# Optimal Control of a Drinking Epidemic Model with Time Delay

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***Abstract*— In this paper, we develop a drinking model that incorporates a temporal delay to account for individuals who have temporarily recovered. This delay reflects the lag in immunity before relapse occurs. The model also considers optimal control strategies, including the impact of education campaigns aimed at reducing alcohol consumption and the availability of treatment for alcoholics. We determine the basic reproduction number and establish the existence of an optimal control pair. Using Pontryagin's Maximum Principle within a delayed framework, we derive the necessary conditions for optimality and confirm the existence of an optimal solution. Additionally, numerical simulations are performed to validate the key findings.**

***Index Terms*— drinking epidemic, optimal control, time delay, numerical simulations**

## I. Introduction

REGULATING alcohol consumption is a major concern in many countries. While alcohol production and consumption have social and economic implications, excessive drinking poses severe health risks, such as liver and brain damage. It also contributes to issues like decreased physical performance, workplace absenteeism, impaired driving leading to accidents, antisocial behavior, crime, and domestic violence. Many individuals are first introduced to alcohol during adolescence or even childhood. Various factors influence the decision to start drinking, including curiosity, lack of awareness, social pressure, and the influence of parents or celebrities.

Epidemiology focuses on the distribution of diseases within populations and the factors influencing their spread. Traditionally, the epidemiological approach has been applied to communicable diseases, such as cholera and measles, to understand their transmission. Several epidemic models and related theoretical studies are available in [1-4]. Although social issues such as drug use [5-6], smoking [7], and alcohol consumption [8] have been examined from an epidemiological perspective, the use of mathematical modeling to analyze these problems is still relatively uncommon. Since the spread of alcohol consumption shares similarities with disease transmission, mathematical epidemiological modeling can offer valuable insights into the progression from initial use to habitual drinking, treatment, relapse, and recovery. It also helps in analyzing equilibrium states. Manthey et al. [9] applied an epidemiological model to examine drinking behavior on college campuses, focusing on student populations.

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Their study found that the reproductive number alone is not enough to determine whether drinking behavior will persist; instead, the pattern of new student recruits plays a crucial role in shaping campus drinking trends.

Benedict [8] analyzed the spread of alcoholism and relapse by examining the effects of reproduction numbers on model dynamics and calculating equilibrium states. Manthey et al. [9] applied an epidemiological model to study campus drinking dynamics, focusing specifically on a college campus and its student population. Their findings suggest that the reproductive number alone is insufficient to predict whether drinking behavior will persist on campus; instead, the pattern of new recruits plays a significant role in shaping campus drinking behavior.

Sharma and Samanta [10] developed an alcohol abuse model that incorporates a treatment program while accounting for potential relapses. They modeled the treatment rate as a time-dependent function, representing treatment control within the drinking model. Huo et al. [11] introduced a binge drinking model with a time delay to represent immunity lag, establishing equilibrium conditions and analyzing their stability. Thamchai and Wu [13] proposed a drinking epidemic model that examines the dynamic behaviors of both drinking-free and persistent drinking equilibria. They also identified long-term optimal treatment strategies for managing both occasional and habitual drinkers. Ma et al. [14] developed an alcoholism model that integrates public health education and three distinct time delays to analyze the dynamics and control mechanisms of alcohol consumption. Djillali et al. [15] investigated the global dynamics of an alcoholism epidemic model with distributed delays. A key feature of their model is the inclusion of social pressure as a contributing factor to drinking alcohol. Wang et al. [16] examined the global dynamics of an alcoholism epidemic model with a saturation incidence rate and two distributed delays: one representing the progression of a susceptible individual to alcoholism and the other representing the relapse of a recovered individual back into alcoholism. Mayengo [17] utilized optimal control theory to model a system of ordinary differential equations describing the dynamics of health risks linked to alcoholism within a community with strong religious beliefs. The study introduced two non-autonomous control variables aimed at mitigating these health risks. Anjam et al. [18] constructed a drinking epidemic model using control variables based on qualitative optimal control principles, with the goal of maximizing the number of susceptible and recovered individuals, minimizing heavy drinkers, and optimizing the dynamics of these groups.

Building on previous research and statistics from [12], which show that nearly one-third of recovering alcoholics relapse within their first year of sobriety, we propose a more realistic drinking model. This model includes a time delay to represent the temporarily recovered population, acknowledging that relapse does not occur immediately but after a certain period. This delay reflects the gap in immunity before individuals return to alcohol consumption.

Our model extends the *SPARS* model presented in [13] and addresses the optimal control problem using two control measures: the level of education campaigns aimed at reducing community alcohol consumption [19] and the level of treatment provided to alcoholics. This paper aims to investigate the influence of education and treatment, incorporating time delays, on alcohol control, thereby advancing and refining the outcomes of previous studies.

The remainder of this paper is organized as follows: First, we establish the *SPARS* drinking epidemic model with a time delay and derive an explicit expression for the basic reproduction number. Next, identify the equilibrium points. We then formulate an optimal control problem to determine the optimal treatment strategy, applying Pontryagin's Maximum Principle to analyze the optimal solution. Subsequently, numerical simulations are performed to validate our analytical results and to illustrate the dynamic movement of populations across different groups. Finally, we present our conclusions.

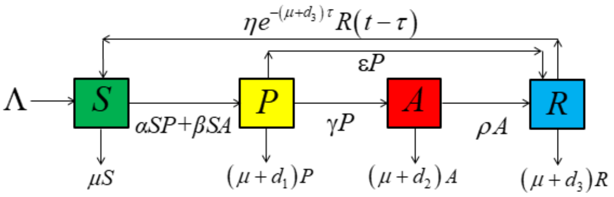



Fig.1. Transfer diagram of the *SPARS* model with time delay.

## II. Methods and Results

Our model focuses on the population aged 16 and over. The population is divided into four classes: susceptible drinkers $(S)$, periodic drinkers $(P)$, alcoholics $(A)$ and recovered drinkers $(R)$, and incorporate time delay we represent a constant $\tau > 0$ which describes the time lag of immunity against drinking in the model. By assuming that the recruitment rate is different from the death rate which indicates.

### A. The SPARS Model with Time Delay

The total population, $N = S + P + A + R$, is to be divided into the following four categories and the flow diagram of the model is depicted in Fig. 1.

- $S(t)$ is the number of susceptible drinkers at time $t$.
- $P(t)$ is the number of occasional or periodic drinkers at time $t$.
- $A(t)$ is the number of alcoholics or hazardous drinkers at time $t$.
- $R(t)$ is the number of temporarily recovered drinkers at time $t$.

The model presented in Fig. 1 may be represented by the following system of equations:

$$\frac{dS(t)}{dt} = \Lambda - \alpha S(t)P(t) - \beta S(t)A(t) - \mu S(t) + \eta e^{-(\mu+d_3)\tau}R(t-\tau), \quad (1)$$

$$\frac{dP(t)}{dt} = \alpha S(t)P(t) + \beta S(t)A(t) - (\gamma + \varepsilon + \mu + d_1)P(t), \quad (2)$$

$$\frac{dA(t)}{dt} = \gamma P(t) - \rho A(t) - (\mu + d_2)A(t), \quad (3)$$

$$\frac{dR(t)}{dt} = \rho A(t) + \varepsilon P(t) - (\mu + d_3)R(t) - \eta e^{-(\mu+d_3)\tau}R(t-\tau). \quad (4)$$

The parameters in the system of equations are as follows:

- $\Lambda$ is the recruitment rate of individuals entering the susceptible group, i.e., the demographic process of individuals reaching age 16 in the modeling time period.
- $\alpha$ is the transmission coefficient of infection for the susceptible individuals from the periodic drinkers.
- $\beta$ is the transmission coefficient of infection for the susceptible individuals from the alcoholics.
- $\gamma$ is the rate that periodic drinkers will become alcoholics.
- $\varepsilon$ is the rate at which periodic drinkers will become the recovered individuals.
- $\rho$ is the rate at which hazardous drinkers will become the recovered individuals.
- $\mu$ is the natural death rate of the general population.
- $\eta$ is the rate of losing immunity against drinking in the recovered population with the time lag $\tau$.
- $d_1$ is the death rate due to alcohol consumption among periodic drinkers.
- $d_2$ is the death rate due to alcohol consumption among alcoholics.
- $d_3$ is the death rate due to alcohol consumption among recovered drinkers.

Using the fact that $N(t) = S(t) + P(t) + A(t) + R(t)$ system of equations (1) - (4) is written as

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t) - d_1 P(t) - d_2 A(t) - d_3 R(t) \le \Lambda - \mu N(t) \quad (5)$$

From inequality (5), we can obtain

$$N(t) \le \frac{\Lambda}{\mu} + Ce^{-\mu t}.$$

So, $\lim_{t\to\infty} N(t) \le \frac{\Lambda}{\mu}$. Hence, the feasible region of system is given by the set

$$\Omega = \left\{ (S, P, A, R) \in R_+^4 : S + P + A + R \le \frac{\Lambda}{\mu} \right\},$$

which is positively invariant with respect to the system.

*B. The Basic Reproduction number and Equilibria*

The basic reproduction number, $\Re_0$, for drinking epidemic model is defined as the number of drinkers produced when a single drinker is introduced into the susceptible population. According to the condition of system (1) - (4), we can define $\Re_0$ as follows:

$$\Re_0 = \frac{\Lambda\left[\alpha\left(\rho+\mu+d_2\right)+\beta\gamma\right]}{\mu\left(\gamma+\varepsilon+\mu+d_1\right)\left(\rho+\mu+d_2\right)}. \tag{6}$$

Next, we will find the equilibriums of the *SPARS* delay model. By setting the RHS of the system of equations (1) - (4) to zero, we get two equilibrium states, namely the drinking-free state $E_0\left(\frac{\Lambda}{\mu},0,0,0\right)$ and the endemic state, $E^*\left(S^*,P^*,A^*,R^*\right)$ where

$$S^* = \frac{\left(\gamma+\varepsilon+\mu+d_1\right)\left(\rho+\mu+d_2\right)}{\alpha\left(\rho+\mu+d_2\right)+\beta\gamma} = \frac{\Lambda}{\mu\Re_0},$$

$$P^* = \frac{\left(\rho+\mu+d_2\right)}{\gamma}A^*,$$

$$A^* = \frac{\Lambda-\mu S^*}{\dfrac{\left(\rho+\mu+d_2\right)\left(\gamma+\varepsilon+\mu+d_1\right)}{\gamma} - \dfrac{\left[\gamma\rho+\varepsilon\left(\rho+\mu+d_2\right)\right]\eta e^{-(\mu+d_3)\tau}}{\gamma\left(\mu+d_3+\eta e^{-(\mu+d_3)\tau}\right)}},$$

and

$$R^* = \frac{\gamma\rho+\varepsilon\left(\rho+\mu+d_2\right)}{\gamma\left(\mu+d_3+\eta e^{-(\mu+d_3)\tau}\right)}A^*.$$

If $\Re_0 > 1$, then $\Lambda-\mu S^* > 0$, so $A^* > 0$.

Hence, if $\Re_0 > 1$ the system of equations (1) - (4) has unique endemic equilibrium $E^*\left(S^*,P^*,A^*,R^*\right)$ regardless of the time delay length.

*C. The Optimal Control Problem*

To begin the optimal control procedure, we let the control variables:

$u_1(t)$ be the education campaign level used to control drinking in a community,

$u_2(t)$ be the level of treatment in the form of drinking.

The goal is to minimize the number of both occasional and hazardous drinkers with minimum control. Hence, the optimal control problem can be constructed as follows:

$$OCP: J\left(u_1,u_2\right) = \int_0^T\left[\delta_1 P(t)+\delta_2 A(t)+\frac{1}{2}\left(\xi_1 u_1^2(t)+\xi_2 u_2^2(t)\right)\right]dt, \tag{7}$$

where $\delta_1,\delta_2$ and $\xi_1,\xi_2$ denote weight factors (positive constants) that balance the size of the terms and subject to

$$\frac{dS(t)}{dt} = \Lambda-\alpha S(t)P(t)-\beta S(t)A(t)-\mu S(t) + \eta e^{-(\mu+d_3)\tau}R(t-\tau), \tag{8}$$

$$\frac{dP(t)}{dt} = \alpha S(t)P(t)+\beta S(t)A(t)-\left(\gamma+\varepsilon+\mu+d_1\right)P(t) - u_1(t)P(t), \tag{9}$$

$$\frac{dA(t)}{dt} = \gamma P(t)-\left(\rho+\mu+d_2\right)A(t)-u_2(t)A(t), \tag{10}$$

$$\frac{dR(t)}{dt} = \rho A(t)+\varepsilon P(t)-(\mu+d_3)R(t)-\eta e^{-(\mu+d_3)\tau}R(t-\tau) + u_1(t)P(t)+u_2(t)A(t), \tag{11}$$

with initial conditions

$$S(0)=S_s \quad, P(0)=P_s \quad, A(0)=A_s \quad, R(0)=R_s.$$

Let $P(t)$ and $A(t)$ be state variables with control variables $u_1(t)$ and $u_2(t)$. Then we can rewrite the system of equations (8) - (11) in the following form:

$$\phi' = B\phi+F(\phi) := G(\phi), \tag{12}$$

where

$$\phi = \begin{bmatrix} S(t) \\ P(t) \\ A(t) \\ R(t) \end{bmatrix},$$

$$F(\phi) = \begin{bmatrix} \Lambda-\alpha S(t)P(t)-\beta S(t)A(t)+\eta e^{-(\mu+d_3)\tau}R_\tau(t) \\ \alpha S(t)P(t)+\beta S(t)A(t) \\ 0 \\ -\eta e^{-(\mu+d_3)\tau}R_\tau(t) \end{bmatrix},$$

$$B = \begin{bmatrix} -\mu & 0 & 0 & 0 \\ 0 & -\left(K_1+u_1(t)\right) & 0 & 0 \\ 0 & \gamma & -\left(K_2+u_2(t)\right) & 0 \\ 0 & \rho & \varepsilon & -K_3 \end{bmatrix},$$

$$R_\tau(t) = R(t-\tau),$$

and

$$K_1 = \gamma+\varepsilon+\mu+d_1,$$
$$K_2 = \rho+\mu+d_2,$$
$$K_3 = \mu+d_3.$$

The second term on the RHS of equation (12) satisfies

$$\begin{aligned} &\left|F\left(\phi_1\right)-F\left(\phi_2\right)\right| \\ &= \left|\left(\Lambda-\alpha S_1P_1-\beta S_1A_1+\eta e^{-(\mu+d_3)\tau}R_{1\tau}\right) - \left(\Lambda-\alpha S_2P_2-\beta S_2A_2+\eta e^{-(\mu+d_3)\tau}R_{2\tau}\right)\right| \\ &\quad + \left|\left(\alpha S_1P_1+\beta S_1A_1\right)-\left(\alpha S_2P_2+\beta S_2A_2\right)\right| \\ &\quad + \left|\left(-\eta e^{-(\mu+d_3)\tau}R_{1\tau}\right)-\left(-\eta e^{-(\mu+d_3)\tau}R_{2\tau}\right)\right| \\ &\le 2\left|\alpha S_1P_1-\alpha S_2P_2\right|+2\left|\beta S_1A_1-\beta S_2A_2\right| \\ &\quad + \left|\eta e^{-(\mu+d_3)\tau}R_{1\tau}-\eta e^{-(\mu+d_3)\tau}R_{2\tau}\right| \end{aligned}$$

$$= 2\alpha\left|\frac{(S_1+S_2)}{2}(P_1-P_2)+\frac{(P_1+P_2)}{2}(S_1-S_2)\right|$$
$$+2\beta\left|\frac{(S_1+S_2)}{2}(A_1-A_2)+\frac{(A_1+A_2)}{2}(S_1-S_2)\right|$$
$$+2\eta e^{-(\mu+d_3)\tau}\left|R_{1\tau}-R_{2\tau}\right|$$
$$\le \alpha|S_1+S_2||P_1-P_2|+\alpha|S_1-S_2||P_1+P_2|$$
$$+\beta|S_1+S_2||A_1-A_2|+\beta|S_1-S_2||A_1+A_2|$$
$$+2\eta e^{-(\mu+d_3)\tau}\left|R_{1\tau}-R_{2\tau}\right|$$
$$\le M_1|S_1-S_2|+M_2|P_1-P_2|$$
$$+M_3|A_1-A_2|+M_4|R_{1\tau}-R_{2\tau}|,$$

where $M_1=\alpha|P_1+P_2|+\beta|A_1+A_2|$,

$M_2=\alpha|S_1+S_2|$,

$M_3=\beta|S_1+S_2|$

and $M_4=2\eta e^{-(\mu+d_3)\tau}$.

Hence, we get

$$\left|G(\phi_1)-G(\phi_2)\right| \le L|\phi_1-\phi_2| \tag{13}$$

where

$$L=\max\{M_1,M_2,M_3,M_4,\|B\|\}<\infty.$$

Therefore, the function $G$ is uniformly Lipschitz continuous. From the definition of control $u_1(t), u_2(t)$ and the control on $S(t), P(t), A(t)$ and $R(t)$, we can conclude that the solution of the system of equations (8) - (11) exists according to [14].

**Theorem 1:** Consider the optimal control problem *OCP*. There exists an optimal control $u^*=(u_1^*,u_2^*)\in U$ such that

$$\min_{(u_1,u_2)\in U} J(u_1,u_2)=J(u_1^*,u_2^*) \tag{14}$$

with the state and control satisfying equations (7) - (11).

*Proof.* For the existence of an optimal control, based on [15], the following conditions must be satisfied.

(i) The set of controls and corresponding state variables are nonempty.

(ii) The control set $U$ is convex and closed.

(iii) The RHS of the state system is bounded by a linear function in the state and the control variables.

(iv) The integrand of the objective functional is concave on $U$.

(v) There exist constants $c_1,\ c_2>0$ and $\sigma>1$ such that the integrand, $L(P,A,u_1,u_2)$, of the objective functional satisfies

$$L(P,A,u_1,u_2)\ge c_2+c_1\left(|u_1|^2+|u_2|^2\right)^{\frac{\sigma}{2}}.$$

In order to verify these conditions, we use the result by [16] to give the existence of solutions of the state equation. (7) with bounded coefficients, which gives a condition (i). The control set is closed and convex by definition and thus satisfies condition (ii). Since our state system is bilinear in $u_1$ and $u_2$, the RHS of the equation (7) satisfies condition (iii), using the boundedness of the solutions.

In addition, the integrand of the objective functional is concave. Also, we can easily see that there exist a constant $\sigma>1$ and $c_1,\ c_2>0$ since $\delta_1,\delta_2,\xi_1,\xi_2>0$, such that

$$\delta_1P(t)+\delta_2A(t)+\frac{1}{2}\left(\xi_1u_1^2(t)+\xi_2u_2^2(t)\right)\ge c_2+c_1\left(|u_1|^2+|u_2|^2\right)^{\frac{\sigma}{2}},$$

which completes the proof for existence of the optimal control. ■

In order to find an optimal solution, we consider the optimal control problem (7). First, we should find the Lagragian and Hamiltonian for the problem. The Lagragian of the optimal control problem is given by

$$L(P,A,u_1,u_2)=\delta_1P(t)+\delta_2A(t)+\frac{1}{2}\left(\xi_1u_1^2(t)+\xi_2u_2^2(t)\right). \tag{15}$$

Applying Pontryagin's Maximum Principle, we form the Hamiltonian and derive the optimality system:

$$H(S,P,A,R,u_1,u_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$$
$$=L(S,P,A,R,u_1,u_2)+\lambda_1(t)\frac{dS(t)}{dt}+\lambda_2(t)\frac{dP(t)}{dt} \tag{16}$$
$$+\lambda_3(t)\frac{dA(t)}{dt}+\lambda_4(t)\frac{dR(t)}{dt}$$

where $\lambda_1(t),\lambda_2(t),\lambda_3(t)$ and $\lambda_4(t)$ are the adjoint functions to be determined. The main tool for the study of optimality on the systems (8) - (11). A necessary condition for optimal control problems can be found in [21]. If we consider $x(t)$ and $x_\tau(t)$, then there exists a continuous function $\lambda(t)$ on $[0, T]$ satisfying the following three equations, that is, the state equation

$$x'(t)=H_\lambda(t,x,x_\tau,u,\lambda), \tag{17}$$

the optimal condition

$$H_u(t,x,x_\tau,u,\lambda)=0, \tag{18}$$

and the adjoint equation

$$-\lambda'(t)=H_x(t,x,x_\tau,u,\lambda)+\lambda(t+\tau)H_{x_\tau}(t,x,x_\tau,u,\lambda), \tag{19}$$

where $H_\lambda, H_u, H_x$ and $H_{x_\tau}$ denotes the derivative with respect to $\lambda, u, x$ and $x_\tau$, respectively. Now we apply the necessary conditions to the Hamiltonian $H$ in (16).

**Theorem 2.** If $(u_1^*,u_2^*)$ is an optimal pair with corresponding states $S^*, P^*, A^*$ and $R^*$, then there exist adjoint variables $\lambda_1(t),\lambda_2(t),\lambda_3(t)$ and $\lambda_4(t)$, which satisfy:

$$\lambda_1'(t)=\lambda_1(t)\left(\mu+\alpha P^*+\beta A^*\right)-\lambda_2(t)\left(\alpha P^*+\beta A^*\right), \tag{20}$$

$$\lambda_2'(t)=-\delta_1+\lambda_1(t)(\alpha S^*)$$
$$-\lambda_2(t)\left(\alpha S^*-(\gamma+\varepsilon+\mu+d_1)-u_1^*(t)\right) \tag{21}$$
$$-\lambda_3(t)\gamma-\lambda_4(t)\left(\varepsilon+u_1^*(t)\right),$$

$$\lambda_3'(t)=-\delta_2+\lambda_1(t)(\beta S^*)-\lambda_2(t)(\beta S^*)$$
$$+\lambda_3(t)\left(\rho+\mu+d_2+u_2^*(t)\right) \tag{22}$$
$$-\lambda_4(t)\left(\rho+u_2^*(t)\right),$$

$$\lambda_4'(t)=\lambda_4(t)(\mu+d_3)-\chi_{[0,t_f-\tau]}\lambda_1(t)(t+\tau)$$
$$+\chi_{[0,t_f-\tau]}\lambda_4(t)(t+\tau)\eta e^{-(\mu+d_3)\tau}, \tag{23}$$

with the transversality conditions

$$\lambda_i(t_{final})=0 \quad ,i=1,2,3,4.$$

Furthermore, the optimal control pair $(u_1^*(t),u_2^*(t))$ are given by

$$u_1^*(t)=\max\left\{\min\left\{\frac{\lambda_2(t)-\lambda_4(t)}{\xi_1}P^*(t),\ \psi_1\right\},0\right\}, \tag{24}$$

$$u_2^*(t)=\max\left\{\min\left\{\frac{\lambda_3(t)-\lambda_4(t)}{\xi_2}A^*(t),\ \psi_1\right\},0\right\}. \tag{25}$$

*Proof.* To determine the adjoint equations and transversality conditions we use the Hamiltonian (16). By using the adjoint equation (19) and differentiating the Hamiltonian with respect to $x(t)$, and $x_\tau(t)$, with setting $x(t)=x^*(t)$, and $x_\tau(t)=x_\tau^*(t)$, we obtain

$$-\lambda_1'(t)=H_{S^*}(t),$$
$$-\lambda_2'(t)=H_{P^*}(t),$$
$$-\lambda_3'(t)=H_{A^*}(t),$$
$$-\lambda_4'(t)=H_{R^*}(t)+\lambda_4(t+\tau)H_{R_\tau^*}(t).$$

Using the optimal conditions and the property of the control space $U$ for the control variables $u_1$ and $u_2$, we get

$$\left.\frac{\partial H}{\partial u_1}\right|_{u_1(t)=u_1^*(t)}=\xi_1u_1^*(t)-(\lambda_2(t)-\lambda_4(t))P^*(t), \tag{26}$$

$$\left.\frac{\partial H}{\partial u_2}\right|_{u_2(t)=u_2^*(t)}=\xi_2u_2^*(t)-(\lambda_3(t)-\lambda_4(t))A^*(t). \tag{27}$$

By using the property of the control space, we obtain

$$u_1^*(t)=\begin{cases}0 & ;\dfrac{(\lambda_2(t)-\lambda_4(t))P^*(t)}{\xi_1}\le 0\\[2ex] \dfrac{(\lambda_2(t)-\lambda_4(t))P^*(t)}{\xi_1} & ;0<\dfrac{(\lambda_2(t)-\lambda_4(t))P^*(t)}{\xi_1}<\psi_1\\[2ex] \psi_1 & ;\dfrac{(\lambda_2(t)-\lambda_4(t))P^*(t)}{\xi_1}\ge\psi_1\end{cases} \tag{28}$$

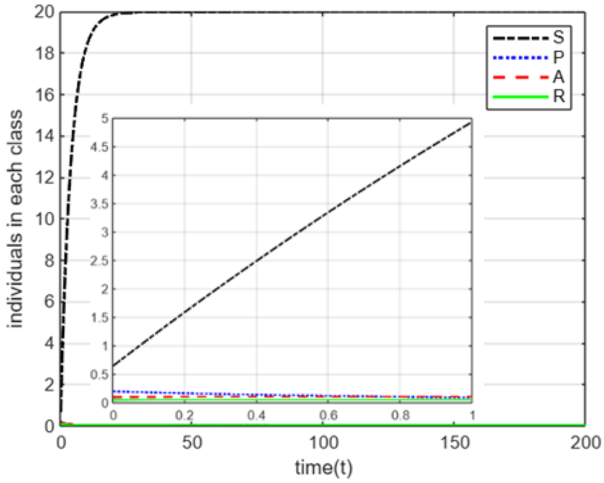
and

$$u_2^*(t)=\begin{cases}0 & ;\dfrac{(\lambda_3(t)-\lambda_4(t))A^*(t)}{\xi_2}\le 0\\[2ex] \dfrac{(\lambda_3(t)-\lambda_4(t))A^*(t)}{\xi_2} & ;0<\dfrac{(\lambda_3(t)-\lambda_4(t))A^*(t)}{\xi_2}<\psi_2\\[2ex] \psi_2 & ;\dfrac{(\lambda_3(t)-\lambda_4(t))P^*(t)}{\xi_2}\ge\psi_2.\end{cases} \tag{29}$$

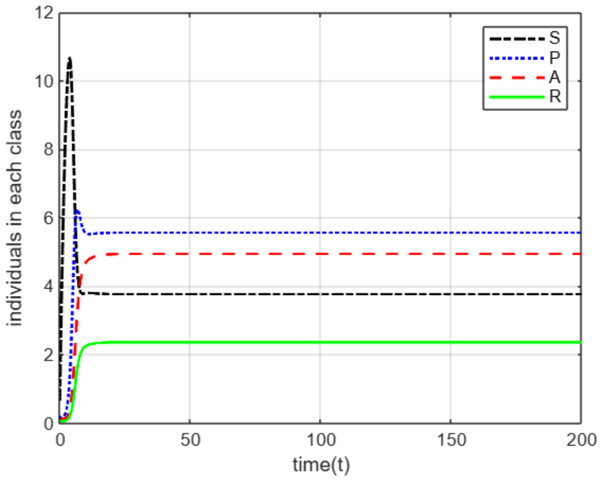
These can be written in compact notation (24) and (25), respectively. ■

## III. Numerical Simulations

In this section, some numerical simulations of system were carried out using MATLAB for supporting the analytic results obtained above. The parameter values used in the numerical simulations are: $\Lambda=5, \gamma=0.4, \mu=0.25$ which are taken from [11] and other parameters are estimated as follows: $\varepsilon=0.2, \rho=0.1, \eta=0.2, d_1=0.05, d_2=0.1$ and $d_3=0.03$.




(a) $\Re_0=0.9630$ and $\tau=0$




(b) $\Re_0=5.3086$ and $\tau=0$

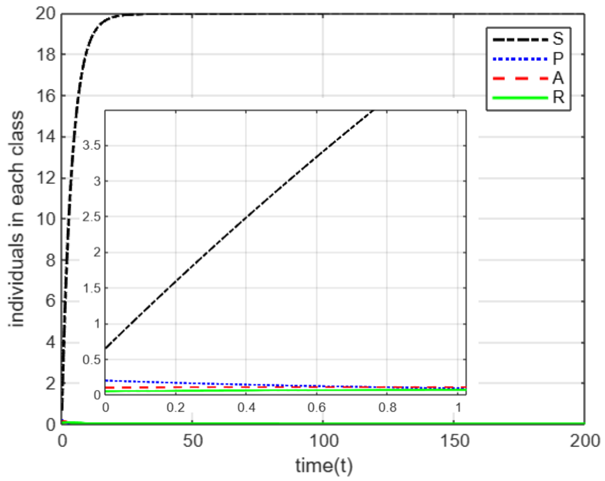
Fig. 2. The evolutions of the four classes of populations when $\tau=0$.

First, we choose $\alpha=0.03$ and $\beta=0.015$ numerical simulation gives $\Re_0=0.9630$ and $\tau=0$, then dynamic of each individual of the drinking-free equilibrium $E_0$ is shown in Fig.2(a).

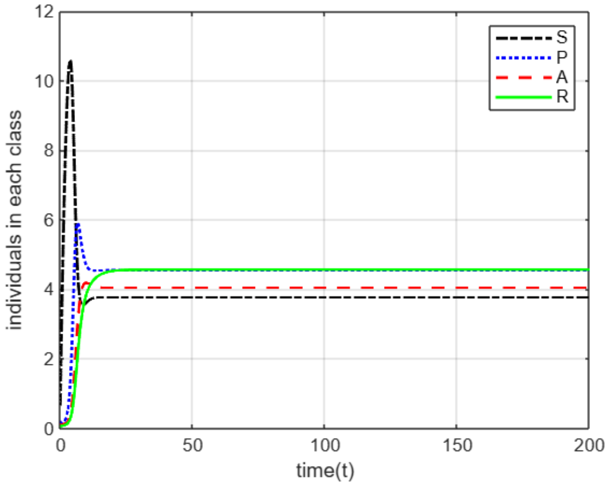
Second, we choose $\alpha=0.15$ and $\beta=0.1$ numerical simulation gives $\Re_0=5.3086$ and $\tau=0$, then dynamic of each individual of the endemic equilibrium $E^*$ is shown in Fig. 2(b).

Third, we choose $\alpha=0.03$ and $\beta=0.015$ numerical simulation gives $\Re_0=0.9630, \tau=14$, then dynamic of each individual of the drinking-free equilibrium $E_0$ is shown in Fig.3(a).

At last, we choose $\alpha=0.15$ and $\beta=0.1$ numerical simulation gives $\Re_0=5.3086, \tau=14$, then dynamic of each individual of the endemic equilibrium $E^*$ is shown in Fig. 3(b).




(a) $\Re_0 = 0.9630$ and $\tau = 14$



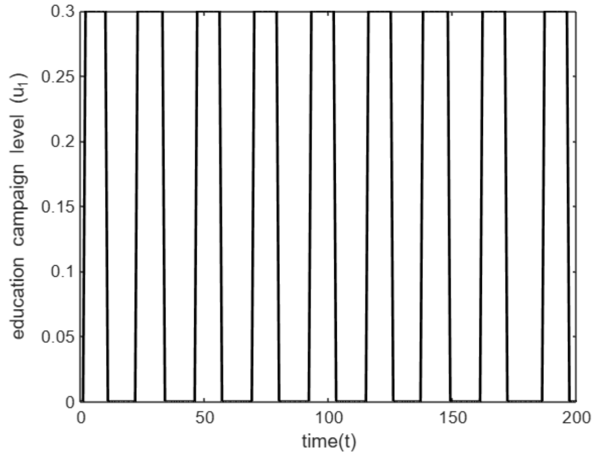

(b) $\Re_0 = 5.3086$ and $\tau = 14$

Fig.3. The evolutions of the four classes of populations when $\tau = 14$ .

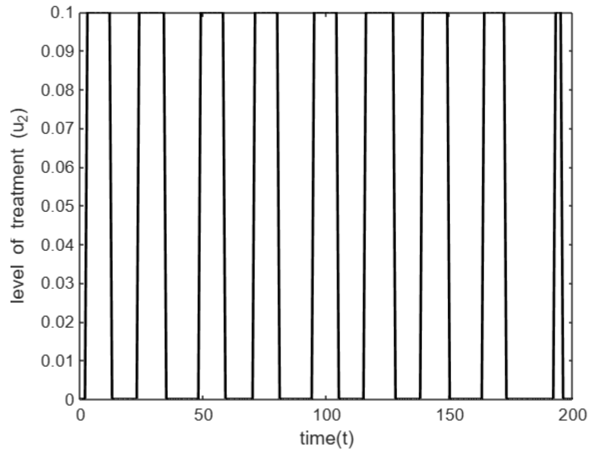
According to Fig.2(a) and Fig.3(a), the disease-free equilibrium points are shown without and with delay, respectively. The two graphs illustrate the population dynamics in a drinking epidemic model with four classes: susceptible drinkers *(S)*, periodic drinkers *(P)*, alcoholics *(A)*, and recovered drinkers *(R)*. In both graphs, we can see that the number of susceptible drinkers rapidly increases initially and then stabilizes. The smaller graph nested within the larger one provides a zoomed-in view of the early stages of the epidemic. These zoomed-in views better illustrate the initial rapid increase in susceptible drinkers and the periodic drinkers, alcoholics, and recovered drinkers are declining nearly zero during the early part of the simulation.

According to Fig.2(b) and Fig.3(b), the endemic equilibrium points are shown without and with delay, respectively. The two graphs illustrate the population dynamics in a drinking epidemic model with four classes: susceptible drinkers *(S)*, periodic drinkers *(P)*, alcoholics *(A)*, and recovered drinkers *(R)*. The key difference is the inclusion of a 14-week time delay in Fig.3(b), which represents the period required for temporarily recovered individuals to relapse and resume alcohol consumption. In Fig.2(b), without a time delay, the population in each class quickly stabilizes, with smooth transitions to equilibrium. In contrast, Fig.3(b), with the time delay, shows slower stabilization and more pronounced initial oscillations, as the delayed relapse disrupts the immediate balance between classes. This delay accounts for the realistic lag in recovery and relapse processes, demonstrating its significant impact on the timing and dynamics of population stabilization.

The optimal control problem for the drinking epidemic model with time delay is solved numerically. The optimality system can be solved by using an iterative method, the Runge-Kutta fourth-order scheme. For the *SPARS* model presented in this work, the state system is given by Eq. (8) - (11). The adjoint system is given by equations (20)-(24) with the characterization of the optimal control by equations (28) and (29). The numerical results for the optimal control problem with $\tau = 14$ are presented in Fig.4-5.




(a) The education campaign level $u_1(t)$



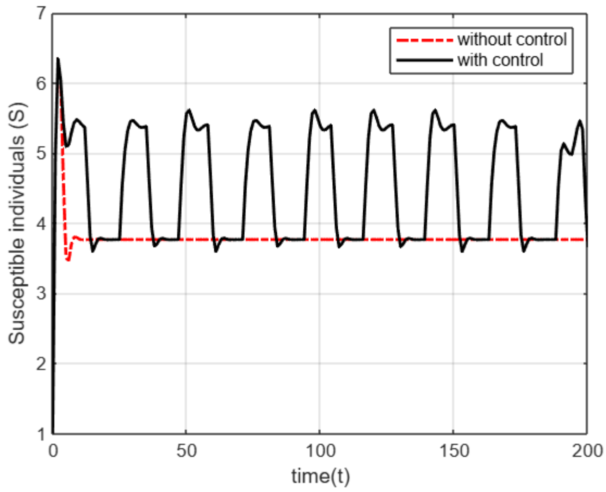

(b) The treatment level $u_2(t)$

Fig.4. Dynamics of the optimal control: the education campaign level $u_1(t)$ and the level of treatment $u_2(t)$ .

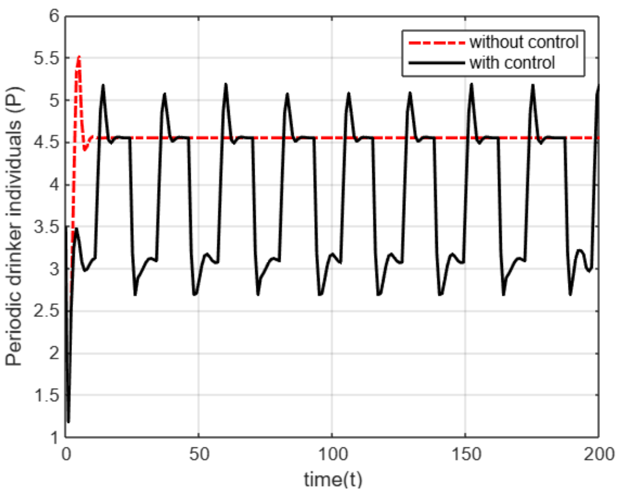
In Fig.4(a) the control variable $u_1^*(t)$ is shown, representing the level of an educational campaign aimed at reducing drinking in a community, plotted over time (from 0 to 200 weeks). The values of $u_1^*(t)$ oscillate between 0 and a maximum level of 0.3, indicating that the campaign is applied periodically rather than continuously. Such a periodic control pattern could be designed to prevent habituation to the campaign or to align with optimal intervention points in the population's behavior cycle. This type of control is common in optimal control strategies, where the goal is to balance effectiveness with cost-efficiency by only using the campaign when it has the most impact.

In Fig.4(b), the graph illustrates the optimal control $u_2^*(t)$ representing the level of treatment administered to individuals struggling with alcoholism in a community. The maximum value of this treatment level is 0.1, and it is applied periodically over a time span of 200 weeks, in a pattern similar to the previously shown education campaign $u_1^*(t)$. Given that there is a time delay, or a time lag of immunity against drinking, this suggests that individuals have a period after treatment during which they are less likely to relapse into drinking. This immunity period acts as a buffer time, where the treatment's influence remains effective for a limited time before the risk of relapse increases.

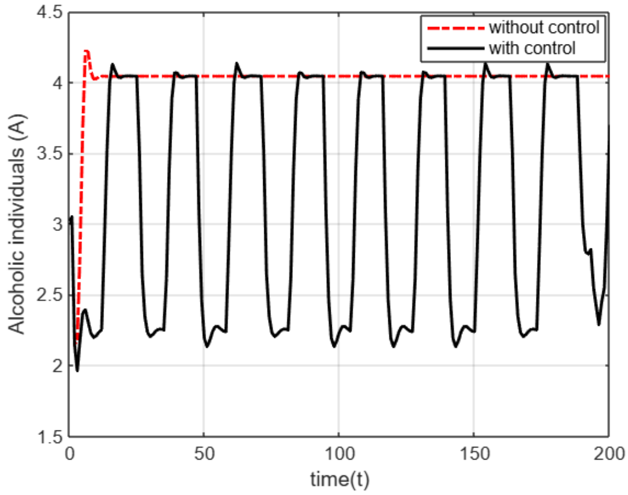
In Fig.5(a)-(d) the effect of time delay for controlling drinking epidemics with $\tau = 14$ is illustrated. The fluctuation of susceptible drinkers and recovered drinkers with control is greater than that without control; conversely, for periodic drinkers and alcoholics, that is, the fluctuation of populations with control is less than or greater than in some periods, which is affected by the fluctuation of the education campaign level with time delay, illustrated in Fig. 4.



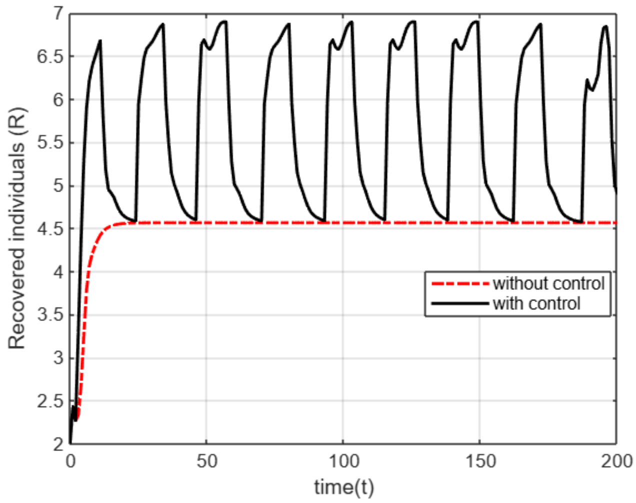

(a) Susceptible individuals




(b) Periodic individuals




(c) Alcoholic individuals




(d) Recovered individuals

Fig.5. The comparison of the effect of drinking control on individuals with and without control when the maximum education campaign level $u_1(t) = 0.3$ and the maximum level of treatment $u_1(t) = 0.1$ are applied, considering drinking with a time delay $\tau = 14$ .

In Fig.5(a), the graph suggests that implementing optimal control measures for education campaigns and treatment can significantly impact the number of susceptible drinkers. While the cyclical pattern of susceptibility might persist even with control, the control measures help to dampen the fluctuations and ultimately lead to a lower baseline level of susceptibility.

In Fig.5(b), the graph demonstrates the effectiveness of optimal control measures in reducing the number of periodic drinkers over time. While the exact mechanisms are not explicitly shown, it is likely that the control strategies have successfully mitigated the impact of time delays in the system.

In Fig.5(c), the graph demonstrates the effectiveness of optimal control measures in reducing the number of alcoholic individuals over time. It is likely that the control strategies have successfully mitigated the impact of time delays in the system.

In Fig.5(d), the graph demonstrates the positive impact of optimal control measures on the number of recovered individuals. While the role of time delay in this specific case appears to be limited, it's essential to consider its potential influence in other scenarios.

The inclusion of control measures significantly reduces the fluctuations in population dynamics and promotes stabilization across all classes. This highlights the effectiveness of education campaigns and treatment interventions in mitigating the spread of drinking behaviors and improving recovery outcomes. The time delay, reflecting realistic relapse behavior, underscores the importance of sustained control strategies to maintain long-term stability and reduce the prevalence of alcohol-related issues in the community. Integrating educational campaigns with robust treatment interventions offers a comprehensive strategy to address alcohol use disorders. While education raises awareness, treatment provides the necessary support and tools for individuals to achieve and maintain sobriety. This combined approach addresses both prevention and recovery, leading to more effective outcomes in reducing alcohol-related harm within the community.

In summary, while educational campaigns play a role in informing the public, their impact on reducing alcohol consumption is limited without accompanying treatment interventions. Structured treatment programs, especially those considering relapse periods, are essential for effective management and reduction of alcohol use disorders.

## IV. CONCLUSION

In this paper, we considered an optimal control problem for the drinking epidemic model with time delay. The main objective developed here is to apply optimal control strategies in order to minimize both the number of drinkers and the resources required for control. The two control functions, $u_1(t)$ and $u_2(t)$, represent the education campaign and the level of treatment, respectively, and are used to control drinking behavior. We discuss the existence of the optimal control and then derive the necessary conditions for it by constructing the Hamiltonian and applying Pontryagin's Maximum Principle to achieve our goal. Finally, to conclude our work, we perform numerical simulations to verify the analytical results derived.

The oscillatory behavior observed in both scenarios, with and without control, suggests the presence of a time delay in the system. This delay could be attributed to factors such as the progression of alcohol dependence, the delayed effectiveness of treatment, or the time it takes for educational campaigns to influence behavior.

The optimal control measures implemented in the controlled scenario effectively reduce the amplitude of these oscillations. This indicates that the control strategies—namely, education campaigns and treatment programs—successfully mitigate the impact of the time delay and stabilize the population of alcoholic individuals at a lower level.

The results presented in this work demonstrate how optimal control theory can be applied to real-world situations. Future research will continue to explore the corresponding parameter thresholds and their implications for practical applications.

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