Ordered Almost n-Interior Ideals in Semigroups Class Fuzzifications

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***Abstract*—O. Grosek and L. Stako initiative studied almost ideals in ordered semigroups in 1980. The concept of n-interior ideals in ordered semigroups in 2022 by N. Tiprachot et al. In this paper, we are interested in the concepts and the properties of ordered almost n-interior ideals and fuzzy ordered almost n-interior ideals of ordered semigroups. We investigate the relationship between ordered almost n-interior ideals and fuzzy ordered almost n-interior ideals of ordered semigroups. Moreover, we extends ordered almost (m,n)-interior ideals and prove basic properties of its.**

***Index Terms*—n-interior ideals, ordered almost n-interior ideals, weakly almost n-interior ideals, fuzzy ordered almost n-interior ideals, fuzzy ordered weakly almost n-interior ideals**

I. Introduction

THE CONCEPTS of ordered semigroups is a generalization of semigroups. The notion of (m,n)-ideals was defined as a generalization of bi-ideals in ordered semigroups by Sanborisoot and Changphas. A particular class of ordered semigroups was also characterized by (m,n)-ideals. Many authors have examined theory in other structures, see, e.g., [1], [2], [3], [4]. The concept of n-interior ideals in ordered semigroups in 2022 by N. Tiprachot et al.[5]. The authors characterized many classes of ordered semigroups by combining -ideals and n-interior ideals. O. Grosek and L. Stako [6] initiative studied almost ideals in ordered semigroups in 1980. In 1981, S. Bogdanovic [7] studied the concept of almost bi-ideals in semigroups by using the notions of almost ideals and bi-ideals in semigroups. In 1965, L. A. Zadeh [8] introduced the concept of FSs. In 1979, N. Kuroki [9] developed various kinds of fuzzy ideals in semigroups. In 2002, N. Kehayopulu and M. Tsingelis [11] used the notion of fuzzy ideals in ordered semigroups. In 2019, K. Wattanatripop et al. [10] discussed fuzzy almost ideals of ternary semigroups. In the same year, S. Suebung et al. [12] gave the concept of almost (m,n)-ideas and fuzzy almost (m,n)-ideals in semigroups. In 2020, Kaopusek et al. [13] defined the concept of introducing almost interior ideals and weakly almost interior ideals in semigroups and studied the relationship between almost interior ideals and weakly almost interior ideals in semigroups. In 2021, W. Krailoet et al. [14] defined the concept of fuzzy almost interior ideals of semigroups. In 2022, S. Suebsung [15] studied ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups. Now, the notion of almost ideals in semigroups was extended to some generalizations of semigroups, for example, almost i-ideal and fuzzy almost i-ideals in n-ary semigroups [16], almost bi-ideal in Γ-semigroup [17], almost interior Γ-ideal in Γ-semigroups [18], almost hyperideals in semihypergroups [19], almost ideals in ternary semigroup [10], almost bi-ideals and almost quasi-ideals in ordered semigroup [20] etc. In addition, T. Gaketem and P. Khamrot extended almost bi-ideal, almost interior ideals, almost ideals and almost quasi-ideals to bipolar fuzzy set, [21], [22], [23], [24]. In 2023, R. Chinarm et al. [25] studied ordered almost ideals and fuzzy ordered almost ideals in ordered ternary semigroups. In the same year, J. Sanborisoot et al. [26] studied almost interior ideal and fuzzy almost interior ideals in ordered semigroups. Recently, R. Chinarm et al. [27] studied almost (m,n)-quasi-ideal and fuzzy almost (m,n)-quasi-ideals in semigroups. In this paper, we are interested in the concepts of ordered almost n-interior ideals, and weakly ordered almost n-interior ideals of ordered semigroups. We study the properties and the relationships between the types of ordred almost n-interior ideals and fuzzy ordered almost n-interior ideals in ordered semigroups. Additional, we extends ordered almost (m,n)-interior ideals and prove basic properties of its.

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II. Preliminaries

In this section, we investigate below necessary notions below and present a few auxiliary results that will be used throughout the paper.

An ordered semigroup $(\mathcal{S},\cdot,\leq)$ is an algebraic structure $(\mathcal{S},\cdot,\leq)$ such that $(\mathcal{S},\cdot)$ is a semigroup, $(\mathcal{S},\leq)$ is a partially ordered set and $a\leq b$ then $ac\leq bc$ and $ca\leq cb$ for all $a,b,c\in\mathcal{S}$.

For a nonempty subset $\mathcal{X}$ and $\mathcal{Y}$ of ordered semigroup $\mathcal{S}$, we write

$(\mathcal{X}] := \{a\in\mathcal{S}\mid a\leq b \text{ for some } b\in\mathcal{X}\}$ and $\mathcal{X}\mathcal{Y} := \{xy \mid x\in\mathcal{X} \text{ and } y\in\mathcal{Y}\}$.

Theorem 2.1. *If $\mathcal{X}$ and $\mathcal{Y}$ are nonempty subsets of an ordered semigroup $\mathcal{S}$, then*

(1) $\mathcal{X}\subseteq(\mathcal{X}]$,
(2) If $\mathcal{X}\subseteq\mathcal{Y}$, then $(\mathcal{X}]\subseteq(\mathcal{Y}]$.
(3) $(\mathcal{X}\cap\mathcal{Y}] = (\mathcal{X}]\cap(\mathcal{Y}]$
(4) $(\mathcal{X}\cup\mathcal{Y}] = (\mathcal{X}]\cup(\mathcal{Y}]$.

Definition 2.2. *A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* subsemigroup *(SG) of $\mathcal{S}$ if $\mathcal{I}^2\subseteq\mathcal{I}$*

Definition 2.3. *A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* left ideal *(LI) of $\mathcal{S}$ if $(\mathcal{I}\mathcal{S}] \subseteq \mathcal{I}$ and $x \in \mathcal{I}$ and $s \in \mathcal{S}$ such that $s \leq x$, then $s \in \mathcal{I}$, that is $(\mathcal{I}] \subseteq \mathcal{I}$.*

Definition 2.4. *A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* right ideal *(RI) of $\mathcal{S}$ if , $(\mathcal{S}\mathcal{I}] \subseteq \mathcal{I}$ and $x \in \mathcal{I}$ and $s \in \mathcal{S}$ such that $s \leq x$, then $s \in \mathcal{I}$, that is $(\mathcal{I}] \subseteq \mathcal{I}$.*

Definition 2.5. *An SG of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called an* interior ideal *(II) of $\mathcal{S}$ if $(\mathcal{S}\mathcal{I}\mathcal{S}] \subseteq \mathcal{I}$ and $s \in \mathcal{I}$ such that $s \leq x$, then $s \in \mathcal{I}$, that is $(\mathcal{I}] \subseteq \mathcal{I}$.*

Definition 2.6. *[5] An SG of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called an n-*interior ideal *(n-II) of $\mathcal{S}$ if $(\mathcal{S}\mathcal{I}^n\mathcal{S}] \subseteq \mathcal{I}$ and $s \in \mathcal{S}$ such that $s \leq x$, then $s \in \mathcal{I}$, that is $(\mathcal{I}] \subseteq \mathcal{I}$ where $n \in \mathbb{N}_0$.*

Definition 2.7. *[12] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a (m,n)-*ideal *of $\mathcal{S}$ if $(\mathcal{I}^m\mathcal{S}\mathcal{I}^n] \subseteq \mathcal{I}$ and $s \in \mathcal{S}$ such that $s \leq x$, then $s \in \mathcal{I}$, that is $(\mathcal{I}] \subseteq \mathcal{I}$ where $m, n \in \mathbb{N}_0$.*

Definition 2.8. *[15] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* left ordered almost ideal *(LOAI) of $\mathcal{S}$ if $(s\mathcal{I}] \cap \mathcal{I} \neq \emptyset$ for all $s \in \mathcal{S}$.*

Definition 2.9. *[15] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* right ordered almost ideal *(ROAI) of $\mathcal{S}$ if $(\mathcal{I}s] \cap \mathcal{I} \neq \emptyset$ for all $s \in \mathcal{S}$.*

Definition 2.10. *[20] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called an* ordered almost bi-ideal *(OABI) of $\mathcal{S}$ if $(\mathcal{I}s\mathcal{I}] \cap \mathcal{I} \neq \emptyset$ for all $s \in \mathcal{S}$.*

Definition 2.11. *[20] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called an* ordered almost quasi-ideal *(OAQI) of $\mathcal{S}$ if $(\mathcal{I}s] \cap (s\mathcal{I}] \cap \mathcal{I} \neq \emptyset$ for all $s \in \mathcal{S}$.*

Definition 2.12. *[26] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* ordered almost interior ideal *(OAII) of $\mathcal{S}$ if $(s\mathcal{I}k] \cap \mathcal{I} \neq \emptyset$ for all $s, k \in \mathcal{S}$.*

A *FS* (FS) φ of nonempty set $\mathcal{S}$ if φ is function into closed interval $[0, 1]$.

For any two FSs φ and ν of a nonempty set $\mathcal{S}$, we define the $\varphi \vee \nu$, $\varphi \wedge \nu$ $\varphi \leq \nu$, $\text{supp}(\varphi)$ as follows: for all $a \in \mathcal{S}$,

(1) $(\varphi \vee \nu)(a) = \max\{\varphi(a), \nu(a)\}$,
(2) $(\varphi \wedge \nu)(a) = \min\{\varphi(a), \nu(a)\}$,
(3) $\varphi \leq \nu \Leftrightarrow \varphi(a) \leq \nu(a)$, and
(4) $\text{supp}(\varphi) \Leftrightarrow \varphi(a) \neq 0$.

For any two FSs φ and ν of a nonempty set $\mathcal{S}$, we define the $\varphi \circ \nu$ as follows: for all $a \in \mathcal{S}$,

$$\varphi \circ \nu(a) = \begin{cases} 1 & \text{if} \quad a \in \mathcal{I} \\ 0 & \text{if} \quad a \notin \mathcal{I}. \end{cases}$$

$$(\varphi \circ \nu)(a) = \begin{cases} \bigvee\limits_{a \leq bc} \{\varphi(b) \wedge \nu(c)\} & \text{if } a \leq bc \ \exists b, c \in \mathcal{S}, \\ 0 & \text{if otherwise.} \end{cases}$$

The characteristic function $\lambda_{\mathcal{I}}(a)$ of a subset $\mathcal{I}$ of a nonempty set $\mathcal{S}$ is a FS of $\mathcal{S}$

$$\lambda_{\mathcal{I}}(a) = \begin{cases} 1 & \text{if} \quad a \in \mathcal{I} \\ 0 & \text{if} \quad a \notin \mathcal{I}. \end{cases}$$

for all $a \in \mathcal{S}$. A fuzzy point r_β of a FS $\mathcal{S}$ defined by

$$r_\beta(a) = \begin{cases} \beta & \text{if} \quad a = r \\ 0 & \text{if} \quad a \neq r. \end{cases}$$

for all $r \in \mathcal{S}$ and $\beta \in (0, 1]$.

Lemma 2.13. *If $\mathcal{I}$ and $\mathcal{L}$ are nonempty subsets of an ordered semigroup $\mathcal{S}$, then the following are true:*

(1) $\lambda_{\mathcal{I}} \vee \lambda_{\mathcal{L}} = \lambda_{\mathcal{I} \cup \mathcal{L}}$.
(2) $\lambda_{\mathcal{I}} \wedge \lambda_{\mathcal{L}} = \lambda_{\mathcal{I} \cap \mathcal{L}}$.
(3) *If $\mathcal{I} \subseteq \mathcal{L}$, then $\lambda_{\mathcal{I}} \leq \lambda_{\mathcal{L}}$.*
(4) $\lambda_{\mathcal{I}} \circ \lambda_{\mathcal{L}} \leq \lambda_{\mathcal{I}\mathcal{L}}$.

Lemma 2.14. *If φ, ν and ξ are FSs of an ordered semigroup $\mathcal{S}$, then the following are true:*

(1) *If $\varphi \leq \nu$, then $\varphi \circ \xi \leq \nu \circ \xi$.*
(2) *If $\varphi \leq \nu$, then $\varphi \vee \xi \leq \nu \vee \xi$.*
(3) *If $\varphi \leq \nu$, then $\varphi \wedge \xi \leq \nu \wedge \xi$.*
(4) *If $\varphi \leq \nu$, then $\text{supp}(\varphi) \leq \text{supp}(\nu)$.*

For a FS φ of an ordered semigroup $\mathcal{S}$, we define $(\varphi] : S \to [0, 1]$ by $(\varphi] := \sup\limits_{a \leq b} \varphi(b)$ for all $a \in \mathcal{S}$.

Lemma 2.15. *If φ, ν and ξ are FSs of an ordered semigroup $\mathcal{S}$, then the following are true:*

(1) $\varphi \leq (\varphi]$.
(2) *If $\varphi \leq \nu$, then $(\varphi] \leq (\xi]$.*
(3) *If $\varphi \leq \nu$, then $(\varphi \circ \xi] \leq (\nu \circ \xi]$ and $(\xi \circ \varphi] \leq (\xi \circ \nu]$.*

Lemma 2.16. *If φ is a FS of an ordered semigroup $\mathcal{S}$, then the following are equivalent.*

(1) *If $a \leq b$, then $\varphi(a) \geq \varphi(b)$ for all $a, b \in \mathcal{S}$.*
(2) $(\varphi] = \varphi$.

Definition 2.17. *A FS φ of an ordered semigroup $\mathcal{S}$ is called*

(1) *a* fuzzy subsemigroup *(FSG) of $\mathcal{S}$ if $\varphi(ab) \leq \varphi(a) \wedge \varphi(b)$ for all $a, b \in S$,*
(2) *a* fuzzy left ideal *(FLI) of $\mathcal{S}$ if $\varphi(ab) \leq \varphi(b)$ and if $a \leq b$, then $\varphi(a) \leq \varphi(b)$ for all $a, b \in \mathcal{S}$,*
(3) *a* fuzzy right ideal *(FRI) of $\mathcal{S}$ if $\varphi(ab) \leq \varphi(a)$ and if $a \leq b$, then $\varphi(a) \leq \varphi(b)$ for all $a, b \in \mathcal{S}$,*
(4) *a* fuzzy ideal *(FI) of $\mathcal{S}$ if it is both a FLI and FRI of $\mathcal{S}$,*
(5) *a* fuzzy interior ideal *(FII) of $\mathcal{S}$ if it is a FSG and $\varphi(acb) \leq \varphi(c)$ and if $a \leq b$, then $\varphi(a) \leq \varphi(b)$ for all $a, b, c \in \mathcal{S}$.*
(6) *a* fuzzy n-interior ideal *(F-n-II) of $\mathcal{S}$ if it is a FSG and $\varphi(ac_1^n b) \leq \varphi(c_1) \wedge \varphi(c_2) \cdots \wedge \varphi(c_n)$ and if $a \leq b$, then $\varphi(a) \leq \varphi(b)$ for all $a, c_i, c \in \mathcal{S}$ where $i \in \{1, 2, \ldots, m\}$.*
(7) *a* weakly fuzzy interior ideal *(WFII) of $\mathcal{S}$ if it is $\varphi(abc) \leq \varphi(b)$ and if $a \leq b$, then $\varphi(a) \leq \varphi(b)$ for all $a, b, c \in \mathcal{S}$.*
(8) *a* fuzzy left ordered almost ideal *(FLOAI) of $\mathcal{S}$ if $(r_\beta \circ \varphi] \wedge \varphi \neq 0$ for all fuzzy point r_β.*
(9) *a* fuzzy right ordered almost ideal *(FROAI) of $\mathcal{S}$ if $(\varphi \circ r_\beta] \wedge \varphi \neq 0$ for all fuzzy point r_β.*
(10) *a* fuzzy ordered almost ideal *(FOAI) of $\mathcal{S}$ if it is both a FLOAI and FROAI of $\mathcal{S}$,*
(11) *a* fuzzy ordered almost bi-ideal *(FOAB) of $\mathcal{S}$ if $(\varphi \circ r_\beta \circ \varphi] \wedge \varphi \neq 0$, for all fuzzy point r_β.*

III. Ordered Almost n-interior ideals of an ordered semigroups

In this section, we define the notions of ordred almost n-interior ideals in ordered semigroups. We also investigate some of their properties.

Definition 3.1. *A nonempty set $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$ is called an* ordered almost n-interior ideal *(OA-n-II) of $\mathcal{S}$ if $(a\mathcal{I}^n b] \cap \mathcal{I} \neq \emptyset$, for all $a, b \in \mathcal{S}$ and $n \in \mathbb{N}_0$.*

Theorem 3.2. *Let $\mathcal{I}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then every n-II of $\mathcal{S}$ is an OA-n-II of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{I}$ is an n-II of $\mathcal{S}$ and $n \in \mathbb{N}_0$. Then $(\mathcal{S}\mathcal{I}^n\mathcal{S}] \subseteq \mathcal{I}$. Let $a, b \in \mathcal{S}$. Then $(a\mathcal{I}^n b] \subseteq (\mathcal{S}\mathcal{I}^n\mathcal{S}] \subseteq \mathcal{I}$. This implied that $\emptyset \neq (a\mathcal{I}^n b] \cap \mathcal{I}$. Hence $\mathcal{I}$ is an OA-n-II of $\mathcal{S}$. ■

Theorem 3.3. *Let $\mathcal{I}$ and $\mathcal{L}$ be nonempty subsets an ordered semigroup of $\mathcal{S}$ with $\mathcal{I} \subseteq \mathcal{L}$. If $\mathcal{I}$ is an OA-n-II of $\mathcal{S}$, then $\mathcal{L}$ is an OA-n-II of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{I}$ is an OA-n-II of $\mathcal{S}$ with $\mathcal{I} \subseteq \mathcal{L}$ and $a, b \in \mathcal{S}, n \in \mathbb{N}_0$. Then $(a\mathcal{I}^n b] \subseteq (a\mathcal{L}^n b]$. Thus, $[a\mathcal{I}^n b) \cap \mathcal{I} \subseteq (a\mathcal{L}^n b] \cap \mathcal{L}$ so $\emptyset \neq (a\mathcal{I}^n b] \cap \mathcal{I} \subseteq (a\mathcal{L}^n b] \cap \mathcal{L}$. Hence, $(a\mathcal{L}^n b] \cap \mathcal{L} \neq \emptyset$. Therefore, $\mathcal{L}$ is an OA-n-II of $\mathcal{S}$. ■

Corollary 3.4. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be OA-n-IIs of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}_1 \cup \mathcal{I}_2$ is an OA-n-II of $\mathcal{S}$.*

Corollary 3.5. *Let $\mathcal{S}$ be an ordered semigroup. Then the finite union OA-n-IIs of $\mathcal{S}$ is an OA-n-II of $\mathcal{S}$.*

Theorem 3.6. *Let $\mathcal{I}$ be an OA-n-II and $\mathcal{H}$ be nonempty subset of an ordered semigroup of $\mathcal{S}$. Then $\mathcal{I} \cup \mathcal{H}$ is an OA-n-II of $\mathcal{S}$.*

Proof: By Theorem 3.3, and $\mathcal{I} \subseteq \mathcal{I} \cup \mathcal{H}$. Thus, $\mathcal{I} \cup \mathcal{H}$ is an OA-n-II of $\mathcal{S}$. ■

Corollary 3.7. *Let $\{\mathcal{I}_i \mid i \in \mathcal{I}\}$ be nonempty subset of semigroup $\mathcal{S}$. Then $\bigcup\limits_{i \in \mathcal{I}} \mathcal{I}_i$ is an OA-n-II of $\mathcal{S}$ if there exists an OA-n-II $\mathcal{I}_i$ for some $i \in \mathcal{I}$.*

Proof: Assume that there exists an OA-n-II $\mathcal{I}_i$ for some $i \in \mathcal{I}$. Then $\mathcal{I}_i \subseteq \bigcup\limits_{i \in \mathcal{I}} \mathcal{I}_i$. By Theorem 3.6, $\bigcup\limits_{i \in \mathcal{I}} \mathcal{I}_i$ is an OA-n-II of $\mathcal{S}$. ■

Definition 3.8. *A nonempty set $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$ is called a ordered weakly almost n-interior ideal (WOA-n-II) of $\mathcal{S}$ if $(a\mathcal{I}^n a] \cap \mathcal{I} \neq \emptyset$, for all $a \in S$ and $n \in \mathbb{N}_0$.*

It is clearly every OA-n-IIs is WOA-n-IIs in ordered semigroups.

Theorem 3.9. *Let $\mathcal{I}$ and $\mathcal{L}$ be nonempty subsets an ordered semigroup of $\mathcal{S}$ with $\mathcal{I} \subseteq \mathcal{L}$. If $\mathcal{I}$ is a WOA-n-II of $\mathcal{S}$, then $\mathcal{L}$ is a WOA-n-II of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{I}$ is a WOA-n-II of $\mathcal{S}$ with $\mathcal{I} \subset \mathcal{L}$ and $a \in \mathcal{S}, n \in \mathbb{N}_0$. Then $(a\mathcal{I}^n a] \subseteq (a\mathcal{L}^n a]$. Thus, $[a\mathcal{I}^n a) \cap \mathcal{I} \subseteq (a\mathcal{L}^n a] \cap \mathcal{L}$ so $\emptyset \neq (a\mathcal{I}^n a] \cap \mathcal{I} \subseteq (a\mathcal{L}^n a] \cap \mathcal{L}$. Hence, $(a\mathcal{L}^n a] \cap \mathcal{L} \neq \emptyset$. Therefore, $\mathcal{L}$ is a WOA-n-II of $\mathcal{S}$. ■

Corollary 3.10. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be WOA-n-IIs of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}_1 \cup \mathcal{I}_2$ is a WOA-n-II of $\mathcal{S}$.*

Corollary 3.11. *Let $\mathcal{S}$ be an ordered semigroup. Then the finite union WOA-n-II of $\mathcal{S}$ is a WOA-n-II of $\mathcal{S}$.*

Theorem 3.12. *Let $\mathcal{I}$ be a WOA-n-II and $\mathcal{H}$ be nonempty subset of an ordered semigroup of $\mathcal{S}$. Then $\mathcal{I} \cup \mathcal{H}$ is a WOA-n-II of $\mathcal{S}$.*

Proof: By Theorem 3.9, and $\mathcal{I} \subseteq \mathcal{I} \cup \mathcal{H}$. Thus, $\mathcal{I} \cup \mathcal{H}$ is a WOA-n-II of $\mathcal{S}$. ■

Corollary 3.13. *Let $\{\mathcal{I}_i \mid i \in \mathcal{I}\}$ be nonempty subset of semigroup $\mathcal{S}$. Then $\bigcup_{i \in \mathcal{I}} \mathcal{I}_i$ is a WOA-n-II of $\mathcal{S}$ if there exists a WOA-n-II $\mathcal{I}_i$ for some $i \in \mathcal{I}$.*

Proof: Assume that there exists a WOA-n-II $\mathcal{I}_i$ for some $i \in \mathcal{I}$ Then $\mathcal{I}_i \subseteq \bigcup\limits_{i \in \mathcal{I}} \mathcal{I}_i$. By Theorem 3.12, $\bigcup_{i \in \mathcal{I}} \mathcal{I}_i$ is a WOA-n-II of $\mathcal{S}$. ■

IV. Fuzzy Ordered Almost n-Interior Ideals of an Ordered Semigroups

In this section, we define the notions of fuzzy ordered almost n-interior ideals, and weakly fuzzy ordered almost n-interior ideals in ordered semigroups and some properties of them are investigated.

Definition 4.1. *A nonzero FS φ of an ordered semigroup $\mathcal{S}$ is called a* fuzzy ordered almost n-interior ideal *(FOA-n-II) of $\mathcal{S}$ if $(r_\beta \circ \varphi^n \circ v_{\beta'}] \wedge \varphi \neq 0$, for fuzzy point r_β and $v_{\beta'}$ of $\mathcal{S}$ and $n \in \mathbb{N}_0$.*

Theorem 4.2. *Let φ be a nonzero FS of an ordered semigroup $\mathcal{S}$. Then every F-n-II of $\mathcal{S}$ is a FOA-n-II of $\mathcal{S}$.*

Proof: Suppose that φ is a F-n-II of $\mathcal{S}$, $r, m \in \mathcal{S}$ and $\beta, \beta' \in (0, 1]$. Since φ is a nonzero, there exists an element $u \in \mathcal{S}$ such that $\varphi(u) \neq 0$.
Let $u = tsk$. Then, for all $n \in \mathbb{N}_0$

$$\begin{aligned}(r_\beta \circ \varphi^n \circ v_{\beta'}](u) &\geq \sup_{u \leq b} (r_\beta \circ \varphi^n \circ v_{\beta'}](u) \\ &\geq (r_\beta \circ \varphi^n \circ v_{\beta'}](b) \\ &= \sup_{u \leq tsk} (r_\beta(t) \wedge \varphi^n(s) \wedge v_{\beta'}(k)] \\ &\geq (r_\beta(t) \wedge \varphi^n(s) \wedge v_{\beta'}(k)] \\ &= \beta \wedge \varphi^n(s) \wedge \beta' \neq 0.\end{aligned}$$

Thus, $(r_\beta \circ \varphi^n \circ v_{\beta'}](u) \geq \beta \wedge \varphi^n(s) \wedge \beta' \neq 0$. Therefore, φ is a FOA-n-II of $\mathcal{S}$. ■

Theorem 4.3. *Let φ be a FOA-n-II and ν be a nonzero FS of an ordered semigroup $\mathcal{S}$ with $\varphi \leq \nu$. Then ν is a FOA-n-II of $\mathcal{S}$.*

Proof: Since φ is a FOA-n-II of $\mathcal{S}$ and ν is a nonzero FS of $\mathcal{S}$ with $\varphi \leq \nu$ and $r \in \mathcal{S}, \beta \in (0, 1]$. Then for all $n \in \mathbb{N}_0$, $(r_\beta \circ \varphi^n \circ v_{\beta'}] \wedge \varphi \neq 0$. By assumption, $(r_\beta \circ \varphi^n \circ v_{\beta'}] \wedge \varphi \leq (r_\beta \circ \nu^n \circ r_{\beta'}] \wedge \nu$ for all $n \in \mathbb{N}_0$. Thus for all $n \in \mathbb{N}_0$, $(r_\beta \circ \nu^n \circ v_{\beta'}] \wedge \nu \neq 0$. Hence ν is a FOA-n-II of $\mathcal{S}$. ■

Theorem 4.4. *Let φ_1 and φ_2 be FOA-n-IIs of $\mathcal{S}$. Then $\varphi_1 \vee \varphi_2$ is a FOA-n-II of $\mathcal{S}$.*

Proof: Since $\varphi_1 \leq \varphi_1 \vee \varphi_2$ we have $\varphi_1 \vee \varphi_2$ is a FOA-n-II of $\mathcal{S}$ by Theorem 4.3. ■

Corollary 4.5. *Let $\mathcal{S}$ be an ordered semigroup. Then the finite union FOA-n-IIs of $\mathcal{S}$ is a FOA-n-II of $\mathcal{S}$.*

Theorem 4.6. *Let φ_1 be a FOA-n-II and φ_2 be FSs of an ordered semigroup of $\mathcal{S}$. Then $\varphi_1 \vee \varphi_2$ is a FOA-n-II of $\mathcal{S}$.*

Proof: By Theorem 4.3 and $\varphi_1 \leq \varphi_1 \vee \varphi_2$. Thus, $\varphi_1 \vee \varphi_2$ is a FOA-n-II of $\mathcal{S}$. ■

Corollary 4.7. *Let φ_i be fuzzy subset of semigroup $\mathcal{S}$. Then $\bigvee_{i\in\mathcal{I}} \varphi_i$ is a FOA-n-II of $\mathcal{S}$ if there exists a FA-n-II φ_i.*

Proof: Assume that there exists a FOA-n-II φ_i for some $i \in \mathcal{I}$ Then $\varphi_i \leq \bigvee_{i\in\mathcal{I}} \varphi_i$. By Theorem 4.6, $\bigvee_{i\in\mathcal{I}} \varphi_i$ is a FOA-n-II of $\mathcal{S}$. ■

Definition 4.8. *A nonzero FS φ of an ordered semigroup $\mathcal{S}$ is called a* weakly fuzzy ordered almost n-interior ideal *(WFOA-n-II) of $\mathcal{S}$ if $(r_\beta \circ \varphi^n \circ r_{\beta'}] \wedge \varphi \neq 0$, for fuzzy point r_β and $r_{\beta'}$ of $\mathcal{S}$ and $n \in \mathbb{N}_0$.*

Every FOA-n-II of an ordered semigroup is a WFOA-n-II of an ordered semigroup.

Theorem 4.9. *Let φ be a WFOA-n-II and ν be a nonzero FS of an ordered semigroup $\mathcal{S}$ with $\varphi \leq \nu$. Then ν is a WFOA-n-II of $\mathcal{S}$.*

Proof: Since φ is a WFOA-n-II of $\mathcal{S}$ and ν is a nonzero FS of $\mathcal{S}$ with $\varphi \leq \nu$ and $r \in \mathcal{I}, \beta \in (0,1]$. Then for all $n \in \mathbb{N}_0$, $(r_\beta \circ \varphi^n \circ r_{\beta'}] \wedge \varphi^n \neq 0$. By assumption, $(r_\beta \circ \varphi^n \circ r_{\beta'}] \wedge \varphi \leq (r_\beta \circ \nu^n \circ r_{\beta'}] \wedge \nu$ for all $n \in \mathbb{N}_0$. Thus for all $n \in \mathbb{N}_0$, $(r_\beta \circ \nu^n \circ r_{\beta'}] \wedge \nu \neq 0$. Hence ν is a WFOA-n-II of $\mathcal{S}$. ■

Theorem 4.10. *Let φ_1 and φ_2 be WFOA-n-IIs of $\mathcal{S}$. Then $\varphi_1 \vee \varphi_2$ is a WFOA-n-II of $\mathcal{S}$.*

Proof: Since $\varphi_1 \leq \varphi_1 \vee \varphi_2$ we have $\varphi_1 \vee \varphi_2$ is a WFOA-n-II of $\mathcal{S}$ by Theorem 4.3. ■

Corollary 4.11. *Let $\mathcal{S}$ be an ordered semigroup. Then the finite union WFOA-n-IIs of $\mathcal{S}$ is a WFOA-n-II of $\mathcal{S}$.*

Theorem 4.12. *Let φ_1 be a WFOA-n-II and φ_2 be FSs of an ordered semigroup of $\mathcal{S}$. Then $\varphi_1 \vee \varphi_2$ is a WFOA-n-II of $\mathcal{S}$.*

Proof: By Theorem 4.9 and $\varphi_1 \leq \varphi_1 \vee \varphi_2$. Thus, $\varphi_1 \vee \varphi_2$ is a WFOA-n-II of $\mathcal{S}$. ■

Corollary 4.13. *Let ρ_i be fuzzy subset of semigroup $\mathcal{S}$. Then $\bigvee_{i\in\mathcal{I}} \varphi_i$ is a WFOA-n-II of $\mathcal{S}$ if there exists a WFOA-n-II φ_i.*

Proof: Assume that there exists a WFOA-n-II φ_i for some $i \in \mathcal{I}$. Then $\varphi_i \leq \bigvee_{i\in\mathcal{I}} \varphi_i$. By Thoerem 4.12, $\bigvee_{i\in\mathcal{I}} \varphi_i$ is a WFOA-n-II of $\mathcal{S}$. ■

In the following theorems, we prove the relationship between almost n-interior ideal and fuzzy almost n-interior ideal in ordered semigroups.

Theorem 4.14. *Let $\mathcal{I}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}$ is an OA-n-II of $\mathcal{S}$ if and only if $\lambda_\mathcal{I}$ is a FOA-n-II of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{I}$ is an OA-n-II of $\mathcal{S}$, $r,k \in \mathcal{S}$ and $\beta, \beta' \in (0,1]$. Then $(r\mathcal{I}^n k] \cap \mathcal{I} \neq \emptyset$, for all $n \in \mathbb{N}_0$. Thus, there exists $p \in \mathcal{S}$ such that $p \in (r\mathcal{I}^n k]$ and $p \in \mathcal{I}$ for all $n \in \mathbb{N}_0$. So $(r_\beta \circ \lambda_\mathcal{I}^n \circ v_{\beta'}](p) \neq 0$ and $\lambda_\mathcal{I}(p) = 1$ for all $n \in \mathbb{N}_0$. It implies that $((r_\beta \circ \lambda_\mathcal{I}^n \circ v_{\beta'}] \wedge \lambda_\mathcal{I})(p) \neq 0$ for all $n \in \mathbb{N}_0$. Hence, $(r_\beta \circ \lambda_I^n \circ v_{\beta'}] \wedge \lambda_\mathcal{I}^n \neq 0$ for all $n \in \mathbb{N}_0$. Therefore, $\lambda_\mathcal{I}$ is a FOA-n-II of $\mathcal{S}$.

For the converse, assume that $\lambda_\mathcal{I}$ is a FOA-n-IIof $\mathcal{S}$ and $r,k \in \mathcal{S}$, $\beta, \beta' \in (0,1]$. Then for all $n \in \mathbb{N}_0$, $(r_\beta \circ \lambda_I^n \circ v_{\beta'}] \wedge \lambda_\mathcal{I} \neq 0$. Thus, there exists $p \in \mathcal{S}$ with $((r_\beta \circ \lambda_I^n \circ v_{\beta'}] \wedge \lambda_\mathcal{I})(p) \neq 0$ for all $n \in \mathbb{N}_0$. This implies that $((r_\beta \circ \lambda_I^n \circ v_{\beta'}](p) \neq 0$ and $\lambda_\mathcal{I}(p) \neq 0$. Hence, $p \in (r\mathcal{I}^n k] \cap \mathcal{I}$ for all $n \in \mathbb{N}_0$. So $(r\mathcal{I}^n k] \cap \mathcal{I} \neq \emptyset$ for all $n \in \mathbb{N}_0$. Therefore, $\mathcal{I}$ is an OA-n-II of $\mathcal{S}$. ■

Theorem 4.15. *Let $\mathcal{I}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}$ is a WOA-n-II of $\mathcal{S}$ if and only if $\lambda_\mathcal{I}$ is a WFOA-n-II of $\mathcal{S}$.*

Proof: It is similar to the proof of Theorem 4.14. ■

Theorem 4.16. *Let φ be a nonzero FS of an ordered semigroup $\mathcal{S}$. Then the φ is a FOA-n-II of $\mathcal{S}$ if and only if* $\text{supp}(\varphi)$ *is an OA-n-II of $\mathcal{S}$.*

Proof: Suppose that φ is a FOA-n-II of $\mathcal{S}$ and $r, v \in S$. Then $(r_\beta \circ \varphi^n \circ v_{\beta'}] \wedge \varphi \neq 0$ for all $n \in \mathbb{N}_0$. Thus, there exist $p \in \mathcal{S}$ such that $((r_\beta \circ \varphi^n \circ v_{\beta'}] \wedge \varphi)(p) \neq 0$ for all $n \in \mathbb{N}_0$ so $\varphi(p) \neq 0$ and there exists $b \in \mathcal{S}$ such that $p = rbw$ and $\varphi(b) \neq 0$. Thus for all $n \in \mathbb{N}$, $(r_\beta \circ \lambda^n_{\text{supp}(\varphi)} \circ v_{\beta'}](p) \neq 0$ and $\lambda_{\text{supp}(\varphi)}(p) \neq 0$. This implies that $((r_\beta \circ \lambda^n_{\text{supp}(\varphi)} \circ v_{\beta'}] \wedge \lambda_{\text{supp}(\varphi)})(p) \neq 0$ for all $n \in \mathbb{N}_0$. Hence, $(r_\beta \circ \lambda^n_{\text{supp}(\varphi)} \circ v_{\beta'}] \wedge \lambda_{\text{supp}(\varphi)} \neq 0$. Therefore, $\lambda_{\text{supp}(\varphi)}$ is a FOA-n-II of $\mathcal{S}$. By Theorem 4.14, $\text{supp}(\varphi)$ is an OA-n-II of $\mathcal{S}$.

For the converse, assume that $\text{supp}(\varphi)$ is an OA-n-II of $\mathcal{S}$. Then $\lambda_{\text{supp}(\varphi)}$ is a FOA-n-II of $\mathcal{S}$. Let $r_\beta, v_{\beta'} \in \mathcal{S}$ and $n \in \mathbb{N}_0$. Then $(r_\beta \circ \lambda^n_{\text{supp}(\varphi)} \circ v_{\beta'}] \wedge \lambda_{\text{supp}(\varphi)} \neq 0$. Thus, there exists $k \in \mathcal{S}$ such that $((r_\beta \circ \lambda^n_{\text{supp}(\varphi)} \circ v_{\beta'}] \wedge \lambda_{\text{supp}(\varphi)})(k) \neq 0$. Hence, $(r_\beta \circ \lambda^n_{\text{supp}(\varphi)} \circ v_{\beta'}](k) \neq 0$ and $\lambda_{\text{supp}(\varphi)}(k) \neq 0$. Then there exists $c \in \text{supp}(\varphi)$ such that $k = rcg$. Thus, $\varphi(k) \neq 0$ and $\varphi(c) \neq 0$. Hence, $(r_\beta \circ \varphi^n \circ v_{\beta'}] \wedge \varphi \neq 0$. Therefore, φ is a FOA-n-II of $\mathcal{S}$. ■

Theorem 4.17. *Let φ be a nonzero FS of an ordered semigroup $\mathcal{S}$. Then the φ is a WFOA-n-II of $\mathcal{S}$ if and only if* $\text{supp}(\varphi)$ *is a WOA-n-II of $\mathcal{S}$.*

Proof: It is similar to the proof of Theorem 4.16. ■

V. Minimal and Maximal Fuzzy Ordered Almost n-Interior Ideals of an Ordered Semigroups

Definition 5.1. *An ordered almost n-interior ideal $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$ without zero is called*

(1) *a* minimal ordered almost n-interior ideal *(MOA-n-II) of $\mathcal{S}$ if for any OA-n-II $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$, we gain that $\mathcal{J} = \mathcal{I}$.*
(2) *a* maximal ordered almost n-interior ideal *(MMOA-n-II) of $\mathcal{S}$ if for any OA-n-II $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{J}$, we gain that $\mathcal{J} = \mathcal{I}$.*

Definition 5.2. *A FOA-n-II φ of a semigroup $\mathcal{S}$ is said to be*

(1) minimal fuzzy ordered almost n-interior ideal *(MFOA-n-II) if for any FOA-n-II ν of $\mathcal{S}$, we have* $\text{supp}(\nu) = \text{supp}(\varphi)$ *whenever $\nu \leq \varphi$.*

(2) maximal fuzzy ordered almost n-interior ideal *(MMFOA-n-II) if for any FOA-n-II ν of $\mathcal{S}$, we have* $\text{supp}(\nu) = \text{supp}(\varphi)$ *whenever* $\varphi \leq \nu$.

Theorem 5.3. *Let $\mathcal{Q}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then the following statements hold:*

(1) *$\mathcal{Q}$ is a MOA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{Q}}$ is a MFOA-n-II of $\mathcal{S}$.*
(2) *$\mathcal{Q}$ is a MMOA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{Q}}$ is a MMFOA-n-II of $\mathcal{S}$.*

Proof: (1) Assume that $\mathcal{Q}$ is a MOA-n-II of $\mathcal{S}$. Then $\mathcal{Q}$ is an A-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\lambda_{\mathcal{Q}}$ is a FOA-n-II of $\mathcal{S}$. Let ν be a FOA-n-II of $\mathcal{S}$ such that $\nu \leq \lambda_{\mathcal{Q}}$. Now, we know, by Theorem 4.16, that $\text{supp}(\nu)$ is an OA-n-II of $\mathcal{S}$ with $\text{supp}(\nu) \subseteq \text{supp}(\lambda_{\mathcal{Q}})$. Since $\text{supp}(\nu) \subseteq \text{supp}(\lambda_{\mathcal{Q}}) = \mathcal{Q}$, by the minimality of $\mathcal{Q}$, we have $\text{supp}(\nu) = \text{supp}(\lambda_{\mathcal{Q}})$. This shows that $\text{supp}(\lambda_{\mathcal{Q}})$ is a MFOA-n-II of $\mathcal{S}$.

Conversely, assume that $\lambda_{\mathcal{Q}}$ is a MFOA-n-II of $\mathcal{S}$. Then $\lambda_{\mathcal{Q}}$ is a FOA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\mathcal{Q}$ is an OA-n-II of $\mathcal{S}$. Let $\mathcal{M}$ be an OA-n-II of $\mathcal{S}$ such that $\mathcal{M} \subseteq \mathcal{Q}$. Then, by Theorem 4.14, λ_M is a FA-n-II of $\mathcal{S}$ such that $\lambda_M \subseteq \lambda_Q$. This implies that $\text{supp}(\lambda_{\mathcal{M}}) \subseteq \text{supp}(\lambda_{\mathcal{Q}})$. By the minimality of $\lambda_{\mathcal{Q}}$, we have $\text{supp}(\lambda_{\mathcal{M}}) = \text{supp}(\lambda_{\mathcal{Q}})$. That is, $\mathcal{M} = \mathcal{Q}$. Therefore, $\mathcal{Q}$ is MOA-n-II.

(2) Assume that $\mathcal{Q}$ is a MMOA-n-II of $\mathcal{S}$. Then $\mathcal{Q}$ is an OA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\lambda_{\mathcal{Q}}$ is a FOA-n-II of $\mathcal{S}$. Let ν be a FOA-n-II of $\mathcal{S}$ such that $\lambda_{\mathcal{Q}} \leq \nu$. Now, we know, by Theorem 4.16, that $\text{supp}(\nu)$ is an almost n-interior ideal of $\mathcal{S}$ with $\text{supp}(\lambda_{\mathcal{Q}}) \subseteq \text{supp}(\nu)$. Since $\mathcal{Q} = \text{supp}(\lambda_{\mathcal{Q}}) \subseteq \text{supp}(\nu)$, by the maximality of $\mathcal{Q}$, we have $\text{supp}(\nu) = \text{supp}(\lambda_{\mathcal{Q}})$. This shows that $\text{supp}(\lambda_{\mathcal{Q}})$ is a MMFOA-n-II of $\mathcal{S}$.

Conversely, assume that $\lambda_{\mathcal{Q}}$ is a MMFOA-n-II of $\mathcal{S}$. Then $\lambda_{\mathcal{Q}}$ is a FOA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\mathcal{Q}$ is an OA-n-II of $\mathcal{S}$. Let $\mathcal{M}$ be an OA-n-II of $\mathcal{S}$ such that $\mathcal{Q} \subseteq \mathcal{M}$. Then, by Theorem 4.14, $\lambda_{\mathcal{M}}$ is a FOA-n-II of $\mathcal{S}$ such that $\lambda_{\mathcal{Q}} \leq \lambda_{\mathcal{M}}$. This implies that $\text{supp}(\lambda_{\mathcal{Q}}) \subseteq \text{supp}(\lambda_{\mathcal{M}})$. By the maximality of $\lambda_{\mathcal{Q}}$, we have $\text{supp}(\lambda_{\mathcal{M}}) = \text{supp}(\lambda_{\mathcal{Q}})$. That is, $\mathcal{M} = \mathcal{Q}$. Therefore, $\mathcal{M}$ is MMOA-n-II. ■

Definition 5.4. *An WOA-n-II $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$ without zero is called*

(1) *a* minimal weakly ordered almost n-interior ideal *(MWOA-n-II) of $\mathcal{S}$ if there is no OA-n-II $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$, we gain that $\mathcal{J} = \mathcal{I}$.*
(2) *a* maximal weakly ordered almost n-interior ideal *(MMWOA-n-II) of $\mathcal{S}$ if there is no OA-n-II $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{J}$, we gain that $\mathcal{J} = \mathcal{I}$.*

Definition 5.5. *A WFOA-n-II φ of a semigroup $\mathcal{S}$ is said to be*

(1) minimal fuzzy weakly ordered almost n-interior ideal *(MFWOA-n-II) if for any FWOA-n-II ν of $\mathcal{S}$, we have* $\text{supp}(\nu) = \text{supp}(\varphi)$ *whenever* $\nu \preceq \varphi$.
(2) maximal fuzzy weakly ordered almost n-interior ideal *(MMFWOA-n-II) if for any FWOA-n-II ν of $\mathcal{S}$, we have* $\text{supp}(\nu) = \text{supp}(\varphi)$ *whenever* $\varphi \preceq \nu$.

Theorem 5.6. *Let $\mathcal{Q}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then the following statements hold:*

(1) *$\mathcal{Q}$ is a MWOA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{Q}}$ is a MFWOA-n-II of $\mathcal{S}$.*
(2) *$\mathcal{Q}$ is a MMWOA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{Q}}$ is a MMFWOA-n-II of $\mathcal{S}$.*

Proof: It is similar to the proof of Theorem 5.3. ■

Corollary 5.7. *Let $\mathcal{S}$ be an ordered semigroup. Then $\mathcal{S}$ has no proper OA-n-II if and only if* $\text{supp}(\vartheta) = \mathcal{S}$ *for every FOA-n-II φ of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{S}$ has no proper OA-n-III and let φ be a FOA-n-II of $\mathcal{S}$. Then by Theorem 4.16, $\text{supp}(\varphi)$ is an OA-n-II of $\mathcal{S}$. By assumption, $\text{supp}(\varphi) = \mathcal{S}$.

Conversely, suppose that $\text{supp}(\vartheta) = \mathcal{S}$ and $\mathcal{K}$ is a proper WOA-n-II of $\mathcal{S}$. Then by Theorem 4.14, $\lambda_{\mathcal{K}}$ is a FWOA-n-II of $\mathcal{S}$. Thus, $\text{supp}(\lambda_{\mathcal{K}}) = \mathcal{K} \neq \mathcal{S}$. It is a contradiction. Hence, $\mathcal{S}$ has no proper OA-n-II. ■

Corollary 5.8. *Let $\mathcal{S}$ be an ordered semigroup. Then $\mathcal{S}$ has no proper WOA-n-II if and only if* $\text{supp}(\vartheta) = \mathcal{S}$ *for every FWOA-n-II φ of $\mathcal{S}$.*

Proof: It is similar to the proof of Corollary 5.7. ■

VI. Prime of fuzzy almost n-interior ideals of an ordered semigroups

In this section, we give a concept of prime ordered almost n-interior ideals and prime fuzzy ordered almost n-interior ideals in ordered semigroups and we prove the property of those.

Definition 6.1. *An OA-n-II $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}$ is said to be:*

(1) prime ordered almost n-interior ideal *(POA-n-II) if $\mathcal{M}\mathcal{L} \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq \mathcal{I}$ or $\mathcal{L} \subseteq \mathcal{I}$, for any OA-$n$-IIs $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$.*
(2) semiprime ordered almost n-interior ideal *(SPOA-n-II) if $\mathcal{M}^2 \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq \mathcal{I}$, for any OA-$n$-II $\mathcal{M}$ of $\mathcal{S}$.*
(3) strongly prime ordered almost n-interior ideal *(SSPOA-n-II) if $\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M} \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq \mathcal{I}$ or $\mathcal{L} \subseteq \mathcal{I}$, for any OA-$n$-IIs $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$.*

Definition 6.2. *Let φ be a FOA-n-II of an ordered semigroup $\mathcal{S}$. Then φ is said to be:*

(1) prime fuzzy ordered almost n-interior ideal *(PFOA-n-II) if $\nu \circ \vartheta \leq \varphi$ implies $\nu \leq \eta$ or $\vartheta \leq \varphi$, for any two FOA-n-IIs ν and ϑ of $\mathcal{S}$.*
(2) semiprime fuzzy ordered almost n-interior ideal *(SPFOA-n-II) if $\nu \circ \nu \leq \varphi$ implies $\nu \leq \varphi$, for any FOA-n-II ν of $\mathcal{S}$.*
(3) strongly prime fuzzy ordered almost n-interior ideal *(SSPFOA-n-II) if $(\nu \circ \vartheta) \wedge (\vartheta \circ \nu) \leq \varphi$ implies $\nu \leq \varphi$ or $\vartheta \leq \varphi$, for any two FOA-n-IIs ν and ϑ of $\mathcal{S}$.*

It is clear that every SSPFOA-n-II is a PFOA-n-II, and every PFOA-n-II is a SPFOA-n-II.

Next, we prove the relationship between the POA-n-II and PFOA-n-II.

Theorem 6.3. *Let $\mathcal{P}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then the following statements hold:*

(1) *$\mathcal{P}$ is a POA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a PFOA-n-II of $\mathcal{S}$.*
(2) *$\mathcal{P}$ is a SPOA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a SPFOA-n-II of $\mathcal{S}$.*

(3) *$\mathcal{P}$ is a SSPOA-n-II of ordered semigroup $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a SSPFOA-n-II of $\mathcal{S}$.*

Proof:

(1) Suppose that $\mathcal{P}$ is a POA-n-II of $\mathcal{S}$. Then $\mathcal{P}$ is an OA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\lambda_{\mathcal{P}}$ is a FOA-n-II of $\mathcal{S}$. Let ϑ and ξ be FOA-n-IIs such that $\vartheta \circ \xi \leq \lambda_{\mathcal{P}}$. Assume that $\vartheta \nleq \lambda_{\mathcal{P}}$ or $\xi \nleq \lambda_{\mathcal{P}}$. Then there exist $h, r \in \mathcal{S}$ such that $\vartheta(h) \neq 0$ and $\xi(r) \neq 0$. While $\lambda_{\mathcal{P}}(h) = 0$ and $\lambda_{P}(r) = 0$. Thus, $h \in \text{supp}(\vartheta)$ and $r \in \text{supp}(\xi)$, but $h, r \notin \mathcal{P}$. So $\text{supp}(\vartheta) \nsubseteq \mathcal{P}$ and $\text{supp}(\xi) \nsubseteq \mathcal{P}$. Since $\text{supp}(\vartheta)$ and $\text{supp}(\xi)$ are OA-n-II s of $\mathcal{S}$ we have $\text{supp}(\vartheta)\,\text{supp}(\xi) \nsubseteq \mathcal{P}$. Thus, there exists $m = de$ for some $d \in \text{supp}(\vartheta)$ and $e \in \text{supp}(\xi)$ such that $m \in \mathcal{P}$. Hence, $\lambda_{P}(m) = 0$ implies that $(\vartheta \circ \xi)(m) = 0$, since $\vartheta \circ \xi \leq \lambda_{\mathcal{P}}$. Since $d \in \text{supp}(\vartheta)$ and $e \in \text{supp}(\xi)$ we have $\vartheta(d) \neq 0$ and $\xi(e) \neq 0$. Thus, $(\vartheta \circ \xi)(m) = \bigvee_{(de) \in F_m} \{\vartheta(d) \wedge \xi(e)\} \neq 0$. It is a contradiction so $\vartheta \leq \lambda_{\mathcal{P}}$ or $\xi \leq \lambda_{\mathcal{P}}$. Therefore, $\lambda_{\mathcal{P}}$ is a PFOA-n-II of $\mathcal{S}$.
Conversely, suppose that $\lambda_{\mathcal{P}}$ is a PFOA-n-II of $\mathcal{S}$. Then $\lambda_{\mathcal{P}}$ is a FOA-n-II of $\mathcal{S}$. Thus by Theorem 4.14, $\mathcal{P}$ is an OA-n-II of $\mathcal{S}$. Let $\mathcal{M}$ and $\mathcal{L}$ be OA-n-II s of $\mathcal{S}$ such that $\mathcal{M}\mathcal{L} \subseteq \mathcal{P}$. Then $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{L}}$ are FOA-n-IIs of $\mathcal{S}$. By Lemma 2.13 $\lambda_{\mathcal{M}} \circ \lambda_{\mathcal{L}} = \lambda_{\mathcal{M}\mathcal{L}} \leq \lambda_{\mathcal{P}}$. By assumption, $\lambda_{\mathcal{M}} \leq \lambda_{\mathcal{P}}$ or $\lambda_{\mathcal{L}} \leq \lambda_{\mathcal{P}}$. Thus, $\mathcal{M} \subseteq \mathcal{P}$ or $\mathcal{L} \subseteq \mathcal{P}$. We conclude that $\mathcal{P}$ is a POA-n-II of $\mathcal{S}$.

(2) Suppose that $\mathcal{P}$ is a SPOA-n-II of $\mathcal{S}$. Then $\mathcal{P}$ is an OA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\lambda_{\mathcal{P}}$ is a FOA-n-II of $\mathcal{S}$. Let ϑ be a OA-n-II such that $\vartheta \circ \vartheta \leq \lambda_{\mathcal{P}}$. Assume that $\vartheta \nleq \lambda_{\mathcal{P}}$. Then there exists $h \in \mathcal{S}$ such that $\vartheta(h) \neq 0$. While $\lambda_{P}(h) = 0$. Thus, $h \in \text{supp}(\vartheta)$, but $h \notin \mathcal{P}$. So $\text{supp}(\vartheta) \nsubseteq \mathcal{P}$. Thus, there exists $m = de$ for some $d \in \text{supp}(\vartheta)$ such that $m \in \mathcal{P}$. Hence, $\lambda_{\mathcal{P}}(m) = 0$ implies that $(\vartheta \circ \xi)(m) = 0$, since $\vartheta \circ \vartheta \leq \lambda_{P}$. Since $d \in \text{supp}(\vartheta)$ and $e \in \text{supp}(\vartheta)$ we have $\vartheta(d) \neq 0$ and $\vartheta(e) \neq 0$. Thus, $(\vartheta \circ \vartheta)(m) = \bigvee_{(de) \in F_m} \{\vartheta(d) \wedge \vartheta(e)\} \neq 0$. It is a contradiction so $\vartheta \leq \lambda_{\mathcal{P}}$. Therefore, $\lambda_{\mathcal{P}}$ is a SPFOA-n-II of $\mathcal{S}$.
Conversely, suppose that $\lambda_{\mathcal{P}}$ is a SPFOA-n-II of $\mathcal{S}$. Then $\lambda_{\mathcal{P}}$ is a FOA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\mathcal{P}$ is an OA-n-II of $\mathcal{S}$. Let $\mathcal{M}$ be an OA-n-II of $\mathcal{S}$ such that $\mathcal{M}^2 \subseteq \mathcal{P}$. Then $\lambda_{\mathcal{M}}$ a FOA-n-II of $\mathcal{S}$. By Lemma 2.13 $\lambda_{\mathcal{M}} \circ \lambda_{\mathcal{M}} = \lambda_{MM} \leq \lambda_{\mathcal{P}}$. By assumption, $\lambda_{\mathcal{M}} \leq \lambda_{\mathcal{P}}$. Thus, $\mathcal{M} \subseteq \mathcal{P}$. We conclude that $\mathcal{P}$ is a SPOA-n-II of $\mathcal{S}$.

(3) Suppose that $\mathcal{P}$ is a SSPOA-n-II of $\mathcal{S}$. Then $\mathcal{P}$ is an OA-n-II of $\mathcal{S}$. Thus by Theorem 4.14, $\lambda_{\mathcal{P}}$ is an OA-n-II of $\mathcal{S}$. Let ϑ and ξ be OA-n-IIs such that $(\vartheta \circ \xi) \wedge (\xi \circ \vartheta) \leq \lambda_{\mathcal{P}}$. Assume that $\vartheta \nleq \lambda_{\mathcal{P}}$ or $\xi \nleq \lambda_{\mathcal{P}}$. Then there exist $h, r \in E$ such that $\vartheta(h) \neq 0$ and $\xi(r) \neq 0$. While $\lambda_{\mathcal{P}}(h) = 0$ and $\lambda_{\mathcal{P}}(r) = 0$. Thus, $h \in \text{supp}(\vartheta)$ and $r \in \text{supp}(\xi)$, but $h, r \notin \mathcal{P}$. So $\text{supp}(\vartheta) \nsubseteq \mathcal{P}$ and $\text{supp}(\xi) \nsubseteq \mathcal{P}$. Hence, there exists $m \in (\text{supp}(\vartheta)\,\text{supp}(\xi)) \cap (\text{supp}(\vartheta)\,\text{supp}(\xi))$ such that $m \notin \mathcal{P}$. Thus $\lambda_{\mathcal{P}}(m) = 0$. Since $m \in \text{supp}(\vartheta)\,\text{supp}(\xi)$ and $m \in \text{supp}(\xi)\,\text{supp}(\vartheta)$ we have $m = d_1e_1$ and $m = e_2d_2$ for some $d_1, d_2 \in \text{supp}(\vartheta)$ and for some $e_1, e_2 \in \text{supp}(\xi)$. Thus, $(\vartheta \circ \xi)(m) = \bigvee_{(d_1e_1) \in F_m} \{\vartheta^p(d_1) \wedge \xi^p(e_1)\} \neq 0$ and $(\xi \circ \vartheta)(m) = \bigvee_{(e_2d_2) \in F_m} \{\xi(e_2) \wedge \vartheta(d_2)\} \neq 0$. So $(\vartheta \circ \xi)(m) \wedge (\vartheta \circ \xi)(m) \neq 0$. It is a contradiction so $(\vartheta \circ \xi)(m) \wedge (\xi \circ \vartheta)(m) = 0$. Hence, $\vartheta \leq \lambda_{\mathcal{P}}$ or $\xi \leq \lambda_{\mathcal{P}}$. Therefore, $\lambda_{\mathcal{P}}$ is a SSPFOA-n-II of $\mathcal{S}$.
Conversely, suppose that $\lambda_{\mathcal{P}}$ is a SSPFOA-n-II of $\mathcal{S}$. Then $\lambda_{\mathcal{P}}$ is a FOA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\mathcal{P}$ is an OA-n-II of $\mathcal{S}$. Let $\mathcal{M}$ and $\mathcal{L}$ be OA-n-IIs of $\mathcal{S}$ such that $\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M} \subseteq \mathcal{P}$. Then $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{L}}$ are FOA-n-II s of $\mathcal{S}$. By Lemma 2.13 $\lambda_{\mathcal{M}\mathcal{L}} = \lambda_{\mathcal{M}} \circ \lambda_{\mathcal{L}}$. Thus, $(\lambda_{\mathcal{M}} \circ \lambda_{\mathcal{L}}) \wedge (\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{M}}) = \lambda_{\mathcal{M}\mathcal{L}} \wedge \lambda_{\mathcal{L}\mathcal{M}} = \lambda_{\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M}} \leq \lambda_{\mathcal{P}}$. By assumption, $\lambda_{\mathcal{M}} \leq \lambda_{\mathcal{P}}$ or $\lambda_{\mathcal{L}} \leq \lambda_{\mathcal{P}}$. Thus, $\mathcal{M} \subseteq \mathcal{P}$ or $\mathcal{L} \subseteq \mathcal{P}$. We conclude that $\mathcal{P}$ is a SSPOA-n-II of S.

■

Definition 6.4. *An WOA-n-II $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}$ is said to be:*

(1) prime weakly ordered almost n-interior ideal *(PWOA-n-II) if $\mathcal{M}\mathcal{L} \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq \mathcal{I}$ or $\mathcal{L} \subseteq \mathcal{I}$, for any WOA-$n$-IIs $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$.*
(2) semiprime weakly ordered almost n-interior ideal *(SPWOA-n-II) if $\mathcal{M}^2 \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq I$, for any WOA-n-II $\mathcal{M}$ of $\mathcal{S}$,*
(3) strongly prime weakly ordered almost n-interior ideal *(SSPWOA-n-II) if $\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M} \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq \mathcal{I}$ or $\mathcal{L} \subseteq \mathcal{I}$, for any WOA-$n$-IIs $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$.*

Definition 6.5. *Let φ be a WFOA-n-II of an ordered semigroup$\mathcal{S}$. Then φ is said to be:*

(1) prime weakly fuzzy ordered almost n-interior ideal *(PWFOA-n-II) if $\nu \circ \vartheta \leq \varphi$ implies $\nu \leq \eta$ or $\vartheta \preceq \varphi$, for any two fuzzy weakly almost n-interior idelas ν and ϑ of $\mathcal{S}$.*
(2) semiprime weakly fuzzy ordered almost n-interior ideal *(SPWFOA-n-II) if $\nu \circ \nu \leq \varphi$ implies $\nu \leq \varphi$, for any WFOA-n-II ν of $\mathcal{S}$.*
(3) strongly prime weakly fuzzy ordered almost n-interior ideal *(SSPWFOA-n-II) if $(\nu \circ \vartheta) \cap (\vartheta \circ \nu) \leq \varphi$ implies $\nu \leq \varphi$ or $\vartheta \leq \varphi$, for any two WFOA-n-IIs ν and ϑ of $\mathcal{S}$.*

Theorem 6.6. *Let $\mathcal{P}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then the following statements hold:*

(1) *$\mathcal{P}$ is a PWOA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a PWFOA-n-II of $\mathcal{S}$.*
(2) *$\mathcal{P}$ is a SPWOA-n-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a SPWFOA-n-II of $\mathcal{S}$.*
(3) *$\mathcal{P}$ is a SSPWOA-n-II of ordered semigroup $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a SSPWFOA-n-II of $\mathcal{S}$.*

Proof: It is similar to the proof of Theorem 7.22. ■

VII. Ordered Almost (m, n)-interior ideals of an ordered semigroups

In this section, we define the concept of ordered almost (m, n)-interior ideals in ordred semigroups and give characterizations of basic properties of its.

Definition 7.1. *A nonempty set $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$ is called an* ordered almost (m, n)-interior ideal *(OA-(m, n)-II) of $\mathcal{S}$ if $(a^m \mathcal{I} b^n] \cap \mathcal{I} \neq \emptyset$, for all $a, b \in \mathcal{S}$ and $m, n \in \mathbb{N}_0$.*

Remark 7.2. *Every OA-n-II in ordered semigroup is OA-(m, n)-II.*

Theorem 7.3. *Let $\mathcal{I}$ and $\mathcal{L}$ be nonempty subsets an ordered semigroup of $\mathcal{S}$ with $\mathcal{I} \subseteq \mathcal{L}$. If $\mathcal{I}$ is an OA-(m,n)-II of $\mathcal{S}$, then $\mathcal{L}$ is an OA-(m,n)-II of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{I}$ is an OA-(m,n)-II of $\mathcal{S}$ with $\mathcal{I} \subseteq \mathcal{L}$ and $a,b \in \mathcal{S}, m,n \in \mathbb{N}_0$. Then $(a^m\mathcal{I}b^n] \subseteq (a^m\mathcal{L}b^n]$. Thus, $[a^n\mathcal{I}b^n) \cap \mathcal{I} \subseteq (a^m\mathcal{L}b^n] \cap \mathcal{L}$ so $\emptyset \neq (a^m\mathcal{I}b^n] \cap \mathcal{I} \subseteq (a^m\mathcal{L}b^n] \cap \mathcal{L}$. Hence, $(a^m\mathcal{L}b^n] \cap \mathcal{L} \neq \emptyset$. Therefore, $\mathcal{L}$ is an OA-(m,n)-II of $\mathcal{S}$. ∎

Corollary 7.4. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be OA-(m,n)-IIs of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}_1 \cup \mathcal{I}_2$ is an OA-(m,n)-II of $\mathcal{S}$.*

Corollary 7.5. *Let $\mathcal{S}$ be an ordered semigroup. Then the finite union OA-(m,n)-IIs of $\mathcal{S}$ is an OA-(m,n)-II of $\mathcal{S}$.*

Theorem 7.6. *Let $\mathcal{I}$ be an OA-(m,n)-II and $\mathcal{H}$ be nonempty subset of an ordered semigroup of $\mathcal{S}$. Then $\mathcal{I} \cup \mathcal{H}$ is an OA-(m,n)-II of $\mathcal{S}$.*

Proof: By Theorem 7.3, and $\mathcal{I} \subseteq \mathcal{I} \cup \mathcal{H}$. Thus, $\mathcal{I} \cup \mathcal{H}$ is an OA-(m,n)-II of $\mathcal{S}$. ∎

Corollary 7.7. *Let $\{\mathcal{I}_i \mid i \in \mathcal{I}\}$ be nonempty subset of semigroup $\mathcal{S}$. Then $\bigcup_{i\in\mathcal{I}} \mathcal{I}_i$ is an OA-(m,n)-II of $\mathcal{S}$ if there exists an OA-(m,n)-II $\mathcal{I}_i$ for some $i \in \mathcal{I}$.*

Proof: Assume that there exists an OA-(m,n)-II $\mathcal{I}_i$ for some $i \in \mathcal{I}$. Then $\mathcal{I}_i \subseteq \bigcup_{i\in\mathcal{I}} \mathcal{I}_i$. By Theorem 7.3, $\bigcup_{i\in\mathcal{I}} \mathcal{I}_i$ is an OA-(m,n)-II of $\mathcal{S}$. ∎

Definition 7.8. *A nonzero FS φ of an ordered semigroup $\mathcal{S}$ is called a* fuzzy ordered almost (m,n)-interior ideal *(FOA-(m,n)-II) of $\mathcal{S}$ if $(r^m_\beta \circ \varphi \circ v^m_{\beta'}] \wedge \varphi \neq 0$, for fuzzy point r_β and $v_{\beta'}$ of $\mathcal{S}$ and $m,n \in \mathbb{N}_0$.*

Theorem 7.9. *Let φ be a FOA-(m,n)-II and ν be a nonzero FS of an ordered semigroup $\mathcal{S}$ with $\varphi \leq \nu$. Then ν is a FOA-(m,n)-II of $\mathcal{S}$.*

Proof: Since φ is a FOA-(m,n)-II of $\mathcal{S}$ and ν is a nonzero FS of $\mathcal{S}$ with $\varphi \leq \nu$ and $r \in \mathcal{S}, \beta \in (0,1]$. Then for all $m,n \in \mathbb{N}_0$, $(r^m_\beta \circ \varphi \circ v^n_{\beta'}] \wedge \varphi \neq 0$. By assumption, $(r^m_\beta \circ \varphi \circ v^m_{\beta'}] \wedge \varphi \leq (r^m_\beta \circ \nu \circ r^n_{\beta'}] \wedge \nu$ for all $n \in \mathbb{N}_0$. Thus for all $m,n \in \mathbb{N}_0$, $(r^m_\beta \circ \nu \circ v^n_{\beta'}] \wedge \nu \neq 0$. Hence ν is a FOA-(m,n)-II of $\mathcal{S}$. ∎

Theorem 7.10. *Let φ_1 and φ_2 be FOA-(m,n)-IIs of $\mathcal{S}$. Then $\varphi_1 \vee \varphi_2$ is a FOA-(m,n)-II of $\mathcal{S}$.*

Proof: Since $\varphi_1 \leq \varphi_1 \vee \varphi_2$ we have $\varphi_1 \vee \varphi_2$ is a FOA-(m,n)-II of $\mathcal{S}$ by Theorem 7.9. ∎

Corollary 7.11. *Let $\mathcal{S}$ be an ordered semigroup. Then the finite union FOA-n-IIs of $\mathcal{S}$ is a FOA-(m,n)-II of $\mathcal{S}$.*

Theorem 7.12. *Let φ_1 be a FOA-(m,n)-II and φ_2 be FSs of an ordered semigroup of $\mathcal{S}$. Then $\varphi_1 \vee \varphi_2$ is a FOA-(m,n)-II of $\mathcal{S}$.*

Proof: By Theorem 7.9 and $\varphi_1 \leq \varphi_1 \vee \varphi_2$. Thus, $\varphi_1 \vee \varphi_2$ is a FOA-(m,n)-II of $\mathcal{S}$. ∎

Corollary 7.13. *Let φ_i be fuzzy subset of semigroup $\mathcal{S}$. Then $\bigvee_{i\in\mathcal{I}} \varphi_i$ is a FOA-n-II of $\mathcal{S}$ if there exists a FOA-(m,n)-II φ_i.*

Proof: Assume that there exists a FOA-(m,n)-II φ_i for some $i \in \mathcal{I}$ Then $\varphi_i \leq \bigvee_{i\in\mathcal{I}} \varphi_i$. By Theorem 7.12, $\bigvee_{i\in\mathcal{I}} \varphi_i$ is a FOA-(m,n)-II of $\mathcal{S}$. ∎

Theorem 7.14. *Let $\mathcal{I}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}$ is an OA-(m,n)-II of $\mathcal{S}$ if and only if $\lambda_\mathcal{I}$ is a FOA-(m,n)-II of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{I}$ is an OA-(m,n)-II of $\mathcal{S}$, $r,k \in \mathcal{S}$ and $\beta, \beta^{'} \in (0,1]$. Then $(r^m\mathcal{I}k^n] \cap \mathcal{I} \neq \emptyset$, for all $m,n \in \mathbb{N}_0$. Thus, there exists $p \in \mathcal{S}$ such that $p \in (r^m\mathcal{I}k^n]$ and $p \in \mathcal{I}$ for all $m,n \in \mathbb{N}_0$. So $(r^m_\beta \circ \lambda_\mathcal{I} \circ v^n_{\beta'}](p) \neq 0$ and $\lambda_\mathcal{I}(p) = 1$ for all $m,n \in \mathbb{N}_0$. It implies that $((r^m_\beta \circ \lambda_\mathcal{I} \circ v^n_{\beta'}] \wedge \lambda_\mathcal{I})(p) \neq 0$ for all $m,n \in \mathbb{N}_0$. Hence, $(r^m_\beta \circ \lambda_I \circ v^n_{\beta'}] \wedge \lambda_\mathcal{I} \neq 0$ for all $m,n \in \mathbb{N}_0$. Therefore, $\lambda_\mathcal{I}$ is a FOA-(m,n)-II of $\mathcal{S}$.

For the converse, assume that $\lambda_\mathcal{I}$ is a FOA-(m,n)-IIof $\mathcal{S}$ and $r,k \in \mathcal{S}$, $\beta, \beta^{'} \in (0,1]$. Then for all $m,n \in \mathbb{N}_0$, $(r^m_\beta \circ \lambda_I \circ v^n_{\beta'}] \wedge \lambda_\mathcal{I} \neq 0$. Thus, there exists $p \in \mathcal{S}$ with $((r^m_\beta \circ \lambda_I \circ v^n_{\beta'}] \wedge \lambda_\mathcal{I})(p) \neq 0$ for all $m,n \in \mathbb{N}_0$. This implies that $((r^m_\beta \circ \lambda_I \circ v^n_{\beta'}](p) \neq 0$ and $\lambda_\mathcal{I}(p) \neq 0$. Hence, $p \in (r^m\mathcal{I}k^n] \cap \mathcal{I}$ for all $m,n \in \mathbb{N}_0$. So $(r^m\mathcal{I}k^n] \cap \mathcal{I} \neq \emptyset$ for all $n,n \in \mathbb{N}_0$. Therefore, $\mathcal{I}$ is an OA-(m,n)-II of $\mathcal{S}$. ∎

Theorem 7.15. *Let φ be a nonzero FS of an ordered semigroup $\mathcal{S}$. Then the φ is a FOA-(m,n)-II of $\mathcal{S}$ if and only if* $\mathrm{supp}(\varphi)$ *is an OA-(m,n)-II of $\mathcal{S}$.*

Proof: Suppose that φ is a FOA-(m,n)-II of $\mathcal{S}$ and $r,v \in S$. Then $(r^m_\beta \circ \varphi \circ v^n_{\beta'}] \wedge \varphi \neq 0$ for all $m,n \in \mathbb{N}_0$. Thus, there exist $p \in \mathcal{S}$ such that $((r^m_\beta \circ \varphi \circ v6n_{\beta'}] \wedge \varphi)(p) \neq 0$ for all $m,n \in \mathbb{N}_0$ so $\varphi(p) \neq 0$ and there exists $b \in \mathcal{S}$ such that $p = rbw$ and $\varphi(b) \neq 0$. Thus for all $m,n \in \mathbb{N}$, $(r^m_\beta \circ \lambda_{\mathrm{supp}(\varphi)} \circ v^n_{\beta'}](p) \neq 0$ and $\lambda_{\mathrm{supp}(\varphi)}(p) \neq 0$. This implies that $((r^m_\beta \circ \lambda_{\mathrm{supp}(\varphi)} \circ v^n_{\beta'}] \wedge \lambda_{\mathrm{supp}(\varphi)})(p) \neq 0$ for all $m,n \in \mathbb{N}_0$. Hence, $(r^m_\beta \circ \lambda_{\mathrm{supp}(\varphi)} \circ v^n_{\beta'}] \wedge \lambda_{\mathrm{supp}(\varphi)} \neq 0$. Therefore, $\lambda_{\mathrm{supp}(\varphi)}$ is a FOA-(m,n)-II of $\mathcal{S}$. By Theorem 7.14, $\mathrm{supp}(\varphi)$ is an OA-(m,n)-II of $\mathcal{S}$.

For the converse, assume that $\mathrm{supp}(\varphi)$ is an OA-(m,n)-II of $\mathcal{S}$. Then $\lambda_{\mathrm{supp}(\varphi)}$ is a FOA-(m,n)-II of $\mathcal{S}$. Let $r^m_\beta, v^n_{\beta'} \in \mathcal{S}$ and $m,n \in \mathbb{N}_0$. Then $(r^m_\beta \circ \lambda_{\mathrm{supp}(\varphi)} \circ v^n\beta^{'}] \wedge \lambda_{\mathrm{supp}(\varphi)} \neq 0$. Thus, there exists $k \in \mathcal{S}$ such that $((r^m_\beta \circ \lambda_{\mathrm{supp}(\varphi)} \circ v^n_{\beta'}] \wedge \lambda_{\mathrm{supp}(\varphi)})(k) \neq 0$. Hence, $(r^m_\beta \circ \lambda_{\mathrm{supp}(\varphi)} \circ v^n_{\beta'}](k) \neq 0$ and $\lambda_{\mathrm{supp}(\varphi)}(k) \neq 0$. Then there exists $c \in \mathrm{supp}(\varphi)$ such that $k = rcg$. Thus, $\varphi(k) \neq 0$ and $\varphi(c) \neq 0$. Hence, $(r^m_\beta \circ \varphi \circ v^n_{\beta'}] \wedge \varphi \neq 0$. Therefore, φ is a FOA-(m,n)-II of $\mathcal{S}$. ∎

Definition 7.16. *An ordered almost (m,n)-interior ideal $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$ without zero is called*

(1) *a* minimal ordered almost (m,n)-interior ideal *(MOA-(m,n)-II) of $\mathcal{S}$ if for any OA-(m,n)-II $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$, we gain that $\mathcal{J} = \mathcal{I}$.*
(2) *a* maximal ordered almost (m,n)-interior ideal *(MMOA-(m,n)-II) of $\mathcal{S}$ if for any OA-(m,n)-II $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{J}$, we gain that $\mathcal{J} = \mathcal{I}$.*

Definition 7.17. *A FOA-(m,n)-II φ of a semigroup $\mathcal{S}$ is said to be*

(1) minimal fuzzy ordered almost n-interior ideal *(MFOA-(m,n)-II) if for any FOA-n-II ν of $\mathcal{S}$, we have* $\mathrm{supp}(\nu) = \mathrm{supp}(\varphi)$ *whenever $\nu \leq \varphi$.*
(2) maximal fuzzy ordered almost (m,n)-interior ideal

(MMFOA-(m,n)-II) if for any FOA-(m,n)-II ν of $\mathcal{S}$, we have $\text{supp}(\nu) = \text{supp}(\varphi)$ *whenever* $\varphi \leq \nu$.

Theorem 7.18. *Let $\mathcal{Q}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then the following statements hold:*

(1) *$\mathcal{Q}$ is a MOA-(m,n)-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{Q}}$ is a MFOA-(m,n)-II of $\mathcal{S}$.*
(2) *$\mathcal{Q}$ is a MMOA-(m,n)-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{Q}}$ is a MMFOA-(m,n)-II of $\mathcal{S}$.*

Proof: (1) Assume that $\mathcal{Q}$ is a MOA-(m,n)-II of $\mathcal{S}$. Then $\mathcal{Q}$ is an A-n-II of $\mathcal{S}$. Thus, by Theorem 7.14, $\lambda_{\mathcal{Q}}$ is a FOA-(m,n)-II of $\mathcal{S}$. Let ν be a FOA-(m,n)-II of $\mathcal{S}$ such that $\nu \leq \lambda_{\mathcal{Q}}$. Now, we know, by Theorem 7.15, that $\text{supp}(\nu)$ is an OA-(m,n)-II of $\mathcal{S}$ with $\text{supp}(\nu) \subseteq \text{supp}(\lambda_{\mathcal{Q}})$. Since $\text{supp}(\nu) \subseteq \text{supp}(\lambda_{\mathcal{Q}}) = \mathcal{Q}$, by the minimality of $\mathcal{Q}$, we have $\text{supp}(\nu) = \text{supp}(\lambda_{\mathcal{Q}})$. This shows that $\text{supp}(\lambda_{\mathcal{Q}})$ is a MFOA-(m,n)-II of $\mathcal{S}$.

Conversely, assume that $\lambda_{\mathcal{Q}}$ is a MFOA-(m,n)-II of $\mathcal{S}$. Then $\lambda_{\mathcal{Q}}$ is a FOA-(m,n)-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\mathcal{Q}$ is an OA-n-II of $\mathcal{S}$. Let $\mathcal{M}$ be an OA-(m,n)-II of $\mathcal{S}$ such that $\mathcal{M} \subseteq \mathcal{Q}$. Then, by Theorem 7.14, λ_M is a FA-n-II of $\mathcal{S}$ such that $\lambda_M \subseteq \lambda_Q$. This implies that $\text{supp}(\lambda_{\mathcal{M}}) \subseteq \text{supp}(\lambda_{\mathcal{Q}})$. By the minimality of $\lambda_{\mathcal{Q}}$, we have $\text{supp}(\lambda_{\mathcal{M}}) = \text{supp}(\lambda_{\mathcal{Q}})$. That is, $\mathcal{M} = \mathcal{Q}$. Therefore, $\mathcal{Q}$ is MOA-(m,n)-II.

(2) Assume that $\mathcal{Q}$ is a MMOA-(m,n)-II of $\mathcal{S}$. Then $\mathcal{Q}$ is an OA-n-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\lambda_{\mathcal{Q}}$ is a FOA-(m,n)-II of $\mathcal{S}$. Let ν be a FOA-(m,n)-II of $\mathcal{S}$ such that $\lambda_{\mathcal{Q}} \leq \nu$. Now, we know, by Theorem 4.16, that $\text{supp}(\nu)$ is an ordreder almost (m,n)-interior ideal of $\mathcal{S}$ with $\text{supp}(\lambda_{\mathcal{Q}}) \subseteq \text{supp}(\nu)$. Since $\mathcal{Q} = \text{supp}(\lambda_{\mathcal{Q}}) \subseteq \text{supp}(\nu)$, by the maximality of $\mathcal{Q}$, we have $\text{supp}(\nu) = \text{supp}(\lambda_{\mathcal{Q}})$. This shows that $\text{supp}(\lambda_{\mathcal{Q}})$ is a MMFOA-(m,n)-II of $\mathcal{S}$.

Conversely, assume that $\lambda_{\mathcal{Q}}$ is a MMFOA-(m,n)-II of $\mathcal{S}$. Then $\lambda_{\mathcal{Q}}$ is a FOA-(m,n)-II of $\mathcal{S}$. Thus, by Theorem 4.14, $\mathcal{Q}$ is an OA-n-II of $\mathcal{S}$. Let $\mathcal{M}$ be an OA-(m,n)-II of $\mathcal{S}$ such that $\mathcal{Q} \subseteq \mathcal{M}$. Then, by Theorem 4.14, $\lambda_{\mathcal{M}}$ is a FOA-n-II of $\mathcal{S}$ such that $\lambda_{\mathcal{Q}} \leq \lambda_{\mathcal{M}}$. This implies that $\text{supp}(\lambda_{\mathcal{Q}}) \subseteq \text{supp}(\lambda_{\mathcal{M}})$. By the maximality of $\lambda_{\mathcal{Q}}$, we have $\text{supp}(\lambda_{\mathcal{M}}) = \text{supp}(\lambda_{\mathcal{Q}})$. That is, $\mathcal{M} = \mathcal{Q}$. Therefore, $\mathcal{M}$ is MMOA-(m,n)-II. ∎

Corollary 7.19. *Let $\mathcal{S}$ be an ordered semigroup. Then $\mathcal{S}$ has no proper OA-(m,n)-II if and only if* $\text{supp}(\vartheta) = \mathcal{S}$ *for every FOA-(m,n)-II φ of $\mathcal{S}$.*

Definition 7.20. *An OA-(m,n)-II $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$. Then $\mathcal{I}$ is said to be:*

(1) prime ordered almost (m,n)-interior ideal *(POA-(m,n)-II) if $\mathcal{ML} \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq \mathcal{I}$ or $\mathcal{L} \subseteq \mathcal{I}$, for any OA-$(m,n)$-IIs $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$.*
(2) semiprime ordered almost n-interior ideal *(SPOA-(m,n)-II) if $\mathcal{M}^2 \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq I$, for any OA-(m,n)-II $\mathcal{M}$ of $\mathcal{S}$,*
(3) strongly prime ordered almost n-interior ideal *(SSPOA-(m,n)-II) if $\mathcal{ML} \cap \mathcal{LM} \subseteq \mathcal{I}$ implies $\mathcal{M} \subseteq \mathcal{I}$ or $\mathcal{L} \subseteq \mathcal{I}$, for any OA-$(m,n)$-IIs $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{S}$.*

Definition 7.21. *Let φ be a fuzzy ordered almost (m,n)-interior ideal of an ordered semigroup $\mathcal{S}$. Then φ is said to be:*

(1) prime fuzzy ordered almost (m,n)-interior ideal *(PFOA-(m,n)-II)if $\nu \circ \vartheta \leq \varphi$ implies $\nu \leq \eta$ or $\vartheta \leq \varphi$, for any two fuzzy oredred almost (m,n)-interior ideals ν and ϑ of $\mathcal{S}$.*
(2) semiprime fuzzy ordered almost (m,n)-interior ideal *(SPFOA-(m,n)-II) if $\nu \circ \nu \leq \varphi$ implies $\nu \leq \varphi$, for any fuzzy ordered almost n-interior ideal ν of $\mathcal{S}$.*
(3) strongly prime fuzzy ordered almost (m,n)-interior ideal *(SSPFOA-(m,n)-II) if $(\nu \circ \vartheta) \wedge (\vartheta \circ \nu) \leq \varphi$ implies $\nu \leq \varphi$ or $\vartheta \leq \varphi$, for any two fuzzy ordered almost (m,n)-interior ideals ν and ϑ of $\mathcal{S}$.*

It is clear that every SSPFOA-(m,n)-II is a PFOA-(m,n)-II, and every PFOA-(m,n)-II is a SPFOA-(m,n)-II.

Next, we prove the relationship between the POA-(m,n)-II and PFOA-(m,n)-II.

Theorem 7.22. *Let $\mathcal{P}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then the following statements hold:*

(1) *$\mathcal{P}$ is a POA-(m,n)-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a PFOA-(m,n)-II of $\mathcal{S}$.*
(2) *$\mathcal{P}$ is a SPOA-(m,n)-II of $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a SPFOA-(m,n)-II of $\mathcal{S}$.*
(3) *$\mathcal{P}$ is a SSPOA-(m,n)-II of ordered semigroup $\mathcal{S}$ if and only if $\lambda_{\mathcal{P}}$ is a SSPFOA-(m,n)-II of $\mathcal{S}$.*

Proof: It is similar to the proof of Theorem 7.22. ∎

VIII. Conclusion

The union of two almost n-interior, ideals is also an OA-n-II in ordered semigroups, and the results in class fuzzifications are the same. In Theorems 4.14, 4.16, 5.3, 7.22 we prove the relationship between OA-n-IIs and class fuzzifications. In the same way, the FOA-n-IIs and WFOA-n-IIs got to same results as OA-n-IIs and WFOA-n-IIs. We extend the concepts of ordered almost (m,n)-interior ideal and fuzzifications. In future work, we can study types of pictures almost ideals and their fuzzifications in ordered semigroups.

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