controllable and observable.

Definition 15 [10] Given a system (A, B, C, D) and matrix $X \in R^{n \times n}$.

a. The Lyapunov equation is defined as
$$A^T X + XA + Q = 0, \text{ where } Q \in R^{n \times n}.$$
b. The controllability Lyapunov equation can be expressed as
$$AX + XA^T + BB^T = 0.$$
c. The observability Lyapunov equation can be expressed as
$$A^T X + XA + C^T C = 0.$$

Theorem 16 [1] Given a system (A, B, C, D) and matrix $Q \in R^{n \times n}$. If all eigenvalues of matrix A have negative real parts, then

a. there exist a solution of Lyapunov equation, namely matrix P, satisfying $A^T P + PA = -Q$;
b. for every given matrix $Q > 0$, there exist matrix $P > 0$ satisfying $A^T P + PA = -Q$.

Definition 17 [11] Let A, Q, and R be real-valued matrices of size $n \times n$ where Q and R are symmetric matrices. The Riccati equation of the system (A, B, C, D) is

$$A^T X + XA + XRX + Q = 0.$$

Definition 18 [12] Let $x = (x_1, x_2, ..., x_m)^T \in R^m$ be a variable, $F(x)$ be a polynomial matrix that is linear with respect to x, and $F_i \in R^{n \times n}$ be symmetric matrices for $i = 0, 1, 2, ..., m$. A Linear Matrix Inequality (LMI) is given by the equation

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i > 0.$$

Lemma 19 [13] Given a symmetric matrix $\psi \in R^{m \times m}$ and two matrices Y and Z with column sizes of m. The following statements are equivalent:

a. There exist a matrix θ of appropriate size such that $\psi + Y^T\theta^T Z + Z^T\theta Y < 0$.

b. $(Y^T)^\perp \psi Y^\perp < 0$ and $(Z^T)^\perp \psi Z^\perp < 0$, where $Y^\perp$ and $Z^\perp$ represent the orthogonal complements of matrices Y and Z, respectively, that implies $Y^\perp Y = 0$ and $Z^\perp Z = 0$.

Definition 20 [14] Let P and Q be the controllability and observability Gramian matrices of size $n \times n$, respectively. The Hankel singular values of the system (A,B,C,D) with transfer function $G(s)$ are defined as $\sigma_i = \sqrt{\lambda_i(PQ)}$, where $\lambda_i(PQ)$ represents the i-th largest eigenvalue of the matrix PQ for i = 1,2, …, n.

Definition 21 [3] Suppose we have controllability Gramian matrix $P > 0$ and observability Gramian matrix $Q > 0$. The system (A,B,C,D) of order n with transfer function $G(s)$ said to be balanced if $P = Q = \Sigma$ so that Σ satisfies Lyapunov equations

$$A\Sigma + \Sigma A^T + BB^T = 0 \text{ and } \Sigma A + A^T\Sigma + C^T C = 0,$$

where $\Sigma = diag(\sigma_1 I_{k_1}, ..., \sigma_j I_{k_j}, \sigma_{j+1} I_{k_{j+1}}, ..., \sigma_m I_{k_m})$,

$$\sigma_1 > ... > \sigma_j > \sigma_{j+1} > ... > \sigma_m > 0,$$

$$k_1 + ... + k_j + k_{j+1} + ... + k_m = n.$$

Definition 22 [10] Let $L_\infty(jR)$ or simply L_∞ is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on jR with norm

$$\|F\|_\infty := ess \sup_{\omega \in R} (\bar{\sigma}[F(j\omega)]).$$

The rational subspace of L_∞, denoted by $\tilde{R}L_\infty(jR)$ or simply $\tilde{R}L_\infty$, consists of all proper and real rational transfer matrices with no poles on the imaginary axis.

Definition 23 [10] H_∞ is a (closed) subspace of L_∞ with analytic and bounded functions in the open right-half plane. The H_∞ norm is defined as

$$\|F\|_\infty := \sup_{\text{Re}(s)>0} \bar{\sigma}[F(s)] = \sup_{\omega \in R} (\bar{\sigma}[F(j\omega)]).$$

The expression $\sup_{\omega \in R}(\bar{\sigma}[F(j\omega)])$ can be regarded as a generalization of the maximum modulus theorem for matrix functions. The real rational subspace of H_∞ is denoted by RH_∞, which consists of all proper and real rational stable transfer matrices.

Lemma 24 [15] (*Bounded Real Lemma*) Let γ be real positive number and transfer function $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of the system (A,B,C,D). The two statements below are equivalent:

a. $\|G(s)\|_\infty < \gamma$ (equivalent to $\gamma^2 I - D^T D > 0$).

b. There exist matrix $P > 0$ such that

$$\begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma^2 I & D^T \\ C & D & -I \end{bmatrix} < 0.$$

A system (A,B,C,D) of order n has a transfer function

$$G(s) = C(sI - A)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Furthermore, the transfer function of reduced system (A_r, B_r, C_r, D_r) with order $r < n$, can be formed as

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right].$$

The errors resulting from model order reduction are

$$E(s) = G(s) - G_r(s) = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right]$$

which correspond to the transfer function $G_e(s) = C_e(sI - A_e)^{-1}B_e + D_e$. Based on the rules of addition operation on the system, the state space realization for $E(s)$ is $E(s) = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_r & B_r \\ \hline C & -C_r & D - D_r \end{array}\right]$, where $A_e = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$, $B_e = \begin{bmatrix} B \\ B_r \end{bmatrix}$, $C_e = [C \quad -C_r]$, $D_e = [D - D_r]$.

III. Result and Discussion

This section presents the result of this research, highlighting the contributions of Theorem 25, Corollary 26, and Algorithm 27 related on model order reduction for heat conduction problem. We discuss the results and their significance, providing insights into how these elements contribute to understanding and applying model order reduction effectively.

Let $\gamma > 0$ and $G(s) - G_r(s) = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_r & B_r \\ \hline C & C_r & D - D_r \end{array}\right]$.

According to Lemma 24, it follows that $\|G(s) - G_r(s)\|_\infty < \gamma$ if and only if there exist matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0$ that holds the matrix inequality:

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & C^T \\ (\Omega_{12})^T & \Omega_{22} & \Omega_{23} & -C_r^T \\ (\Omega_{13})^T & (\Omega_{23})^T & -\gamma^2 I & (D^T - D_r^T) \\ C & -C_r & D - D_r & -I \end{bmatrix} < 0, \quad (2)$$

where

$\Omega_{11} = P_{11}A + A^T P_{11}$,

$\Omega_{12} = P_{12}A_r + A^T P_{12}$,

$\Omega_{13} = P_{11}B + P_{12}B_r$,

$(\Omega_{12})^T = P_{12}^T A + A_r^T P_{12}^T$,

$\Omega_{22} = P_{22}A_r + A_r^T P_{22}$,

$\Omega_{23} = P_{12}^T B + P_{22}B_r$,

$(\Omega_{13})^T = B^T P_{11} + B_r^T P_{12}{}^T$, and

$(\Omega_{23})^T = B^T P_{12} + B_r^T P_{22}$.

The inequality (2) represents a nonlinear matrix inequality. In order to implement the model order reduction using the LMI method, this nonlinear matrix inequality (2) needs to be transformed into a linear matrix inequality using algebraic manipulation. Hence, the following theorem which states the necessary and sufficient conditions for the existence of model order reduction using the LMI method, can be applied.

Theorem 25 Given a system (A,B,C,D) of order n with the transfer function $G(s)\in R\mathrm{H}_\infty$ and $G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal state space realization. There is a reduced system (A_r,B_r,C_r,D_r) of order $r<n$ with a transfer function $G_r(s)\in R\mathrm{H}_\infty$ where $\|G(s)-G_r(s)\|_\infty<\gamma$ if and only if there are $X_{11}=(P_{11}-P_{12}P_{22}^{-1}P_{12}^T)^{-1}\in\varsigma_n$, $P_{11}\in\varsigma_n$, $P_{12}\in R^{n\times r}$, and $P_{22}\in\varsigma_r$ which satisfy the following two matrix inequalities:

$$AX_{11}+X_{11}A^T+\frac{1}{\gamma^2}BB^T<0 \text{ and } P_{11}A+A^TP_{11}+C^TC<0.$$

Proof:

It is known that

$$E(s)=G(s)-G_r(s)=\left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right]=\left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_r & B_r \\ \hline C & -C_r & D-D_r \end{array}\right].$$

By algebraic manipulation, the matrices A_e, B_e, C_e, and D_e of the state space realization $E(s)$ are given by

$$A_e=\tilde{A}+\tilde{B}_2\tilde{G}\tilde{C}_2$$
$$=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix},$$

$$B_e=\tilde{B}_1+\tilde{B}_2\tilde{G}\tilde{D}_{21}$$
$$=\begin{bmatrix} B \\ 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} 0 \\ I_p \end{bmatrix}=\begin{bmatrix} B \\ B_r \end{bmatrix},$$

$$C_e=\tilde{C}_1+\tilde{D}_{12}\tilde{G}\tilde{C}_2$$
$$=\begin{bmatrix} C & 0 \end{bmatrix}+\begin{bmatrix} 0 & -I_q \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} C & -C_r \end{bmatrix},$$

$$D_e=\tilde{D}_{11}+\tilde{D}_{12}\tilde{G}\tilde{D}_{21}$$
$$=D+\begin{bmatrix} 0 & -I_q \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} 0 \\ I_p \end{bmatrix}=D-D_r,$$

where

$$\tilde{A}=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}_{(n+r)\times(n+r)}, \tilde{B}_1=\begin{bmatrix} B \\ 0 \end{bmatrix}_{(n+r)\times p},$$

$$\tilde{B}_2=\begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix}_{(n+r)\times(r+q)}, \tilde{C}_1=\begin{bmatrix} C & 0 \end{bmatrix}_{q\times(n+r)},$$

$$\tilde{C}_2=\begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix}_{(r+p)\times(n+r)}, \tilde{D}_{11}=D_{q\times p}, \tilde{D}_{12}=\begin{bmatrix} 0 & -I_q \end{bmatrix}_{q\times(r+q)},$$

$$\tilde{D}_{21}=\begin{bmatrix} 0 \\ I_p \end{bmatrix}_{(r+p)\times p}, \tilde{G}=\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}_{(r+q)\times(r+p)}.$$

From matrix inequality (2) we have

$$\begin{bmatrix} P_{11}A+A^TP_{11} & A^TP_{12}+P_{12}A_r & P_{11}B+P_{12}B_r & C^T \\ P_{12}^TA+A_r^TP_{12}^T & P_{22}A_r+A_r^TP_{22} & P_{12}^TB+P_{22}B_r & -C_r^T \\ B^TP_{11}+B_r^TP_{12}^T & B^TP_{12}+B_r^TP_{22} & -\gamma^2I_p & D^T-D_r^T \\ C & -C_r & D-D_r & -I_q \end{bmatrix}<0$$

$$\Leftrightarrow\begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}+\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right)^T & \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix} & \begin{bmatrix} C^T \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix}\right)^T & -\gamma^2I_p & D^T \\ \begin{bmatrix} C^T \\ 0 \end{bmatrix}^T & D & -I_q \end{bmatrix}$$
$$+\begin{bmatrix} \begin{bmatrix} P_{12} & 0 \\ P_{22} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & -I_q \end{bmatrix} \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_p \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$
$$+\left(\begin{bmatrix} \begin{bmatrix} P_{12} & 0 \\ P_{22} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & -I_q \end{bmatrix} \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_p \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}\right)^T<0$$

$$\Leftrightarrow\begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}+\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right)^T & \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix} & \begin{bmatrix} C^T \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix}\right)^T & -\gamma^2I_p & D^T \\ \begin{bmatrix} C^T \\ 0 \end{bmatrix}^T & D & -I_q \end{bmatrix}$$
$$+\begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -I_q \end{bmatrix} \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_p \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$
$$+\left(\begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -I_q \end{bmatrix} \end{bmatrix}\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_p \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}\right)^T<0$$

$$\Leftrightarrow\begin{bmatrix} P\tilde{A}+(P\tilde{A})^T & P\tilde{B}_1 & (\tilde{C}_1)^T \\ \tilde{B}_1^TP & -\gamma^2I_p & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -I_q \end{bmatrix}$$
$$+\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12} \end{bmatrix}\tilde{G}\begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(r+p)\times q} \end{bmatrix}$$
$$+\left(\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12} \end{bmatrix}\tilde{G}\begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(r+p)\times q} \end{bmatrix}\right)^T<0$$

$$\Leftrightarrow\begin{bmatrix} \mathrm{He}\{P\tilde{A}\} & P\tilde{B}_1 & (\tilde{C}_1)^T \\ \tilde{B}_1^TP & -\gamma^2I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -I \end{bmatrix}$$

$$+\mathrm{He}\left\{\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12} \end{bmatrix}\tilde{G}\begin{bmatrix}\tilde{C}_2 & \tilde{D}_{21} & 0_{(r+p)\times q}\end{bmatrix}\right\}<0. \quad (3)$$

Matrix $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{12}{}^T & P_{22}\end{bmatrix}$ is positive definite of size $(n+r)\times(n+r)$ thus $P\in\varsigma_{n+r}$. By applying Theorem 4, we get $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{12}{}^T & P_{22}\end{bmatrix}>0$ is equivalent to $P_{22}>0$ and $P_{11}-P_{12}P_{22}^{-1}P_{12}^T>0$. Suppose $P_{11}-P_{12}P_{22}^{-1}P_{12}^T$ is denoted by Δ. Since $P_{22}>0$ and $\Delta>0$ then P_{22} and Δ are nonsingular. Therefore, based on the Matrix Inversion Formula in Theorem 3, it is obtained

$$P^{-1}=\begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P_{12}P_{22}^{-1} \\ -P_{22}^{-1}P_{12}{}^T\Delta^{-1} & P_{22}^{-1}+P_{22}^{-1}P_{12}{}^T\Delta^{-1}P_{12}P_{22}^{-1}\end{bmatrix}_{(n+r)\times(n+r)}$$

and we have $\begin{bmatrix} P_{12} & 0_{n\times q} \\ P_{22} & 0_{r\times q} \\ 0_{p\times r} & 0_{p\times q} \\ 0_{q\times r} & -I_q\end{bmatrix}=\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12}\end{bmatrix}$.

Let

$$F_1=\begin{bmatrix} \begin{bmatrix} I_n & 0_{n\times r}\end{bmatrix}P^{-1} & 0_{n\times p} & 0_{n\times q} \\ 0_{p\times(n+r)} & I_p & 0_{p\times q}\end{bmatrix}$$

$$=\begin{bmatrix} \begin{bmatrix} I_n & 0_{n\times r}\end{bmatrix}\begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P_{12}P_{22}^{-1} \\ -P_{22}^{-1}P_{12}^T\Delta^{-1} & P_{22}^{-1}+P_{22}^{-1}P_{12}^T\Delta^{-1}P_{12}P_{22}^{-1}\end{bmatrix} & 0_{n\times p} & 0_{n\times q} \\ 0_{p\times(n+r)} & I_p & 0_{p\times q}\end{bmatrix}$$

$$=\begin{bmatrix} \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P_{12}P_{22}^{-1}\end{bmatrix} & 0_{n\times p} & 0_{n\times q} \\ 0_{p\times(n+r)} & I_p & 0_{p\times q}\end{bmatrix},$$

and

$$F_1\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12}\end{bmatrix}=\begin{bmatrix} \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P_{12}P_{22}^{-1}\end{bmatrix} & 0_{n\times p} & 0_{n\times q} \\ 0_{p\times(n+r)} & I_p & 0_{p\times q}\end{bmatrix}\begin{bmatrix} P_{12} & 0_{n\times q} \\ P_{22} & 0_{r\times q} \\ 0_{p\times r} & 0_{p\times q} \\ 0_{q\times r} & -I_q\end{bmatrix}$$

$$=\begin{bmatrix} 0_{n\times r} & 0_{n\times q} \\ 0_{p\times r} & 0_{p\times q}\end{bmatrix}_{(n+p)\times(r+q)}.$$

Thus, the matrix F_1 is the orthogonal complement of $\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12}\end{bmatrix}$, that is $F_1=\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12}\end{bmatrix}^{\perp}$.

In the same way, $F_2\begin{bmatrix} \tilde{C}_2{}^T \\ \tilde{D}_{21}^T \\ 0_{q\times(r+p)}\end{bmatrix}=\begin{bmatrix} 0_{n\times r} & 0_{n\times p} \\ 0_{q\times r} & 0_{q\times p}\end{bmatrix}_{(n+q)\times(r+p)}$.

So matrix F_2 is the orthogonal complement of matrix $\begin{bmatrix} \tilde{C}_2{}^T \\ \tilde{D}_{21}^T \\ 0_{q\times(r+p)}\end{bmatrix}$ that is $F_2=\begin{bmatrix} \tilde{C}_2{}^T \\ \tilde{D}_{21}^T \\ 0_{q\times(r+p)}\end{bmatrix}^{\perp}$.

Let $\theta^T=\tilde{G}$, $\Psi=\begin{bmatrix} \mathrm{He}\{P\tilde{A}\} & P\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_1^T & -\gamma^2 I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -I\end{bmatrix}_{(n+r+p+q)\times(n+r+p+q)}$,

$Y^T=\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12}\end{bmatrix}_{(n+r+p+q)\times(r+q)}$, $Z^T=\begin{bmatrix} \tilde{C}_2{}^T \\ \tilde{D}_{21}^T \\ 0_{q\times(r+p)}\end{bmatrix}_{(n+r+p+q)\times(r+p)}$.

By applying Lemma 19, the matrix inequality (3) is satisfied if and only if

$$(Y^T)^{\perp}\Psi Y^{\perp}<0,$$

$$\Leftrightarrow (Y^T)^{\perp}\Psi((Y^T)^{\perp})^T<0,$$

$$\Leftrightarrow \begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12}\end{bmatrix}^{\perp}\Psi\left(\begin{bmatrix} P\tilde{B}_2 \\ 0_{p\times(r+q)} \\ \tilde{D}_{12}\end{bmatrix}^{\perp}\right)^T<0,$$

$$\Leftrightarrow \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P_{12}P_{22}^{-1} & 0_{n\times p} & 0_{n\times q} \\ 0_{p\times n} & 0_{p\times r} & I_p & 0_{p\times q}\end{bmatrix}\begin{bmatrix} \mathrm{He}\{P\tilde{A}\} & P\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_1^T & -\gamma^2 I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -I\end{bmatrix}$$

$$\cdot\begin{bmatrix} (\Delta^{-1})^T & 0_{n\times p} \\ -(P_{22}^{-1})^T P_{12}^T(\Delta^{-1})^T & 0_{r\times p} \\ 0_{p\times n} & I_p \\ 0_{q\times n} & 0_{q\times p}\end{bmatrix}<0, \quad (4)$$

and

$$(Z^T)^{\perp}\Psi Z^{\perp}<0 \Leftrightarrow (Z^T)^{\perp}\Psi((Z^T)^{\perp})^T<0,$$

$$\Leftrightarrow \begin{bmatrix} \tilde{C}_2{}^T \\ \tilde{D}_{21}{}^T \\ 0_{q\times(r+p)}\end{bmatrix}^{\perp}\Psi\left(\begin{bmatrix} \tilde{C}_2{}^T \\ \tilde{D}_{21}{}^T \\ 0_{q\times(r+p)}\end{bmatrix}^{\perp}\right)^T<0,$$

$$\Leftrightarrow \begin{bmatrix} I_n & 0_{n\times r} & 0_{n\times p} & 0_{n\times q} \\ 0_{q\times n} & 0_{q\times r} & 0_{q\times p} & I_q\end{bmatrix}\begin{bmatrix} \mathrm{He}\{P\tilde{A}\} & P\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_1^T & -\gamma^2 I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -I\end{bmatrix}$$

$$\cdot\begin{bmatrix} I_n & 0_{n\times q} \\ 0_{r\times n} & 0_{r\times q} \\ 0_{p\times n} & 0_{p\times q} \\ 0_{q\times n} & I_q\end{bmatrix}<0. \quad (5)$$

With some algebraic manipulation, it follows that

$$\begin{bmatrix} P_{11}A+A^TP_{11} & A^TP_{12} & P_{11}B & C^T \\ P_{12}^TA & 0 & P_{12}^TB & 0 \\ B^TP_{11} & B^TP_{12} & -\gamma^2 I_p & D^T \\ C & 0 & D & -I_q\end{bmatrix}=\begin{bmatrix} \mathrm{He}\{P\tilde{A}\} & P\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_1^T & -\gamma^2 I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -I\end{bmatrix}.$$

Therefore, matrix inequality (4) is equivalent to the following matrix inequality:

$$\begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P_{12}P_{22}^{-1} & 0_{n\times p} & 0_{n\times q} \\ 0_{p\times n} & 0_{p\times r} & I_p & 0_{p\times q}\end{bmatrix}\begin{bmatrix} P_{11}A+A^TP_{11} & A^TP_{12} & P_{11}B & C^T \\ P_{12}^TA & 0 & P_{12}^TB & 0 \\ B^TP_{11} & B^TP_{12} & -\gamma^2 I_p & D^T \\ C & 0 & D & -I_q\end{bmatrix}$$

$$\cdot\begin{bmatrix} (\Delta^{-1})^T & 0_{n\times p} \\ -(P_{22}^{-1})^T P_{12}^T(\Delta^{-1})^T & 0_{r\times p} \\ 0_{p\times n} & I_p \\ 0_{q\times n} & 0_{q\times p}\end{bmatrix}<0. \quad (6)$$

From the multiplication of the first two matrices in matrix inequality (6), namely

$$\begin{aligned} M_{11} &= \Delta^{-1}(P_{11}A + A^T P_{11}) - \Delta^{-1}P_{12}P_{22}^{-1}P_{12}^T A, \\ M_{12} &= \Delta^{-1}A^T P_{12}, \\ M_{13} &= \Delta^{-1}P_{11}B - \Delta^{-1}P_{12}P_{22}^{-1}P_{12}^T B, \\ M_{14} &= \Delta^{-1}C^T, \\ M_{21} &= B^T P_{11}, \\ M_{22} &= B^T P_{12}, M_{23} = -\gamma^2 I_p, \\ M_{24} &= D^T, \end{aligned} \tag{7}$$

then matrix inequality (6) can be expressed as follows:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \end{bmatrix} \cdot \begin{bmatrix} (\Delta^{-1})^T & 0_{n\times p} \\ -(P_{22}^{-1})^T P_{12}^T (\Delta^{-1})^T & 0_{r\times p} \\ 0_{p\times n} & I_p \\ 0_{q\times n} & 0_{q\times p} \end{bmatrix} < 0,$$

$$\Leftrightarrow \begin{bmatrix} M_{11}\Delta^{-1} - M_{12}P_{22}^{-1}P_{12}^T\Delta^{-1} & M_{13} \\ M_{21}\Delta^{-1} - M_{22}P_{22}^{-1}P_{12}^T\Delta^{-1} & M_{23} \end{bmatrix} < 0 \cdot \tag{8}$$

Then, let

$$\begin{aligned} N_{11} &= M_{11}\Delta^{-1} - M_{12}P_{22}^{-1}P_{12}^T\Delta^{-1}, \\ N_{12} &= M_{13}, \\ N_{21} &= M_{21}\Delta^{-1} - M_{22}P_{22}^{-1}P_{12}^T\Delta^{-1}, \\ N_{22} &= M_{23}. \end{aligned} \tag{9}$$

Thus matrix inequality (8) eqivalent to

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}_{(n+p)\times(n+p)} < 0 \cdot \tag{10}$$

By substituting equations (7) into equations (9), then matrices $N_{11}, N_{12}, N_{21},$ and N_{22} can be expressed as the following matrices:

$$\begin{aligned} N_{11} &= M_{11}\Delta^{-1} - M_{12}P_{22}^{-1}P_{12}^T\Delta^{-1}, \\ &= \left(\Delta^{-1}(P_{11}A + A^T P_{11}) - \Delta^{-1}P_{12}P_{22}^{-1}P_{12}^T A\right)\Delta^{-1} \\ &\quad - \left(\Delta^{-1}A^T P_{12}\right)P_{22}^{-1}P_{12}^T\Delta^{-1}, \\ &= \Delta^{-1}(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)A\Delta^{-1} + \Delta^{-1}A^T\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)\Delta^{-1}, \\ &= \Delta^{-1}\Delta A\Delta^{-1} + \Delta^{-1}A^T\Delta\Delta^{-1}, \\ &= A\Delta^{-1} + \Delta^{-1}A^T, \\ &= A\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1} + \left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1}A^T, \\ &= \mathrm{He}\left\{A\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1}\right\}; \\ N_{12} &= M_{13} = \Delta^{-1}P_{11}B - \Delta^{-1}P_{12}P_{22}^{-1}P_{12}^T B, \\ &= \Delta^{-1}(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)B, \\ &= \Delta^{-1}\Delta B, \\ &= B; \\ N_{21} &= M_{21}\Delta^{-1} - M_{22}P_{22}^{-1}P_{12}^T\Delta^{-1}, \\ &= B^T P_{11}\Delta^{-1} - B^T P_{12}P_{22}^{-1}P_{12}^T\Delta^{-1}, \\ &= B^T(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)\Delta^{-1}, \\ &= B^T\Delta\Delta^{-1}, \\ &= B^T; \\ N_{22} &= M_{23}, \\ &= -\gamma^2 I_p. \end{aligned}$$

Hence, matrix inequality (10) can be written as

$$\begin{bmatrix} \mathrm{He}\left\{A\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1}\right\} & B \\ B^T & -\gamma^2 I_p \end{bmatrix} < 0 \cdot \tag{11}$$

In the same way, matrix inequalities (5) is equivalent to

$$\begin{bmatrix} I_n & 0_{n\times r} & 0_{n\times p} & 0_{n\times q} \\ 0_{q\times n} & 0_{q\times r} & 0_{q\times p} & I_q \end{bmatrix} \begin{bmatrix} P_{11}A + A^T P_{11} & A^T P_{12} & P_{11}B & C^T \\ P_{12}^T A & 0 & P_{12}^T B & 0 \\ B^T P_{11} & B^T P_{12} & -\gamma^2 I_p & D^T \\ C & 0 & D & -I_q \end{bmatrix}$$

$$\cdot \begin{bmatrix} I_n & 0_{n\times q} \\ 0_{r\times n} & 0_{r\times q} \\ 0_{p\times n} & 0_{p\times q} \\ 0_{q\times n} & I_q \end{bmatrix} < 0,$$

$$\Leftrightarrow \begin{bmatrix} P_{11}A + A^T P_{11} & A^T P_{12} & P_{11}B & C^T \\ C & 0 & D & -I_q \end{bmatrix} \cdot \begin{bmatrix} I_n & 0_{n\times q} \\ 0_{r\times n} & 0_{r\times q} \\ 0_{p\times n} & 0_{p\times q} \\ 0_{q\times n} & I_q \end{bmatrix} < 0,$$

$$\Leftrightarrow \begin{bmatrix} P_{11}A + A^T P_{11} & C^T \\ C & -I_q \end{bmatrix}_{(n+q)\times(n+q)} < 0 \cdot \tag{12}$$

By applying Theorem 4, we have

a. from matrix inequality (11), we get

$$\begin{aligned} &\mathrm{He}\left\{A\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1}\right\} - B\left(-\gamma^2 I_p\right)^{-1}B^T < 0, \\ &\Leftrightarrow A\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1} + \left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1}A^T \\ &\quad - B\left(-\frac{1}{\gamma^2}I_p\right)B^T < 0, \\ &\Leftrightarrow AX_{11} + X_{11}A^T + \frac{1}{\gamma^2}BB^T < 0, \end{aligned}$$

where $X_{11} = \left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1}$;

b. from matrix inequality (12), we obtain

$$\begin{aligned} &P_{11}A + A^T P_{11} - C^T(-I_q)^{-1}C < 0, \\ &\Leftrightarrow P_{11}A + A^T P_{11} - C^T(-I_q)C < 0, \\ &\Leftrightarrow P_{11}A + A^T P_{11} + C^T C < 0. \end{aligned}$$

Because $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \in \varsigma_{n+r}$ with $P_{11}, P_{12},$ and P_{22} are matrices of size $n\times n, n\times r$ and $r\times r$ respectively, then by applying Theorem 4, it is obtained $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0$ if and only if $P_{11} > 0, P_{22} > 0$ and $P_{11} - P_{12}P_{22}^{-1}P_{12}^T > 0$.Then we obtain $\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1} > 0$. In other words, $X_{11} > 0$.

Because $P_{11}, P_{22},$ and X_{11} are positive definite matrices with sizes $n\times n, r\times r,$ and $n\times n$ respectively, then it can be concluded that $P_{11} \in \varsigma_n, P_{22} \in \varsigma_r,$ and $X_{11} \in \varsigma_{n+r}$. Furthermore, because P_{12} is a real matrix of size $n\times r$ then $P_{12} \in R^{n\times r}$. This completes the proof. ■

The proof of Theorem 25 above guarantees the existence of a reduced order system while representing the characteristics of the original system. Consequently, the calculation of a quantitative measure leading to the determination of a lower bound on the error of the model order reduction formally presented in Corollary 26.

Corollary 26 Given a system (A,B,C,D) of order n with a transfer function $G(s)=\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]\in RH_\infty$ where the singular Hankel values $\sigma_1\ge\ldots\ge\sigma_r\ge\sigma_{r+1}\ge\ldots\ge\sigma_n>0$. Then, for every reduced system (A_r,B_r,C_r,D_r) of order $r<n$ with the transfer function $G_r(s)=\left[\begin{array}{c|c}A_r & B_r\\ \hline C_r & D_r\end{array}\right]\in RH_\infty$ implies $\|G(s)-G_r(s)\|_\infty\ge\sigma_{r+1}$.

From Theorem 25 and Corollary 26, necessary and sufficient conditions for the existence of the reduced system have been obtained with a lower bound of error that depends on the Hankel singular value. Furthermore, the practical computation of this reduced system, characterized by the transfer function $G_{n-k_m}(s)$, can be efficiently achieved by leveraging the systematic approach outlined in Algorithm 27. This algorithm simplifies the process of obtaining an optimally reduced system by directly addressing the minimization of $\|G(s)-G_{n-k_m}(s)\|_\infty$, ensuring both accuracy and computational efficiency.

Algorithm 27

1. Form matrices A, B, C, and D from the system (A,B,C,D) of order n.
2. Investigate the stability, controllability and observability properties of the system (A,B,C,D).
3. If the system (A,B,C,D) is stable, controllable, and observable, then go to Step 4. If the system (A,B,C,D) does not meet one of the properties, either stable, controllable or observed, then the process stop. In other words, no reduced system (A_r,B_r,C_r,D_r) of order $r=n-k_m$ can be found.
4. Determine the state space and transfer function $G(s)$ of the system (A,B,C,D).
5. Form a balanced realization of the transfer function $G(s)$ as $G_b(s)=\left[\begin{array}{c|c}A_b & B_b\\ \hline C_b & D_b\end{array}\right]$.
6. Set the matrix $P_{12}=\begin{bmatrix}I_{n-k_m}\\ 0_{k_m\times(n-k_m)}\end{bmatrix}$.
7. Minimize the value of γ^2 that satisfies three matrix inequalities:
$$\begin{bmatrix}P_{11} & P_{12}Q_{22}\\ Q_{22}P_{12}^T & Q_{22}\end{bmatrix}>0,$$
$$\begin{bmatrix}(P_{11}-P_{12}Q_{22}P_{12}^T)A+A^T(P_{11}-P_{12}Q_{22}P_{12}^T) & (P_{11}-P_{12}Q_{22}P_{12}^T)B\\ B^T(P_{11}-P_{12}Q_{22}P_{12}^T) & -\gamma^2 I\end{bmatrix}<0,$$
and $P_{11}A+A^TP_{11}+C^TC<0$,
where $P_{11}\in\varsigma_n$ and $Q_{22}\in\varsigma_{n-k_m}$ are matrices that serve as variables.
8. Determine the matrices $X_{11}=\left(P_{11}-P_{12}Q_{22}P_{12}^T\right)^{-1}$ and $\left(Q_{22}\right)^{-1}$.
9. Define the matrix $\tilde{P}=\begin{bmatrix}P_{11} & P_{12}\\ P_{12}^T & Q_{22}^{-1}\end{bmatrix}$ and denote the optimal value γ from step 7 with γ_{opt}.
10. Substitute the matrix $P=\begin{bmatrix}P_{11} & P_{12}\\ P_{12}^T & P_{22}\end{bmatrix}$ with the matrix $\tilde{P}$ and also the value of γ with γ_{opt} in the matrix inequality
$$\begin{bmatrix}P_{11}A+A^TP_{11} & P_{12}A_r+A_rP_{12} & P_{11}B+P_{12}B_r & C^T\\ P_{12}^TA+A_r^TP_{12}^T & P_{22}A_r+A_r^TP_{22} & P_{12}^TB+P_{22}B_r & -C_r^T\\ B^TP_{11}+B_r^TP_{12}^T & B^TP_{12}+B_r^TP_{22} & -\gamma^2 I & (D^T-D_r^T)\\ C & -C_r & D-D_r & -I\end{bmatrix}<0$$
11. Determine matrices A_r,B_r,C_r and D_r as solutions to the linear matrix inequalities obtained from Step 10 so that a reduced system (A_r,B_r,C_r,D_r) is also obtained.
12. Set the matrices $MI_1=A_bX_{11}+X_{11}A_b^T+\frac{1}{\gamma^2}B_bB_b^T$ and $MI_2=P_{11}A_b+B_b^TP_{11}+C_b^TC_b$.
13. If $X_{11}\in\varsigma_n, P_{11}\in\varsigma_n, P_{12}\in R^{15\times5}$ and $P_{22}\in\varsigma_r$ can be found to satisfy:
$MI_1=A_bX_{11}+X_{11}A_b^T+\frac{1}{\gamma^2}B_bB_b^T<0$ and
$MI_2=P_{11}A_b+B_b^TP_{11}+C_b^TC_b<0$, then the existence of a reduced-order system (A_r,B_r,C_r,D_r) with order r and $\|G(s)-G_r(s)\|_\infty<\gamma$ are guaranteed.

In the next section, we will provide an example to demonstrate the existence of a reduced model order by applying Theorem 25. Our simulation is applied to a real-world problem involving heat conduction in a rod. Before carrying out the simulation, the heat conduction problem will be explained.

We consider a wire rod with unit length l and heat conduction coefficient α^2. Suppose $T(\zeta,t)$ represents the temperature of the wire rod at position ζ and time t. According to [16], the heat conduction equation can be expressed as
$$T_t(\zeta,t)=\alpha^2T_{\zeta\zeta}(\zeta,t), \quad (17)$$
where $T_t(\zeta,t)=\frac{\partial T(\zeta,t)}{\partial t}$ and $T_{\zeta\zeta}(\zeta,t)=\frac{\partial^2T(\zeta,t)}{\partial\zeta^2}$.

In this problem, we assume the end of the rod at position $\zeta=0$ given input in the form of a time-dependent heat source, denoted by $u(t)$, while the end of the rod at position $\zeta=l$ not isolated. So that the boundary conditions for the heat conduction equation can be written as
$$T(0,t)=u \text{ and } T_\zeta(l,t)=0. \quad (18)$$

The heat conduction problem has a given initial condition, that is $T(\zeta,0)=f(\zeta)$ for $0<\zeta<l$ and $0<t<\infty$ where $f(\zeta)$ denotes the function in ζ. To further clarify the problem of heat conduction in wire rod with regard to boundary conditions and initial conditions, a schematic is presented in Figure 1. The arrow in Figure 1 indicate the direction of heat flow.

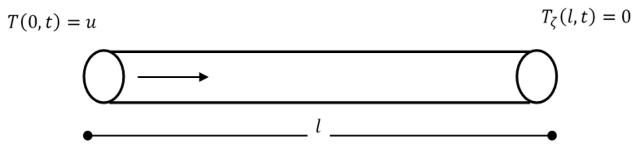

Figure 1. Heat conduction scheme in wire rod

To obtain the state space equation from the heat conduction model, the location parameter intervals will be discretized into n interval parts, each of which has a length of $h=\frac{l}{n}$ units. The discretization form can be seen in Figure 2 below.

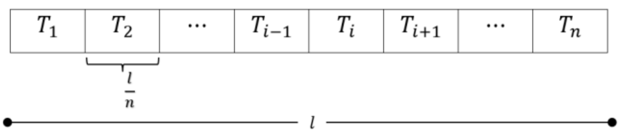

Figure 2. Discretization process

On Figure 2, T_i represents the temperature in the interval $\frac{(i-1)}{n}l \le \zeta \le \frac{i}{n}l$ where $i=1,2,...,n$. Next, we review the boundary conditions in equation (18). From the boundary conditions $T(0,t)=u$, it is obtained that $T_0=u$. By using the Forward Finite Difference method, the derivative of T_i with location ζ can be approximated by $\frac{\partial T_i}{\partial \zeta}=\frac{T_{i+1}-T_i}{h}$ for $i=1,2,...,n$. Thus from the boundary condition $T_\zeta(l,t)=0$ in equation (18), it is obtained

$$\frac{\partial T_n}{\partial \zeta}=0 \Leftrightarrow \frac{T_{n+1}-T_n}{h}=0 \Leftrightarrow T_{n+1}=T_n \cdot$$

The second derivative form of T_i with location ζ in equation (17) can be approximated by using the Finite Central Difference method to obtain

$$\frac{\partial^2 T_i}{\partial \zeta^2}=\frac{T_{i+1}-2T_i+T_{i-1}}{h^2}$$

for $i=1,2,...,n$. By substituting $h=\frac{l}{n}$ into $\frac{\partial^2 T_i}{\partial \zeta^2}$, then

$$\frac{\partial^2 T_i}{\partial \zeta^2}=\left(\frac{n}{l}\right)^2\left(T_{i+1}-2T_i+T_{i-1}\right)\cdot$$

Because $T_0=u$ and $T_{n+1}=T_n$, the second derivative of T_i in location ζ for $i=1,2,...,n$ can be described as follows

$$\frac{\partial^2 T_1}{\partial \zeta^2}=\left(\frac{n}{l}\right)^2(T_2-2T_1+T_0)=\left(\frac{n}{l}\right)^2(T_2-2T_1+u),$$

$$\frac{\partial^2 T_2}{\partial \zeta^2}=\left(\frac{n}{l}\right)^2(T_3-2T_2+T_1),$$

$$\vdots$$

$$\frac{\partial^2 T_{n-1}}{\partial \zeta^2}=\left(\frac{n}{l}\right)^2(T_n-2T_{n-1}+T_{n-2}),$$

$$\frac{\partial^2 T_n}{\partial \zeta^2}=\left(\frac{n}{l}\right)^2(T_{n+1}-2T_n+T_{n-1})=\left(\frac{n}{l}\right)^2(T_n-2T_n+T_{n-1}),$$

$$\frac{\partial^2 T_n}{\partial \zeta^2}=\left(\frac{n}{l}\right)^2(-T_n+T_{n-1}).$$

Based on equation (17) and the description $\frac{\partial^2 T_n}{\partial \zeta^2}$ above, then it is obtained a linear system of order n as follows

$$\frac{1}{\alpha^2}\frac{d}{dt}\begin{bmatrix}T_1\\T_2\\T_3\\\vdots\\T_{n-2}\\T_{n-1}\\T_n\end{bmatrix}=\left(\frac{n}{l}\right)^2\begin{bmatrix}1\\0\\0\\\vdots\\0\\0\\0\end{bmatrix}u$$
$$+\left(\frac{n}{l}\right)^2\begin{bmatrix}-2&1&0&0&\cdots&0&0&0\\1&-2&1&0&\cdots&0&0&0\\0&1&-2&1&\cdots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&-2&1&0\\0&0&0&0&\cdots&1&-2&1\\0&0&0&0&\cdots&0&1&-1\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\\vdots\\T_{n-2}\\T_{n-1}\\T_n\end{bmatrix}$$

Furthermore, the wire rod used in this simulation is a wire rod made of aluminum with a heat conduction coefficient $\alpha^2=0{,}86$ cm^2/second. Suppose the length of the wire rod l is 15 cm and the discretization is carried out by taking $n=15$, then the n order linear system can be expressed as

$$\frac{d}{dt}\begin{bmatrix}T_1\\T_2\\\vdots\\T_{13}\\T_{14}\\T_{15}\end{bmatrix}=0{,}86\begin{bmatrix}1\\0\\\vdots\\0\\0\\0\end{bmatrix}u+0{,}86\begin{bmatrix}-2&1&\cdots&0&0&0\\1&-2&\cdots&0&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&\cdots&-2&1&0\\0&0&\cdots&1&-2&1\\0&0&\cdots&0&1&-1\end{bmatrix}\begin{bmatrix}T_1\\T_2\\\vdots\\T_{13}\\T_{14}\\T_{15}\end{bmatrix}$$

Next, let

$$A=0{,}86\begin{bmatrix}-2&1&\cdots&0&0&0\\1&-2&\cdots&0&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&\cdots&-2&1&0\\0&0&\cdots&1&-2&1\\0&0&\cdots&0&1&-1\end{bmatrix},$$

$$B=0{,}86\begin{bmatrix}1\\0\\\vdots\\0\\0\\0\end{bmatrix},\ T=\begin{bmatrix}T_1\\T_2\\\vdots\\T_{13}\\T_{14}\\T_{15}\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_{13}\\x_{14}\\x_{15}\end{bmatrix}=x,$$

thus $\frac{dT}{dt}=\frac{dx}{dt}\cdot$ Then, it is obtained $\frac{dx}{dt}=\dot{x}(t)=Ax(t)+Bu(t)$ where $x(t)\in R^{15}$ and $u(t)\in R$.

We assume that the temperature of a wire rod is only carried out at ζ_1, ζ_{15}, and ζ_i where i are even numbers between 1 until 15. This means that observations are made for ζ_1, ζ_{15}, and ζ_i with $i = 2,4,...,14$ so that the output vector $y(t)$ can be written as $y(t) = Cx(t)$ with $y(t) \in R$ and $C = [1 \;\; 1 \;\; 0 \;\; \cdots \;\; 0 \;\; 1 \;\; 1]$. The state-space equation for the heat conduction problem is

$$\dot{x}(t) = Ax(t) + Bu(t),\; y(t) = Cx(t).$$

Next, a simulation will be carried out to show the existence of model order reduction of a continuous-time linear system of order 15 using the LMI method and the upper bound of reduction error γ is chosen to be 7,2. Based on simulation results, it is found that the system (A, B, C) with order 15 is an asymptotically stable because all the eigenvalues of matrix A have a negative real parts. Furthermore, the minimal realization order of the transfer function $G(s)$ is 15. In this case, the minimal realization order of $G(s)$ is the same with the order of the state space, so it can be concluded that the state space realization of the transfer function $G(s)$ is a minimal realization. Additionally, it is also found that the order of the reduced system (A_r, B_r, C_r) with the desired transfer function $G_r(s)$ is $r = 5$.

To guarantee the existence of the reduced system of order $r = 5$ which satisfies $\|G(s) - G_r(s)\|_\infty < \gamma$ correspond to Theorem 25, we need to obtain positive definite matrices P_{11}, P_{22}, X_{11} of sizes 15×15, 5×5, 15×15, respectively, as well as a real matrix P_{12} of size 15×5. From the simulation results, we obtain the matrices P_{11}, P_{22}, X_{11} are positive definite matrices because all the eigenvalues of matrices P_{11}, P_{22}, X_{11} are positive, and the matrix P_{12} is a real matrix.

Let $MI_1 = A_b X_{11} + X_{11} A_b^T + \frac{1}{\gamma^2} B_b B_b^T < 0$ and $MI_2 = P_{11} A_b + B_b^T P_{11} + C_b^T C_b < 0$, then by substituting the matrices A, B, C, P_{11}, and X_{11} into matrices MI_1 and MI_2, it is found that all the eigenvalues of the MI_1 and MI_2 are negative. Hence, it can be concluded that MI_1 and MI_2 are respectively negative definite matrices.

Next, we consider a system (A, B, C) of order 15 with a transfer function $G(s)$ and a minimal realization. Hence, it is obtained matrices P_{11}, P_{22}, and X_{11} are a positive definite matrices of size 15×15, 5×5, and 15×15, and real matrix P_{12} of size 15×5 that satisfies the matrix inequalities $AX_{11} + X_{11}A^T + \frac{1}{\gamma^2} BB^T < 0$ and $P_{11}A + A^T P_{11} + C^T C < 0$ where $X_{11} = (P_{11} - P_{12} P_{22}^{-1} P_{12}^T)^{-1}$. Therefore, based on Theorem 25, it can be guaranteed the existence of a reduced system (A_r, B_r, C_r) of order $r = 5$ with transfer function $G_r(s)$ that satisfies $\|G(s) - G_r(s)\|_\infty < 7,2$.

IV. Conclusion

This paper focuses on determining the necessary and sufficient conditions for the existence of model order reduction of a continuous-time linear system using LMI method. Based on the previous discussion, the following conclusions were drawn:

1. The necessary and sufficient conditions for the existence of a reduced system (A_r, B_r, C_r, D_r) of order $r < n$ with a transfer function $G_r(s) \in R\mathrm{H}_\infty$ satisfying $\|G(s) - G_r(s)\|_\infty < \gamma$ are the existence of $X_{11} \in \varsigma_n$, $P_{11} \in \varsigma_n$, $P_{12} \in R^{n\times r}$, $P_{22} \in \varsigma_r$ that satisfy $AX_{11} + X_{11}A^T + \frac{1}{\gamma^2} BB^T < 0$ and $P_{11}A + A^T P_{11} + C^T C < 0$ where $X_{11} = (P_{11} - P_{12} P_{22}^{-1} P_{12}^T)^{-1}$.
2. For any reduced system (A_r, B_r, C_r, D_r) of order $r < n$ with the transfer function $G(s), G_r(s) \in R\mathrm{H}_\infty$, the inequality $\|G(s) - G_r(s)\|_\infty \ge \sigma_{r+1}$ holds and σ_{r+1} is a lower bound for the error resulting from model order reduction using the LMI method where $\sigma_1 \ge ... \ge \sigma_r \ge \sigma_{r+1} \ge ... \ge \sigma_n > 0$.
3. The numerical simulations of the model order reduction for a heat conduction problem were conducted, with the choice of $\gamma = 7.2$. The results demonstrated that the existence of a reduced system (A_r, B_r, C_r) of order $r = 5$ from a system (A, B, C) of order $n = 15$ satisfies $\|G(s) - G_r(s)\|_\infty < 7,2$. This assurance is achieved by finding $X_{11} \in \varsigma_{15}$, $P_{11} \in \varsigma_{15}$, $P_{12} \in R^{15\times5}$, and $P_{22} \in \varsigma_5$ that fulfill the conditions $AX_{11} + X_{11}A^T + \frac{1}{\gamma^2} BB^T < 0$ and $P_{11}A + A^T P_{11} + C^T C < 0$, with $X_{11} = (P_{11} - P_{12} P_{22}^{-1} P_{12}^T)^{-1}$.

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