T-Ordering and Minus Ordering on Picture Fuzzy Matrices

V. Kamalakannan, P. Murugadas and M.Kavitha

Abstract—**The primary objective of this article is to comprehensively investigate and elucidate the characteristics associated with both T-ordering and minus-ordering within the realm of picture fuzzy matrices. Furthermore, we aim to explore and elucidate various properties pertaining to T-ordering and minus-ordering when applied to picture fuzzy matrices, leveraging the framework of generalized inverses for a more comprehensive understanding of these ordering techniques. Additionally, the article explores a novel concept termed reverse T ordering and reverse Minus ordering and Left-T and Right-T Partial Orderings on Picture Fuzzy Matrix, introducing and elucidating these concepts through illustrative examples. Finally, we apply picture fuzzy to a decision-making case study.**

Index Terms—**Picture Fuzzy Set(PicFS),Picture Fuzzy Matrices(PicFMs), T-ordering, Minus ordering, Moore-Penrose inverse.**

I. Introduction

THe increasing complexity of problems encountered in fields such as Economics, Engineering, Environmental Sciences, and Social Sciences, which elude resolution through classical mathematical approaches, presents a significant challenge in the contemporary practical landscape. To address such intricate scenarios, like those encountered in fuzzy Mathematics, the research community has witnessed a continuous surge in scholars dedicated to exploring the theoretical underpinnings and practical applications of fuzzy sets since the seminal work introduced by Zadeh [1].
The inception of the concept of fuzzy matrices dates back to 1977 when Thomasan introduced this novel idea [2]. Since its initial introduction, fuzzy matrices have undergone continuous development, thanks to the contributions of numerous researchers [3], [4], [5]. One significant stride in this domain was made by Jianmiao Cen [6], who led the way in establishing partial orderings for fuzzy matrices, a concept akin to the star ordering used in complex matrices. Subsequently, this groundbreaking work opened the door for a plethora of research endeavors exploring the implications and applications of these partial orderings in the context of fuzzy matrices. Building upon this foundation, A.R. Meenachi made noteworthy advancements by providing a characterization of the minus ordering applied to matrices, a characterization framed within the context of their generalized inverses [7]. This insightful work added depth and clarity to our understanding of how matrices can be ordered and compared, particularly in relation to their generalized inverses.
Conventional fuzzy sets, while powerful, sometimes encounter formidable obstacles when it comes to assigning precise membership values. In response to this limitation, Intuitionistic Fuzzy Sets (IFS), introduced by Atanassov [8], have emerged as a fitting alternative. Sriram S and Murugadas P examined the Moore-Penrose Inverse of Intuitionistic Fuzzy Matrices[9]. Susanta K Khan and Anita Pal conducted research on the Generalized Inverse of Intuitionistic Fuzzy Matrices [10]. Subsequently, Pradhan R and Pal M investigated Some Results on the Generalized Inverse of Intuitionistic Fuzzy Matrices [11], [12]. Bhowmik M and Pal M studied Some results on Generalized Interval-Valued Intuitionistic Fuzzy Sets [13], Generalized Interval-Valued Intuitionistic Fuzzy Sets [14]. IFS excels in managing incomplete information by considering both truth membership (simple membership) and falsity-membership (or non-membership) values. However, it is essential to note that IFS, while capable of handling incomplete data, does not address the challenges posed by indeterminate and inconsistent information that often permeate belief systems. Across a spectrum of disciplines spanning the social sciences and medical sciences, researchers have encountered instances where employing just two factors proves inadequate for adequately characterizing specific types of data. In these particular scenarios, it becomes imperative to incorporate an additional component to ensure a comprehensive representation of the data. It is in response to this need that the concept of Picture Fuzzy Sets (PicFS) was pioneered by Cuong and Kreinovich in the year 2013, serving as an extension and generalization of Intuitionistic Fuzzy Sets (IFS) [15], [16]. Moreover, in more contemporary developments in 2020, the investigation into Picture Fuzzy Matrix (PicFM) and its real-world applications was brought into focus by the research efforts of Shovan Dogra and Madhumangal Pal [17], illuminating the capabilities and significance of PicFS in diverse contexts. P. Murugadas focused on Implication operations on Picture Fuzzy Matrices [18]. Subsequently, V. Kamalakannan, P. Murugadas, and M. Kavitha delved into the Generalized Inverse of Picture Fuzzy Matrices [19], while V. Kamalakannan and P. Murugadas concentrated on Modal Operators on Picture Fuzzy Matrices [20] and Some Results on Generalized Inverse of Picture Fuzzy Matrix [21]
In our research, we delve into the application of T-ordering and minus ordering to Picture Fuzzy Matrices, employing a diverse set of generalized inverses, which encompass the g-inverse and the Moore-Penrose Inverse. Our investigation aims to scrutinize the intricate interplay between these ordering schemes. Additionally, we undertake the task of deducing equivalent conditions for each ordering method through the

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utilization of generalized inverses. Through our rigorous analysis, we demonstrate that both T-ordering and Minus ordering adhere to the principles of partial ordering within the encompassing set of all regular Picture Fuzzy Matrices. Furthermore, we take a step further in our study by providing a comprehensive characterization of the minus ordering when applied to matrices. This characterization is framed within the context of their generalized inverses, offering a valuable perspective on how these ordering criteria relate to the inherent properties of matrices.

II. Preliminaries

Throughout the manuscript, $\mathscr{P}_{jk}$ denotes PicFMs of order $j \times k$ and $\mathscr{P}_j$ denotes PicFMs of order $j \times j$.

Definition II.1. For $M \in \mathscr{P}_{(m,n)}$, $X \in \mathscr{P}_{(n,m)}$ is said to be
(i) $\{1,2\}$-inverse or semi-inverse of M, if $MXM = M$ and $XMX = X$.
(ii) $\{1,3\}$-inverse or a least square g-inverse of M, if $MXM = M$ and $(MX)^T = MX$.
(iii) $\{1,4\}$-inverse or a minimum norm g-inverse of M, if $MXM = M$, and $(XM)^T = XM$.
(iv) a Moore-Penrose inverse of M, if $MXM = M$, $XMX = X$, $(MX)^T = MX$ and $(XM)^T = XM$.
The Moore-Penrose inverse of M is denoted by M^+.

Definition II.2. $M\{\lambda\}$ is the collection of all λ-inverse of M, where λ is an element of $\{1,2,3,4\}$.

III. T-Ordering on Picture Fuzzy Matrix

Definition III.1. Let $M, N \in \mathscr{P}_{(m,n)}$. The T-ordering $M \overset{T}{\leq} N$ in $\mathscr{P}_{(m,n)}$ is defined as $M \overset{T}{\leq} N \Leftrightarrow M^TM = M^TN$ and $MM^T = NM^T$.

Example III.1. Let $M = \begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle \end{bmatrix}$ and $N = \begin{bmatrix} \langle 0.4,0.4,0.0\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle \end{bmatrix}$ then,
$M^TM = \begin{bmatrix} \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle \end{bmatrix}$ ....(1)
$M^TN = \begin{bmatrix} \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle \end{bmatrix}$ ....(2)

Hence from (1) and (2), we get $M^TM = M^TN$.
Similarly we have $MM^T = NM^T$.
Therefore $M \overset{T}{\leq} N$.

Theorem III.1. Let $M, N \in \mathscr{P}_{(m,n)}$ and M^+ exists. Then the following conditions are equivalent.
(i) $M \overset{T}{\leq} N$
(ii) $M^+M = M^+N$ and $MM^+ = NM^+$
(iii) $MM^+N = M = NM^+M$

Proof. (i)⇒(ii)
By (i) we have $M^TM = M^TN$ and $MM^T = NM^T$. Then

$$\begin{aligned} M^+M &= M^+MM^+M \\ &= M^+(M^+)^TM^TM \\ &= M^+(M^+)^TM^TN \\ &= M^+MM^+N \\ &= M^+N. \end{aligned}$$

Similarly we have, $MM^+ = NM^+$.
(ii)⇒(iii)
$M^+M = M^+N$ implies $M = MM^+M = MM^+N$ and $MM^+ = NM^+$ implies $M = MM^+M = NM^+M$.
(iii)⇒(i)
By $M = MM^+N$, $(MM^+)^TM = (MM^+)^TN$.
Then $M^T(M^+)^TM^TM = M^T(M^+)^TM^TN$.
Hence $M^TM = M^TN$.
Similarly we have $MM^T = NM^T$ by $M = NM^+M$.

Theorem III.2. Let $M, N \in \mathscr{P}_{(m,n)}$. If M^+ and N^+ both exist, then the following conditions are equivalent.
(i) $M \overset{T}{\leq} N$
(ii) $M^+M = N^+M$ and $MM^+ = MN^+$
(iii) $N^+MM^+ = M^+ = M^+MN^+$
(iv) $M^TMN^+ = M^T = N^+MM^T$

Proof: (i)⇒(iv)
$M^TM = M^TN$ implies $M^TM = M^TNN^+N$,
Then $M^TM = (M^TM)^T = (N^+N)^T(M^TN^T$
$= N^+NM^TM$.
Hence, $M^TMM^+ = N^+NM^TMM^+$
and $M^T(MM^+)^T = N^+NM^T(MM^+)^T$.
Therefore, $M^T = N^+NM^T = N^+MM^T$.
Similarly, $M^T = M^TMN^+$ by $MM^T = NM^T$.
(iv)⇒(ii)
By $M^T = N^+MM^T, M^T(M^+)^T = N^+MM^T(M^+)^T$.
Then, $M^+M = N^+MM^+M = N^+M$.
Similarly we have, $MM^+ = MN^+$ by $M^T = M^TMN^+$.
(ii)⇒ (i)

$$\begin{aligned} M^+M &= (M^+M)^T \\ &= (N^+M)^T \\ &= (N^+NN^+M)^T \\ &= (N^+M)^T(N^+N)^T \\ &= (M^+M)^TN^+N \\ &= M^+MN^+N \\ &= M^+MM^+N \\ &= M^+N. \end{aligned}$$

Similarly we have, $MM^+ = NM^+$.
Thus, (i) holds by Theorem III.1(ii).
(ii)⇒(iii)
By $M^+M = N^+M$, $M^+ = M^+MM^+ = N^+MM^+$.
Similarly we have,
$MM^+ = MN^+$ implies $M^+ = M^+MN^+$.
(iii)⇒(ii)
$N^+MM^+ = M^+ = M^+MN^+$ implies
$M^+M = N^+MM^+M = N^+M$
and $MM^+ = MM^+MN^+ = MN^+$.

Theorem III.3. In $\mathscr{P}^+_{(m,n)}$, the set of all matrices $M \in \mathscr{P}_{(m,n)}$ for which M^+ exists, $\overset{T}{\leq}$ is a partial ordering.

Proof. $M \overset{T}{\leq} M$ obvious. If $M \overset{T}{\leq} N$, $N \overset{T}{\leq} M$, then
$M = NM^+M$, $N = NN^+M$ by Theorem III.1.(iii).
Thus, by Theorem III.2.(ii), $N = NN^+M = NM^+M = M$.
If $M \overset{T}{\leq} N$, $N \overset{T}{\leq} L$, then $M = NM^+M$ and $N = LN^+N$ by Theorem III.1.(iii). By Theorem III.2(ii), we have
$M = NM^+M = LN^+NM^+M = LN^+M = LM^+M$.

Similarly, we have $M = MM^+L$. Thus, $M \overset{T}{\leq} L$, by Theorem III.1.(iii).

Example III.2. Let $M = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$, $N = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ and $L = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.2, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ then,

$M^TM = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(1)

$M^TN = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(2)

Hence from (1) and (2), we get $M^TM = M^TN$.
Similarly we have $MM^T = NM^T$. Therefore $M \overset{T}{\leq} N$.

$N^TN = \begin{bmatrix} \langle 0.4, 0.5, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(3)

$N^TL = \begin{bmatrix} \langle 0.4, 0.5, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(4)

Hence from (3) and (4), we get $N^TN = N^TL$.
Similarly we have $NN^T = LN^T$. Therefore $N \overset{T}{\leq} L$.

$MM^T = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(5)

$LM^T = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(6)

Hence from (5) and (6), we get $MM^T = LM^T$.
Similarly we have $M^TM = M^TL$. Therefore $M \overset{T}{\leq} L$.

Theorem III.4. If $M \overset{T}{\leq} N$, then we have
(i) $N^+M = M^+N$ and $MN^+ = NM^+$
(ii) $N^TM = M^TN$ and $NM^T = MN^+$
that is N^TM and NM^T are symmetric
(iii) $MN^+M = M = NN^+M = NMN^+ = N^+MN$, $NM^+M = M = MM^+N = MNM^+ = M^+NM$
(iv) $NM^TM = MM^TN = M^TNM = MNM^T$, $MN^TN = NN^TM = N^TMN = NMM^T$

Theorem III.5. If $M \overset{T}{\leq} N$, then we have
(i) $M^T \overset{T}{\leq} N^T$
(ii) $M^+ \overset{T}{\leq} N^+$
(iii) $N^TM \overset{T}{\leq} N^TN$, $MN^T \overset{T}{\leq} NN^T$
(iv) $N^+M \overset{T}{\leq} N^+N$, $MN^+ \overset{T}{\leq} NN^+$
(v) $M^TM \overset{T}{\leq} N^TN$, $MM^T \overset{T}{\leq} NN^T$
(vi) $M^+M \overset{T}{\leq} N^+N$, $MM^+ \overset{T}{\leq} NN^+$
(vii) If $N^TN^+ = N^+N^T$ then $M^TM^+ = M^+M^T$

Proof. (i) and (ii) hold clearly.
(iii)

$$\begin{aligned}(N^TM)^TN^TM &= M^TNN^TM \\ &= M^TMM^TM \\ &= M^TMM^TN \\ &= M^TMN^TN \\ &= M^TNN^TN.\end{aligned}$$

Similarly we have $N^TM(M^TM)^T = N^TN(N^TM)^T$ thus $N^TM \overset{T}{\leq} N^TN$.
Similarly we have $MN^T \overset{T}{\leq} NN^T$. Thus (iii) holds.
(iv)

$$\begin{aligned}(N^+M)^TN^+M &= (M^+M)^TM^+M \\ &= M^T(M^+)^TM^+M \\ &= M^T(M^+)^TM^+N \\ &= M^T(M^+)^TN^+N \\ &= (M^+M)^TN^+N \\ &= (N^+M)^TN^+N,\end{aligned}$$

and $N^+M(N^+M)^T = N^+N(N^+M)^T$.
Thus, $N^+M \overset{T}{\leq} N^+N$. Similarly we have $MN^+ \overset{T}{\leq} NN^+$.
(v)

$$\begin{aligned}(M^TM)^TM^TM &= M^TMM^TN \\ &= M^TMN^TN \\ &= (M^TM)^TN^TN.\end{aligned}$$

and $M^TM(M^TM)^T = N^TN(M^TM)^T$.
Thus, $M^TM \overset{T}{\leq} N^TN$.
Similarly we have $MM^T \overset{T}{\leq} NN^T$. Thus (v) holds
(vi)

$$\begin{aligned}(M^+M)^TN^+N &= M^T(M^+)^TN^+N \\ &= M^T(M^+)^TM^+N \\ &= M^T(M^+)^TM^+M \\ &= (M^+M)^TM^+M.\end{aligned}$$

and $N^+N(M^+M)^T = M^+M(M^+M)^T$.
Then $M^+M \overset{T}{\leq} N^+N$.
Similarly we have $MM^+ \overset{T}{\leq} NN^+$.
Thus (vi) holds Similarly we can prove (vii).

IV. Minus Ordering on Picture Fuzzy Matrix

In this section we define minus-ordering of PicFM. We establish that in the set of all regular picture fuzzy matrices, the minus ordering is a partial ordering.
Through out this section, let $\mathscr{P}^-_{(m,n)}$ denote the set of all regular PicFMs in $\mathscr{P}_{(m,n)}$.

Definition IV.1. If $M \in \mathscr{P}_{(m,n)}$ and $X \in \mathscr{P}_{(n,m)}$ satisfies the relation $MXM = M$ then X is called a generalized inverse(g-inverse) of M which is denoted by M^-. The g-inverse of a PicFM is not necessarily unique. We denote the set of all g-inverses of M by $M\{1\}$.

Definition IV.2. For $M \in \mathscr{P}^-_{(m,n)}$ and $N \in \mathscr{P}_{(m,n)}$, the minus ordering denoted as $\bar{<}$ is define as $M \bar{<} N \Leftrightarrow M^-M = M^-N$ and $MM^- = NM^-$ for some $M^- \in M\{1\}$.

Example IV.1. Let $M = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$, then $M^- = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ and $N = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
Now,
$M^-M = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.3, 0.3, 0.2\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(1)

$$M^-N = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.3, 0.3, 0.2\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} \text{...........(2)}$$

Hence from (1) and (2), we get $MM^- = M^-N$.
Similarly we have $MM^- = NM^-$.
Therefore $M \overset{-}{<} N$.

Lemma IV.1. For $M \in \mathscr{P}^-_{(m,n)}$ and $N \in \mathscr{P}_{(m,n)}$, the following are equivalent
(i) $M \overset{-}{<} N$.
(ii) $M = MM^-N = NM^-M = NM^-N$.
Proof.
(i) $\Rightarrow$ (ii)
$M \overset{-}{<} N \Rightarrow MM^- = NM^-$ and $M^-M = M^-N$
for some $M^- \in M\{1\}$.
Now, $M = M(M^-M) = MM^-N$
$M = (MM^-)M = NM^-M$
$M = N(M^-M) = NM^-N$.
(ii) $\Rightarrow$(i) Let $X = M^-MM^-$

$$\begin{aligned} MXM &= M(M^-MM^-)M \\ &= (MM^-M)M^-M \\ &= M \end{aligned}$$

$\Rightarrow X \in M\{1\}$.
Now

$$\begin{aligned} XM &= (M^-MM^-)MM^-N \\ &= M^-(MM^-M)M^-N \\ &= (M^-MM^-)N \\ &= XN. \end{aligned}$$

Similarly we have $MX = NX$.
Hence $M \overset{-}{<} N$ with respect to $X \in M\{1\}$.

Theorem IV.1. Let $M,\ N \in \mathscr{P}^-_{(m,n)}$. If $M \overset{-}{<} N$, then $N\{1\} \subseteq M\{1\}$.
Proof: By Lemma IV.1,
$M \overset{-}{<} N \Rightarrow M = MM^-N = NM^-M$.
For $N^- \in N\{1\}$,

$$\begin{aligned} MN^-M &= (MM^-N)N^-(NM^-M) \\ &= MM^-(NN^-N)MM^- \\ &= (MM^-N)M^-M \\ &= MM^-M \\ &= M \end{aligned}$$

Hence $MN^-M = M$ for each $N^- \in N\{1\}$.
Therefore, $N\{1\} \subseteq M\{1\}$.

Lemma IV.2. For $M,\ N \in \mathscr{P}^-_{(m,n)}$.
(i) $R(M) \subseteq R(N) \Leftrightarrow M = MN^-N$ for each N^- of N.
(ii) $C(M) \subseteq C(N) \Leftrightarrow M = NN^-M$ for each N^- of N.

Theorem IV.2. If $M,\ N \in \mathscr{P}^-_{(m,n)}$, then the following are equivalent.
(i) $M \overset{-}{<} N$
(ii) $R(M) \subseteq R(N),\ C(M) \subseteq C(N)$ and $MN^-M = M$.
Proof. (i)$\Rightarrow$(ii)
By Lemma IV.1,

$$\begin{aligned} M &= NM^-N \\ &= NM^-(NN^-N) \\ &= (NM^-N)N^-N \\ &= MN^-N \end{aligned}$$

Therefore $M = MN^-N$ for each $N^- \in N\{1\}$
$\Rightarrow R(M) \subseteq R(N)$
Similarly we have $M = NN^-M$ for each $N^- \in N\{1\}$
$\Rightarrow C(M) \subseteq C(N)$
(ii)$\Rightarrow$(i)
Let $X = N^-MN^-$,

$$\begin{aligned} MXM &= M(N^-MN^-)M \\ &= (MN^-M)N^-M \\ &= MN^-M \\ &= M \end{aligned}$$

$\Rightarrow X \in M\{1\}$.
Now By Lemma IV.2

$$\begin{aligned} MX &= M(N^-MN^-) \\ &= NN^-M(N^-MN^-) \\ &= NN^-(MN^-M)N^- \\ &= NX \end{aligned}$$

Similarly we have $XM = XN$ and $MN^-M = M$ by Lemma IV.2
Hence $M \overset{-}{<} N$ for $X \in M\{1\}$.

Theorem IV.3. In $\mathscr{P}^-_{(m,n)}$ the minus ordering $\overset{-}{<}$ is a partial ordering.
Proof: (i) $M \overset{-}{<} M$ is obvious. Hence $\overset{-}{<}$ is reflexive.
(ii) By Lemma IV.1 $M \overset{-}{<} N \Rightarrow M = NM^-N$.
$N \overset{-}{<} M \Rightarrow N = NN^-M = MN^-N$.

$$\begin{aligned} M &= NM^-N \\ &= (NN^-M)M^-(MN^-N) \\ &= NN^-(MN^-N) \\ &= NN^-N \\ &= N. \end{aligned}$$

$A \overset{-}{<} N$ and $N \overset{-}{<} M \Rightarrow M = N$. Hence $\overset{-}{<}$ is antisymmetric.
(iii) By Theorem IV.2
$M \overset{-}{<} N \Rightarrow M = MN^-M$ and $M = MN^-N = NN^-M$
by Lemma IV.1 $N \overset{-}{<} L \Rightarrow N = NN^-L = LN^-N$.
Let $X = N^-MN^-$

$$\begin{aligned} MXM &= M(N^-MN^-)M \\ &= (MN^-M)N^-M \\ &= MN^-M \\ &= M \end{aligned}$$

$\Rightarrow X \in M\{1\}$.
Since $M \overset{-}{<} N$ and $N \overset{-}{<} L$, by applying Theorem IV.2, repeatedly. We have,

$$\begin{aligned} MX &= M(N^-MN^-) \\ &= NN^-M(N^-MN^-) \\ &= NN^-(MN^-M)N^- \\ &= NN^-MN^- \\ &= (LN^-N)N^-MN^- \\ &= LN^-(NN^-M)N^- \\ &= L(N^-MN^-) \\ &= LX \end{aligned}$$

Similarly we have $XM = XC$.
Since $X \in M\{1\}$ with $MX = LX$ and $XM = XL$, it follows that $M \overset{-}{<} L$.

Theorem IV.4. For $M,\ N \in \mathscr{P}_{m,n}$ and M^+ exists, then the following are equivalent.
(i) $M \bar{<} N$.
(ii) $M^+M = M^+N$; $MM^+ = NM^+$.
(iii) $MM^+N = M = NM^+M$.
Proof: (i) $\Rightarrow$ (ii)
$M \bar{<} N \Rightarrow MM^- = NM^-$ and $M^-M = M^-N$
for some $M^- \in M\{1\}$.
Now, $M = M(M^-M) = MM^-N$ as $M^- \in M\{1\}$.
So $M^+M = M^+MM^+N = M^+N$.
Similarly, $MM^+ = NM^+$.
(ii) $\Rightarrow$ (iii)
$M^+M = M^+N$. This gives $M = MM^+M = MM^+N$.
Also, from $MM^+ = NM^+$ implies
$M = MM^+M = NM^+M$.
Thus $M = MM^+N = NM^+M$.
(iii) $\Rightarrow$ (i)
Let $X = M^+MM^+$.

$$\begin{aligned} MXM &= M(M^+MM^+)M \\ &= (MM^+M)M^+M \\ &= MM^+M \\ &= M \end{aligned}$$

Thus, X is a g-inverse of M.
Now,

$$\begin{aligned} XM &= (M^+MM^+)MM^+N \\ &= M^+(MM^+M)M^+N \\ &= (M^+MM^+)N \\ &= XN \end{aligned}$$

Similarly, $MX = NX$.
Hence $M \bar{<} N$ for some $X \in M\{1\}$.

Example IV.2. Let $M = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$, then
$M^+ = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
$M^- = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ and
$N = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
Now,
$MM^- = \begin{bmatrix} \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(1)
and
$NM^- = \begin{bmatrix} \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(2)

Hence from (1) and (2), we get $MM^- = NM^-$.

$M^-M = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(3)
$M^-N = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(4)

Hence from (3) and (4), we get $M^-M = M^-N$.
Therefore $M \bar{<} N$.

$M^+M = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(5)
$M^+N = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ ....(6)

Hence from (5) and (6), we get $M^+M = M^+N$.
$MM^+ = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(7)
$NM^+ = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(8)
Hence from (7) and (8), we get $MM^+ = NM^+$.
Also,
$MM^+N = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} = M$ .....(9)
$NM^+M = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} = M$ ....(10)
Hence from (9) and (10), we get $MM^+N = M = NM^+M$.

Lemma IV.3. If M and N are two PicFMs, then
$M \bar{<} N \Rightarrow M^T \bar{<} N^T$.
Proof: $M \bar{<} N \Rightarrow MM^- = NM^-$ and $M^-M = M^-N$.
Here,
$MM^- = NM^- \Leftrightarrow (MM^-)^T = (NM^-)^T$
$\Leftrightarrow (M^-)^T M^T = (M^-)^T N^T$
$\Leftrightarrow (M^T)^- M^T = (M^T)^- N^T$.
Hence, $MM^- = NM^- \Leftrightarrow (M^T)^- M^T = (M^T)^- N^T$.
Similarly, $M^-M = M^-N \Leftrightarrow M^T(M^T)^- = N^T(M^T)^-$.
Hence, $M \bar{<} N \Rightarrow M^T \bar{<} N^T$.

Example IV.3. Let $M = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$, then
$M^- = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ and
$N = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
$M^T = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ and
$N^T = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$

Here, $M \bar{<} N$ and $M^T \bar{<} N^T$.

Lemma IV.4. Let M and N are two PicFMs, If $M \bar{<} N$ and N is idempotent then M is also idempotent.
Proof:

$$\begin{aligned} M^2 &= M.M \\ &= (MM^-N)(NM^-M) \\ &= MM^-N^2M^-M \\ &= (MM^-N)M^-M \\ &= MM^-M \\ &= M. \end{aligned}$$

Example IV.4. Let $M = \begin{bmatrix} \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$, then
$M^- = \begin{bmatrix} \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ and
$N = \begin{bmatrix} \langle 0.5, 0.4, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$

then $N^2 = N$, which is an idempotent PicFM.

$M^-N = \begin{bmatrix} \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(1)
and
$NM^- = \begin{bmatrix} \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ ....(2)

$MM^{-}=\begin{bmatrix}\langle 0.5,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.5,0.1\rangle\end{bmatrix}$ ....(3)

Hence from (1), (2) and (3), we get $M^{-}N=MM^{-}$. and $NM^{-}=MM^{-}$.

Therefore, $M\lesssim N$.

Also, $M^2=M$ is also an idempotent PicFM.

Theorem IV.5. Let $M,\ N\in\mathscr{P}_{m,n}$. If M^{+} and N^{+} both are exists and $M\in N\{2\}$, then the following are equivalent.

(i) $M\lesssim N$.

(ii) $M^{+}M=N^{+}M$ and $MM^{+}=MN^{+}$.

(iii) $N^{+}MM^{+}=M^{+}=M^{+}MN^{+}$.

(iv) $M^{T}MN^{+}=M^{T}=M^{T}NM^{+}$.

Proof:

(i) $\Rightarrow$ (ii)

Let $M\lesssim N$, then $MM^{-}=NM^{-}$ and $M^{-}M=M^{-}N$

We know that,

$$\begin{aligned}M&=MN^{+}M\\ M^{+}M&=M^{+}MN^{+}M\\ (M^{+}M)^{T}&=(M^{+}MN^{+}M)^{T}\\ M^{+}M&=(N^{+}M)^{T}(M^{+}M)^{T}\\ M^{+}M&=N^{+}MM^{+}M\\ M^{+}M&=N^{+}M\end{aligned}$$

Similarly, $MM^{+}=MN^{+}$.

(ii) $\Rightarrow$ (iii)

$M^{+}M=N^{+}M$ and $MM^{+}=MN^{+}$.

Now, $M^{+}=M^{+}MM^{+}=N^{+}MM^{+}$.

Similarly, $M^{+}=M^{+}MM^{+}=M^{+}MN^{+}$.

(iii) $\Rightarrow$ (iv)

$$\begin{aligned}M&=MM^{+}M\\ M^{T}&=M^{T}(MM^{+})^{T}\\ M^{T}&=M^{T}MM^{+}\\ M^{T}&=M^{T}M(M^{+}MN^{+})\\ M^{T}&=M^{T}(MM^{+}M)N^{+}\\ M^{T}&=M^{T}MN^{+}\end{aligned}$$

Similarly, $M^{T}NM^{+}=M^{T}$.

Therefore, $M^{T}MN^{+}=M^{T}=M^{T}NM^{+}$.

(iv) $\Rightarrow$ (i) Take $M^{T}=M^{T}MN^{+}$

$$\begin{aligned}M&=MN^{+}M\\ MN^{+}&=(MN^{+}M)N^{+}\\ MN^{+}&=M(N^{+}M)N^{+})\\ MN^{+}&=MM^{+}\end{aligned}$$

Similarly, $M^{+}M=M^{+}N$.

Therefore, by Theorem IV.4, (iv) $\Rightarrow$ (i) holds.

Example IV.5. Let $M=\begin{bmatrix}\langle 0.4,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$, then

$M^{+}=\begin{bmatrix}\langle 0.4,0.4,0.1\rangle & \langle 0.4,0.5,0.1\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$

$M^{-}=\begin{bmatrix}\langle 0.3,0.3,0.2\rangle & \langle 0.4,0.5,0.1\rangle\\ \langle 0.5,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ and

$N=\begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$

Now,

$MM^{-}=NM^{-}=\begin{bmatrix}\langle 0.4,0.3,0.2\rangle & \langle 0.4,0.4,0.1\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.5,0.1\rangle\end{bmatrix}$ .....(1)

and

$M^{-}M=M^{-}N=\begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ .....(2)

Hence from (1) and (2), we get $M^{-}N=MM^{-}$ and $NM^{-}=MM^{-}$. Therefore, $M\lesssim N$.

Now,

$M^{+}M=M^{+}N=\begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ ......(3)

$MM^{+}=NM^{+}=\begin{bmatrix}\langle 0.4,0.4,0.1\rangle & \langle 0.4,0.4,0.1\rangle\\ \langle 0.4,0.4,0.1\rangle & \langle 0.4,0.5,0.1\rangle\end{bmatrix}$ ......(4)

Hence from (3) and (4), we get

$M^{+}N=MM^{+}$ and $NM^{+}=MM^{+}$.

Also,

$N^{+}MM^{+}=M^{+}MN^{+}=\begin{bmatrix}\langle 0.4,0.4,0.1\rangle & \langle 0.4,0.5,0.1\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$

$=M^{+}$

and

$M^{T}MN^{+}=M^{+}NM^{+}=\begin{bmatrix}\langle 0.4,0.4,0.1\rangle & \langle 0.4,0.5,0.1\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$

$=M^{T}$.

V. Reverse T-Ordering on Picture Fuzzy Matrix

Definition V.1. Let $M,\ N\in\mathscr{P}_{(m,n)}$. The T- Reverse ordering $M\overset{T}{\geq}N$ in $\mathscr{P}_{(m,n)}$ is defined as $M\overset{T}{\geq}N\Leftrightarrow N^{T}N=N^{T}M$ and $NN^{T}=MN^{T}$.

Example V.1. Let $M=\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.45,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\end{bmatrix}$ and $N=\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\end{bmatrix}$ then,

$N^{T}N=\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\end{bmatrix}$ ....(1)

$N^{T}N=\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\end{bmatrix}$ ....(2)

Hence from (1) and (2), we get $N^{T}M=N^{T}N$.

Similarly we have $MN^{T}=NN^{T}$.

Therefore $M\overset{T}{\geq}N$.

Theorem V.1. Let $M,\ N\in\mathscr{P}_{(m,n)}$ and N^{+} exists. Then the following conditions are equivalent.

(i) $M\overset{T}{\geq}N$

(ii) $N^{+}N=N^{+}M$ and $NN^{+}=MN^{+}$

(iii) $NN^{+}M=N=MN^{+}N$

Proof. (i)$\Rightarrow$(ii)

By (i) we have $N^{T}N=N^{T}M$ and $NN^{T}=MN^{T}$.

Then

$$\begin{aligned}N^{+}N&=N^{+}NN^{+}N\\ &=N^{+}(N^{+})^{T}N^{T}N\\ &=N^{+}(N^{+})^{T}N^{T}M\\ &=N^{+}NN^{+}M\\ &=N^{+}M\end{aligned}$$

Similarly we have, $NN^{+}=MN^{+}$.

(ii)$\Rightarrow$(iii)

$N^{+}N=N^{+}M$ implies $N=NN^{+}N=NN^{+}M$ and $NN^{+}=MN^{+}$ implies $N=NN^{+}N=MN^{+}N$.

(iii)$\Rightarrow$(i)

By $N = NN^+M,\ (NN^+)^TN = (NN^+)^TM$.
Then $N^T(N^+)^TN^TN = N^T(N^+)^TN^TM$.
Hence $N^TN = N^TM$.
Similarly we have $NN^T = MN^T$ by $N = MN^+N$.

Theorem V.2. Let $M,\ N \in \mathscr{P}_{(m,n)}$. If M^+ and N^+ both exist, then the following conditions are equivalent.
(i) $M \overset{T}{\geq} N$
(ii) $N^+N = M^+N$ and $NN^+ = NM^+$
(iii) $M^+NN^+ = N^+ = N^+NM^+$
(iv) $N^TNM^+ = N^T = M^+NN^T$
Proof: (i)⇒(iv)
$N^TN = N^TM$ implies $N^TN = N^TMM^+M$,
Then $N^TN = (N^TN)^T = (M^+M)^T(N^TM^T)$
$= M^+MN^TN$.
Hence, $N^TNN^+ = M^+MN^TNN^+$
and $N^T(NN^+)^T = M^+MN^T(NN^+)^T$.
Therefore, $N^T = M^+MN^T = M^+NN^T$.
Similarly, $N^T = N^TNM^+$ by $NN^T = MN^T$.
(iv)⇒(ii)
By $N^T = M^+NN^T, N^T(N^+)^T = M^+NN^T(N^+)^T$.
Then, $N^+N = M^+NN^+N = M^+N$.
Similarly we have, $NN^+ = NM^+$ by $N^T = N^TNM^+$.
(ii)⇒ (i)

$$\begin{aligned}N^+N &= (N^+N)^T\\ &= (M^+N)^T\\ &= (M^+MM^+N)^T\\ &= (M^+N)^T(M^+M)^T\\ &= (N^+N)^TM^+M\\ &= N^+NM^+M\\ &= N^+NN^+M\\ &= N^+M.\end{aligned}$$

Similarly we have, $NN^+ = MN^+$.
Thus, (i) holds by Theorem V.2(ii).
(ii)⇒(iii)
By $N^+N = M^+N,\ N^+ = N^+NN^+ = M^+NN^+$.
Similarly we have,
$NN^+ = NM^+$ implies $N^+ = N^+NM^+$.
(iii)⇒(ii)
$M^+NN^+ = N^+ = N^+NM^+$ implies
$N^+N = M^+NN^+N = M^+N$
and $NN^+ = NN^+NM^+ = NM^+$.

Theorem V.3. In $\mathscr{P}^+_{(m,n)}$, the set of all matrices $M \in \mathscr{P}_{(m,n)}$ for which M^+ exists, $\overset{T}{\geq}$ is a partial ordering.
Proof. Similar to Theorem III.3

Example V.2. Let $M = \begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$,
$N = \begin{bmatrix}\langle 0.4,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ and
$L = \begin{bmatrix}\langle 0.4,0.4,0.1\rangle & \langle 0.4,0.2,0.2\rangle\\ \langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ then,
$N^TM = \begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ ....(1)
$N^TN = \begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ ....(2)

Hence from (1) and (2), we get $N^TM = N^TN$.
Similarly we have $NN^T = MN^T$.
Therefore $M \overset{T}{\geq} N$.

$L^TN = \begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ ....(3)
$L^TL = \begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ ....(4)

Hence from (3) and (4), we get $L^TN = L^TL$.
Similarly we have $NL^T = LL^T$.
Therefore $N \overset{T}{\geq} L$.

$L^TM = \begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ ....(5)
$L^TL = \begin{bmatrix}\langle 0.4,0.5,0.1\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix}$ ....(6)

Hence from (5) and (6), we get $L^TM = L^TL$.
Similarly we have $ML^T = LL^T$.
Therefore $M \overset{T}{\geq} L$.

VI. Reverse Minus Ordering on Picture Fuzzy Matrix

Definition VI.1. For $M \in \mathscr{P}^-_{(m,n)}$ and $N \in \mathscr{P}_{(m,n)}$, the reverse minus ordering denoted as $\bar{>}$ is define as $M \bar{>} N \Leftrightarrow N^-N = N^-M$ and $NN^- = MN^-$ for some $N^- \in N\{1\}$.

Example VI.1. Let $M = \begin{bmatrix}\langle 0.0,0.0,0.5\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$,
then
$N = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
Now,
$N^- = \begin{bmatrix}\langle 0.0,0.0,0.5\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\end{bmatrix}$ and
$N^-N = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\end{bmatrix}$ ....(1)
$N^-M = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\end{bmatrix}$ ....(2)

Hence from (1) and (2), we get $NN^- = N^-M$.
Similarly we have $NN^- = MN^-$.
Therefore $M \bar{>} N$.

Lemma VI.1. For $N \in \mathscr{P}^-_{(m,n)}$ and $M \in \mathscr{P}_{(m,n)}$, the following are equivalent
(i) $M \bar{>} N$.
(ii) $N = NN^-M = MN^-N = MN^-M$.
Proof.
(i) ⇒ (ii)
$M \bar{>} N \Rightarrow NN^- = MN^-$ and $N^-N = N^-M$
for some $N^- \in N\{1\}$.
Now,

$$\begin{aligned}N &= N(N^-N) = NN^-M\\ N &= (NN^-)N = MN^-N\\ N &= M(N^-N) = MN^-M\end{aligned}$$

(ii) ⇒(i)
Let $X = N^-NN^-$

$$\begin{aligned}NXN &= N(N^-NN^-)N\\ &= (NN^-N)N^-M\\ &= N\end{aligned}$$

$\Rightarrow X \in N\{1\}$.
Now

$$\begin{aligned} XN &= (N^-NN^-)NN^-M \\ &= N^-(NN^-N)N^-M \\ &= (N^-NN^-)M \\ &= XM. \end{aligned}$$

Similarly we have $NX = MX$. Hence $M \overset{-}{>} N$ with respect to $X \in N\{1\}$.

Theorem VI.1. Let $M,\ N \in \mathscr{P}^-_{(m,n)}$. If $M \overset{-}{>} N$, then $M\{1\} \subseteq N\{1\}$.
Proof: By Lemma VI.1,
$M \overset{-}{>} N \Rightarrow N = NN^-M = MN^-N$.
For $M^- \in M\{1\}$,

$$\begin{aligned} NM^-N &= (NN^-M)M^-(MN^-N) \\ &= NN^-(MM^-M)NN^- \\ &= (NN^-M)N^-N \\ &= NN^-N \\ &= N. \end{aligned}$$

Hence $NM^-N = N$ for each $M^- \in M\{1\}$.
Therefore, $M\{1\} \subseteq N\{1\}$.

VII. Left-T and Right-T Partial Orderings on Picture Fuzzy Matrix

Definition VII.1. Let $M, N \in \mathscr{P}_{(m,n)}$. We say that M is below N with respect to left-T ordering if $M^TM = M^TN$ and $C(M) \subseteq C(N)$ and is denoted by $Mt < N$. We say that M is below N with respect to right-T ordering if $MM^T = NM^T$ and $R(M) \subseteq R(N)$ and is denoted by $M < tN$.
In general there is no relation between these two orderings. This is illustrated in the following example.

Example VII.1. Let $M = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$ and $N = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$ then
$M^TM = M^TN = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.5, 0.5, 0.0\rangle \end{bmatrix}$
Therefore, $M^TM = M^TN$ and

Let $Y = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.0, 0.0, 0.5\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.5, 0.5, 0.0\rangle \end{bmatrix}$ then
$NY = M \Rightarrow C(M) \subseteq C(N)$
Hence $Mt < N$. But $M \not< tN$ Since $MM^T \neq NM^T$.

In particular, for $M \in \mathscr{P}^+_{(m,n)}$ since $M^+ = M^T$ the above definition is equivalent to the following
$Mt < N \Leftrightarrow M^+M = M^+N$ and $C(M) \subseteq C(N)$
$M < tN \Leftrightarrow MM^+ = NM^+$ and $R(M) \subseteq R(N)$

Theorem VII.1. Let $M \in \mathscr{P}^+_{(m,n)}$, $N \in \mathscr{P}_{(m,n)}$. $Mt < N$ and $M < tN \Leftrightarrow M \overset{T}{<} N$
Proof. $Mt < N$ and $M < tN$
$\Rightarrow M^TM = M^TN$ and $MM^T = NM^T$
$\Rightarrow M \overset{T}{<} N$.
Conversely, $M \overset{T}{<} N$
$\Rightarrow M^TM = M^TN$ and $MM^T = NM^T$
$\Rightarrow M^+M = M^+N$ and $MM^+ = NM^+$
Now, $M \overset{T}{<} N \Rightarrow M^+M = M^+N$
$\Rightarrow MM^+M = MM^+N$ (premultiply by M)
$\Rightarrow M = XN$ where $X = MM^+$
$\Rightarrow R(M) \subseteq R(N)$
Similarly, $MM^+ = NM^+$
$\Rightarrow M = NM^+M$
$\Rightarrow M = NY$
$\Rightarrow C(M) \subseteq C(N)$
Thus $M \overset{T}{<} N \Leftrightarrow Mt < N$ and $M < tN$.

Example VII.2. Let $M = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ and $N = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ then
$M^TM = M^TN = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$ and
$MM^T = NM^T = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$
$\Rightarrow M \overset{T}{<} N$

Let $X = \begin{bmatrix} \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.2, 0.3\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix}$ and
$Y = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.0, 0.0, 0.5\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.5, 0.5, 0.0\rangle \end{bmatrix}$ then

$XN = M \Rightarrow R(M) \subseteq R(N)$
$NY = M \Rightarrow C(M) \subseteq C(N)$
Hence $Mt < N$.

Theorem VII.2. Let $M \in \mathscr{P}^+_{(m,n)}$ and $N \in \mathscr{P}_{(m,n)}$, if either $Mt < N$ or $M < tN$ then $M \overset{-}{<} N$
Proof. $Mt < N \Rightarrow M^TM = M^TN$
$\Rightarrow M^+M = M^+N$
$\Rightarrow MM^+M = MM^+N$
$\Rightarrow M = (MM^+)N$
$\Rightarrow R(M) \subseteq R(N)$
Now, $M^TM = M^TN$
$\Rightarrow M^TMN^-M = M^TNN^-M$
$\Rightarrow M^+MN^-M = M^+NN^-M$
$\Rightarrow (MM^+M)N^-M = MM^+(NN^-M)$
$\Rightarrow MN^-M = M$
Hence $Mt < N \Rightarrow C(A) \subseteq C(B)$, $R(M) \subseteq R(N)$ and $MN^-M = M$
$\Rightarrow M \overset{-}{<} N$
Proof of $M < tN \Rightarrow M \overset{-}{<} N$ can be proved in the same manner.

Theorem VII.3. For $M, N \in \mathscr{P}^+_{(m,n)}$ we have
(i) $Mt < N \Leftrightarrow M^+t < N^+$.
(ii) $M < tN \Leftrightarrow M^+ < tN^+$.
Proof. $Mt < N$
$\Rightarrow M^TM = M^TN$ and $C(M) \subseteq C(N)$.
Now, $C(M) \subseteq C(N)$
$\Rightarrow M = NN^+M$
$\Rightarrow M = NN^TM$
$\Rightarrow M^T = M^TNN^T$
$\Rightarrow M^T = (N^TM)N^T$
$\Rightarrow M^T = N^T(MN^T)$
$\Rightarrow C(M) \subseteq C(N)$
Now,$M^TM = M^TN \Rightarrow M(M^TM)N^T = M(M^TN)N^T$
$\Rightarrow MN^T = M(NN^TM)^T$

$\Rightarrow MN^T = MM^T$
$\Rightarrow MN^+ = MM^+$
Thus $(M^+)^T N^+ = (M^+)^T M^+$ and $C(M^+) \subseteq C(N^+)$
$\Rightarrow M^+ t < N^+$.
Converse follows from above part by using $(M^+)^+ = M$.
Proof of (ii) is similar to the same manner.

Theorem VII.4. For $M, N \in \mathscr{P}^+_{(m,n)}$ we have
(i) $Mt < N \Leftrightarrow MM^+ = MN^+$ and
$R(M) \subseteq R(N) \Leftrightarrow M < tN$.
(ii) $M < tN \Leftrightarrow M^+ M = N^+ M$ and
$C(M) \subseteq C(N) \Leftrightarrow Mt < N$.
Proof. (i) $Mt < N \Leftrightarrow M^+ t < N^+$
$\Leftrightarrow (M^+)^T M^+ = (M^+)^T N^+$ and $C(M^+) \subseteq C(N^+)$

VIII. Decision-making Using Picture Fuzzy Matrix

ALGORITHM:
Step 1 Define the PFM: Identify the criteria and alternatives, and assign the corresponding picture fuzzy numbers $(\langle \mu, \eta, \nu \rangle)$ to each alternative under each criterion.
Step 2 Calculate the Weighted Sum: Assign weights to each criterion based on their importance. Then, compute the weighted sum of the picture fuzzy numbers for each alternative.
Step 3 Defuzzification: Convert the picture fuzzy numbers into crisp values for easier comparison.
Step 4 Rank the Alternatives: Compare the crisp values of the alternatives and rank them.

Case Study: Choosing the Optimal Car Based on Multi-Criteria Decision Making.
Background:
You are in the market for a new car and need to make a decision based on multiple criteria. The decision-making problem involves evaluating three car options against three key criteria: Price, Fuel Efficiency, and Safety.

This case study will guide you through the process of selecting the best car by analyzing these criteria systematically.

Criteria for Evaluation:
1. Price (C1): The cost of purchasing the car.
2. Fuel Efficiency (C2): The cars efficiency in terms of fuel consumption, typically measured in miles per gallon (MPG) or liters per 100 kilometers.
3. Safety (C3): The cars safety features and ratings, including crash test ratings, airbags, and advanced safety technologies.

Normalize the data for each criterion to ensure comparability. This can be done by scaling the data so that it falls within a common range (e.g., 0 to 1).

Objective: To identify which of the three cars – Car A, Car B, and Car C, provides the highest overall value considering the criteria of Price, Fuel Efficiency, and Safety.

Step 1 Define the PFM:
Consider the following picture fuzzy numbers assigned to each car for each criterion:

For Car–A:
Price (C1): $\langle 0.7, 0.1, 0.2 \rangle$
Fuel Efficiency (C2): $\langle 0.6, 0.1, 0.3 \rangle$
Safety (C3): $\langle 0.8, 0.1, 0.1 \rangle$

For Car–B:
Price (C1): $\langle 0.6, 0.1, 0.3 \rangle$
Fuel Efficiency (C2): $\langle 0.7, 0.1, 0.2 \rangle$
Safety (C3): $\langle 0.7, 0.1, 0.2 \rangle$

For Car–C:
Price (C1): $\langle 0.8, 0.1, 0.1 \rangle$
Fuel Efficiency (C2): $\langle 0.6, 0.1, 0.3 \rangle$
Safety (C3): $\langle 0.6, 0.1, 0.3 \rangle$

Step 2 Calculate the Weighted Sum:
Assume the weights for the criteria are as follows:
- W1 (Price) = 0.4
- W2 (Fuel Efficiency) = 0.3
- W3 (Safety) = 0.3

To calculate the weighted sum for Car A, for example:
$WSCarA = 0.4 \times \langle 0.7, 0.2, 0.1 \rangle + 0.3 \times \langle 0.6, 0.3, 0.1 \rangle + 0.3 \times \langle 0.8, 0.1, 0.1 \rangle$
$WSCarA_\mu = 0.7$
$WSCarA_\eta = 0.1$
$WSCarA_\nu = 0.2$
So, the weighted sum for Car A is $\langle 0.7, 0.1, 0.2 \rangle$.

Step 3 Defuzzification:
To convert the picture fuzzy numbers into crisp values, use a defuzzification formula. One simple method is:
For Car A:
$DCarA = WSCarA_\mu - WSCarA_\nu = 0.7 - 0.2 = 0.5$
Similarly, we get
$DCarB = WSCarB_\mu - WSCarB_\nu = 0.44$
$DCarC = WSCarC_\mu - WSCarC_\nu = 0.57$
Step 4 Rank the Alternatives:
Now, compare the defuzzified values:
- Car A: 0.5
- Car B: 0.44
- Car C: 0.57

Ranking:
1. Car C (0.57)
2. Car A (0.5)
3. Car B (0.44)

Based on this decision-making algorithm, Car C is the best option.

IX. Conclusion

This article has undertaken a comprehensive exploration of T-ordering and Minus-ordering characteristics in the domain of Picture Fuzzy Matrices. We have studied the relationship of T-ordering on Moore-Penrose Inverse, as well as the relationship between minus ordering and different g-inverses of Picture Fuzzy Matrices. We have also provided proof that T-ordering and Minus ordering satisfy the partial ordering relation. Furthermore, we have conducted a study of several properties of T-ordering and Minus ordering with various g-inverses, supported by suitable examples. Towards the end, we also delve into a new concept called reverse T ordering and reverse Minus ordering and Left-T and Right-T Partial Orderings on Picture Fuzzy Matrix, introducing and elucidating these concepts through illustrative examples. Finally, we apply picture fuzzy to a decision-making case study.

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