Green's and Stoke's Theorems Redefined with SFVCP

Manisha Kankarej and Jai Pratap Singh

***Abstract*—In this research we have explored the concept of circulation and flux of a vector field using the definition of standard fractional vector cross product(SFVCP). We have then applied this novel definition to Green's (tangential and normal form) and Stoke's theorem which has not been explored previously. Green's theorem is applied in context of complex variable to redefine Cauchy's Integral theorem. It is evident that for $\gamma = 1$ all theorems reduces to standard theorems. The unique perspective of the paper lies in applying SFVCP to explain Green's and Stoke's theorem. It is evident that SFVCP offers more practical applications, accuracy and is more powerful in electromagnetic field and fluid mechanics, where they are used to relate field quantities over a region of space to those over its boundary, providing both conceptual insights and practical tools for solving physical problems.**

***Index Terms*—Standard Fractional Vector Cross Product(SFVCP), Circulation, Flux, Fractional Flux, Green's Theorem, Cauchy's Integral Theorem, Stoke's Theorem**

I. Introduction

Divergence and curl are two fundamental characteristics of vector fields that play a crucial role in numerous applications. Both concepts are most easily grasped by envisioning the vector field for depicting the flow of a liquid or gas; that is, each vector in the vector field can be interpreted as a velocity vector, describing the motion of liquid and gas. Roughly speaking, divergence measures the tendency of the fluid to converge or disperse at a point, and curl measures the tendency of the fluid to rotate or circulate around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. [14]

Engheta [5] laid the foundation of curl operators in the study of electro magnetics, providing a framework to analyze electromagnetic fields and its interaction. Naqvi at.el. [15], [16] studied the application of fractional cross product in Maxwell's equation and waveguides, adding insights to the behaviour of electromagnetic fields in complex system. Later Kankarej and Singh [9] developed a new definition of fractional vector cross product(FVCP) in 2023 which was an extension to the definition of Das [3], [4]. This definition was further modified to develop the definition of standard fractional vector cross product(SFVCP) to study more properties of electromagnetic theory, fluid dynamics, and related areas [10]–[12].

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Inspired by above study, the author of this paper Kankarej and Singh, used definition of SFVCP to redefine most critical theorems and explained its nature. The main idea of the paper is to define the flux of vector field using standard fractional vector cross product(SFVCP) and use it to redefine Green's and Stoke's theorem. Following is the flow of the paper:
* In the second section, definition of SFVCP is introduced.
* Third and fourth sections introduces the concept of circulation and flux.
* Section five, defined fractional flux of a vector field.
* Section six and seven, introduces tangential and normal forms of Greens theorem for different values of γ
* Section eight and nine covers Green's theorem for complex variable with different values of γ. * Section ten, introduces Cauchy's integral form.
* In section eleven and twelve, Stoke's theorem along with different values of γ are discussed.
* In section thirteen and fourteen, different examples and conclusion covering Green's and Stoke's theorem are discussed using new definition of SFVCP.

For the first time the concept of SFVCP is used to define Green's and Stokes's theorem which brings novelty to paper. It is amazing to notice that all the standard theorems are particular case of SFVCP definition. Hence new definition of SFVCP is more accurate to explain concepts in electromagnetics and fluid mechanics.

Application of this novel definition can be extended as in [13] Mishra and Patnaik to study fractional vector cross product (FVCP) in the micro strip antenna. They defined FVCP to apply it to the transmission of electric and magnetic fields in micro strip antenna. Similar to this work new definition of SFVCP can be used to explore different types of antenna and their radiation pattern. Idrissi and Essoufi [7], [8] studied global existence of weak solutions for a three-dimensional magnetoelastic interaction model. In their model they combined a fractional harmonic map heat flow with an evolution equation for displacement. This is another concept to brainstorm, apply new definition and explore this study. The fractional factor theory of graphs originated from the feasible flow problem in communication networks. Altai [1] introduced a theory of fractional calculus by using a map $k(x)$ instead of x in the definitions of the classical derivative and the classical integral where x is a variable and $k : R \to R$ is continuously differentiable function. Gao and Shi [6] studied the sufficient conditions for the existence of fractional factors in the different setting of network from a theoretical perspective. These theoretical results provide the basis for the initial network designing. New definition introduced in this paper can be used further to study the properties as mentioned in [1], [6].

II. Mathematical Formulation

From [10]–[12], let (e_1, e_2, e_3) be standard orthonormal basis of R^3 and $\gamma \in [0, 1]$ a real number. We have the SFVCP defined as,

$$e_i \times^{\gamma} e_j = \cos\left(\frac{\gamma\pi}{2}\right)e_j + \sin\left(\frac{\gamma\pi}{2}\right)e_k - \cos\left(\frac{\gamma\pi}{2}\right)e_i \tag{1}$$

$$e_j \times^{\gamma} e_i = \cos\left(\frac{\gamma\pi}{2}\right)e_i - \sin\left(\frac{\gamma\pi}{2}\right)e_k - \cos\left(\frac{\gamma\pi}{2}\right)e_j \tag{2}$$

$$e_l \times^{\gamma} e_l = 0 \ for \ l = \{1, 2, 3\} \tag{3}$$

where (i, j, k) is a cyclic permutation of (1, 2, 3).

III. Circulation

By [18], let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $F(x, y) = M(x, y)i + N(x, y)j$ be a vector field with M(x,y) and N(x,y) having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $F(x, y)$ around C equals the double integral of (curl $F(x, y)$).k over R.

$$\int_C F(x, y).T \ ds = \int_C M(x, y)dx + N(x, y)dy$$

This is also called as a circulation density of a vector field $F(x, y)$ at a point. To explain this let us consider following Fig 1 where we assume both components of F i.e. M(x,y) and N(x,y) are positive.

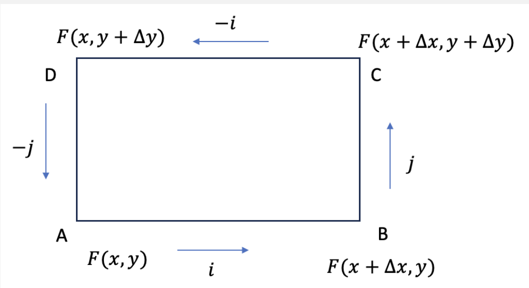

Fig 1. Circulation density of a vector field $F(x, y)$

The rate at which the fluid flows along the bottom edge of a rectangular region in the direction i is positive for the vector field F shown above. To calculate the circulation we calculate the flow rate along each edge in the direction of the arrows.

Top: $F(x, y + \Delta y).(-i) \ \Delta x = -M(x, y + \Delta y) \ \Delta x$

Bottom: $F(x, y).(i) \ \Delta x = M(x, y) \ \Delta x$

Right: $F(x + \Delta x, y).(j) \ \Delta y = N(x + \Delta x, y) \ \Delta y$

Left: $F(x, y).(-j) \ \Delta y = -N(x, y) \ \Delta y$

We sum opposite pairs to get:

Top and bottom:

$-(M(x, y + \Delta y) - M(x, y)) \ \Delta x \approx -(\frac{\partial M}{\partial y}\Delta y) \ \Delta x$

Right and left:

$(N(x + \Delta x, y) - N(x, y)) \ \Delta y \approx (\frac{\partial N}{\partial x}\Delta x) \ \Delta y$

Adding them we get:

$$\int_c F(x, y).T \ ds = \int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \ dx \ dy \tag{4}$$

Thus the circulation density of a vector field $F(x, y)$ is the line integral in tangential direction which is equal to surface integral of the difference of partial derivatives of different components of $F(x, y)$.

IV. Flux

By [18], for piecewise smooth, simple closed curve C and vector field $F(x, y) = M(x, y)i + N(x, y)j$ the outward flux of $F(x, y)$ around C over R is given as:

$$\int_C F(x, y).n \ ds = \int_C M(x, y)dy - N(x, y)dx$$

This is also called as a circulation density of a vector field $F(x, y)$ at a point. To explain this let us consider following Fig 2 where we assume both components of F are positive.

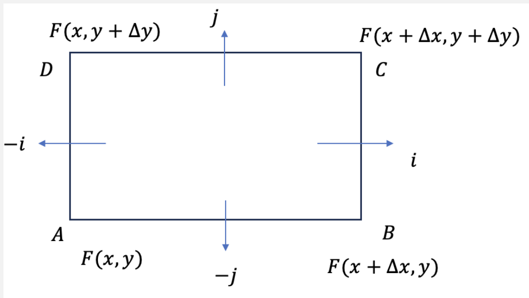

Fig 2. Flux of a vector field $F(x, y)$

The rate at which the fluid flows outwards is shown by the direction of arrows. The directions of the rectangular region are $i, -i, j$ and $-j$ for the vector field F as shown in the diagram above. To calculate the flux we calculate the flow rate along each edge in the direction of the arrows.

Top: $F(x, y + \Delta y).(j) \ \Delta x = N(x, y + \Delta y) \ \Delta x$

Bottom: $F(x, y).(-j) \ \Delta x = -N(x, y) \ \Delta x$

Right: $F(x + \Delta x, y).(i) \ \Delta y = M(x + \Delta x, y) \ \Delta y$

Left: $F(x, y).(-i) \ \Delta y = -M(x, y) \ \Delta y$

We sum opposite pairs to get:

Top and bottom:

$(N(x,y+\Delta y)-N(x,y))\ \Delta x \approx (\frac{\partial N}{\partial y}\Delta y)\ \Delta x$

Right and left:

$(M(x+\Delta x,y)-M(x,y))\ \Delta y \approx (\frac{\partial M}{\partial x}\Delta x)\ \Delta y$

Adding them we get:

$$\int_c F(x,y).n\ ds = \int\int_R (\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y})\ dx\ dy \tag{5}$$

V. Fractional Flux

By [18], for piecewise smooth, simple closed curve C and vector field $F(x,y)=M(x,y)i+N(x,y)j$, the flux of $F(x,y)$ is given in the direction normal to the plane of $F(x,y)$.

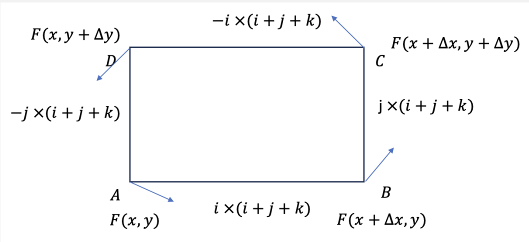

Fig 3: Graphical representation for fractional flux in counterclockwise direction

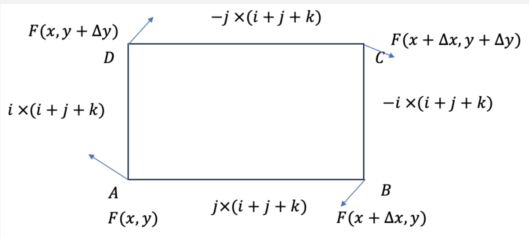

Fig 4: Graphical representation for fractional flux in clockwise direction

$$F(x,y)\times k = F(x,y).n = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

$$\int_c F(x,y).n\ ds = \int_c (M\frac{dy}{ds} - N\frac{dx}{ds})ds \tag{6}$$

The rate at which the fluid flows outwards in counterclockwise direction at the bottom edge of a rectangular region for the vector field F shown is above. To calculate the fractional flux we calculate the flow rate along each edge in the direction of the arrows.

Top:
$F(x,y+\Delta y).(-i\times^{\gamma}(i+j+k)\ \Delta x$

$=(-i\times^{\gamma}j - i\times^{\gamma}k)N(x,y+\Delta y)\ \Delta x$

Bottom:
$F(x,y).(i\times^{\gamma}(i+j+k))\ \Delta x$

$=(i\times^{\gamma}j + i\times^{\gamma}k)N(x,y)\ \Delta x$

Right:
$F(x+\Delta x,y).(j\times^{\gamma}(i+j+k))\ \Delta y$

$=(j\times^{\gamma}i + j\times^{\gamma}k)M(x+\Delta x,y)\ \Delta y$

Left:
$F(x,y).(-j\times^{\gamma}(i+j+k))\ \Delta y$

$=(-j\times^{\gamma}i - j\times^{\gamma}k)M(x,y)\ \Delta y$

We sum opposite pairs to get:
Top and bottom:

$F(x,y+\Delta y).(-i\times^{\gamma}(i+j+k)$

$+F(x,y).(i\times^{\gamma}(i+j+k))\ \Delta x$

$=k\big[\cos\frac{\gamma\pi}{2}-\sin\frac{\gamma\pi}{2}\big]\frac{\partial N}{\partial y}\Delta y\ \Delta x$

Adding right and left:

$F(x+\Delta x,y).(j\times^{\gamma}(i+j+k))\ \Delta y$

$+F(x,y).(-j\times^{\gamma}(i+j+k))\ \Delta y$

$=k\big[\cos\frac{\gamma\pi}{2}-\sin\frac{\gamma\pi}{2}\big]\frac{\partial M}{\partial x}\Delta x\ \Delta y$

Adding them we get

$\int_c F(x,y).n\ ds =$

$\big[k\big[\cos\frac{\gamma\pi}{2}-\sin\frac{\gamma\pi}{2}\big]\frac{\partial N}{\partial y}$

$+k\big[\cos\frac{\gamma\pi}{2}-\sin\frac{\gamma\pi}{2}\big]\frac{\partial M}{\partial x}\big]\Delta y\ \Delta x$

Now applying this definition on eqn (1) for x-y plane we have

$$\begin{aligned}&\int_c F(x,y).n\ ds\\ &=\int\int[(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y})\cos\frac{\gamma\pi}{2}\\ &-((\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y})\sin\frac{\gamma\pi}{2}]dx\ dy\end{aligned} \tag{7}$$

where $0\le\gamma\le 1$.

By the definition of fractional flux the line integral in normal direction is equal to the surface integral as mentioned in eqn (7).

VI. Greens theorem

We now come to the one of the most important theorem that extend the ‘Fundamental Theorem of Calculus’ to higher dimension. Green’s theorem says that the sum of the ”microscopic” swirls over the region is the same as the ”macroscopic” swirl around the boundary. It means to compute a single integral over an interval, we do a computation on the boundary of the region that involves one fewer integrations. [14]

There are two forms of Green's theorem as given below:

A. Circulation - Curl or Tangential form

By [10], [18], for piecewise smooth, simple closed curve C and vector field $F(x,y) = M(x,y)i + N(x,y)j$, the counterclockwise circulation of $F(x,y)$ around C equals the double integral of (curl $F(x,y)$).k over R.

$$\int_C F(x,y).T\ ds = \int_C M(x,y)dx + N(x,y)dy$$

$\int_C F(x,y).T\ ds = \int\int_R \nabla \times F\ dA$

Using SFVCP it is given by

$\int_C F(x,y).T\ ds = \int\int_R \nabla \times^{\gamma} F\ dA$

As proved above we have

$$\int_C F(x,y).T\ ds = \int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \sin\frac{\gamma\pi}{2}\ dx\ dy \quad (8)$$

B. Flux - Divergence or Normal form

By [10], [18], for piecewise smooth, simple closed curve C and vector field $F(x,y) = M(x,y)i + N(x,y)j$, the outward flux of F across C equals the double integral of div F over the region R enclosed by C.

$\int_C F(x,y).n\ ds = \int\int_R \nabla.F\ dA$

Using equation(7) we have

$$\begin{aligned} &\int_c F(x,y).n\ ds \\ &= \int\int [(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}) \cos\frac{\gamma\pi}{2} \\ &-(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}) \sin\frac{\gamma\pi}{2}]dx\ dy \end{aligned} \quad (9)$$

VII. Different forms of Greens Theorem for $0 \leq \gamma \leq 1$

A. Case 1

For $\gamma = 0$ equations (8) and (9) will take the form

$\int_C F(x,y).T\ ds = 0$

$\int_c F(x,y).n\ ds$

$= \int\int [(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})\ dx\ dy$

B. Case 2

For $\gamma = \frac{1}{3}$ equations (8) and (9) will take the form

$\int_C F(x,y).T\ ds = \frac{1}{2}\int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\ dx\ dy$

$\int_c F(x,y).n\ ds$

$= \int\int [\frac{\sqrt{3}-1}{2}(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})]dx\ dy$

C. Case 3

For $\gamma = \frac{1}{2}$ equations (8) and (9) will take the form

$\int_C F(x,y).T\ ds = \frac{1}{\sqrt{2}}\int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\ dx\ dy$

$\int_c F(x,y).n\ ds = 0$

D. Case 4

For $\gamma = \frac{2}{3}$ equations (8) and (9) will take the form

$\int_C F(x,y).T\ ds = \frac{\sqrt{3}}{2}\int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\ dx\ dy$

$\int_c F(x,y).n\ ds$

$= \int\int [\frac{-\sqrt{3}+1}{2}(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})]dx\ dy$

E. Case 5

For $\gamma = 1$ equations (8) and (9) will take the form

$\int_C F(x,y).T\ ds = \int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\ dx\ dy$

$\int_c F(x,y).n\ ds$

$= -\int\int [(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})\ dx\ dy$

Negative sign indicates that the direction of flux is in direction opposite to the plane. This also confirms that new definition is more generalized form and at $\gamma = 1$ it represents a particular condition which satisfies the theorem.

VIII. Greens theorem in complex variable

Similar to the notion of a line integral in planar vector field we can apply the concept of Greens theorem to a complex variable in complex plane. [2] Let $f(z) = P(z) + iQ(z)$. Then we define the complex line integral as

$$\begin{aligned} &\int_c f(z)dz = \\ &\int \big[P(z) + iQ(z)\big](x'(t) + iy(t))dt \\ &= \int P(z(t))x'(t) - Q(z(t))y'(t)dt \\ &+i\int P(z(t))y'(t) + Q(z(t))x'(t)dt \end{aligned} \quad (10)$$

where $z(t) = x(t) + iy(t)$

we can still identify each integral as being dot product of a certain vector field with the velocity vector $v(t) = r'(t)$ and we get line integrals

$\int(P,-Q).dr + i\int(Q,P).dr$ the corresponding vector fields are $F_1 = (P,-Q)$ and $F_2 = (Q,P)$.

Suppose C is the boundary of a region R in the plane such that $f(z)$ is defined not just on the boundary but on all of R, then Green's theorem applies.

We compute

$\nabla \times^{\gamma} F_1 = (\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}) \cos\frac{\gamma\pi}{2} i$

$+\left(\frac{-\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)\cos\frac{\gamma\pi}{2}j+\left(\frac{-\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\sin\frac{\gamma\pi}{2}k$
and

$\nabla\times^{\gamma}F_2=\left(\frac{\partial Q}{\partial y}-\frac{\partial P}{\partial x}\right)\cos\frac{\gamma\pi}{2}i$

$+\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)\cos\frac{\gamma\pi}{2}j+\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)\sin\frac{\gamma\pi}{2}k$

and we get

$$\begin{aligned}\int_c f(z)dz=&\int\int_R\left(\frac{-\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\sin\frac{\gamma\pi}{2}\ dA\\&+i\int\int_R\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)\sin\frac{\gamma\pi}{2}\ dA\end{aligned}\tag{11}$$

This result is same as original value of analytic function for $\gamma=1$ and would not help us evaluate the line integral if F were irrotational or incompressible. By Cauchy Riemann equation we know

$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y},\frac{\partial Q}{\partial x}=-\frac{\partial P}{\partial y}$,

and that a complex function $f(z)=P(x+iy)+iQ(x+iy)$ is holomorphic on a region R. Therefore for holomorphic functions both integrands are identically zero, so certainly the integrals are zero. Thus we have

IX. Different forms of Green's theorem for complex variable

A. Case 1

For $\gamma=0$ equation (11) will take the form

$$\int_c f(z)dz=0$$

B. Case 2

For $\gamma=\frac{1}{3}$ equation (11) will take the form

$$\begin{aligned}\int_c f(z)dz=&\frac{1}{2}\Big[\int\int_R\left(\frac{-\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA\\&+i\int\int_R\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)dA\Big]\end{aligned}$$

C. Case 3

For $\gamma=\frac{1}{2}$ equation (11) will take the form

$$\begin{aligned}\int_c f(z)dz=&\frac{1}{\sqrt{2}}\Big[\int\int_R\left(\frac{-\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA\\&+i\int\int_R\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)dA\Big]\end{aligned}$$

D. Case 4

For $\gamma=\frac{2}{3}$ equation (11) will take the form

$$\begin{aligned}\int_c f(z)dz=&\frac{\sqrt{3}}{2}\Big[\int\int_R\left(\frac{-\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA\\&+i\int\int_R\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)dA\Big]\end{aligned}$$

E. Case 5

For $\gamma=1$ equation (11) will take the form

$$\begin{aligned}\int_c f(z)dz=&\int\int_R\left(\frac{-\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA\\&+i\int\int_R\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)dA\end{aligned}$$

X. Cauchy's Integral Theorem

Let R be a closed region in the plane with smooth boundary ∂R (we allow the boundary to be any finite number of simple closed curves). [2] Then if $f(z)$ is any complex function which is defined and holomorphic on all of R then
$\int_{\partial R}f(z)dz=0.$

In particular, if $f(z)$ is holomorphic on entire complex plane then $\int_c f(z)dz=0$ for all curves. Moreover if the region has multiple components $C_1\ U\ C_2\ U\ ...U\ C_r$ then we get relation over these curves as
$\int_{C_1}f(z)dz+\int_{C_2}f(z)dz....+\int_{C_r}f(z)dz=0.$

Residue theorem Let $g(z)$ be a function holomorphic on a region R except for finitely many singularities at points $z_1,z_2,....z_n$. Let C be a curve in the region R, not passing through any of the singularities, which has winding numbers $n_1,n_2,...n_n$ with respect to each $z_1,z_2,..z_n$. Let $R_1,R_2....R_n$ be the residues of$g(z)$ at these points. Then
$\int_c g(z)dz=\sum_{i=1}^{n}2\pi iR_i.$

XI. Stokes theorem

By [18] Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. let $F=Mi+Nj+Pk$ be a vector field whose components have continuous first partial derivatives in an open region containing S. Then the circulation of F around C in the direction counterclockwise with respect to the surface's unit normal vector n equals integral of $\nabla\times F.n$ over S.

$\int_C F(x,y)\ dr=\int\int_S\nabla\times F.n\ d\sigma$

$\int_C F(x,y)\ dr=\int\int_S\nabla\times^{\gamma}F.n\ d\sigma$

By [10] we have

$$\begin{aligned}&\int_C F(x,y)\ dr\\&=\int\int_S\sin\frac{\gamma\pi}{2}\Big[i\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)\\&\quad+j\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)\\&\quad+k\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big]\\&+\cos\frac{\gamma\pi}{2}\Big[i\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}+\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)\\&\quad+j\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}-\frac{\partial P}{\partial y}\right)\\&\quad+k\left(\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z}+\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)\Big].n\ d\sigma\end{aligned}\tag{12}$$

XII. Different forms of Stoke's theorem

A. Case 1

For $\gamma = 0$ equation (12) will take the form

$$\int_C F(x,y)\ dr = [i(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} + \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) +j(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} - \frac{\partial P}{\partial y}) +k(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} + \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})].n\ d\sigma$$

B. Case 2

For $\gamma = \frac{1}{3}$ equation (12) will take the form

$$\int_C F(x,y)\ dr = \int\int_S \frac{1}{2}[i(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}) +j(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) + k(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})] +\frac{\sqrt{3}}{2}[i(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} + \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) +j(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} - \frac{\partial P}{\partial y}) +k(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} + \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})].n\ d\sigma$$

C. Case 3

For $\gamma = \frac{1}{2}$ equation (12) will take the form

$$\int_C F(x,y)\ dr = \int\int_S \frac{1}{\sqrt{2}}[i(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}) +j(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) + k(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})] +\frac{1}{\sqrt{2}}[i(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} + \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) +j(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} - \frac{\partial P}{\partial y}) +k(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} + \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})].n\ d\sigma$$

D. Case 4

For $\gamma = \frac{2}{3}$ equation (12) will take the form

$$\int_C F(x,y)\ dr = \int\int_S \frac{\sqrt{3}}{2}[i(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}) +j(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) + k(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})] +\frac{1}{2}[i(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} + \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) +j(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} - \frac{\partial P}{\partial y}) +k(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} + \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})].n\ d\sigma$$

E. Case 5

For $\gamma = 1$ equation (12) will take the form

$$\int_C F(x,y)\ dr = \int\int_S [i(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}) +j(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) + k(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})]$$

This result is similar to Stoke's theorem.

XIII. Examples

Example 1: Calculate the fractional flux of the field $F = (x-y)i + xj$ across the circle $x^2 + y^2 = 1$ in the xy-plane .

Solution: Parametric form for circle is $r(t) = \cos ti + \sin tj$,

$dr(t) = -\sin ti + \cos tj$

With $M = x - y$, $\frac{\partial M}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$

$N = x$, $\frac{\partial N}{\partial x} = 1$, $\frac{\partial N}{\partial y} = 0$

Fractional flux using eqn (7) is:

$\int_c F(x,y).n\ ds = \int\int \big[\big(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\big)\cos\frac{\gamma\pi}{2}$

$-\big(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\big)\sin\frac{\gamma\pi}{2}\big]dx\ dy$

which gives

$\int_c F(x,y).n\ ds$

$= \int\int \big[\big(1+0\big)\cos\frac{\gamma\pi}{2} - \big(1+0\big)\sin\frac{\gamma\pi}{2}\big]dx\ dy$

$\int_c F(x,y).n\ ds = \int\int \big[-\sin\frac{\gamma\pi}{2} + \cos\frac{\gamma\pi}{2}\big]dx\ dy$

For $\gamma = 1$ we have

$\int_c F(x,y).n\ ds$

$= \int\int -1 dx\ dy$

$= -\pi$ (area around the circle is $dx\ dy = \pi r^2$).

Example 2: Apply Greens theorem (tangential form) to calculate the circulation density on the field $F = (x-y)i + xj$ across the circle $x^2 + y^2 = 1$ in the xy-plane .

Solution: With $M = x - y$, $\frac{\partial M}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$

$N = x$, $\frac{\partial N}{\partial x} = 1$, $\frac{\partial N}{\partial y} = 0$

Using eqn (8) we have:

$\int_c M\ dx + N\ dy = \int\int \big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\big)\sin\frac{\gamma\pi}{2}dx\ dy$

which gives

$\int_c M\ dx + N\ dy = \int\int \big(1+1\big)\sin\frac{\gamma\pi}{2}dx\ dy$

For $\gamma = 1$
$\int_c M\ dx + N\ dy = \int\int 2\ dx\ dy = 2\pi$

(area around the circle is $dx\ dy = \pi r^2$).

Example 3: Apply Greens theorem (normal form) to calculate the flux density on the field $F = (x-y)i + xj$ across the circle $x^2+y^2 = 1$ in the xy-plane .

Solution: From equation 9 and example 1 we have,

$\int_c F(x,y).n\ ds$

$= \int\int \big[\big(1+0\big)\cos\frac{\gamma\pi}{2} - \big(1+0\big)\sin\frac{\gamma\pi}{2}\big]dx\ dy$

$\int_c F(x,y).n\ ds = \int\int \cos\frac{\gamma\pi}{2} - \sin\frac{\gamma\pi}{2}\ dx\ dy$

For $\gamma = 1$

$\int_c F(x,y).n\ ds = \int\int -1\ dx\ dy = -\pi$

(area around the circle is $dx\ dy = \pi r^2$).

Example 4: Find the line integral of the vector field $F(x,y) = (3y - e^{\sin x}, 7x + \sqrt{y^2+1})$ along the path $x^2 + y^2 = 9$.

Solution: With $M = 3y - e^{\sin x}$, $\frac{\partial M}{\partial y} = 3$

$N = 7x + \sqrt{y^2+1}$, $\frac{\partial N}{\partial x} = 7$,

Using eqn (8) we have:

$\int_c M\ dx + N\ dy = \int\int \big(7-3\big)\sin\frac{\gamma\pi}{2}dx\ dy$

which gives

$\int_c M\ dx + N\ dy = \int\int \big(4\big)\sin\frac{\gamma\pi}{2}dx\ dy$

For $\gamma = 1$
$\int_c M\ dx + N\ dy = \int\int 4\ dx\ dy = 4\times 9\pi = 36\pi$

(area around the circle is $dx\ dy = \pi r^2$).

Example 5: Apply Stoke's theorem on the hemisphere $S: x^2+y^2+z^2 = 9, z \leq 0$, its bounding circle $C: x^2+y^2 = 9, z = 0$, and the field $F = yi - xj$.

Solution: With $M = y$, $\frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial M}{\partial z} = 0$

$N = -x$, $\frac{\partial N}{\partial x} = -1$, $\frac{\partial N}{\partial y} = 0$, $\frac{\partial N}{\partial z} = 0$

$P = 0$, $\frac{\partial P}{\partial x} = 0$, $\frac{\partial P}{\partial y} = 0$, $\frac{\partial P}{\partial z} = 0$

For the curl integral of F from equation (9), we have

$\nabla \times^\gamma F$

$= \sin\frac{\gamma\pi}{2}\big[i\big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\big) + j\big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\big) + k\big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\big)\big]$

$+ \cos\frac{\gamma\pi}{2}\big[i\big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} + \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\big)$

$+j\big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} - \frac{\partial P}{\partial y}\big)$

$+k\big(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} + \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\big)\big]$

substituting the values we have,

$\nabla \times^\gamma F$

$= \sin\frac{\gamma\pi}{2}\big[i\big(0-0\big) + j\big(0-0\big) + k\big(-1-1\big)\big]$

$+ \cos\frac{\gamma\pi}{2}\big[i\big(1+1+0-0\big)$

$+j\big(-1-1+0-0\big)$

$+k\big(0-0+0-0\big)\big]$

$\nabla \times^\gamma F = \sin\frac{\gamma\pi}{2}(-2k) + \cos\frac{\gamma\pi}{2}(2i-2j)$

$n = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}} = \frac{xi+yj+zk}{3}$

$d\sigma = \frac{3}{z}dA$

$\nabla \times^\gamma F.nd\sigma$

$= \frac{2x\ \ \cos\frac{\gamma\pi}{2} - 2y\ \ \cos\frac{\gamma\pi}{2} - 2z\ \ \sin\frac{\gamma\pi}{2}}{3}\frac{3}{z}\ dA$

$\nabla \times^\gamma F.nd\sigma$

$= \frac{2x}{z}\ \ \cos\frac{\gamma\pi}{2} - \frac{2y}{z}\ \ \cos\frac{\gamma\pi}{2} - 2\ \ \sin\frac{\gamma\pi}{2}\ dA$

$\int\int_s \nabla \times^\gamma F.n\ d\sigma$

$= \int\int_{x^2+y^2\leq 9}\big[\frac{2x}{z}\ \ \cos\frac{\gamma\pi}{2} - \frac{2y}{z}\ \ \cos\frac{\gamma\pi}{2} - 2\ \ \sin\frac{\gamma\pi}{2}\big]dA$

For $\gamma = 1$, we have,

$\int\int_s \nabla \times^\gamma F.n\ d\sigma$

$= \int\int_{x^2+y^2\leq 9} -2dA$

(For the given circle, $dA = \pi r^2$)

$\int\int_s \nabla \times^\gamma F.n\ d\sigma = -18\pi$

Example 6 Use Cauchy's theorem to integrate $f(z) = \frac{1}{z}$. [2]

Solution: Let $z(t) = \cos t + i\sin t$

$\int_c f(z)dz = \int_0^{2\pi}\frac{1}{\cos t + i\sin t}(-\sin t + i\cos t)dt$

By rationalization of denominator it becomes 1 and we get

$\int_0^{2\pi}\cos t - i\sin t(-\sin t + i\cos t)dt = \int_0^{2\pi} i\ dt = 2\pi i$

Thus if C is any curve which winds n times counterclockwise around the origin then,

$\int_c \frac{dz}{z} = 2\pi in$

XIV. Conclusion

In this research authors have explored different forms of Green's and Stokes's theorem which are most critical characteristics of electromagnetism with new definition of SFVCP. Different forms of these theorems for different fractional angle gives an overview that Green's and Stokes's theorem can be visualised and understood in three dimension rather than tangential and normal components. Authors have supported the study with various examples and also discussed ideas to explore further application of SFVCP in areas of electromagnetic theory, fluid dynamics and related areas.

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