Blaschke-Minkowski Homomorphisms in Complex Vector Spaces

Na Fu and Bin Chen

Abstract—**This article extends the concept of Blaschke-Minkowski homomorphisms operator in n-dimensional Euclidean space to complex vector space, and establishes some important geometric inequalities for this purpose. Moreover, the Shephard-type problem of complex Blaschke-Minkowski homomorphisms is also studied.**

Index Terms—**Blaschke-Minkowski homomorphism; complex vector space; Brunn-Minkowski inequality; Shephard peoblem**

I. Introduction

LET K^n be the set of convex bodies (compact, convex subset with non-empty interiors) in n -dimensional Euclidean space R^n . Denoted by $B \in R^n$ the unit ball, and S^{n-1} the unit sphere. We write $V(\cdot)$ for the n -dimensional volume.

For $K \in K^n$, the support function, $h_K = h(K,\cdot): R^n \to R$, is given by ([19])

$$h(K,u) = \max\{u \cdot Y : Y \in K\},\ \ u \in S^{n-1}.$$

Here "." denotes the standard scalar product.

For $K \in K^n$, the projection body, denoted by ΠK, is an origin-symmetric convex body whose support function is given by ([19])

$$h(\Pi K, v) = \frac{n}{2} V(K, \cdots, K, [-v, v]),$$

for $v \in S^{n-1}$. Here $V(K,\cdots,K,[-v,v])$ is the mixed volume of $n-1$ copies of K and one copy of the segment $[-v,v]$. The mixed projection body of convex bodies $K_1,\cdots,K_{n-1}$ is defined by (see[4]): for $u \in S^{n-1}$

$$h(\Pi(K_1,\cdots,K_{n-1}), v) = \frac{n}{2} V(K_1,\cdots,K_{n-1},[-v,v]) \cdot$$

The projection body was introduced by Minkowski at the beginning of the last century. Through the works of Petty ([16]), Schneider ([18]), Bolker ([3]), Lutwak ([13]) and Zhang ([25]), projection bodies have attracted widespread attention.

Manuscript received September 25, 2024; revised March 19, 2025.

This work was supported by the Youth Science and Technology Foundation of Gansu Province under Grant No. 25JRRA914 and the Doctor Starting Foundation of Lanzhou University of Technology under Grant No. 062408.

Na Fu is is a teacher of Wuwei Sixth Middle School, Wuwei, 733000 China. (e-mail: funa930323@163.com).

Bin Chen is an associate professor of School of Science, Lanzhou University of Technology, Lanzhou, 730050, China. (corresponding author to provide e-mail: chenb121223@163.com).

Based on the properties of projection operators, Schuster [20] established a special valuations:

Definition 1.A. The mapping $\Phi: K^n \to K^n$ is said to be a Blaschke-Minkowski homomorphism (abbreviated as BMH) if it satisfies: for $K, L \in K^n$,

(1) Φ is continuous.

(2) $\Phi(K \mp L) = \Phi(K) + \Phi(L)$, where $\mp$ and $+$ denote the Blaschke sum and Minkowski sum, respectively.

(3) For every $\vartheta \in SO(n)$, $\Phi(\vartheta K) = \vartheta\Phi(K)$, where $SO(n)$ denotes the group of rotations in n -dimensions.

The mapping Φ is an even BMH if and only if there is a centrally symmetric figure of revolution $F \in R^n$, which is not a singleton, so that

$$h(\Phi K, \cdot) = S_{n-1}(K,\cdot) * h(F,\cdot), \tag{1}$$

where set F is unique up to translations, and $S_{n-1}(K,\cdot)$ denotes the surface area measure. Moreover, the BMH has paid considerable attentions, see e.g., [5-7], [20-22], [26-29].

Denoted by $K^n(C^n)$ and $K^n_{os}(C^n)$ the set of convex bodies and the set of origin-symmetric convex bodies in complex vector space C^n. Let $\|\cdot\|_L$ the norm corresponding to $L \in K^n(C^n)$:

$$L = \{\zeta \in C^n : \|\zeta\|_L \le 1\}.$$

We identify C^n with R^n using the standard mapping

$$\lambda = (\lambda_1,\cdots,\lambda_n) = (\lambda_{11} + i\lambda_{12},\cdots,\lambda_{n1} + i\lambda_{n2}) \mapsto (\lambda_{11},\lambda_{12},\cdots,\lambda_{n1},\lambda_{n2}).$$

The unit ball B in C^n is given by

$$B = \{\lambda \in C^n : \sum_{k=1}^{n} (\lambda_{i1}^2 + \lambda_{i2}^2) \le 1\}.$$

Let ω_{2n} and S^{2n-1} denote the volume and surface of B.

Until recently, Abardia-Bernig [2] extended the concept of projection body in R^n to C^n . Let $K_1,\cdots,K_{2n-1} \in K^n(C^n)$ and $C \in C^1$ be a convex subset. The complex projection body $\Pi^C(K_1,\cdots,K_{2n-1})$ is a convex body is defined through the support function as

$$h(\Pi^C(K_1,\cdots,K_{2n-1}),\omega) = \frac{1}{2n}\int_{S^{2n-1}} h(C\cdot\omega,\xi)dS(K_1,\cdots,K_{2n-1},\xi),$$

where $C\cdot\omega := \{c\omega : c \in C\}, \omega \in C^n$, and support function $h_K : C^n \to R$ (see Section 2 for details). Later the case of complex convex bodies began to attract attention, see e.g., [1], [2], [9-11], [15], [17], [24], [30] .

The main goal of this paper is to extend the BMH operator to complex vector space. Motivated by (1), we present the complex BMH, $\Phi^C : K^n(C^n) \to K^n(C^n)$, as the complex convex body:

Definition 1.1. *Let* $K_1,\cdots,K_{2n-1} \in K^n(C^n)$. *The complex mixed BMH, denoted by* $\Phi^C(K_1,\cdots,K_{2n-1}) \in K^n(C^n)$, *is defined by the support function as*

$$h(\Phi^C(K_1,\cdots,K_{2n-1}),\omega) = S(K_1,\cdots,K_{2n-1},\cdot) * h(C\cdot\omega,\cdot), \omega \in C^n. \tag{2}$$

For Definition 1.1, we have the following result.

Proposition 1.2. *For* $K, L \in K^n(C^n)$. *The operator* Φ^C *satisfies:*

(1) Φ^C *is continuous.*

(2) $\Phi^C(K \mp_c L) = \Phi^C(K) + \Phi^C(L)$.

(3) For every $\vartheta \in SO(2n)$, $\Phi^C(\vartheta K) = \vartheta\Phi^C(K)$.

Based on Proposition 1.2, we call $\Phi^C K$ the complex BMH of K. Let $K_1 = \cdots = K_{2n-i-1} = K$ and $K_{2n-i} = \cdots = K_{2n-1} = L$ in Definition 1.1. Then $\Phi^C(K_1,\cdots,K_{2n-1})$ can be written as $\Phi_i^C(K,L)$. If $L = B$, we write $\Phi_i^C(K,B) = \Phi_i^C K$ and $\Phi_0^C K = \Phi^C K$.

We now establish several important inequalities for complex BMH as follows:

Theorem 1.3. *(Minkowski type inequality) Suppose that* $K, L \in K^n(C^n)$. *Then*

$$V(\Phi_1^C(K,L))^{2n-1} \ge V(\Phi^C K)^{2n-2} V(\Phi^C L),$$

with equality iff K *and* L *are homothetic.*

Theorem 1.4. *(Aleksandrov-Fenchel type inequality) Suppose that* $K_1,\cdots,K_{2n-1} \in K^n(C^n)$, *while* $1 \le r \le 2n-2$, *then*

$$V(\Phi^C(K_1,\cdots,K_{2n-1}))^r \ge \prod_{j=1}^{r} V(\Phi^C(\underbrace{K_j,\cdots,K_j}_{r},K_{r+1},\cdots,K_{2n-1})).$$

Theorem 1.5. *(Brunn-Minkowski type inequality) Suppose that* $K, L \in K^n(C^n)$. *Then*

$$V(\Phi^C(K \mp_c L))^{\frac{1}{2n}} \ge V(\Phi^C K)^{\frac{1}{2n}} + V(\Phi L)^{\frac{1}{2n}},$$

with equality iff K *is a translation of* L.

Another main goal of this paper is to research the Shephard problem for complex BMH.

Problem 1.6. *For* $K \in K^n(C^n)$ *and a translate of* L *is contained in* $\Phi^C K^n$, *is there the implication*

$$\Phi^C K \subseteq \Phi^C L \;\Rightarrow\; V(K) \subseteq V(L)?$$

Obviously, Problem 1.6 is a more general Shephard-type problem than classical Shephard problem (see[23]).

Now, we give an affirmative answer of Problem 1.6.

Theorem 1.7. *For* $K \in K^n(C^n)$, $L \in Z^n$. *If* $\Phi^{\overline{C}} K \subseteq \Phi^{\overline{C}} L$, *then*

$$V(K) \le V(L),$$

with equality iff $\Phi^{\overline{C}} K = \Phi^{\overline{C}} L$. *Here* $\overline{C}$ *is the complex conjugate of* C, *and* Z^n *is the set of complex BMH.*

The negative form of Problem 1.6 is obtained:

Theorem 1.8. Suppose that $L \in K^n(C^n)$. *If* Φ^C *is an even operator and* L *is not an origin-symmetric, then there is a* $K \in K_{os}^n(C^n)$, *such that* $\Phi^C K \subset \Phi^C L$, *but*

$$V(K) > V(L).$$

The organization of this paper is as follows. In Section II, we introduce some notations and basic facts in Section II. In Section III, we give the proof of Proposition 1.2 and introduce some characterizations of complex mixed BMH. In Section IV, we shall obtain some important geometric inequalities. Section V, we further study the Shephard type problem.

II. Preliminaries

For a complex number $c \in C$. Denote by "$\cdot$" the standard Hermitian inner product on C^n being conjugate linear. We write ι for canonical isomorphism between C^n and R^{2n}, that is,

$$\iota(z) = (\Re[z_1],...,\Re[z_n],\Im[z_1],...,\Im[z_n]), z \in C^n.$$

Here $\Re$ and $\Im$ denote the real and imaginary part. Note that

$$\Re[c\cdot z] = \iota c \cdot \iota z, \tag{3}$$

for all $c, z \in C^n$. Here the inner product on the right hand side is the standard inner product on R^{2n}.

For $K \in K^n(C^n)$, the support function, $h_K : C^n \to R$, is uniquely determined by ([8])

$$h_K(z) = \max\{\Re[z\cdot Y] : Y \in K\}, z \in C^n.$$

It is an easy consequence of (3) that

$$h_K = h_{\iota K} \circ \iota,$$

where $h_{\iota K}$ is the usual real support function.

For all $\alpha \ge 0$ and $\phi \in GL(n,C)$

$$h_{\alpha K} = \alpha h_K \;\; and \;\; h_{\phi K} = h_K \circ \phi^*,$$

where ϕ^* denotes the conjugate transpose of ϕ if $\phi \in C^{m\times n}$.

For two real numbers $\alpha, \lambda \ge 0$, the Minkowski sum $\alpha K + \lambda L$ is given by

$$\alpha K + \lambda L = \{\alpha x + \lambda y : x \in K \;\; and \;\; y \in L\},$$

or equivalently,

$$h_{\alpha K + \lambda L} = \alpha h_K + \lambda h_L. \tag{4}$$

For $K \in K^n(C^n)$, the surface area measure S_K is the Borel measure on S^{2n-1} defined, for Borel set $\omega \in S^{2n-1}$, by

$$S_K(\omega) = \mathrm{H}^{2n-1}(\iota\{x \in K : \exists u \in \omega \;\; with \;\; \Re[x\cdot u] = h_K(u)\}),$$

where H^{2n-1} denotes $(2n-1)$-dimensional Hausdorff measure on R^{2n}. By the definition of S_K, it is not difficult to show that surface area measures are translation invariant. Up to translations, a body $K \in K^n(C^n)$ is uniquely determined by its surface area measure, that is,

$$S_K = S_L \;\Leftrightarrow\; K = L + x, \;\; x \in C^n.$$

By the Minkowski's existence theorem (see[14]), we can give the concept of the complex Blaschke combination as follows: For $K, L \in K^n(C^n)$ and $\alpha, \lambda \ge 0$ (not both zero), the complex Blaschke combination is given by

$$S(\alpha \otimes K \mp_c \lambda \otimes L,\cdot) = \alpha S(K,\cdot) + \lambda S(L,\cdot). \tag{5}$$

Let $\alpha = \lambda = 1/2$ in (5), this allows us to define the Blaschke body ∇K as the unique origin-symmetric convex body with

$$\nabla K = \frac{1}{2} \otimes S(K,\cdot) \mp_c \frac{1}{2} \otimes S(-K,\cdot). \qquad (6)$$

For $K_1, \cdots, K_{2n} \in K^n(C^n)$, the mixed volume has the following integral expression (see [2], [15], [24]):

$$V(K_1,...,K_{2n}) = \frac{1}{2n} \int_{S^{2n-1}} h_{K_{2n}} dS(K_1,...,K_{2n-1},\cdot). \qquad (7)$$

Here $S(K_1, \cdots, K_{2n-1}, \cdot)$ denotes the mixed surface area measure of $K_1, \cdots, K_{2n-1}$. We take $K_1 = \cdots = K_{2n-1} = K$, then $S(K_1,...,K_{2n-1},\cdot)$ is written as $S_{2n-1}(K,\cdot)$. If $K_1 = \cdots = K_{2n-i} = K$, $K_{2n-i+1} = \cdots = K_{2n} = L$, then we write

$$V(K_1, \cdots, K_{2n}) = V(K, 2n-i; L, i) = V_i(K, L). \qquad (8)$$

Obviously,

$$V_i(K, L) = V(K). \qquad (9)$$

For $K_1, \cdots, K_{2n} \in K^n(C^n)$ and $1 \le r \le 2n$, due to the work of Lutwak [12], it follows that

$$V(K_1, \cdots, K_{2n})^r \ge \prod_{j=1}^{r} V(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{2n}). \qquad (10)$$

Let $K_1 = \cdots = K_{2n-i-1} = K$, $K_{2n-i} = \cdots = K_{2n-1} = B$ and $K_{2n} = L$ in (8). Then call $W_i(K, L) = V(K, 2n-i-1; B, i; L)$ is mixed quermassintegrals, and

$$W_i(K, L) = \int_{S^{2n-1}} h(L,u) dS_i(K,u), \qquad (11)$$

for $0 \le i \le 2n-1$ and $u \in S^{2n-1}$. Obviously,

$$W_i(K, K) = W_i(K) \ \textit{and} \ W_0(K) = V(K), \qquad (12)$$

A special case of (10) is the Minkowski type inequality: Let $K_1, \cdots, K_{2n} \in K^n(C^n)$. If $0 \le i \le 2n-1$, then

$$W_i(K, L)^{2n-i} \ge W_i(K)^{2n-i-1} W_i(L), \qquad (13)$$

with equality iff K and L are homothetic.

Using (11), we may immediately get the following result: Suppose that $K, L \in K^n(C^n)$, and any $Q \in K^n(C^n)$

$$W_i(Q, K) = W_i(Q, L), \qquad (14)$$

then K is a translation of L.

III. The complex Blaschke-Minkowski Homomorphism

In this section, we give a proof of Proposition 1.2. As easy identify for complex BMH, which involves mixed volumes, will facilitate a number of proofs given later.

Lemma 3.1. Let $K_1, \cdots, K_{2n-1}, L_1, \cdots, L_{2n-1} \in K^n(C^n)$. *Then*

$$V(K_1,...,K_{2n-1}, \Phi^C(L_1,...,L_{2n-1})) = V(L_1,...,L_{2n-1}, \Phi^{\overline{C}}(K_1,...,K_{2n-1})).$$

Proof. Let $M = (K_1, \cdots, K_{2n-1})$, $N = (L_1, \cdots, L_{2n-1})$ and C is a convex subset. By the definition of Φ^C and (7), we get

$$V(M, \Phi^C N) = \frac{1}{2n} \left\langle h(\Phi^C N, \xi), S(M, \xi) \right\rangle$$
$$= \frac{1}{2n} \left\langle S(N, \omega) * h(C \cdot \xi, \omega), S(M, \xi) \right\rangle$$
$$= \frac{1}{2n} \left\langle S(M, \xi) * h(C \cdot \xi, \omega), S(N, \omega) \right\rangle$$
$$= \frac{1}{2n} \left\langle S(M, \xi) * h(\overline{C} \cdot \xi, \omega), S(N, \omega) \right\rangle.$$

The statement of the proposition now follows from Fubini's theorem and the relation

$$h(C \cdot \xi, \omega) = h(\overline{C} \cdot \omega, \xi),$$

for $\xi, \omega \in C^n$. The proof is completed.

If $K_1 = \cdots = K_{2n-i-1} = K$, while $K_{2n-i} = \cdots = K_{2n-1} = B$, then Lemma 3.1 reduces to following result.

Lemma 3.2. Let $K, L_1, \cdots, L_{2n-1} \in K^n(C^n)$. *Then*

$$W_i(K, \Phi^C(L_1, \cdots, L_{2n-1})) = V(L_1, \cdots, L_{2n-1}, \Phi_i^{\overline{C}} K).$$

If $L_1 = \cdots = L_{2n-j-1} = L$ and $L_{2n-j} = \cdots = L_{2n-1} = B$, then Lemma 3.2 states:

Lemma 3.3. *Let* $K, L \in K^n(C^n)$. *If* $0 \le i \le 2n-1$, *while* $0 \le j \le 2n-2$, *then*

$$W_i(K, \Phi_j^C L) = W_j(L, \Phi_i^{\overline{C}} K). \qquad (15)$$

In the next Lemma, we will further summarize the special case of identity (2). These make use of the fact that the image of a ball under a complex BMH is again a ball. Note that $dS_{2n-1}(B, \xi) = d\xi$, where $d\xi$ is the ordinary spherical Lebesgue measure. Thus, by (2)

$$h(\Phi^C B, \omega) = S_{2n-1}(B, \xi) * h(C \cdot \omega, \xi)$$
$$= \int_{S_{2n-1}} h(C \cdot \omega, \xi) d\xi = r_C.$$

Let r_C denote the radius of ball $\Phi^C B$, equivalently, $\Phi^C B = r_C B$. Note that $h(C \cdot \xi, \omega) = h(\overline{C} \cdot \omega, \xi)$ and the surface area $S(B,\cdot)$ is constant in S^{2n-1}. From above argument, we obtain $\Phi^C B = \Phi^{\overline{C}} B = r_C B$. Here r_C is a constant which depends only on C.

Take $K_1 = \cdots = K_{2n-1} = B$ in Lemma 3.1 and use $\Phi^C B = \Phi^{\overline{C}} B = r_C B$ to get following result:

Lemma 3.4. Let $L_1, \cdots, L_{2n-1} \in K^n(C^n)$. *Then*

$$W_{2n-1}(\Phi^C(L_1, \cdots, L_{2n-1})) = r_C V(L_1, \cdots, L_{2n-1}, B). \qquad (16)$$

In Lemma 3.4, if $L_1 = \cdots = L_{2n-2} = K$ and $L_{2n-1} = L$, then (16) becomes

$$W_{2n-1}(\Phi_1^C(K, L)) = r_C W_1(K, L). \qquad (17)$$

If $L_1 = \cdots = L_{2n-i-1} = L$ and $L_{2n-i} = \cdots = L_{2n-1} = B$, then (16) becomes

$$W_{2n-1}(\Phi_i^C L) = r_C W_{i+1}(L). \qquad (18)$$

Proof of Proposition 1.2.

(1) Support that a map $\Phi^C : K_o^n(C^n) \to K_o^n(C^n)$ satisfies $h(\Phi^C K, \omega) = S(K,\cdot) * h(C \cdot \omega, \cdot)$ is a nonnegative measure. The continuity of Φ^C follows from the fact that the support function $h(K,\cdot)$ is continuous.

(2) From (4), (5) and (2), we obtain

$$h(\Phi^C K + \Phi^C L, \omega) = h(\Phi^C K, \omega) + h(\Phi^C L, \omega)$$
$$= S_{2n-1}(K,\cdot) * h(C \cdot \omega, \cdot) + S_{2n-1}(L,\cdot) * h(C \cdot \omega, \cdot)$$
$$= (S_{2n-1}(K,\cdot) + S_{2n-1}(L,\cdot)) * h(C \cdot \omega, \cdot)$$
$$= S_{2n-1}(K \mp_c L, \cdot) * h(C \cdot \omega, \cdot)$$
$$= h(\Phi^C(K \mp_c L), \cdot).$$

i.e., $h(\Phi^C K + \Phi^C L, \cdot) = h(\Phi^C (K \mp_c L), \cdot)$.

(3) For $\vartheta \in SO(2n)$ and $\vartheta^{-1} \in SO(2n)$

$$\begin{aligned} h(\Phi^C \vartheta K, \omega) &= S_{2n-1}(\vartheta K, \cdot) * h(C\cdot\omega, \cdot) \\ &= S_{2n-1}(K, \vartheta^{-1}) * h(C\cdot\omega, \cdot) \\ &= h(\Phi^C K, \vartheta^{-1}\omega) = h(\vartheta \Phi^C K, \omega). \end{aligned}$$

IV. Geometric Inequalities

In this part, we give proofs of Theorems 1.3-1.5. Firstly, we obtain a general form of Theorem 1.3.

Theorem 4.1. *Let* $K, L \in K^n(C^n)$. *If* $0 \le i \le 2n-1$, *then*

$$W_i(\Phi_1^C(K,L))^{2n-1} \ge W_i(\Phi^C K)^{2n-2} W_i(\Phi^C K), \tag{19}$$

with equality iff K *and* L *are homothetic.*

Proof. From Lemma 3.2, (10) and (13), for $Q \in K^n(C^n)$, we have

$$\begin{aligned} W_i(Q, \Phi_1^C(K,L))^{2n-1} &= V(K,\cdots,K,L,\Phi_i^{\overline{C}} Q)^{2n-1} \\ &\ge V_1(K, \Phi_i^{\overline{C}} Q)^{2n-2} V_1(L, \Phi_i^{\overline{C}} Q) \\ &= W_i(Q, \Phi^C K)^{2n-2} W_i(Q, \Phi^C L) \\ &\ge W_i(Q)^{\frac{(2n-i-1)(2n-1)}{2n-i}} W_i(\Phi^C K)^{\frac{2n-2}{2n-i}} W_i(\Phi^C L)^{\frac{1}{2n-i}}, \end{aligned} \tag{20}$$

by the equality condition of (13), equality holds in (20) iff Q, $\Phi^C K$ and $\Phi^C L$ are homothetic. Let $Q = \Phi_1^C(K,L)$. By (13), we obtain the desired inequality (19).

If there is equality in (19), then there exists $\alpha_1, \alpha_2 > 0$, so that

$$\Phi_1^C(K,L) = \alpha_1 \Phi^C K = \alpha_2 \Phi^C L. \tag{21}$$

This together with the equality in (19), it follows that

$$\alpha_1^{2n-2}\alpha_2 = 1. \tag{22}$$

On the other hand, (17), (18) and (20) imply

$$\alpha_1 = \frac{W_1(K,L)}{W_1(K)} \quad \textit{and} \quad \alpha_2 = \frac{W_1(K,L)}{W_1(L)}. \tag{23}$$

Combined with (22) and (23), we get

$$W_1(K,L)^{2n-1} = W_1(K)^{2n-2} W_1(L).$$

According to the equality condition of (13), we see that equality holds in (19) iff K and L are homothetic. Note that Theorem 1.3 is special case of $i = 0$.

Then, we establish a more general result of Theorem 1.4.

Theorem 4.2. *Let* $K_1, \cdots, K_{2n-1} \in K^n(C^n)$. *If* $0 \le i \le 2n-1$, *while* $2 \le r \le 2n-2$, *then*

$$\begin{aligned} &W_i(\Phi^C(K_1,\cdots,K_{2n-1}))^r \\ &\ge \prod_{j=1}^r W_i(\Phi^C(\underbrace{K_j,\cdots,K_j}_{r}, K_{r+1},\cdots,K_{2n-1})). \end{aligned} \tag{24}$$

Proof. Let $Q \in K^n(C^n)$, by Lemma 3.2 and (10), we get

$$\begin{aligned} W_i(Q, \Phi^C(K_1,\cdots,K_{2n-1}))^r &= V(K_1,\cdots,K_{2n-1},\Phi_i^{\overline{C}} Q)^r \\ &\ge \prod_{j=1}^r V(\underbrace{K_j,\cdots,K_j}_{r}, K_{r+1},\cdots,K_{2n-1},\Phi_i^{\overline{C}} Q) \\ &= \prod_{j=1}^r W_i(Q, \Phi^C(\underbrace{K_j,\cdots,K_j}_{r}, K_{r+1},\cdots,K_{2n-1})). \end{aligned}$$

Let $\Phi_{k'}^C(K_j, N) = \Phi^C(K_j,\cdots,K_j,K_{r+1},\cdots,K_{2n-1})$, where $k' = 2n-r-1$. By (13), we have

$$W_i(Q, \Phi_{k'}^C(K_j,N))^{2n-i} \ge W_i(Q)^{2n-i-1} W_i(\Phi_{k'}^C(K_j,N)).$$

Thus, we obtain

$$\begin{aligned} &W_i(Q, \Phi^C(K_1,\cdots,K_{2n-1}))^{(2n-i)r} \\ &\ge W_i(Q)^{(2n-i-1)r} \prod_{j=1}^r W_i(\Phi_{k'}^C(K_j,N)). \end{aligned} \tag{25}$$

Let $Q = \Phi^C(K_1,\cdots,K_{2n-1})$ in (25), it yields the desired inequality. Note that Theorem 1.4 is the special case of $i = 0$.

The following result provide the general Brunn-Minkowski inequality.

Theorem 4.3. *Let* $K, L \in K^n(C^n)$. *If* $0 \le i \le 2n-1$, *and* $2 \le j \le 2n-1$, *then*

$$W_i(\Phi_j^C(K \mp_c L))^{\frac{1}{2n-i}} \ge W_i(\Phi_j^C K)^{\frac{1}{2n-i}} + W_i(\Phi_j^C L)^{\frac{1}{2n-i}}, \tag{26}$$

with equality iff K *is a translation of* L.

Proof. Let $Q \in K^n(C^n)$. From (5), (11) and (15), we get

$$\begin{aligned} W_i(Q, \Phi_j^C(K \mp_c L)) &= W_j(K \mp_c L, \Phi_i^{\overline{C}} Q) \\ &= W_j(K, \Phi_i^{\overline{C}} Q) + W_j(L, \Phi_i^{\overline{C}} Q) \\ &= W_i(Q, \Phi_j^C K) + W_i(Q, \Phi_j^C L). \end{aligned}$$

By inequality (13), we have

$$W_i(Q, \Phi_j^C K)^{2n-i} \ge W_i(Q)^{2n-i-1} W_i(\Phi_j^C K),$$

with equality iff Q and $\Phi_j^C K$ are homothetic; and

$$W_i(Q, \Phi_j^C L)^{2n-i} \ge W_i(Q)^{2n-i-1} W_i(\Phi_j^C L),$$

with equality iff Q and $\Phi_j^C L$ are homothetic. Therefore,

$$\begin{aligned} &W_i(Q, \Phi_j^C(K \mp_c L)) W_i(Q)^{-\frac{2n-i-1}{2n-i}} \\ &\ge W_i(\Phi_j^C K)^{\frac{1}{2n-i}} + W_i(\Phi_j^C L)^{\frac{1}{2n-i}}, \end{aligned}$$

with equality iff Q, $\Phi_j^C K$ and $\Phi_j^C L$ are homothetic. Taking $Q = \Phi_j^C(K \mp_c L)$, then (26) is obtained. For any real β, β_1, β_2, then

$$\Phi_j^C(K \mp_c L) = \beta_1 \Phi_j^C K = \beta_2 \Phi_j^C L$$

yields $\Phi_j^C K = \beta \Phi_j^C L$. This together with the definition of surface area measure, and (2), we see that equality holds in (26) iff K is a translation of L.

We easily know that Theorem 1.5 is the special case $i, j = 0$ of Theorem 4.3.

V. Shephard-type problems for complex mixed Blaschke-Minkowski homomorphisms

In this part, we further study the Shephard type problem for complex mixed BMH. Firstly, we give an affirmative answer.

Theorem 5.1. *For* $K \in K^n(C^n)$, $L \in Z^n$. *If* $0 \le i \le 2n-1$, *and* $\Phi_i^{\overline{C}} K \subseteq \Phi_i^{\overline{C}} L$, *then*

$$W_i(K) \le W_i(L). \tag{27}$$

with equality iff $\Phi_i^{\overline{C}} K = \Phi_i^{\overline{C}} L$.

We now give a general form of Theorem 5.1.

Lemma 5.2. *For* $K, L \in K^n(C^n)$. *If* $\Phi_i^{\overline{C}} K \subseteq \Phi_i^{\overline{C}} L$ *and*

$0 \le i \le 2n-1$, *for every* $Q \in Z^n$, *then*

$$W_i(K,Q) \le W_i(L,Q), \tag{28}$$

with equality iff $\Phi_i^{\bar{C}} K = \Phi_i^{\bar{C}} L.$

Proof. From (11) and (15), taking $Q = \Phi_j^C M$, if $\Phi_i^{\bar{C}} K \subseteq \Phi_i^{\bar{C}} L$ we have

$$\begin{aligned} W_i(K,Q) &= W_i(K,\Phi_j^C M) = W_j(M,\Phi_i^{\bar{C}} K) \\ &= \frac{1}{2n}\int_{S^{2n-1}} h(\Phi_i^{\bar{C}} K,u)dS_j(K,u) \\ &\le \frac{1}{2n}\int_{S^{2n-1}} h(\Phi_i^{\bar{C}} L,u)dS_j(K,u) \\ &= W_j(M,\Phi_i^{\bar{C}} L) = W_i(L,\Phi_j^C M) = W_i(L,Q). \end{aligned}$$

This gives (28). According to the support function and (14), equality holds in (28) iff $\Phi_i^{\bar{C}} K = \Phi_i^{\bar{C}} L.$

Proof of Theorem 5.1. Since $L \in Z^n$, let $Q = L$ in Lemma 5.2, combining with (12) and (13), we get

$$W_i(L) \ge W_i(K,L) \ge W_i(K)^{\frac{2n-i-1}{2n-i}} W_i(L)^{\frac{1}{2n-i}},$$

i.e.,

$$W_i(K) \le W_i(L).$$

According to equality condition of (28), then $W_i(K) = W_i(L)$ iff K is a translation of L. Note that Theorem 1.7 is the special case of $i = 0$.

The following negative form of Shephard-type problem is obtained:

Theorem 5.3. *For* $L \in K^n(C^n)$ *and* $0 \le i \le 2n-1$. *If* Φ^C *is an even operator and* L *is not an origin-symmetric, then there is a* $K \in K_{os}^n(C^n)$, *so that* $\Phi_i^C K \subseteq \Phi_i^C L$, *but*

$$W_i(K) > W_i(L). \tag{29}$$

We give the following lemmas to support the proof of Theorem 5.3.

Lemma 5.4. *For* $K \in K^n(C^n)$ *and* $0 \le i \le 2n-1$, *then*

$$W_i(\nabla K) \ge W_i(K), \tag{30}$$

with equality iff K *is origin-symmetric.*

Proof. From (5), (11) and (13), for $Q \in K^n(C^n)$, we get

$$\begin{aligned} W_i(\nabla K,Q) &= \frac{1}{2n}\int_{S^{2n-1}} h(Q,u)dS_i(\nabla K,u) \\ &= \frac{1}{2}W_i(K,Q) + \frac{1}{2}W_i(-K,Q) \\ &\ge W_i(Q)^{\frac{1}{2n-i}}\left(\frac{1}{2}W_i(K)^{\frac{2n-i-1}{2n-i}} + \frac{1}{2}W_i(-K)^{\frac{2n-i-1}{2n-i}}\right). \end{aligned}$$

Taking $Q = \nabla K$, note that $W_i(K) = W_i(-K)$, then we obtain (30). According to Blaschke body, we know that equality holds in (30) iff K is origin-symmetric.

Lemma 5.5. Let $K \in K^n(C^n)$ *and* $0 \le i \le 2n-1$. *If* C *convex subset, then*

$$\Phi_i^C(\nabla K) = \Phi_i^C K. \tag{31}$$

Proof. By (2), (5) and Blaschke body, we get

$$\begin{aligned} h(\Phi_i^C(\nabla K),\omega) &= S_i(\nabla K,\cdot) * h(C\cdot\omega,\cdot) \\ &= \frac{1}{2}S_i(K,\cdot) * h(C\cdot\omega,\cdot) + \frac{1}{2}S_i(-K,\cdot) * h(C\cdot\omega,\cdot) \\ &= \frac{1}{2}h(\Phi_i^C K,\omega) + \frac{1}{2}h(\Phi_i^C(-K),\omega), \end{aligned}$$

for any $\omega \in S^{2n-1}$. Since Φ_i^C is even, then $\Phi_i^C K = \Phi_i^C(-K)$. Thus, we obtain $\Phi_i^C(\nabla K) = \Phi_i^C K.$

Proof of Theorem 5.3 Let $L \in K^n(C^n)$. Since L is not an origin-symmetric, by Lemma 5.4, we know that

$$W_i(\nabla K) > W_i(L).$$

Choose $\varepsilon > 0$, such that $W_i((1-\varepsilon)\nabla L) > W_i(L)$, and let $(1-\varepsilon)\nabla L = K$, then

$$W_i(K) > W_i(L).$$

But associate with Lemma 5.5, and the fact $\Phi_i^C(\lambda K) = \lambda^{2n-i-1}\Phi_i^C K$ $(\lambda > 0)$, we obtain

$$\begin{aligned} \Phi_i^C K &= \Phi_i^C((1-\varepsilon)\nabla L) = (1-\varepsilon)^{2n-i-1}\Phi_i^C(\nabla L) \\ &= (1-\varepsilon)^{2n-i-1}\Phi_i^C L \subseteq \Phi_i^C L. \end{aligned}$$

We easily know that Theorem 1.8 is the special case $i = 0$ of Theorem 5.3.

REFERENCES

[1] J. Abardia, "Difference bodies in complex vector spaces," *J. Funct. Anal*, vol. 263, no. 11, pp. 3588-3603, 2012.

[2] J. Abardia and A. Bernig, "Projection bodies in complex vector spaces," *Adv. Math*, vol. 227, no. 2, pp. 830-846, 2011.

[3] E. D. Bolker, "A class of convex bodies," *Trans. Amer. Math. Soc*., vol. 145, pp. 323-345, 1969.

[4] T. Bonnesen and W. Fenchel, "Theorie der konvexen Korper," Speinger-Verlag, Berlin, 1934.

[5] Y. Feng, W. Wang and J. Yuan, "Differences of quermass- and dual quermassintegral of Blaschke-Minkowski and Radial Blaschke-Minkowski homomorphisms," *Bull. Belg. Math. Soc*., vol. 21, no. 4, pp. 577-592, 2014.

[6] Y. Feng, W. Wang and J. Yuan, "Inequalities of quermassintegral about mixed Blaschke-Minkowski homomorphisms," *Tamk. J. Math*., vol. 46, no. 3, pp. 217-227, 2015.

[7] Y. Feng and W. Wang, "Blaschke-Minkowski homomorphisms and affine surface area," *Publ. Math. Debrecen*., vol. 85, no. 3-4, pp. 297-308, 2014.

[8] C. Haberl, "Complex affine isoperimetric inequalities," *Calc.Var. Partial Defferential Equations*, vol. 58, No. 169, 2019.

[9] Q. Huang, B. He and G. Wang, "The Busemann theorem for complex p-convex bodies," *Arch. Math*., vol. 99, no. 3, pp. 289-299, 2012.

[10] A. Koldobsky, H. Konig and M. Zymonopoulou, "The complex Busemann-Petty problem on sections of convex bodies," *Adv. Math*., vol. 218, no. 2, pp. 352-367, 2008.

[11] A. Koldobsky, G. Paouris and M. Zymonopoulou, "Complex intersection bodies," *J. Lond. Math. Soc*., vol. 88, no. 2, pp. 538-562, 2013.

[12] E. Lutwak, "Inequalities for mixed projection bodies," *Trans. Amer. Math. Soc*., vol. 339, no. 2, pp. 901-916, 1993.

[13] E. Lutwak, "Mixed projection inequalities," *Trans. Amer. Math. Soc*., vol. 287, no. 1, pp. 91-106, 1985.

[14] E. Lutwak, "The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem," *J. Differential Geom*., vol. 38, pp. 131-150, 1993.

[15] L. Liu, W. Wang and Q. Huang, "On polars of mixed complex projection bodies," *Talwanese. J. Math*., vol. 17, no. 6, pp. 1887-1899, 2013.

[16] C. M. Petty, "Projection bodies," *Proc. Coll. Convexity*, Copenhagen, 1965, *Kobenhacns Univ. Math. Inst*., pp. 234-241, 1967.

[17] B. Rubin, "Comparison of volumes of convex bodies in real, complex, and quaternionic spaces," *Adv. Math*., vol. 225, no. 3, pp. 1461-1498, 2010.

[18] R. Schneider, "Zu einem problem von shephard uber die projectionen konvexer korper," *Math. Z*., vol. 101, pp. 71-82, 1967.

[19] R. Schneider, "Convex Bodies: The Brunn-Minkowski theory," 2nd edn, *Cambridge Univ. Press*, Cambridge, 2014.

[20] F. E. Schuster, "Volume inqualities and additive maps of convex bodies," *Mathematika*, vol. 149, no. 2, pp. 211-234, 2006.

[21] F. E. Schuster, “Convolutions and multiplier transformations of convex bodies,” *Trans. Amer. Math. Soc.,* vol. 395, no. 11, pp. 5567-5591, 2007.
[22] F. E. Schuster, “Valuations and Busemann-Petty type problems,” *Adv. Math*., vol. 219, no. 1, pp. 344-368, 2008.
[23] G. C. Shephard, “Shadow systems of convex bodies,” *Israel J. Math*., vol. 2, pp. 229-236, 1964.
[24] W. Wang and R. He, “Inequalities for mixed complex projection bodies,” *Ann. Polon. Math*., vol. 113, no. 17, pp. 243-253, 2015.
[25] G. Zhang, “Restricted chord projection and affine inequalities,” *Geom. Dedicata,* vol. 39, no. 2, pp. 213-222, 1991.
[26] C. Zhao, “On polars of Blaschke-Minkowski homomorphisms,” *Math. Scan.*, vol. 111, no. 1, pp. 147-160, 2012.
[27] C. Zhao, “On Blaschke-Minkowski homomorphisms,” *Geom. Dedicata*, vol. 149, no. 1, pp. 373-378, 2010.
[28] C. Zhao, “On radial and polar Blaschke-Minkowski homomorphisms,” *Proc. Amer. Math. Soc.*, vol. 141, no. 2, pp. 667-676, 2013.
[29] C. Zhao, “Volume sums of polar Blaschke-Minkowski homomorphism,” *Proc. Indian Acad. Math*., vol. 125, no. 2, pp. 209-219, 2015.
[30] M. Zymonopoulou, “The complex Busemann-Petty problem for arbitrary measures,” *Arch. Math*., vol. 91, no. 5, pp. 664-678, 2008.