Vertex Reducible Total Labeling of Cyclic Graphs and Their Associated Graphs

Xin Gao, Jingwen Li, Liangjing Sun

Abstract—**For an undirected connected graph *G (p, q)*, where *p* is the number of vertices and *q* is the number of edges, if there exists a single mapping f: $V(G) \cup E(G) \rightarrow \{1,2,\cdots, p+q\}$, such that the sum of labels of all vertices and their associated edges with the same degree in the graph is constant, then the mapping relationship is called Vertex Reduced Total Labeling (VRTL). On the basis of existing reducible algorithms and combined with practical problems, a new heuristic search algorithm is designed. This algorithm uses preprocessing and adjustment functions, and employs a cyclic iterative optimization method to label the vertices and edges of a random graph, ensuring that the total sum of labels for the same degree vertices is the same. By conducting in-depth analysis and research on the experimental results of finite vertices, the labeling rules for the graphs related to cycle graphs were summarized, and the labeling rules for infinite vertex cyclic graphs were derived. By summarizing these rules, a series of theorems about cyclic graphs and their connected graphs have been obtained, and corresponding proofs have been provided. Finally, two conjectures are proposed.**

Index Terms—**cycle graphs, vertex reducible total labeling, heuristic search algorithm, recurrent iteration;**

I. Introduction

GRAPH theory studies complex problems by abstracting them into the graphical forms. In 1736, the Swedish mathematician Euler proved that there is no method for traversing the Seven Bridges of Königsberg, thereby paving the way for the development of graph theory. In recent years, with the advancement of information technology, researchers have approached graph theory problems from a novel perspective, utilizing computer design algorithms. This not only addressed the limitations of traditional graph labeling research but also introduced various new labeling concepts based on existing concepts. In 1970, Rosa [1] and Kotzig [2] proposed the concepts of edge magic total labeling, establishing a foundation for the labeling theory within graph theory. Subsequently, in 1997, Burris A. C. and Schelp R. H. introduced the concept of vertex-distinguishing edge coloring [3]-[5] along with related conjectures, marking a significant advancement in the exploration of vertex and edge coloring problems. Furthermore, in 2009, Professor Zhang Zhongfu [6] and other scholars proposed a series of concepts related to reducible coloring [7]-[8], based on different perspectives within distinguishable coloring theory. This expansion not only broadened the theoretical scope of graph labeling and coloring but also opens up the way for introducing more derivative concepts. Later, in 2023, researchers proposed and designed a novel algorithm for adjacent vertex reducible total labeling [9]-[11], resulting in several relevant theorems and conjectures. This progression indicates that the study of graph labeling continues to evolve, meeting the demand for a deeper exploration of graph structures.

Graph labeling problems have widely applications in computer networks, logistics and transportation, social networks, wireless spectrum analysis, and other fields. For instance, in computer networks, vertex and edge labeling can be utilized for routing and path selection. In social networks, vertex labeling may represent different users or specific social relationships. This paper introduces an innovative algorithm that integrates practical issues with the idea of vertex reducible total labeling. This algorithm addresses the issue of vertex reducible total labeling in finite vertex inner cyclic graphs, and is employed to conduct experiments on the atlas of unicyclic and bicyclic graphs. By analyzing the experimental results, this article summarizes the labeling rules for the graphs related to cycle graphs, and derives the labeling rules for infinite vertex cyclic graphs. Through this summarization, a series of theorems regarding cyclic graphs and their connected graphs is established, accompanied by corresponding proofs. Finally, two conjectures are proposed.

II. Basic Knowledge

Definition 1: For a connected graph $G(p, q)$, where p represents the number of vertices and q represents the number of edges, if there exists a single mapping $f{:}V(G) \cup E(G) \rightarrow \{1,2,\cdots, p+q\}$, such that the sum of labeling for all vertices of the same degree and their associated edges satisfies the equation $Sum(u)=f(u)+\sum_{uv \in E(G)} f(uv)=K$, where K is a constant, then this mapping relationship is referred to as Vertex Reduced Total Labeling (VRTL).

Definition 2: For path, cyclic, star, fan, and wheel graphs, let G_1 and G_2 be one of the above graphs, and agree to use the symbol a to represent the central node of the star, fan, and wheel graphs, the 1-degree vertex of the path graph, and any vertex of the circle graph. The symbol b represents non-central nodes in star and wheel graphs, 2-degree vertices

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in fan graphs, and 2-degree vertices in road graphs. The joint graph $G_1 \uparrow_{aa} G_2$ connects the node a of graph G_1 to the node a of graph G_2, as shown in Fig. 1.

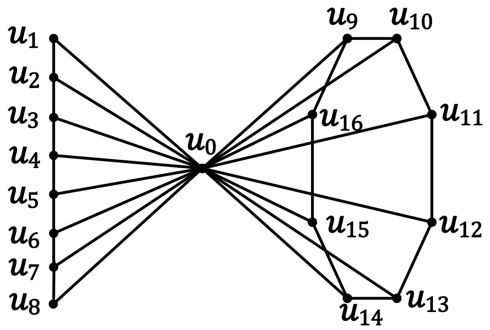


Fig. 1. $F_8 \uparrow_{aa} W_8$

Definition 3: Let G_1 and G_2 be two simple graphs. The joint graph $G_1 \downarrow G_2$ represents the graph formed by bonding any edge in graph G_1 to any edge in graph G_2. For the number of vertices, graph $G_1 \downarrow G_2$ is 2 less than the sum of G_1 and G_2, and for the number of edges, graph $G_1 \downarrow G_2$ is 1 less than the sum of G_1 and G_2.

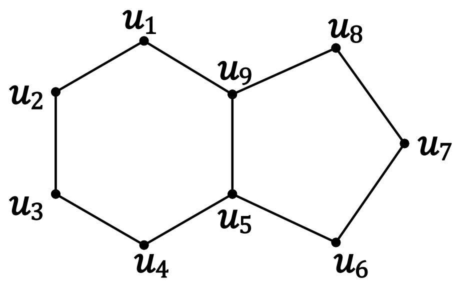


Fig. 2. $G_6 \downarrow G_5$

Definition 4: Let G_1 and G_2 be two simple graphs. The corona graph $G_1 \circ G_2$ is formed by connecting all vertices of a copied graph G_2 to each vertex of graph G_1. For example, the corona graph $C_5 \circ P_2$ as shown in Fig. 3.

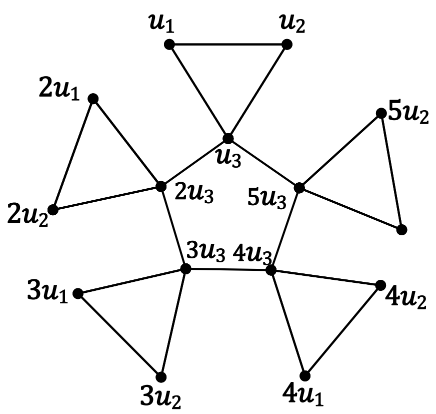


Fig. 3. $C_5 \circ P_2$

Definition 5: Let G_1 and G_2 be two simple graphs. The generalized corona graph $G_1^a \circ G_2^a$ is formed by connecting a copied a node of the graph G_2 to each a node of the graph G_1. For example, the graph $C_{10}^a \circ S_3^a$ as shown in Fig. 4.

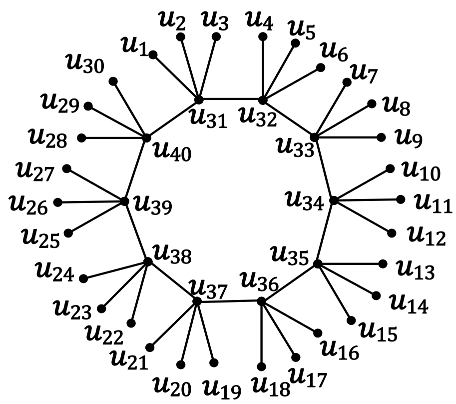


Fig. 4. $C_{10}^a \circ S_3^a$

III. ALGORITHM

A. Fundamentals of the algorithm

The fundamental concept of the VRTL algorithm is to initialize the adjacency matrix of graph G, then perform a complete permutation of the solution space of graph G and recursively search for a valid state. If a state that satisfies the balance function is found during the search process, the current state M is recorded and output, and terminate the algorithm. If no matrix satisfies the equilibrium state after traversing the solution space of graph G, it indicates that graph G is not a VRTL graph, and the algorithm terminates. The implementation of the algorithm is primarily divided into the following processes:

Step 1: Preprocess the initialization matrix by calculating the number of edges and vertices, the degree of each vertices, the degree sequence, and other relevant information about graph G.

Step 2: Set up a classification function *Classify* to group vertices of the same degree in graph G into one set, while placing all other vertices into a separate set.

Step 3: Define a balance function *IsBalance* to determine whether graph G satisfies vertex reducible total labeling during the search of the solution space.

Step 4: Recursively search the solution space of the graph set and utilize the balance function to determine whether there exists a labeled matrix that satisfies vertex reducible total labeling during the search of the solution space. Finally, select the graph set that meets the specified conditions and output it as a labeled matrix. The algorithm ends here. If no solution that satisfies the constraint of vertex reducible total labeling within the solution space of the graph set, the output graph G is an N-VRTL graph, and the algorithm terminates.

B. Pseudocode

Input	The adjacency matrix of the graph $G(p,q)$
output	The matrix satisfying the labeling requirements
1	Read the initial adjacency matrix.
2	Get *p q degree IsBalance* $\varphi(p\ q)$ *Classify* /* p is the number of vertices, q is the number of edges, *degree* represents the degree sequence, *IsBalance* represents the balance operator that determines whether the label is satisfied, $\varphi(p\ q)$

	represents the solution space, and *Classify* uses a classification function to filter the graph set for the existence of similarity vertices */
3	while(φ(p, q) !=null)
4	search φ(p, q)→update G. Adjust Matrix
5	If G. Is Balance ← true
6	Label Adjust ← Adjust Matrix
7	end if
8	If G. Is Balance ← false
9	Output This graph is a vertex reduced total labeling graph
10	end if
11	end while
12	else
13	Output Label Adjust
14	end else
15	end

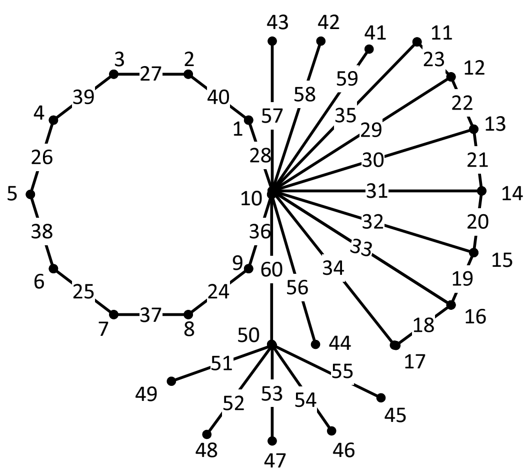
C. Analyze experimental results

By using this algorithm to validate all unicyclic graphs and bicyclic graphs within 3-9 vertices, it is possible to count the number of graphs that satisfy the vertex reduced total labeling. From this analysis, it can be concluded that when the number of vertices is $3\leq n\leq 9$, all unicyclic graphs and bicyclic graphs are VRTL graphs, as shown in Table I.

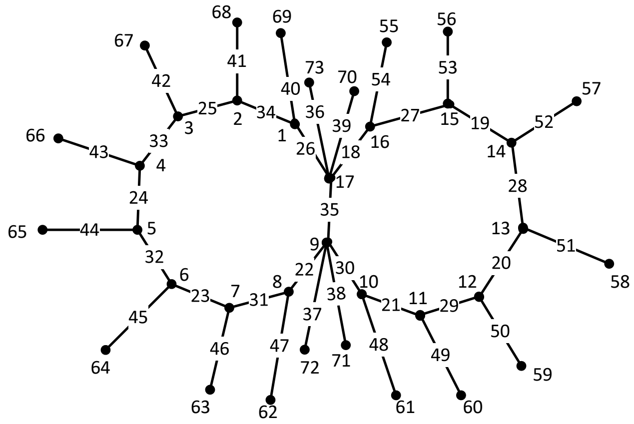
TABLE I
STATISTICS OF UNICYCLIC AND BICYCLIC GRAPHS WITHIN 9 VERTICES

unicyclic graphs (p,q)	The total number of graphs	The number of VRTL graphs	bicyclic graph (p,q)	The total number of graphs	The number of VRTL graphs
(3,3)	1	1	(3,4)	0	0
(4,4)	2	2	(4,5)	1	1
(5,5)	5	5	(5,6)	5	5
(6,6)	13	13	(6,7)	19	19
(7,7)	33	33	(7,8)	67	67
(8,8)	89	89	(8,9)	236	236
(9,9)	240	240	(9,10)	797	797

The labeling results of graph *G* (27, 33) and graph *G* (36, 37) are shown in Fig. 5.



(a) *G* (27, 33)



(b) *G* (36, 37)

Fig. 5. The VRTL of *G* (27, 33) and *G* (36, 37).

IV. THEOREM AND PROOF

Theorem 1: For a circle graph C_n, when $n\geq 3\cap n\equiv 1(mod 2)$ or $4\leq n\leq 8\cap n\equiv 0(mod 2)$, it is a VRTL graph.

Proof: Let the vertex set of graph C_n be $\{u_1,u_2,\ldots,u_n\}$ and the edge set be $\{u_1u_n\cup u_iu_{i+1}\mid 1\leq i\leq n-1\}$. The graph C_n contains n edges and n vertices in total.

Case 1: when $n\geq 3\cap n\equiv 1(mod 2)$, the VRTL of C_n is:

$$f\left(u_i\right)=i,(1\leq i\leq n)$$

$$f\left(u_1u_n\right)=2n-\frac{n-1}{2}$$

$$f\left(u_iu_{i+1}\right)=\begin{cases}2n-\dfrac{i-1}{2},(i=1,3,5\cdots)\\ \dfrac{3n-i+1}{2},(i=2,4,6\cdots)\end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of C_n are:

$$f(V)=\{1,2,\cdots,n\}$$
$$f(E)=\{n+1,n+2,\cdots,2n\}$$

Currently, in the vertex set $\{u_1,u_2,\ldots,u_n\}$, all vertices are 2-degree vertices. It is essential to ensure that the sum of labeling for all vertices is the same in the graph C_n.

The total sum of the labeling for all vertices is:

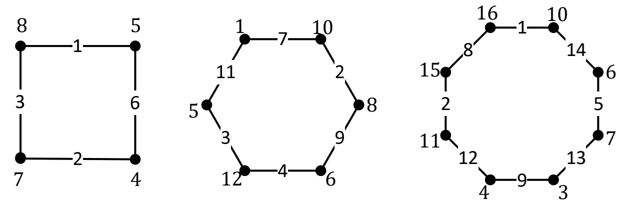
$$Sum_2=\{f(u_i)+\sum\nolimits_{uv\in E(u_i)}f\left(uv\right)\mid 1\leq i\leq n\}$$
$$=f(u_1u_2)+f(u_1)+f(u_1u_n)f(u_i)\,\|\cdots\|$$
$$f(u_1u_n)+f(u_n)+f(u_{n-1}u_n)$$
$$=\frac{7n+3}{2}$$

The symbol "||" appearing throughout the text represents a logical OR.

According to the definition of VRTL, f is a single mapping function from the edge set $E(C_n)$ and vertex set $V(C_n)$ to $\{1,2,\cdots,2n\}$, and the sum of labeling for all the same degree vertices is a constant.

Thus, when $n\geq 3\cap n\equiv 1(mod 2)$, the circle graph C_n is proven to be a VRTL graph, as shown by case 1.

Case 2: when $4\leq n\leq 8\cap n\equiv 0(mod 2)$, the VRTL of C_n is shown in Fig. 6.



(a) C_4 (b) C_6 (c) C_8

Fig. 6. The VRTL of C_4 , C_6 and C_8 .

Due to $f(V)\bigcup f(E)\to[1,2n]$, $f(V)\bigcap f(E)=\varnothing$ and the fact that all the labeling sum of the same degree vertices is a constants, the circle graph C_n is proven to be a VRTL graph when $4\le n\le 8\bigcap n\equiv 0(mod2)$.

To sum up, when $n\ge 3\bigcap n\equiv 1(mod2)$, $4\le n\le 8\bigcap n\equiv 0(mod2)$, the circle graph C_n is a VRTL graph.

Conjecture 1: The circle graph C_n are VRTL graphs.

Theorem 2: The joint graph $C_n\uparrow_{aa}P_m(n\ge 3,m\ge 2)$ is a VRTL graph.

Proof: Set the vertex set of the joint graph $C_n\uparrow_{aa}P_m$ as $\{u_1,u_2,\ldots,u_n,u_{n+1},\cdots,u_{n+m-1}\}$ and the edge set as $\{u_1u_n\bigcup u_iu_{i+1}\mid 1\le i\le n+m-2\}$. The graph $C_n\uparrow_{\mathrm{aa}}P_m$ contains $n+m-1$ edges and $n+m-1$ vertices, as shown in Fig. 7.

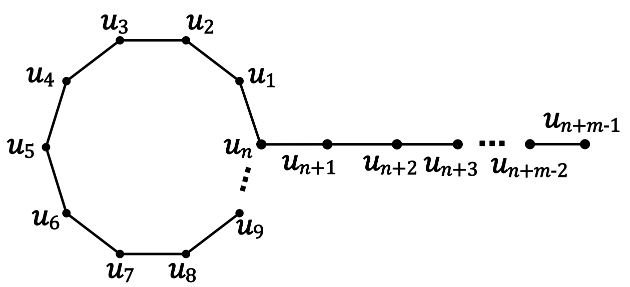


Fig. 7. $C_n\uparrow_{aa}P_m$

Firstly, the VRTL of $C_n\uparrow_{aa}P_m$ is:

$$f(u_i)=i,1\le i\le n+m-1$$

$$f(u_1u_n)=2m+2n-2-\left\lfloor\frac{m+n-1}{2}\right\rfloor$$

$$f(u_iu_{i+1})=\begin{cases}2m+2n-2-\dfrac{i-1}{2},(i=1,3,5\cdots)\\[2mm]2m+2n-2-\left\lfloor\dfrac{m+n-1}{2}\right\rfloor-\dfrac{i}{2},(i=2,4,6\cdots)\end{cases}$$

The symbol $\lfloor\ \rfloor$ appearing throughout the text represents rounding down when the calculation result is not an integer.

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n\uparrow_{aa}P_m$ are:

$$f(V)=\{1,2,\cdots,\ n+m-1\}$$

$$f(E)=\{n+m,n+m+1,\cdots,2n+2m-2\}$$

Currently, in the vertex set $\{u_1,u_2,\cdots,u_n,u_{n+1},\cdots,u_{n+m-1}\}$, u_{n+m-1} is a 1-degree vertex, $u_1,u_2,\cdots,u_{n-1}\bigcup u_{n+1},u_{n+2}\cdots,u_{n+m-2}$ are 2-degree vertices, and u_n is a 3-degree vertex. There are no vertices of the same degree as u_n and u_{n+m-1}, so it is not necessary to consider the sum of its labeling. It is essential to ensure that the sum of labeling for all 2-degree vertices labeling are the same in the graph $C_n\uparrow_{aa}P_m$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2=\{f(u_i)+\sum\nolimits_{uv\in E(u_i)}f(uv)\mid 1\le i\le n+m-2,i\ne n\}$$

$$=f(u_i)+f(u_iu_{i+1})+f(u_{i-1}u_i)\mid 2\le i\le n+m-2\bigcap i\ne n\ \|$$

$$f(u_1u_2)+f(u_1)+f(u_1u_n)$$

$$=4n+4m-3-\left\lfloor\frac{m+n-1}{2}\right\rfloor$$

According to the definition of VRTL, f is a single mapping function from the edge set $E(C_n\uparrow_{aa}P_m)$ and vertex set $V(C_n\uparrow_{aa}P_m)$ to $\{1,2,\cdots,2n+2m-2\}$, and the sum of labeling for all the same degree vertices is a constant.

Thus, the joint graph $C_n\uparrow_{\mathrm{aa}}P_m(n\ge 3,m\ge 2)$ is proven to be a VRTL graph.

Theorem 3: The joint graph $C_n\uparrow_{aa}C_m(n\ge 3,m\ge 3)$ is a VRTL graph.

Proof: Set the vertex set of the joint graph $C_n\uparrow_{aa}C_m$ as $\{u_1,u_2,\cdots,u_n,u_{n+1},\cdots,u_{n+m-1}\}$ and the edge set as $\{u_1u_n\bigcup u_iu_{i+1}\mid 1\le i\le n+m-2\}\bigcup\{u_nu_{n+m-1}\}$. The graph $C_n\uparrow_{aa}C_m$ contains $n+m$ edges and $n+m-1$ vertices, as shown in Fig. 8.

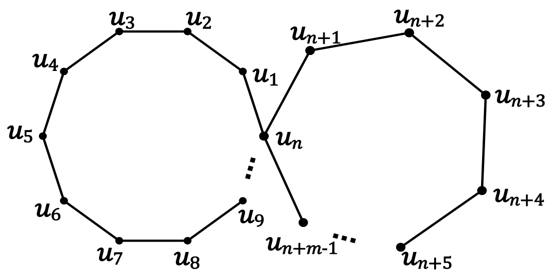


Fig. 8. $C_n\uparrow_{aa}C_m$

Case 1: when $n\ge 3,m\ge 3\bigcap m+n\equiv 0(mod2)$, the VRTL of $C_n\uparrow_{aa}C_m$ is:

$$f(u_i)=i,0<i<n+m$$

$$f(u_1u_n)=2n+2m-1$$

$$f(u_nu_{n+m-1})=n+m$$

$$f(u_iu_{i+1})=\begin{cases}2n+2m-1-\dfrac{i}{2},(i=2,4,6\cdots)\\[2mm]2n+2m-1-\dfrac{m+n+i-1}{2},(i=1,3,5\cdots)\end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n\uparrow_{aa}C_m$ are:

$$f(V)=\{1,2,\cdots,\ n+m-1\},$$

$$f(E)=\{n+m,n+m+1,\cdots,2n+2m-2\}。$$

Currently, in the vertex set $\{u_1,u_2,\cdots,u_n,u_{n+1},\cdots,u_{n+m-1}\}$, $u_1,u_2,\cdots,u_{n-1}\bigcup u_{n+1},u_{n+2}\cdots,u_{n+m-1}$ are all 2-degree vertices, and u_n is a 4-degree vertex. Since there are no vertices with the same degree as u_n , calculating the sum of its labeling is completely unnecessary. It is essential to ensure that the sum of labeling for all 2-degree vertex are the same in the graph $C_n\uparrow_{\mathrm{aa}}C_m$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2 = \{ f(u_i) + \sum_{uv\in E(u_i)} f(uv) \mid 1 \le i \le n+m-2, i \ne n\}$$

$$= f(u_i) + f(u_i u_{i+1}) + f(u_{i-1}u_i) \mid 2 \le i \le n+m-2 \cap i \ne n \,\|$$

$$\{ f(u_1u_2) + f(u_1) + f(u_1u_n)$$

$$= \{4n + 4m - 1 - \frac{m+n}{2}\}$$

According to the definition of VRTL, f is a one-to-one mapping function from the vertex set $V(C_n \uparrow_{aa} C_m)$ and edge set $E(C_n \uparrow_{aa} C_m)$ to $\{1,2,\cdots,2n+2m-1\}$, and the sum of labeling for all the same degree vertices is a constant.

Thus, when $n \ge 3, m \ge 3 \cap m+n \equiv 0(mod 2)$, the joint graph $C_n \uparrow_{aa} C_m$ is proven to be a VRTL graph, as shown by case 1.

Case 2: When $n \ge 3, m \ge 3 \cap m+n \equiv 1(mod 2)$, the VRTL of $C_n \uparrow_{aa} C_m$ is:

$$f(u_i) = i, 0 < i < n+m$$

$$f(u_1 u_n) = 2n + 2m - 1$$

$$f(u_n u_{n+m-1}) = 2n + 2m - 1 - \frac{m+n-1}{2}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+2m-1-\frac{m+n+i}{2}, (i = 1,3,5\cdots) \\ 2n+2m-1-\frac{i}{2}, (i = 2,4,6\cdots) \end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n \uparrow_{aa} C_m$ are:

$$f(V) = \{1, 2, \cdots, n+m-1\}$$

$$f(E) = \{n+m, n+m+1, \cdots, 2n+2m-2\}$$

Currently, in the vertex set $\{u_1, u_2, \cdots, u_n, u_{n+1}, \cdots, u_{n+m-1}\}$, $u_1, u_2, \cdots, u_{n-1} \cup u_{n+1}, u_{n+2} \cdots, u_{n+m-1}$ are all 2-degree vertices, and u_n is a 4-degree vertex. Since there are no vertices with the same degree as u_n, calculating the sum of its labeling is completely unnecessary. It is essential to ensure that the sum of labeling for all 2-degree vertices are the same in the graph $C_n \uparrow_{aa} C_m$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2 = \{ f(u_i) + \sum_{uv\in E(u_i)} f(uv) \mid 1 \le i \le n+m-1, i \ne n\}$$

$$= f(u_i) + f(u_i u_{i+1}) + f(u_{i-1}u_i) \mid 2 \le i \le n+m-2 \cap i \ne n \,\|$$

$$f(u_{n+m-1}) + f(u_n u_{n+m-1}) + f(u_{n+m-2} u_{n+m-1}) \,\|$$

$$f(u_1u_2) + f(u_1) + f(u_1u_n)$$

$$= \{4n + 4m - 1 - \frac{m+n+1}{2}\}$$

According to the definition of VRTL, f is a single mapping function from the edge set $E(C_n \uparrow_{aa} C_m)$ and vertex set $V(C_n \uparrow_{aa} C_m)$ to $\{1,2,\cdots,2n+2m-1\}$, and the sum of labeling for all the same degree vertices is a constant.

Thus, when $n \ge 3, m \ge 3 \cap m+n \equiv 1(mod 2)$, the joint graph $C_n \uparrow_{aa} C_m$ is proven to be a VRTL graph, as shown by case 2.

To sum up, the joint graph $C_n \uparrow_{aa} C_m (n \ge 3, m \ge 3)$ is a VRTL graph.

Theorem 4: The joint graph $C_n \uparrow_{aa} S_m (n \ge 3, m \ge 3)$ is a VRTL graph.

Proof: Set the vertex set of the graph $C_n \uparrow_{aa} S_m$ as $\{u_1, u_2, \cdots, u_n \cup v_1, v_2, \cdots, v_m\}$ and the edge set as $\{u_1u_n \cup u_iu_{i+1}(1 \le i < n) \cup v_0 v_j (1 \le j \le m)\}$. The graph $C_n \uparrow_{aa} S_m$ contains $n+m$ edges and $n+m$ vertices, as shown in Fig. 9.

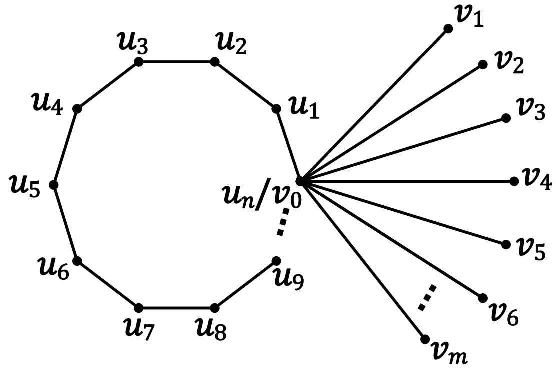


Fig. 9. $C_n \uparrow_{aa} S_m$

Firstly, the VRTL of $C_n \uparrow_{aa} S_m$ is:

$$f(u_i) = i, (1 \le i \le n)$$

$$f(v_j) = 2n + j, (1 \le j \le m)$$

$$f(u_1 u_n) = 2n$$

$$f(v_0 v_j) = 2n + 2m + 1 - j, (j = 1, 2, \cdots, m)$$

$$f(u_i u_{i+1}) = \begin{cases} \left\lfloor \frac{3n}{2} \right\rfloor - \frac{i-1}{2}, (i = 1,3,5\cdots) \\ 2n - \frac{i}{2}, (i = 2,4,6\cdots) \end{cases} (1 \le i < n)$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n \uparrow_{aa} S_m$ are:

$$f(V) = \{1, 2, \cdots, n \cup 2n+1, 2n+2, \cdots, 2n+m\}$$

$$f(E) = \{n+1, \cdots, 2n \cup 2n+m+1, 2n+m+2 \cdots, 2n+2m\}$$

Currently, in the vertex set $\{u_1, u_2, \cdots, u_n \cup v_1, v_2, \cdots, v_m\}$, $v_1, v_2, \cdots, v_m$ are all 1-degree vertices, $u_1, u_2, \cdots, u_{n-1}$ are all 2-degree vertices, and u_n is $m+2$-degree vertex. Since there are no vertices of the same degree as u_n, calculating the sum of its labeling is completely unnecessary. It is essential to ensure that the sum of labeling for all 1-degree vertex and 2-degree vertex labeling are the same in the graph $C_n \uparrow_{aa} S_m$.

The total sum of the labeling for the 1-degree vertices is:

$$Sum_1 = \{ f(v_j) + \sum_{uv\in E(v_j)} f(uv) \mid 1 \le j \le m\}$$

$$= f(v_j) + f(v_0 v_j) \mid 1 \le j \le m$$

$$= 4n + 2m$$

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2 = \{ f(u_i) + \sum_{uv\in E(u_i)} f(uv) \mid 1 \le i \le n-1\}$$

$$= f(u_i) + f(u_i u_{i+1}) + f(u_{i-1}u_i) \mid 2 \le i \le n-1 \,\|$$

$$\{ f(u_1) + f(u_1u_2) + f(u_1u_n)$$

$$= \left\lfloor \frac{7n}{2} \right\rfloor + 1$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(C_n \uparrow_{aa} S_m)$

and vertex set $V(C_n \uparrow_{aa} S_m)$ to $\{1,2,\cdots,2n+2m\}$, and the sum of labeling for all vertices of the same degree is a constant. Based on the definition of VRTL, the joint graph $C_n \uparrow_{aa} S_m (n \geq 3, m \geq 3)$ is proven to be a VRTL graph.

Theorem 5: The joint graph $C_n \uparrow_{aa} S_m (n \geq 3, m \geq 3)$ is a VRTL graph.

Proof: Connecting any vertex of the graph C_n to the center vertex of the graph F_m to form the joint graph $C_n \uparrow_{aa} F_m$. Set the vertex set as $\{u_1, u_2, \ldots, u_{n+m}\}$ and the edge set as $\{u_i u_{i+1} \mid 1 \leq i \leq n+m-1 \bigcup u_n u_{n+i} \mid 1 \leq i \leq m \bigcup u_1 u_n\}$. The graph $C_n \uparrow_{aa} F_m$ contains $n+2m-1$ edges and $n+m$ vertices in total, as shown in Fig. 10.

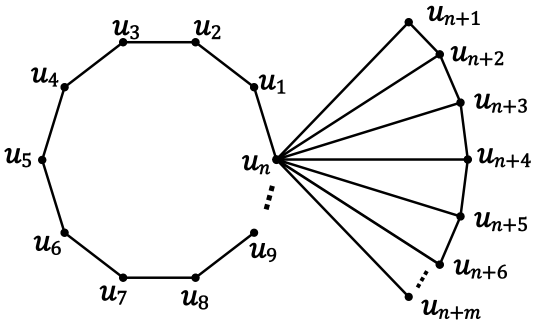


Fig. 10. $C_n \uparrow_{aa} F_m$

Firstly, the VRTL of $C_n \uparrow_{aa} F_m$ is:

$$f(u_i) = i, 1 \leq i \leq n+m$$

$$f(u_1 u_n) = 2n - \left\lfloor \frac{n}{2} \right\rfloor + 2m - 1$$

$$f(u_n u_{n+1}) = 2n - \left\lfloor \frac{n}{2} \right\rfloor + 3m - 1, i = 1$$

$$f(u_n u_i) = n - \left\lfloor \frac{n}{2} \right\rfloor + 2m - 2 + i, n+1 < i \leq n+m$$

$$f(u_i u_{i+1}) = \begin{cases} 2n + 3m - \dfrac{i+1}{2}, 1 \leq i < n \bigcap i \equiv 1 (mod 2) \\ 2n - \left\lfloor \dfrac{n}{2} \right\rfloor + 2m - 1 - \dfrac{i}{2}, 1 \leq i < n \bigcap i \equiv 0 (mod 2) \\ 2n + 2m - i, n < i < n+m \end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n \uparrow_{aa} F_m$ are:

$$f(V) = \{1, 2, \cdots, n+m\}$$

$$f(E) = \{n+m+1, n+m+2, \cdots, 2n+3m-1\}$$

Currently, in the vertex set $\{u_1, u_2, \cdots, u_n, u_{n+1}, \cdots, u_{n+m}\}$, $u_1, u_2, \cdots, u_{n-1} \bigcup u_{n+1} \bigcup u_{n+m}$ are all 2-degree vertices, and $u_{n+2}, u_{n+3}, \cdots, u_{n+m-1}$ are all 3-degree vertices, and u_n is $m+2$-degree vertex. Since there are no vertices of the same degree as u_n, calculating the sum of its labeling is completely unnecessary. It is essential to ensure that the sum of labeling for all 2-degree vertex and 3-degree vertex labeling are the same in the graph $C_n \uparrow_{aa} F_m$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2 = \{ f(u_i) + \textstyle\sum_{uv \in E(u_i)} f(uv) \mid 1 < i < n \bigcup i = n+1 \bigcup i = n+m \}$$
$$= f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) \mid 1 < i < n \parallel f(u_1 u_n) + f(u_1) + f(u_1 u_2) \parallel \{ f(u_n u_{n+m}) + f(u_{n+m}) + f(u_{n+m-1} u_{n+m})$$
$$= 4n + 5m - 1 - \left\lfloor \frac{n}{2} \right\rfloor$$

The total sum of the labeling for the 3-degree vertices is:

$$Sum_3 = \{ f(u_i) + \textstyle\sum_{uv \in E(u_i)} f(uv) \mid n+1 < i < n+m-2 \}$$
$$= f(u_i) + f(u_{i-1} u_i) + f(u_i u_{i+1}) + f(u_n u_i) \mid n+1 < i < n+m-2$$
$$= 5n + 6m - 1 - \left\lfloor \frac{n}{2} \right\rfloor$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(C_n \uparrow_{aa} F_m)$ and vertex set $V(C_n \uparrow_{aa} F_m)$ to $\{1,2,\cdots,2n+3m-1\}$, and the sum of labeling for all vertices of the same degree is a constant. According to the definition of VRTL, the joint graph $C_n \uparrow_{aa} F_m (n \geq 3, m \geq 3)$ is proven to be a VRTL graph.

Theorem 6: The joint graph $C_n \uparrow_{aa} W_m (n \geq 3, m \geq 3)$ is a VRTL graph.

Proof: Set the vertex set of the graph $C_n \uparrow_{aa} W_m$ as $\{u_1, u_2, \cdots, u_n \bigcup v_1, v_2, \cdots, v_m\}$ and the edge set as $\{u_1 u_n \bigcup u_i u_{i+1} (1 \leq i < n) \bigcup v_j v_{j+1} (1 \leq j < m) \bigcup v_1 v_m \bigcup v_0 v_j (1 \leq j \leq m)\}$. The graph $C_n \uparrow_{aa} W_m$ contains $n+2m$ edges and $n+m$ vertices, as shown in Fig. 11.

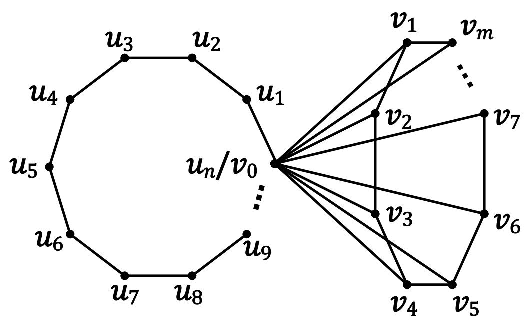


Fig. 11. $C_n \uparrow_{aa} W_m$

Firstly, the VRTL of $C_n \uparrow_{aa} W_m$ is:

$$f(u_i) = i, (1 \leq i \leq n)$$

$$f(v_j) = 2n + j, (1 \leq j \leq m)$$

$$f(u_1 u_n) = 2n$$

$$f(u_i u_{i+1}) = \begin{cases} \left\lfloor \dfrac{3n}{2} \right\rfloor - \dfrac{i-1}{2}, (i = 1,3,5\cdots) \\ 2n - \dfrac{i}{2}, (i = 2,4,6\cdots) \end{cases} (1 \leq i < n)$$

$$f(v_1 v_m) = 2n + 2m,$$

$$f(v_j v_{j+1}) = 2n + 2m - i, \quad (j = 1,2,\cdots,m-1)$$

$$f(v_0 v_j) = \begin{cases} 2n + 2m + 1 + j, & (j = 1,2,\cdots,m-1) \\ 2n + 2m + 1, & (j = m) \end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n \uparrow_{aa} W_m$ are:

$f(V)=\{1,2,\cdots,n\bigcup 2n+1,2n+2,\cdots,2n+m\}$
$f(E)=\{n,n+1,\cdots,2n\bigcup 2n+m+1,2n+m+2\cdots,2n+3m\}$

Currently, in the vertex set $\{u_1,u_2,\cdots,u_n\bigcup v_1,v_2,\cdots,v_m\}$, $u_1,u_2,\cdots,u_{n-1}$ are all 2-degree vertices, $v_1,v_2,\cdots,v_m$ are all 3-degree vertices, and u_n is m+2-degree vertex. Since there are no vertices of the same degree as u_n, calculating the sum of its labeling is completely unnecessary. It is essential to ensure that the sum of labeling for all 2-degree and 3-degree vertices are the same in the graph $C_n\uparrow_{aa}W_m$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2=\{f(u_i)+\sum_{uv\in E(u_i)}f(uv)\mid 1\le i\le n-1\}$$
$$=f(u_i)+f(u_iu_{i+1})+f(u_{i-1}u_i)\mid 1<i\le n-1\,\|$$
$$f(u_1)+f(u_1u_n)+f(u_1u_2)$$
$$=\left\lfloor\frac{7n}{2}\right\rfloor+1$$

The total sum of the labeling for the 3-degree vertices is:

$$Sum_3=\{f(v_j)+\sum_{uv\in E(v_j)}f(uv)\mid 1\le j\le m\}$$
$$=f(v_j)+f(v_jv_{j+1})+f(v_{j-1}v_j)+f(v_0v_j)\mid 1<j<m\,\|$$
$$f(v_1)+f(v_1v_m)+f(v_1v_2)+f(v_0v_1)\,\|$$
$$f(v_m)+f(v_{m-1}v_m)+f(v_1v_m)+f(v_0v_m)$$
$$=8n+6m+2$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(C_n\uparrow_{aa}W_m)$ and vertex set $V(C_n\uparrow_{aa}W_m)$ to $\{1,2,\cdots,2n+3m\}$, and the labeling sum of the same degree vertices is a constant. According to the definition of VRTL, the joint graph $C_n\uparrow_{aa}W_m(n\ge 3,m\ge 3)$ is proven to be a VRTL graph.

Theorem 7: For the joint graph $C_n\uparrow_{aa}P_2\uparrow_{aa}C_n(n\ge 3)$, when $n\equiv 0(mod 2)$, it is a VRTL graph.

Proof: Set the vertex set of the graph $C_n\uparrow_{aa}P_2\uparrow_{aa}C_n$ as $\{u_1,u_2,\cdots,2u_n\}$ and the edge set as $\{u_1u_n\bigcup u_nu_{2n}\bigcup u_{n+1}u_{2n}\bigcup u_iu_{i+1}(1\le i\le n-1\mid i\ne n)\}$. The graph $C_n\uparrow_{aa}P_2\uparrow_{aa}C_n$ contains $2n+1$ edges and $2n$ vertices, as shown in Fig. 12.

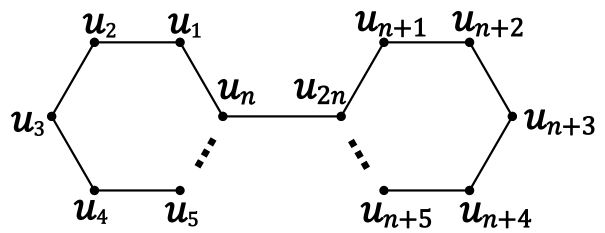


Fig. 12. $C_n\uparrow_{aa}P_2\uparrow_{aa}C_n$

When $n\ge 3\bigcap n\equiv 0(mod 2)$, the VRTL of $C_n\uparrow_{aa}P_2\uparrow_{aa}C_n$ is:

$$f(u_i)=i,(1\le i\le 2n)$$
$$f(u_1u_n)=4n$$
$$f(u_nu_{2n})=4n+1$$
$$f(u_{n+1}u_{2n})=4n-\frac{n}{2}$$
$$f(u_iu_{i+1})=\begin{cases}3n-\dfrac{i-1}{2},(i=1,3,\cdots,2n-1)\\[4pt]4n-\dfrac{i}{2},(i=2,4,\cdots 2n-2\bigcap i\ne n)\end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n\uparrow_{aa}P_2\uparrow_{aa}C_n$ are:

$$f(V)=\{1,2,\cdots,2n\}$$
$$f(E)=\{2n+1,2n+2,\cdots,4n+1\}$$

Currently, in the vertex set $\{u_1,u_2,\cdots,2u_n\}$, u_n and u_{2n} are 3-degree vertices, and the remaining vertices are all 2-degree vertices. It is essential to ensure that the sum of labeling for all 2-degree and 3-degree vertices are the same in the graph $C_n\uparrow_{aa}F_m$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2=\{f(u_i)+\sum_{uv\in E(u_i)}f(uv)\mid 1\le i\le 2n-1,i\ne n\}$$
$$=f(u_i)+f(u_iu_{i+1})+f(u_{i-1}u_i)\mid 2\le i\le n-1\bigcup n+2\le i\le 2n-1\,\|$$
$$f(u_1u_2)+f(u_1)+f(u_1u_n)\,\|\,f(u_{n+1})+f(u_{n+1}u_{n+1})+f(u_{n+1}u_{2n})$$
$$=7n+1$$

The total sum of the labeling for the 3-degree vertices is:

$$Sum_3=\{f(u_i)+\sum_{uv\in E(u_i)}f(uv)\mid i=n\bigcup i=2n\}$$
$$=f(u_n)+f(u_1u_n)+f(u_{n-1}u_n)+f(u_nu_{2n})\,\|$$
$$f(u_{2n})+f(u_nu_{2n})+f(u_{n+1}u_{2n})+f(u_{2n-1}u_{2n})$$
$$=\frac{23n}{2}+2$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(C_n\uparrow_{aa}P_2\uparrow_{aa}C_n)$ and vertex set $V(C_n\uparrow_{aa}P_2\uparrow_{aa}C_n)$ to $\{1,2,\cdots,4n+1\}$, and the labeling sum of the same degree vertices is a constant. According to the definition of VRTL, when $n\ge 3$ and $n\equiv 0(mod 2)$, the joint graph $C_n\uparrow_{aa}P_2\uparrow_{aa}C_n$ is proven to be a VRTL graph.

Theorem 8: For the joint graph $C_n\downarrow C_m(n\ge 3,m\ge 3)$, when, $n+m\equiv 1(mod 2)$ it is a VRTL graph.

Proof: Set the vertex set of the graph $C_n\downarrow C_m$ as $V=\{u_1, u_2,\ldots,u_n,u_{n+1},\cdots,u_{n+m-2}\}$ and the edge set as $E=\{u_1u_{n+m-2}\bigcup u_{n-1}u_{n+m-2}\bigcup u_iu_{i+1}\mid 1\le i\le n+m-2\}$. The graph $C_n\downarrow C_m$ contains $n+m-1$ edges and $n+m-2$ vertices in total, as shown in Fig. 13.

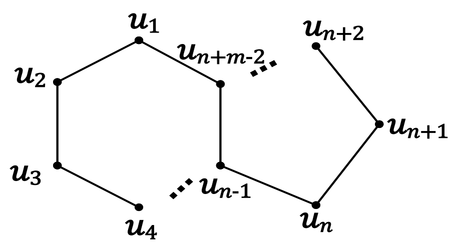


Fig. 13. $C_n\downarrow C_m$

When $n\ge 3$, $m\ge 3$, and $n+m\equiv 1(mod 2)$, the VRTL of $C_n\downarrow C_m$ is:

$$f(u_i)=i,1\le i\le n+m-2$$
$$f(u_{n-1}u_{n+m-2})=2n+2m-3$$

$$f\left(u_1u_{n+m-2}\right)=2n+2m-2-\frac{n+m+1}{2}$$

$$f\left(u_iu_{i+1}\right)=\begin{cases}2n+2m-3-\dfrac{i+1}{2}, i=1,3,5\cdots\\ 2n+2m-3-\dfrac{n+m-1+i}{2}, i=2,4,6\cdots\end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n\downarrow C_m$ are:

$$f(V)=\{1,2,\cdots,\ n+m-2\}$$

$$f(E)=\{n+m-1,n+m,\cdots,2n+2m-3\}$$

Currently, in the vertex set $\{u_1,u_2,\ldots,u_n,u_{n+1},\cdots,u_{n+m-1}\}$, both u_{n-1} and u_{n+m-2} are vertices of 3-degrees. The remaining vertices are all 2-degree vertices. It is essential to ensure that the sum of labeling for all 2-degree and 3-degree vertices are the same in the graph $C_n\downarrow C_m$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2=\{f\left(u_i\right)+\sum_{uv\in E(u_i)}f\left(uv\right)\mid 1\le i\le n+m-3, i\ne n-1\}$$

$$==f(u_i)+f(u_iu_{i+1})+f(u_{i-1}u_i)\mid 2\le i\le n+m-3\,\|$$

$$f(u_1)+f(u_1u_2)+f(u_1u_{n+m-2})$$

$$=4n+4m-5-\frac{m+n+1}{2}$$

The total sum of the labeling for the 3-degree vertices is:

$$Sum_3=\{f(u_i)+\sum_{uv\in E(u_i)}f(uv)\mid i=n-1\cup i=n+m-2\}$$

$$=f(u_{n-1})+f(u_{n-2}u_{n-1})+f(u_{n-1}u_n)+f(u_{n-1}u_{n+m-2})\,\|$$

$$f(u_{n+m-2})+f(u_1u_{n+m-2})+f(u_{n-1}u_{n+m-2})+f(u_{n+m-3}u_{n+m-2})$$

$$=6n+6m-12-\frac{m+3n}{2}$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(C_n\downarrow C_m)$ and vertex set $V(C_n\downarrow C_m)$ to $\{1,2,\cdots,2n+2m-2\}$, and the labeling sum of the same degree vertices is a constant. According to the definition of VRTL, the joint graph $C_n\downarrow C_m(n\ge 3,m\ge 3)$, when $n+m\equiv 1(mod 2)$, it is proven to be a VRTL graph.

Theorem 9: For the corona graph $C_n\circ P_2(n\ge 3)$, when $n\equiv 1(mod 2)$, it is a VRTL graph.

Proof: The graph formed by connecting all vertices of a copy of the graph P_2 to each vertex of the graph C_n is called the corona graph $C_n\circ P_2$, as shown in Fig. 14.

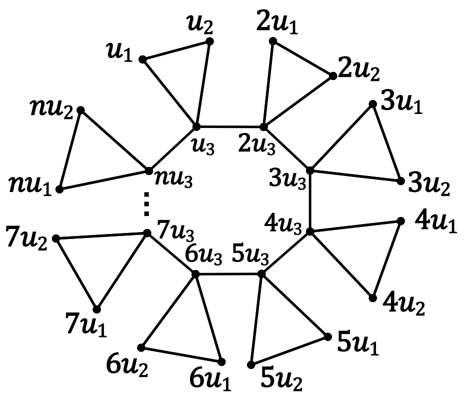


Fig. 14. $C_n\circ P_2$

Set the vertex set of the corona graph $C_n\circ P_2$ as $\{u_1,u_2,u_3,2u_1,2u_2,2u_3,\cdots,nu_1,nu_2,nu_3\}$ and the edge set as $\{u_1u_n\cup u_iu_{i+1}\mid 1\le i\le n+m-2\}$. The graph $C_n\circ P_2$ contains $4n$ edges and $3n$ vertices in total.

When $n>3$, and $n\equiv 1(mod 2)$, the VRTL of $C_n\circ P_2$ is:

$$f(hu_i)=3(h-1)+i(1\le i\le 3, 1\le h\le n)$$

$$f(u_3nu_3)=6n+\frac{n+1}{2}$$

$$f(hu_1hu_2)=6n+1-h(1\le h\le n)$$

$$f(hu_2hu_3)=5n+1-2h,(1\le h\le n)$$

$$f(hu_1hu_3)=5n+2-2h,(1\le h\le n)$$

$$f(hu_3(h+1)u_3)=\begin{cases}6n+\dfrac{h+1}{2}, h\equiv 1(mod 2)\\ 6n+\dfrac{h+n+1}{2}, h\equiv 0(mod 2)\end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n\circ P_2$ are:

$$f(V)=\{1,2,\cdots,3n\}$$

$$f(E)=\{3n+1,3n+2,\cdots,7n\}$$

Currently, in the vertex set $\{u_1,u_2,u_3,2u_1,2u_2,2u_3,\cdots,nu_1,nu_2,nu_3\}$, $u_3,2u_3,\cdots,nu_3$ are all 4 degree vertices, and the remaining vertices are all 2-degree vertices. It is essential to ensure that the sum of labeling for all 2-degree and 4-degree vertices are the same in the graph $C_n\circ P_2$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2=f(hu_1)+f(hu_1hu_2)+f(hu_1hu_3)\mid 1\le h\le n\,\|$$

$$f(hu_2)+f(hu_1hu_2)+f(hu_2hu_3)\mid 1\le h\le n$$

$$=11n+1$$

The total sum of the labeling for the 4-degree vertices is:

$$Sum_4=\{f(hu_3)+\sum_{uv\in E(hu_3)}f(uv)\mid 1\le h\le n\}$$

$$=f(hu_3)+f(hu_1hu_3)+f((h-1)u_3hu_3)+f(hu_3(h+1)u_3)+$$

$$f(hu_2hu_3)\,\|\,f(u_3)+f(u_1u_3)+f(u_2u_3)+f(u_32u_3)+f(u_3nu_3)\,\|$$

$$f(nu_3)+f(nu_1nu_3)+f(nu_2nu_3)+f(u_3nu_3)+f((n-1)u_3nu_3)$$

$$=\frac{45n+7}{2}$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(C_n\circ P_2)$ and vertex set $V(C_n\circ P_2)$ to $\{1,2,\cdots,7n\}$, and t the labeling sum of the same degree vertices is a constant. According to the definition of VRTL, when $n\equiv 1(mod 2)$, the corona graph $C_n\circ P_2(n\ge 3)$ is proven to be a VRTL graph.

Theorem 10: The generalized corona graph $S_n^b\circ C_3^a(n\ge 4)$ is a VRTL graph.

Proof: Let the vertex set of generalized corona graph $S_n^b\circ C_3^a$ be $\{u_0,u_1,u_2,u_3,2u_1,2u_2,2u_3,\cdots,nu_1,nu_2,nu_3,\}$ and the edge set be $\{\{hu_1hu_2\cup hu_2hu_3\cup hu_1hu_3\mid 1\le h\le n\}\cup u_0hu_3\mid 1\le h\le n\}$. The graph $S_n^b\circ C_3^a$ contains $4n$ edges and $3n+1$ vertices in total, as shown in Fig. 15.

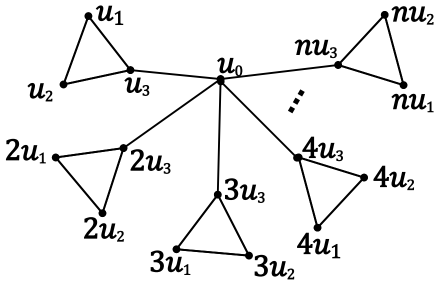


Fig. 15. $S_n^b \circ C_3^a$

Firstly, the VRTL of $S_n^b \circ C_3^a$ is:

$$f(u_0) = 7n+1$$
$$f(hu_i) = 3(h-1) + i(1 \le i \le 3, 1 \le h \le n)$$
$$f(hu_1hu_2) = 6n+1-h(1 \le h \le n)$$
$$f(hu_2hu_3) = 5n+1-2h, (1 \le h \le n)$$
$$f(hu_1hu_3) = 5n+2-2h, (1 \le h \le n)$$
$$f(u_0hu_3) = 6n+h(1 \le h \le n)$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $S_n^b \circ C_3^a$ are:

$$f(V) = \{1, 2, \cdots, 3n, 7n+1\}$$
$$f(E) = \{3n+1, 3n+2, \cdots, 7n\}$$

Currently, in the vertex set $\{u_0, u_1, u_2, u_3, 2u_1, 2u_2, 2u_3, \cdots, nu_1, nu_2, nu_3\}$, $u_3, 2u_3, \ldots, nu_3$ are all 3-degree vertices, u_0 is n-degree vertex, and the remaining vertex are all 2-degree vertices. Since there are no vertices with the same degree as u_0, calculating the sum of its labels is completely unnecessary. It is essential to ensure that the sum of labeling for all 2-degree and 3-degree vertices are the same in the graph $S_n^b \circ C_3^a$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2 = f(hu_1) + f(hu_1hu_2) + f(hu_1hu_3) \mid 1 \le h \le n \,\|$$
$$f(hu_2) + f(hu_1hu_2) + f(hu_2hu_3) \mid 1 \le h \le n$$
$$= 11n+1$$

The total sum of the labeling for the 3-degree vertices is:

$$Sum_3 = \{f(hu_3) + \textstyle\sum_{uv \in E(hu_3)} f(uv) \mid (1 \le h \le n)\}$$
$$= f(hu_3) + f(hu_1hu_3) + f(hu_2hu_3) + f(u_0hu_3) \mid (1 \le h \le n)$$
$$= 16n+3$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(S_n^b \circ C_3^a)$ and vertex set $V(S_n^b \circ C_3^a)$ to $\{1, 2, \cdots, 7n+1\}$, and the labeling sum of the same degree vertices is a constant. Based on the definition of vertex reduced total labeling, the generalized corona graph $S_n^b \circ C_3^a (n \ge 4)$ is proven to be a VRTL graph.

Theorem 11: The generalized corona graph $F_n^{bc} \circ C_3^a (n \ge 4)$ is a VRTL graph.

Proof: Set the vertex set of the generalized corona graph $F_n^{bc} \circ C_3^a$ as $\{u_0, u_1, u_2, u_3, 2u_1, 2u_2, 2u_3, \cdots, nu_1, nu_2, nu_3\}$ and the edge set as $\{\{hu_1hu_2 \bigcup hu_2hu_3 \bigcup hu_1hu_3 \mid 1 \le h \le n\} \bigcup u_0hu_3 \mid 1 \le h \le n\}$. The graph $F_n^{bc} \circ C_3^a$ contains $5n-1$ edges and $3n+1$ vertices in total, as shown in Fig. 16.

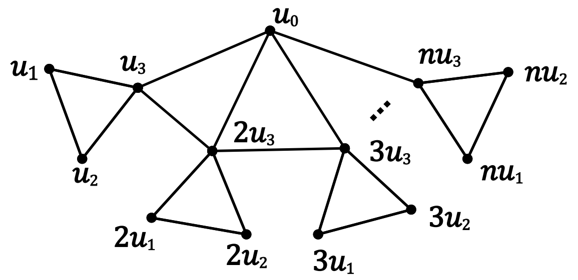


Fig. 16. $F_n^{bc} \circ C_3^a$

Firstly, the VRTL of $F_n^{bc} \circ C_3^a$ is:

$$f(u_0) = 7n+1$$
$$f(hu_i) = 3(h-1) + i(1 \le i \le 3, 1 \le h \le n)$$
$$f(hu_1hu_2) = 6n+1-h(1 \le h \le n)$$
$$f(hu_2hu_3) = 5n+1-2h, (1 \le h \le n)$$
$$f(hu_1hu_3) = 5n+2-2h, (1 \le h \le n)$$
$$f(u_0hu_3) = \begin{cases} 7n(h=1) \\ 8n+1-h(2 \le h \le n) \end{cases}$$
$$f(hu_3(h+1)u_3) = 6n+h(1 \le h \le n-1)$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $F_n^{bc} \circ C_3^a$ are:

$$f(V) = \{1, 2, \cdots, 6n, 8n\}$$
$$f(E) = \{6n+1, 6n+2, \cdots, 8n-1\}$$

Currently, in the vertex set $\{u_0, u_1, u_2, u_3, 2u_1, 2u_2, 2u_3, \cdots, nu_1, nu_2, nu_3\}$, $2u_3, 3u_3, \ldots, (n-1)u_3$ are all 5-degree vertices, u_3 and nu_3 are all 4-degree vertices, u_0 is n-degree vertex, and the remaining vertices are all 2-degree vertices. Since there are no vertices of the same degree as u_0, calculating the sum of its labeling is completely unnecessary. It is essential to ensure that the sum of labeling for all 2-degree, 4-degree and 5-degree vertices are the same in the graph $F_n^{bc} \circ C_3^a$.

The total sum of the labeling for the 2-degree vertices is:

$$Sum_2 = f(hu_1) + f(hu_1hu_2) + f(hu_1hu_3) \mid 1 \le h \le n \,\|$$
$$f(hu_2) + f(hu_1hu_2) + f(hu_2hu_3) \mid 1 \le h \le n$$
$$= 11n+1$$

The total sum of the labeling for the 4-degree vertices is:

$$Sum_4 = f(u_3) + f(u_1u_3) + f(u_2u_3) + f(u_0u_3) + f(u_3 2u_3) \,\|$$
$$f(nu_3) + f(nu_1nu_3) + f(nu_2nu_3) + f(u_0nu_3) + f((n-1)u_3nu_3)$$
$$= 23n+24$$

The total sum of the labeling for the 2-degree vertices is:

$$Sum_5 = \{f(hu_3) + \textstyle\sum_{uv \in E(hu_3)} f(uu) \mid (2 \le h \le n-1)\}$$
$$= f(hu_3) + f(hu_1hu_3) + f(hu_2hu_3) + f(u_3hu_3) +$$
$$f(hu_3(h+1)u_3) + f((h-1)u_3hu_3) \mid 2 \le h \le n-1$$
$$= \{30n+30\}$$

From the above proof process, it can be determined that f is a single mapping function from the edge set $E(F_n^{bc} \circ C_3^a)$ and vertex set $V(F_n^{bc} \circ C_3^a)$ to $\{1, 2, \cdots, 7n\}$, and the sum of labeling for all vertices with the same degree is a constant. Based on the definition of VRTL, the generalized corona graph $F_n^{bc} \circ C_3^a (n \ge 4)$ has been proven to be a VRTL graph.

Theorem 12: For the generalized corona graph $C_n^a \circ S_m^a$ $(n \ge 3, m \ge 3)$, when $m = 1(mod 2)$, $m = 0(mod 2) \cap n = 1(mod 2)$ and $m = 0(mod 2) \cap n = 4,6,8$, it is a VRTL graph.

Proof: The generalized corona graph $C_n^a \circ S_m^a$ is formed by connecting center vertex of the graph S_m to each vertex of the graph C_n. Set the vertex set of graph $C_n^a \circ S_m^a$ as $\{u_1, u_2, \ldots, u_n \cup v_1, v_2 \cdots, v_{nm}\}$ and the edge set as $\{u_i u_{i+1} \mid 1 \le i \le n \cup u_1 u_n \cup u_{i+1} v_{im+h} \mid 0 \le i \le n-1, 1 \le h \le m\}$. The graph $C_n^a \circ S_m^a$ contains $n + nm$ edges and $n + nm$ vertices in total, as shown in Fig. 17.

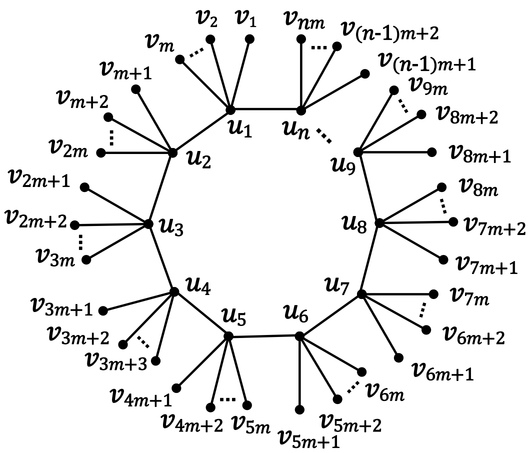


Fig. 17. $C_n^a \circ S_m^a$

Case 1: When $m = 1(mod 2)$, the VRTL of $C_n^a \circ S_m^a$ is:

$$f(u_i)\begin{cases} 2nm+1 (i=1) \\ 2nm+n+2-i (i=2,3,\cdots,n) \end{cases}$$

$$f(v_{im+h}) = \begin{cases} i+1+(h-1)n, (i=0,1,\cdots,n-1, h=1,3,\cdots,m) \\ hn-i, (i=0,1,\cdots,n-1, h=2,4,\cdots,m-1) \end{cases}$$

$$f(u_1u_n) = 2nm+2n$$

$$f(u_iu_{i+1}) = 2nm+n+i (i=1,2,\cdots,n-1)$$

$$f(u_{i+1}v_{im+h}) = \begin{cases} 2nm-i-(h-1)n, (i=0,1,\cdots,n-1, h=1,3,\cdots,m) \\ 2nm+1+i-hn, (i=0,1,\cdots,n-1, h=2,4,\cdots,m-1) \end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n^a \circ S_m^a$ are:

$$f(V) = \{1,2,\cdots,2nm\}$$

$$f(E) = \{2nm+1, 2nm+2, \cdots, 2nm+2n\}$$

Currently, in the vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2 \cdots, v_{nm}\}$, $v_1, v_2, \ldots, v_{nm}$ are all 1-degree vertices, and $u_1, u_2, \ldots, u_n$ are all *m+2*-degree vertices. It is essential to ensure that the sum of labeling for all 1-degree and *m+2*-degree vertices are the same in the graph $C_n^a \circ S_m^a$.

The total sum of the labeling for the 1-degree vertices is:

$$Sum_1 = \{f(v_i) + \textstyle\sum_{uv \in E(v_i)} f(uv) \mid 1 \le i \le nm\}$$
$$= f(v_{im+h}) + f(u_{i+1}v_{im+h}) \mid 0 \le i \le n-1, 1 \le h \le m$$
$$= 2nm+1$$

The total sum of the labeling for the *m+2*-degree vertices is:

$$Sum_{m+2} = \{f(u_i) + \textstyle\sum_{uv \in E(u_i)} f(uv) \mid 1 \le i \le n\}$$
$$= f(u_1) + f(u_1u_2) + f(u_1u_n) + f(u_1v_1) + f(u_1v_2) + \cdots + f(u_1v_m) \,\|\, f(u_n) + f(u_{n-1}u_n) + f(u_1u_n) + f(u_nv_{(n-1)m+1}) + \cdots + f(u_nv_{nm})$$
$$= \frac{3}{2}nm^2 + 6nm + \frac{7n+m+3}{2}$$

According to the definition of VRTL, f is a one-to-one mapping function from the vertex set $V(C_n^a \circ S_m^a)$ and edge set $E(C_n^a \circ S_m^a)$ to $\{1,2,\cdots,2n+2m\}$, and the sum of labeling for all the same degree vertices is a constant.

Thus, when $m = 1(mod 2)$, the generalized corona graph $C_n^a \circ S_m^a$ is proven to be a VRTL graph, as shown by case 1.

Case 2: When $n = 1(mod 2)$ and $m = 0(mod 2)$, the VRTL of $C_n^a \circ S_m^a$ is:

$$f(u_i) = 2nm + i (i=1,2,\cdots,n)$$

$$f(v_{im+h}) = \begin{cases} i+1+(h-1)n, (i=0,1,\cdots,n-1, h=1,2,\cdots,m-1) \\ hn-i, (i=0,1,\cdots,n-1, h=2,4,\cdots,m) \end{cases}$$

$$f(u_1u_n) = 2nm+2n-\frac{n-1}{2}$$

$$f(u_iu_{i+1}) = \begin{cases} 2nm+2n-\frac{i-1}{2}, (i=1,3,\cdots,n-2) \\ 2nm+2n-\frac{n-1+i}{2}, (i=2,4,\cdots,n-1) \end{cases}$$

$$f(u_{i+1}v_{im+h}) = \begin{cases} 2nm-i-(h-1)n, (i=0,1,\cdots,n-1, h=1,2,\cdots,m-1) \\ 2nm+1+i-hn, (i=0,1,\cdots,n-1, h=2,4,\cdots,m) \end{cases}$$

At this time, the vertex labeling set $f(V)$ and edge labeling set $f(E)$ of $C_n^a \circ S_m^a$ are:

$$f(V) = \{1,2,\cdots,2nm\}$$

$$f(E) = \{2nm+1, 2nm+2, \cdots, 2nm+2n\}$$

Currently, in the vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2 \cdots, v_{nm}\}$, $v_1, v_2, \ldots, v_{nm}$ are all 1-degree vertices, and $u_1, u_2, \ldots, u_n$ are all *m+2*-degree vertices. It is essential to ensure that the sum of labeling for all 1-degree and *m+2*-degree vertices are the same in the graph $C_n^a \circ S_m^a$.

The total sum of the labeling for the 1-degree vertices is:

$$Sum_1 = \{f(v_i) + \textstyle\sum_{uv \in E(v_i)} f(uv) \mid 1 \le i \le nm\}$$
$$= f(v_{im+h}) + f(u_{i+1}v_{im+h}) \mid 0 \le i \le n-1, 1 \le h \le m$$
$$= 2nm+1$$

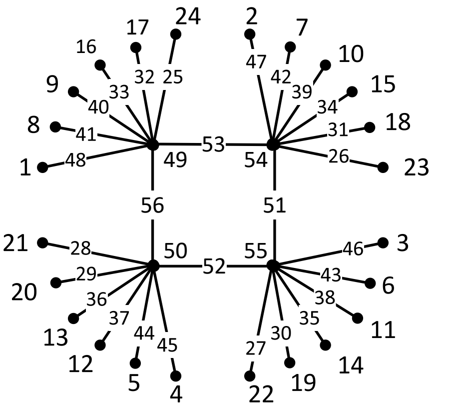
The total sum of the labeling for the 2-degree vertices is:

$$Sum_{m+2} = \{f(u_i) + \textstyle\sum_{uv \in E(u_i)} f(uv) \mid 1 \le i \le n\}$$
$$= f(u_1) + f(u_1u_2) + f(u_1u_n) + f(u_1v_1) + f(u_1v_2) + \cdots + f(u_1v_m) \,\|\, f(u_n) + f(u_{n-1}u_n) + f(u_1u_n) + f(u_nv_{(n-1)m+1}) + \cdots + f(u_nv_{nm})$$
$$= \frac{m}{2}(3nm+1) + 6nm + 4n - \frac{n-1}{2} + 1$$

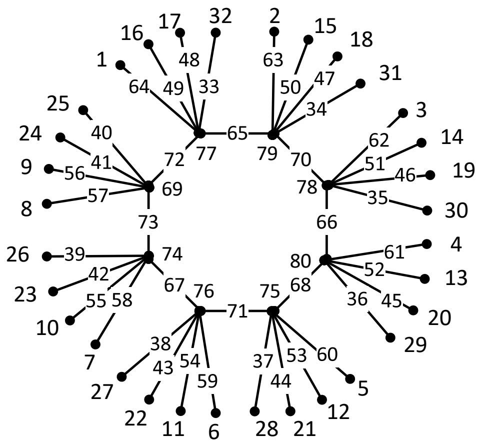
According to the definition of VRTL, f is a single mapping function from the vertex set $V(C_n^a \circ S_m^a)$ and edge set $E(C_n^a \circ S_m^a)$ to $\{1,2,\cdots,2n+2m\}$, and the sum of labeling for all the same degree vertices is a constant.

Thus, when $n = 1(mod 2), m = 0(mod 2)$, the generalized corona graph $C_n^a \circ S_m^a$ has been proven to be a VRTL graph, as shown by case 2.

Case 3: When $m=0(mod2)$ and $n=4,6,8$, the labeling of part of graph $C_n^a \circ S_m^a$ are shown in Fig. 18.



(a) The VRTL of $C_4^a \circ S_6^a$.



(b) The VRTL of $C_8^a \circ S_4^a$.

Fig. 18. The two labeled results of graph $C_n^a \circ S_m^a$.

Due to $f(V)\bigcup f(E)\to[1,2nm+2n]$, $f(V)\bigcap f(E)=\varnothing$, and the fact that the sum of labeling for the same degree vertices is a constant, the generalized corona graph $C_n^a \circ S_m^a$ is proven to be a VRTL graph when $n\equiv 4,6,8, m\equiv 0(mod2)$.

To sum up, for the generalized corona graph $C_n^a \circ S_m^a$ $(n\geq 3, m\geq 3)$, when $m=1(mod2)$, $m=0(mod2)\bigcap n=1(mod2)$ and $m=0(mod2)\bigcap n=4,6,8$, it is proven to be a VRTL graph.

Conjecture 2: All unicyclic graphs are VRTL graphs.

V. Conclusion

This article proposes a VRTL algorithm based on vertex reduction total labeling constraint conditions. The algorithm first organizes the solution space of the input graph, then traverses the solution space of the entire arrangement, and determines whether the graph satisfies the vertex reduction total labeling through a recursive search. Finally, it outputs a labeling matrix that satisfies the constraint conditions. By analyzing the experimental results of cyclic graphs, this paper proposes 12 theorems about cyclic graphs and their related graphs, verifies their correctness, and proposes two conjectures.

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