$L(2,1)$ Colouring and Radio Colouring of Some Graphs and its Parametrized Graphs

A.Vimala Rani

Abstract—**Graph Theory would not be what it is today without graph colouring. Graph colouring is the major part of discrete mathematics which is majorly used in network analysis. Radio colouring and L(2,1) colouring are cardinal topics in graph theory that associate many real life situation.The dominant role of graph colouring is to partition the independent components which in turn is applied in practical solution of networks. The interference reduction problem is modeled as a graph coloring problem which is the principal keynote of study. There are several well-known colouring parameters in graph theory. Here we deal with the following colouring parameters, radio number and span.The colouring parameters correlates the constrains of networks. In this article the span of sunlet, bistar, pencil graph families and radio number of Mycielski of cycle, bistar graphs are examined. Also, the comparison of span and radio number of some graph families are examined.**

Keywords: Distance two colouring, Radio colouring, Sunlet graph, Bistar graph, Pencil graph, Mycielskian graph.

1 INTRODUCTION

As bigger and more advanced wireless networks are continuously deployed, the issue of mitigating interference in wireless networks is becoming more and more significant. An important drawback of wire-less networks is the interference of signals. Preventing nearby and related nodes connected by radio signals from receiving and sending signals which interfere or blend together is the goal of reducing interference. Thus, in a wireless network, interference arises when one or more nodes receive conflicting broadcasts over the same radio frequency. This hinders the receiver's capacity to interpret incoming signals. Using the graph coloring problem to lessen interference in wireless networks is a more feasible solution than embedding a system or spending a lot of money on radio transmitter technology. Because it turns out to be a highly challenging topic, simpler network topologies have been studied in order to reduce interference in random networks. There are several ways to lessen interference, including channel assignment, power control, and temperature regulation. Reducing interference in a network can be achieved by carefully allocating communication channels to individual nodes. It's crucial to remember that there is a limited supply of radio frequencies, therefore it makes sense to look closely into the issue of decreasing the number of channels allotted to a particular network. Channel overlap may be required in some situations where a network's allotted number of channels is insufficient to connect every node. I have chosen to focus on radio colouring and distance two colouring for my research because they both originate from the idea of interference reduction problems. The constraints for the radio and distance two coloring of graphs were examined in this work. The Frequency Assignment Problem served as inspiration for both the radio and distance two coloring concepts.

As radio colouring and distance two colouring are based on the concept of interference reduction problem these topics has been chosen for my research work. This work analyzed the bounds for the radio and distance two colouring of graphs. The radio colouring and distance two colouring is inspired from the Frequency Assignment Problem.

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A.Vimala Rani is an assistant professor of Mathematics Department, Academy of Maritime Education and Training, Chennai, Tamil Nadu, 603112, India. (e-mail: vimalaraniarul@ametuniv.ac.in).

2 BACKGROUND

Let $G = (V, E)$ be a simple, connected and undirected graph. The $L(2,1)$- colouring problem proposed by Griggs and Roberts is a variation of the frequency assignment problem introduced by Hale. Griggs and Yeh showed that the $L(2,1)$-problem is NP-complete for general graphs. A Distance two ($L(2,1)$) Colouring of a graph is a function c from the vertex set $V(G)$ to the set of all non negative integers such that $|c(u) - c(v)| \geq 2$ if $d(u,v) = 1$ and $|c(u) - c(v)| \geq 1$ if $d(u,v) = 2$, where $d(u,v)$ denotes the distance between u and v in G. The $L(2,1)$-colouring number or span number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-colouring with $\max\{c(v) : v \epsilon V(G)\} = k$.

A radio colouring of a graph is defined as colouring the vertices of G with positive integers such that the distance two vertices are assigned different colours and adjacent vertices are coloured with difference at least two. A radio colouring that uses k-colours is a k-radio colouring.

The minimum number of colours used is the radio chromatic number or radio number rn(G) of G.

Difference between Radio Colouring and $D(2,1)$ Colouring: The distance two colouring uses non-negative integers for colouring and radio colouring uses positive integers for colouring. The span is the minimum value among the largest colour assigned to vertices of the graph and the radio number is the minimum number of colours used.

R.Kalfakakou et al. [2] defined the Radio Colouring of graphs and radio chromatic number of graph in 2003.

Yeh [1991] and then Griggs and Yeh [1992] first considered L(2,1) Colouring. Griggs showed that $\lambda(T) = \Delta + 1$ or $\Delta + 2$ and also the span of path, cycle and upper bound for connected graphs and conjectured that $\lambda(G) \leq \triangle^2$ for any simple graph with maximum degree at least 2 [1]. Griggs and Yeh showed that the $L(2,1)$-problem is NP-complete for general graphs.

David. W. Mauro et al. contributed the results of the complete graph, generalized Petersen graph and trees [3-5]. Peter Bella et al. proposed the result for planar graph [6]. Sakai. D explored the results of chordal graphs [7]. S. K. Vaidya et al. discussed the results of graph operations on cycle and total graph of the path, cycle and star graph, n-ary k-regular cactus [8–10].

Zhendong Shao, in his thesis, studied the bounds of total graph of star free graph, Mycielski of the complete graph and improved the upper bounds of standard products of graphs [11]. Pingli. Lv et al. discussed the bounds of the cartesian sum of graphs [12].

Zhendong Shao et al. discussed the bounds of Kneser graphs and squares of Kneser graphs [13]. Christopher Schwarz, Denise Sakai Troxell, studied the bound for the cartesian product of cycles [14]. Pranava K. Jha et al. explored the results of direct product of paths and cycles [14].

3 PRELIMINARIES

Definition 3.1. A *k-colouring* of a graph is an assignment of colours(positive integers) to the nodes of G using k colours. A proper colouring is assigning colours to the vertices of G, such that every two adjacent vertices are assigned different colours. The minimum number of colours used in proper colouring is said to be chromatic number of graph and is denoted by $\chi(G)$.

Definition 3.2. A *radio colouring* of a graph is defined as colouring the vertices of G with positive integers such that the distance two vertices are assigned different colours and adjacent vertices are coloured with difference at least two. A radio colouring that uses k-colours is a k-radio colouring. The minimum number of colours used is the radio chromatic number or radio number rn(G) of G.

Definition 3.3. An *L(2,1) colouring* of a graph G is an assignment of colours(non-negative integers) to the vertices of G such that

(i) colours assigned to adjacent vertices differ by at least two,

(ii) colours assigned to vertices at distance two must differ, and

(iii) no restriction is placed on colours assigned to vertices at distance three or more.

Definition 3.4. For a graph G, the *Mycielskian of G* is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{x'y : xy\} \cup \{x'u : x''\}$. We call x' the twin of x in $\mu(G)$ and vice versa and u, the root of $\mu(G)$.

Definition 3.5. The *n-sunlet graph* on 2n vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.

Definition 3.6. The *line graph of a graph G*, denoted by L(G), is a graph whose vertices are the edges of G and if u,v ∈E(G) then uv ∈E(L(G)) if u and v share a vertex in G.

Definition 3.7. Let G be a graph with vertex set V(G) and edge set E(G). *The middle graph of G* denoted by M(G) is defined as follows. The vertex set of M(G) is V(G)∪E(G). Two vertices x,y of M(G) are adjacent in M(G) in case one of the following holds: (i) x,y are in E(G) and x,y are adjacent in G, and (ii) x is in V(G), y is in E(G), and x,y are incident in G.

Definition 3.8. Let G be a graph with vertex set V(G) and edge set E(G). *The total graph of G* denoted by T(G) is defined in the following way. The vertex set of T(G) is V(G)∪E(G) . Two vertices x,y of T(G) are adjacent in T(G) in case one of the following holds: (i) x,y are in V(G) and x is adjacent to y in G, (ii) x,y are in E(G) and x,y are adjacent in G, and (iii)x is in V(G), y is in E(G), and x,y are incident in G.

Definition 3.9. The graph acquired by joining the centre vertices of two copies of $K_{1,n}$ is called bistar graph $B_{n,n}$.

Definition 3.10. The *square graph* of a simple connected graph G is defined by consisering the similar vertex set as of V(G) and edge set is obtained by joining two vertices if they are at a distance 1 or 2 away from each other in G and is represented by G^2.

Definition 3.11. Let G' *and* G'' be two copies of connected graph G. The *shadow graph* $D_2(G)$ is obtained by joining each vertex v′ in G′ to the neighbours of the corresponding vertex v″ in G″.

Definition 3.12. *Splitting graph of G* is constructed by including a new vertex equivalent to each vertex of G with the property that the adjacent vertices of vertex in G is the same to the adjacent vertices of newly added vertices and is indicated by $S'(G)$.

Definition 3.13. Let G = (V(G),E(G)) be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \cdots S_i \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and T = V \(∪ S_i). The *degree splitting graph* of G denoted by DS(G) is obtained from G by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of S_i for $1 \le i \le t$.

Definition 3.14. The *comb*(CB_k) is a graph obtained by joining a single pendant edge to each vertex of a path P_k.

Definition 3.15. A *caterpillar graph G* is a tree having a central path P_t on t vertices, namely $\{v_1, v_2, v_3, ..., v_i, ..., v_t\}$, where the leaf vertices $\{u_1, u_2, u_3, ..., u_{mi}\}$, $m_i \ge 1$ are attached to every vertex v_i for $1 \le i \le t$ of the central path P_t.

Definition 3.16. Let n be a positive integer with n ≥ 2. *A pencil graph* with 2n+2 vertices, denoted by Pc_n, is a graph with vertex set and edge set as follows. $V(Pc_n) = \{u_i, v_i | i \in [0,n]\}$, and $E(Pc_n) = \{u_i u_{i+1}, v_i v_{i+1} | i \in [1,n-1]\} \cup \{u_i v_i | i \in [0,n]\} \cup \{u_1 u_0, v_1 u_0, u_n v_0, v_n v_0\}$.

Definition 3.17. A *uniform caterpillar* is a caterpillar with each vertex is either of degree 1 or of degree m where $m = \Delta(G)$. We denote a uniform caterpillar with n vertices on the spine by $Cat_{n,m-1}$.

Definition 3.18. A *broom* is a tree obtained from a path by adding pendant edges at exactly one of the end-vertices of the path.

4 L(2,1) COLOURING OF SUNLET AND ITS PARAMETRIZED GRAPHS

In this section the distance two colouring of sunlet families of graphs are explored.

Lemma 4.1. For any graph G, the lower bound for $\lambda(G)$ is $\Delta + 1$ and $\lambda(G) \le \Delta^2 + \Delta - 2$. Also, if G is a graph of order n, then $\lambda(G) \le n + \chi(G) - 2$[1].

Theorem 4.1. The span of line graph of sunlet graph is $\lambda(L(S_k)) = 7$, where $k \ge 3$.

Proof: Let $V(L(S_k)) = E(S_k) = \{u_i : 1 \le i \le k\} \cup \{v_i : 1 \le i \le k\} and E(L(S_k)) = \{u_i v_i, v_i v_{i+1}, u_{i+1} v_i / 1 \le i \le k\}$. In this graph $u_{k+1} = u_1$ and $v_{k+1} = v_1$.
In line graph of sunlet graph a maximum degree vertex is adjacent to two maximum degree vertices. Therefore by distance conditions $\Delta + 3$ is required. Hence $\lambda(L(S_k)) \ge 7$.

The following colouring shows that $\lambda(L(S_k)) \le 7$.

Case(i). When $k \equiv 2 \pmod 3$

The colouring pattern is as follows

$$c(u_i) = \begin{cases} 7 & i \equiv 1 (\text{mod } 3), i \ne k-1 \\ 5 & i \equiv 2 \pmod 3;\ i \ne k \qquad 1 \le i \le k-2 \\ 6 & i \equiv 0 \pmod 3 \end{cases}$$

$$c(u_k) = 1,\ c(u_{k-1}) = 0$$

$$c(v_i) = \begin{cases} 0 & i \equiv 1 (\text{mod } 3), i \ne k-1 \\ 2 & i \equiv 2 \pmod 3;\ i \ne k \qquad 1 \le i \le k-2 \\ 4 & i \equiv 0 \pmod 3 \end{cases}$$

$$c(v_k) = 3,\ c(v_{k-1})$$

Case(ii). When $k \equiv 0 \pmod 3$

The colouring pattern is as follows

$$c(u_i) = \begin{cases} 7 & i \equiv 1 \pmod 3 \\ 5 & i \equiv 2 \pmod 3 \\ 6 & i \equiv 0 \pmod 3 \end{cases}$$

$$c(v_i) = \begin{cases} 0 & i \equiv 1 \pmod 3 \\ 2 & i \equiv 2 \pmod 3 \qquad 1 \le i \le k \\ 4 & i \equiv 0 \pmod 3 \end{cases}$$

Case(iii). When $k \equiv 1 \pmod 3$
The colouring pattern is as follows

$$c(u_i) = \begin{cases} 6 & i \equiv 0 (\text{mod } 3), i \ne k-1 \\ 5 & i \equiv 2 \pmod 3; \qquad 2 \le i \le k-2 \\ 7 & i = \text{k} - 1,\ i \equiv 1 \pmod 3 \quad i \ne k \end{cases}$$

$$c(u_1) = 3,\ c(u_k) = 1$$

$$c(v_i) = \begin{cases} 0 & i \equiv 1 (\text{mod } 3), i \ne k \\ 2 & i \equiv 2 \pmod 3 \qquad 1 \le i \le k-1 \\ 4 & i \equiv 0 \pmod 3 \end{cases}$$

$$c(v_k) = 6$$

Therefore $\lambda(\mathrm{L(S}_k)) = 7$. □

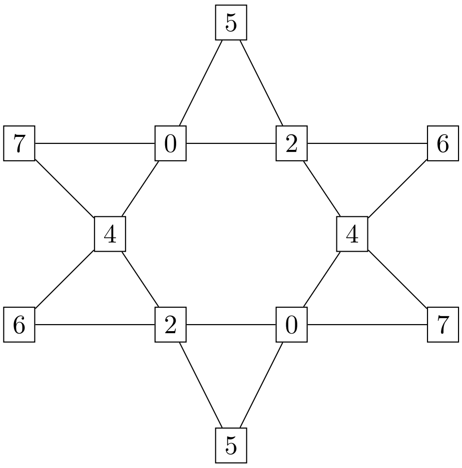

Figure 1: $L(2,1)$ colouring of line graph of sunlet graph on 6 vertices.

Theorem 4.2. If $\lambda(M(S_k))$ is the span of middle graph of sunlet graph, then for $k \geq 10$ the span is

$$\lambda(M(S_k)) = \begin{cases} 10 & k \equiv 2, 4 \text{ (mod 5) and} \\ 9 & \text{otherwise.} \end{cases}$$

Proof: Let $V(G) = V(M(S_k)) = \{v_i : 1 \leq i \leq 2k, u_i, u'_i : 1 \leq i \leq k\}$. In this graph a maximum degree vertex is adjacent to two maximum degree vertices. Therefore by lemma 4.1. $\Delta + 3$ is required. Hence $\lambda(\mathrm{M(S}_k)) \geq 9$.

The following colouring shows that $\lambda(\mathrm{M(S}_k)) \leq 9$.

Case(i).

Subcase(i.1). $k \equiv 4 \pmod 5$

The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively for $i = 1$ to $2k - 3$ and $c(v_{2k-2}) = 2$, $c(v_{2k-1}) = 5$, $c(v_{2k}) = 3$ respectively.

The vertices u'_i are coloured using the set $\{7, 1, 5, 9, 3\}$ consecutively for $i = 1$ to $k - 2$ and $c(u'_{k-1}) = 10$, $c(u'_k) = 9$.

The vertices of u_i are coloured 3 if $c(u'_j)$ is 1 and $c(u_i) = 1$, for all other vertices of u_i.

Subcase(i.2). $k \equiv 2 \pmod 5$

The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively for $i = 1$ to $2k - 4$ and $c(v_{2k-3}) = 1$, $c(v_{2k-2}) = 3$, $c(v_{2k-1}) = 5$, $c(v_{2k}) = 7$ respectively.

The vertices u'_i are coloured using the set $\{1, 5, 9, 3, 7\}$ consecutively for $i = 2$ to $k-5$ and $c(u'_k) = c(u'_{k-2}) = 10$, $c(u'_{k-1}) = (u'_1) = 9$.

The vertices of u_i are coloured 1 if $c(u'_i)$ is 3 and $c(u_{k-3}) = c(u_{k-2}) = 0$ and other vertices are coloured as 1.

Case(ii). Subcase (ii.1). $k \equiv 0 \pmod 5$

The vertices v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively.

The vertices u'_i are coloured using the set $\{7, 1, 5, 9, 3\}$ consecutively.

The vertices of u_i are coloured 3 if $c(u'_i)$ is 1 and the other vertices of u_i are coloured as 1.

Subcase (ii.2). $k \equiv 3 \pmod 5$

The vertices v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively except $i = 2k$ and $c(v_{2k}) = 3$.

The vertices u'_i are coloured using the set $\{7, 1, 5, 9, 3\}$ consecutively and $c(u'_{k-2}) = 7$, $c(u'_{k-1}) = 1$, $c(u'_k) = 5$.

The vertices of u_i are coloured 3 if $c(u'_i)$ is 1 and the other vertices of u_i are coloured as 1.

Subcase(ii.3). $k \equiv 1 \pmod 5$

The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively for $i = 1$ to $2k - 2$ and $c(v_{2k-1}) = 3$, $c(v_{2k}) = 5$.

The vertices u'_i are coloured using the set $\{7, 1, 5, 9, 3\}$ consecutively for $i = 1$ to $k - 2$ and $c(u'_{k-1}) = 1$, $c(u'_k) = 9$.

The vertices of u_i are coloured 3 if $c(u'_i)$ is 1 and $c(u_{k-1}) = 4$, $c(u_i) = 1$ for all other vertices of u_i. □

Theorem 4.3. If $k \geq 7$, then the span of total graph of sunlet graph is

$$\lambda(\mathrm{T(S}_k)) = \begin{cases} 9 & k \equiv 0 \text{ (mod 5)and} \\ 10 & \textit{otherwise.} \end{cases}$$

Proof: Let $V(G) = V(T(S_k)) = \{v_i : 1 \leq i \leq 2k, u_i, u'_i : 1 \leq i \leq k.\}$

In this graph a maximum degree vertex is adjacent to two maximum degree vertices. Therefore by lemma 4.1. $\Delta + 3$ is required. Hence $\lambda(\mathrm{T(S}_k)) \geq 9$. The following colouring shows that $\lambda(\mathrm{T(S}_k)) \leq 9$.

Case(i). $k \equiv 0$ (mod 5)

The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively.

The vertices u'_i are coloured using the set $\{1, 5, 9, 3, 7\}$ consecutively.

The vertices of u_i are coloured using the set $\{5, 3, 3, 1, 1\}$ consecutively.

Case(ii).

Subcase(ii.1). $k \equiv 1$ (mod 5)

The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively for $1 \leq i \leq 2k - 2$ and $c(v_{2k-1}) = 3$, $c(v_{2k}) = 5$.

The vertices u'_i are coloured using the set $\{7, 1, 5, 9, 3\}$ consecutively for $1 \leq i \leq k - 2$ and $c(u'_{k-1}) = 1$, $c(u'_k) = 9$.

The vertices of u_i are coloured using the set $\{3, 3, 1, 1, 5\}$ consecutively for $2 \leq i \leq k - 2$ and $c(u_{k-1}) = 10$, $c(u_k) = 1$, $c(u_1) = 9$.

Subcase(ii.2). $k \equiv 3 \pmod 5$
The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively for $2 \le i \le 2k$ and $c(v_1) = 10$.
The vertices u'_i are coloured using the set $\{5, 9, 3, 7, 1\}$ consecutively for $1 \le i \le k-1$ and $c(u'_k) = 1$.
The vertices of u_i are coloured using the set $\{3, 1, 1, 5, 3\}$ consecutively for $2 \le i \le k$ and $c(u_1) = 3$.

Subcase(ii.3). $k \equiv 4 \pmod 5$
The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively for $1 \le i \le 2k-3$ and $c(v_{2k-2}) = 1$, $c(v_{2k-1}) = 5$, $c(v_{2k}) = 7$.
The vertices u'_i are coloured using the set $\{1, 5, 9, 3, 7\}$ consecutively for $2 \le i \le k-3$ and $c(u'_k) = 9$, $c(u'_1) = c(u'_{k-2}) = 10$, $c(u'_{k-1}) = 3$.
The vertices of u_i are coloured using the set $\{5, 3, 3, 1, 1\}$ consecutively for $1 \le i \le k-3$ and $c(u_{k-2}) = 0$, $c(u_{k-1}) = 9$, $c(u_k) = 3$.

Subcase(ii.4). $k \equiv 2 \pmod 5$
The vertices of v_i are coloured using the set $\{0, 2, 4, 6, 8\}$ consecutively for $1 \le i \le 2k-4$ and $c(v_{2k-3}) = 1$, $c(v_{2k-2}) = 3$, $c(v_{2k-1}) = 5$, $c(v_{2k}) = 7$.
The vertices u'_i are coloured using the set $\{1, 5, 9, 3, 7\}$ consecutively for $2 \le i \le k-3$ and $c(u'_k) = c(u'_{k-2}) = 10$, $c(u'_1) = c(u'_{k-1}) = 9$.
The vertices of u_i are coloured using the set $\{5, 3, 3, 1, 1\}$ consecutively for $1 \le i \le k-3$ and $c(u_{k-2}) = c(u_{k-1}) = 0$, $c(u_k) = 1$. □

Theorem 4.4. The span of subdivision graph of sunlet graph $\lambda(S(S_k)) = 4$ for $k \ge 3$.

Proof: The subdivision graph of S_k is defined as a graph on $4k$ vertices. Let $V(S(S_k)) = \{u_i, v_i, a_i, b_i / 1 \le i \le k\}$ and $E(S(S_k))=\{a_iv_i,\ b_iv_i,\ b_iu_i,\ a_iv_{i+1}\ /\ 1 \le i \le k\}$. Here $v_{k+1} = v_1$. By lemma 4.1., $\lambda(S(S_k)) \ge 4$.
The vertices of $S(S_k)$ is coloured as follows.

Case (i) When $k \equiv 0\,(mod\,3)$

$$c(v_i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 2 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3 \\ 0 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c(a_i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 3 \\ 0 & \text{if } i \equiv 2 \pmod 3 \\ 4 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c(b_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

Case (ii) When $k \equiv 1\,(mod\,3)$

$$c(v_i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod 3, i \neq k \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 2 & \text{if } i \equiv 0 \pmod 3 \\ 1 & \text{if } i = k \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3, i \neq k \\ 3 & \text{if } i \equiv 2 \pmod 3 \\ 0 & \text{if } i \equiv 0 \pmod 3, i \neq k \end{cases}$$

$$c(a_i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 3, i \neq k \\ 0 & \text{if } i \equiv 2 \pmod 3 \\ 4 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i = k \end{cases}$$

$$c(b_i) = \begin{cases} 1 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3, i \neq 1, i = k \\ 3 & \text{if } i \equiv 1 \pmod 3, i \neq k \\ 4 & \text{if } i = 1 \end{cases}$$

Case (iii) When $k \equiv 2\,(mod\,3)$

$$c(v_i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod 3, i = k \\ 4 & \text{if } i \equiv 2 \pmod 3, i \neq k \\ 2 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3, i \neq k-1 \\ 3 & \text{if } i \equiv 2 \pmod 3, i \neq k \\ 0 & \text{if } i \equiv 0 \pmod 3, i = k \\ 4 & \text{if } i = k-1 \end{cases}$$

$$c(a_i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 3, i \neq k-1 \\ 0 & \text{if } i \equiv 2 \pmod 3, i \neq k \\ 4 & \text{if } i \equiv 0 \pmod 3, i = k \\ 3 & \text{if } i = k-1 \end{cases}$$

$$c(b_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3, i \neq k-1 \\ 1 & \text{if } i \equiv 2 \pmod 3, i \neq k \\ 5 & \text{if } i \equiv 0 \pmod 3, i = k \\ 2 & \text{if } i = k-1 \end{cases}$$

Hence the above colouring pattern shows that $\lambda(S(S_K)) \le 4$. Hence $\lambda(S(S_K)) = 4$. This concludes the proof. □

5 L(2,1) COLOURING OF BISTAR FAMILIES OF GRAPH

In this section the distance two colouring of bistar families of graphs are attained.

Theorem 5.1. If $k \geq 1$, then the span of bistar graph $\lambda(B_{k,k}) = \Delta + 1$.

Proof: Let $V(B_{k,k}) = \{u, v, u_i, v_i : 1 \leq i \leq k\}$ and $E(B_{k,k}) = \{uv, uu_i, vv_i : 1 \leq i \leq k\}$. By Lemma 4.1. $\lambda(B_{k,k}) \geq \Delta + 1$. The colouring of the graph is explained below.
Let u and v be the vertices with maximum degree. Since the maximum degree vertices are adjacent, they are assigned 2 distinct colours, namely $0, k+2$ to u,v respectively. This shows that $\lambda(B_{k,k}) \leq \Delta + 1$. Hence, $\lambda(B_{k,k}) = \Delta + 1$.
The pendant vertices adjacent to u are coloured as $c(u_i) = i + 1$ and the pendant vertices adjacent to v are coloured as $c(v_i) = i$. □

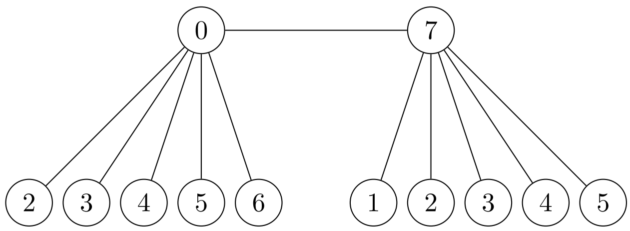

Figure 2: $L(2,1)$ colouring of Bistar graph $B_{5,5}$.

Theorem 5.2. For $k \geq 1$, the span of square graph of bistar graph is $\lambda(B^2_{k,k}) = \Delta + 2$.

Proof: Let $V(B^2_{k,k}) = \{u, v, u_i, v_i : 1 \leq i \leq k\}$ and $E(B^2_{k,k}) = \{uv, uu_i, vv_i, vu_i, uv_i : 1 \leq i \leq k\}$. In this graph all the vertices are either adjacent or at distance two from each other vertices and hence $\Delta + 2$ colours are required. Here is the colouring pattern for span of $B^2_{k,k}$. $c(u) = 2$, $c(v) = 0$, $c(u_i) = 2k - 1 + i$, $c(v_i) = k - 1 + i : 1 \leq i \leq k$. □

Theorem 5.3. The span of shadow graph of bistar graph $\lambda(D_2(B_{k,k})) = \Delta + 3$, where $k \geq 1$.

Proof: Let $V(D_2(B_{k,k}) = \{u, v, u_i, v_i, u', v', u'_i, v'_i\}$ and $E(D_2(B_{k,k})) = \{uv, u'v', uv', vu', uu_i, vv_i, u'u'_i, u_iu', uu'_i, v_iv', vv'_i\}$. In this graph, the vertices u, v, u', v' are of maximum degree and each maximum degree vertex is adjacent to two maximum degree vertices like u is adjacent to v and v', v is adjacent to u and u' and so on.

Hence on colouring, $\Delta + 2$ colours are required to colour $\{u, u_i, u'_i, v, v'\}$ as they are adjacent.
Since u' is either adjacent or at distance two from the coloured vertices, it requires a different colour. Therefore $\Delta + 3$ colours are required.
The colouring of the vertices are done in the following manner:
$c(u) = 2k + 1$, $c(u') = 2k + 2$, $c(v) = 2k + 4$, $c(v') = 2k + 5$, $c(u_i) = i - 1$, $c(u_i{'}) = k + i - 1$, $c(v_i) = i - 1$, $c(v_i{'}) = k + i - 1$. □

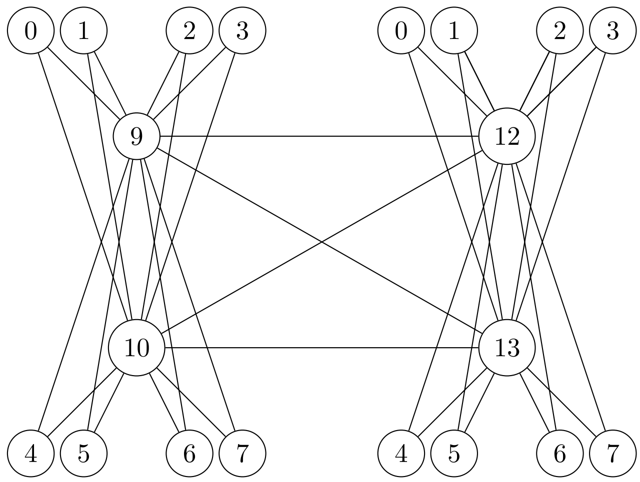

Figure 3: $L(2,1)$ colouring of Shadow graph of Bistar graph $D_2(B_{4,4})$.

Theorem 5.4. For $k \geq 2$ the span of splitting graph of bistar graph $\lambda(S'(B_{k,k})) = \Delta + 2$.

Proof: Let $V(S'(B_{k,k})) = \{u, u_i, v, v_i, u', u'_i, v', v'_i\}$ and $E(S'(B_{k,k})) = \{uv, uu'_i, vv_i, u'u_i, v_iv', vv'_i, uv'_i\}$.
By lemma 4.1. $\lambda(S'(B_{k,k})) \geq \Delta + 1$.

In this graph, there are 2 vertices of maximum degree, $2k$ pendant vertices, $2k$ degree two vertices and 2 degree k vertices. Since, the two maximum degree vertices are adjacent $\Delta + 2$ colours are required for colouring and so $\lambda(S'(B_{k,k})) \neq \Delta + 1$. Therefore, $\lambda(S'(B_{k,k})) \geq \Delta + 2$. The two degree k vertices are assigned the colour zero, the pendant vertices are assigned colours from 0 to $k - 1$ and maximum degree vertices are coloured as $k + 1$ and $k + 3$ respectively. The degree two vertices, adjacent to vertex coloured $k + 1$ are assigned colours distinctly from $k + 4$ to $2k + 3$ and degree two vertices adjacent to vertex coloured $k + 3$ are assigned colours distinctly from $k + 5$ to $2k + 3$ and k respectively.
$c(u) = k + 1$, $c(v) = k + 3$, $c(u') = c(v') = 0$
$c(u'_i) = c(v'_i) = i - 1$, $c(u_i) = k + 3 + i$: $1 \leq i \leq k$

$$c(v_i) = \begin{cases} k + 4 + \text{i} & 1 \leq i \leq k - 1, \text{ and} \\ k & i = k. \end{cases}$$

This colouring pattern shows that $\lambda(S'(B_{k,k})) \leq \Delta + 2$. This concludes the proof. □

Theorem 5.5. For $k \geq 3$, the span of degree splitting graph of bistar graph $\lambda(DS(B_{k,k})) = \Delta + 3$.

Proof: Let $V(DS(B_{k,k})) = (w_i, u, v, u_i, v_{i,})$ and $E(DS(B_{k,k})) = \{uv, uw_2, vw_2, w_1u_i, w_1v_i, uu_i, vv_i\}$. In this graph, w_1 is the maximum degree vertex and is adjacent to u_i, v_i. Therefore to colour w_1, u_i, v_i $\Delta+1$ colours are required, since we use non-negative integers. Also, the vertices u and v are adjacent to each other and are either adjacent or at distance two from the coloured vertices. Therefore each must be assigned distinct colours other than the above used $\Delta + 1$ colours. Therefore $\lambda(DS(B_{k,k})) \geq \Delta + 3$.

The colouring is given by: $c(w_1) = 1$, $c(w_2) = 2k + 3$ $c(v) = 2$, $c(u_i) = 2 + i$, $c(u) = 0$, $c(v_i) = k + 2 + i$: $1 \leq i \leq k$. The above colouring pattern shows that $\lambda(DS(B_{k,k})) \leq \Delta + 3$. Hence $\lambda(DS(B_{k,k})) = \Delta + 3$.

□

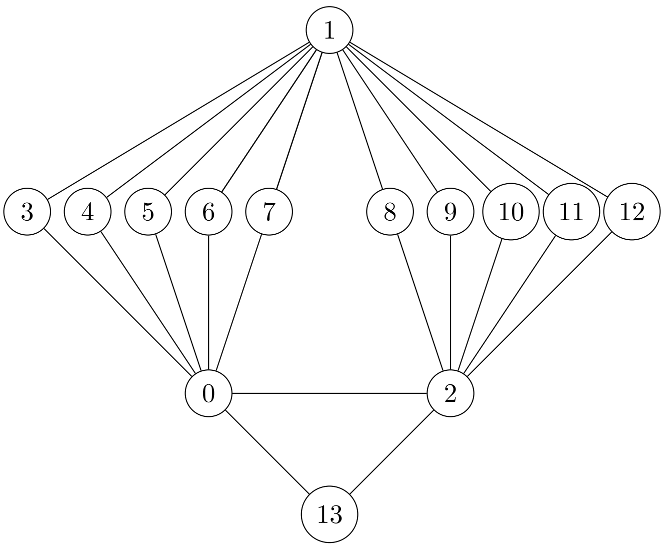

Figure 4: $L(2,1)$ colouring of Degree Splitting graph of Bistar graph $DS(B_{5,5})$.

Theorem 5.6. For $k \geq 4$, the span of comb graph $\lambda(CB_k) = 5$.

Proof: The comb graph is acquired by attaching a pendant edge to each vertex of the path. Let $V(CB_k)$=(u_1,u_2,...,u_k,v_1,v_2,...,v_k) where e_i=u_iv_i for $1 \leq i \leq k$ and e_{k+i}=u_iu_{i+1} for $1 \leq i \leq k-1$.
Here, the vertices of path $\{u_1, u_2, ..., u_k\}$ are coloured as $(0, 2, 4, 0, 2, 4, ..., 0, 2, 4)$. Since by the definition of $L(2,1)$ colouring, adjacent vertices should have colour difference at least 2, the vertex adjacent to vertex coloured 2 cannot be coloured as $0, 1, 2, 3, 4$. Therefore colour 5 must be assigned to v_2. Hence $\lambda(CB_k) \geq 5$.

$$c(v_i) = \begin{cases} 3 & \text{if } i = 1,\, i \equiv 1 \pmod 3 \\ 5 & \text{if } i = 2,\, i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c(u_i) = \begin{cases} 0 & \text{if } i = 1,\, i \equiv 1 \pmod 3 \\ 2 & \text{if } i = 2,\, i \equiv 2 \pmod 3 \\ 4 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

The above colouring shows that $\lambda(CB_k) \leq 5$.
Hence $\lambda(CB_k) = 5$. □

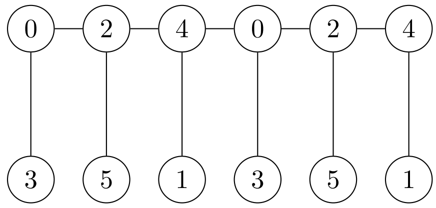

Figure 5: $L(2,1)$ colouring of Comb graph CB_6.

Theorem 5.7. The span of complete graph is $\lambda(K_n) = 2\Delta$, where $n \geq 2$.

Proof: In Complete graph, every 2 vertices are adjacent. As per the concept of $L(2,1)$, adjacent vertices should have colour difference at least 2. Therefore adjacent colours cannot be assigned and different even numbers should be used as colours. This implies that 2Δ colours are required.

Let a vertex of $V(K_n)$ be assigned the colour 0. Let it be v_1 and the consecutive vertices are labelled as v_2,v_3,v_4,...,v_n. Since each vertex in K_n is adjacent to every other vertices v_2 cannot be coloured with $c(v_1)$ and $c(v_1)+1$ as per definition. So, v_2 is assigned the colour 2 and every other vertices are assigned with distinct even number as colours as they are adjacent to each other and totally $2n+2 = 2\Delta$ colours are essential to colour the vertices of K_n and are coloured in the subsequent manner. $c(v_i)$=$c(v_{i-1})$+2 for $2 \leq i \leq n-1$. □

Theorem 5.8. The span of ST_n is 7, where $n \geq 3$. □

6 RADIO COLOURING OF SOME GRAPHS

In this section the radio colouring of some graphs is discussed.

Observation 6.1. If G is a graph G(not totally disconnected) with maximum degree $\Delta(G)$, then $rn(G) \geq 1+\Delta(G)$. In particular if G is an r-regular graph for some integer $r \geq 2$, then $rn(G) \geq 1 + r$.

Proof: Let G be a graph that is not totally disconnected. Let v be a vertex with maximum degere and $v_1, v_2, \cdots, v_\Delta$ be the vertices adjacent to v. As the vertices adjacent to v are at distance two from each other they should be assigned distinct colours as per the definition of radio colouring and therefore Δ colours are used, and the vertex v is assigned a colour that is not already used as it a proper colouring. Therefore totally $\Delta + 1$ colours are required for colouring. This shows that G requires at least $\Delta + 1$ colours for radio colouring. □

Theorem 6.1. The radio chromatic number of Mycielski of bistar graph is $\Delta + 3 = 2n + 5$ where $n \geq 3$.

Proof: Mycielski of bistar graph contains three vertices(u, v, w) of maximum degree, $4n$ vertices $(u_i, v_i, u'_i, v'_i : 1 \leq i \leq n)$ vertices of two degree and two vertices(u', v') of degree $n + 2$. The three maximum degree vertices are adjacent to the vertices of degree $n+2$ and are at distance two from each other.

Since radio colouring is a proper colouring and by distance two condition atleast Δ colours are required. Therefore $rn(\mu(B_n, n)) \geq \Delta \geq 2n + 2$ and since the maximum degree vertices are at distance two from each other, three more colours are required for radio colouring. Therefore totally $2n + 5$ colours are required. The following shows the radio colouring of Mycielski of bistar graph.

$c(w) = 1, c(u) = 2n + 6, c(v) = 2n + 8,$

$c(u') = n + 3, c(v') = n + 4$

$c(u'_i) = c(v_i) = i + n, c(v'_i) = c(u_i) = i + n + 4$

□

Theorem 6.2. The radio chromatic number of regular caterpillar graph is $\Delta + 1$, where $n \geq 3$.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of the path and $v_{i1}, v_{i2}, ..., v_{in}$ be the vertices incident with $v_1, v_2, ..., v_n$. In caterpillar graph all the vertices of the path are of same degree$(n - 1)$ and it is maximum. According to observation 6.1 $rn(Cat_{n,m-1}) \geq \Delta + 1$. The following colouring format shows that $rn(Cat_{n,m-1}) \leq \Delta + 1$. Hence $rn(Cat_{n,m-1}) = \Delta + 1$.

$c(v_{i1}) = c(v_{i2}) = ... = c(v_{in}) = i$

$$c(v_i) = \begin{cases} m+1 & \text{if } i = 1,\, i \equiv 1 \pmod 3 \\ m+3 & \text{if } i = 2,\, i \equiv 2 \pmod 3 \\ m+5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

□

Theorem 6.3. The radio chromatic number of broom graph $T_{n,n}$ is $\Delta + 1$, where $n \geq 3$.

Proof: Assume that T is not a path and so $\Delta(T) = \Delta \geq 3$. Suppose that $T_{n,n}$ is obtained from the path $P = (v_1, v_2, ..., v_n)$ by adding n pendant edges $u_i v_n (1 \leq i \leq)$ at the end-vertex v_n. By Observation 6.1, $rn(T_{n,n}) \geq \Delta + 1$. The pendant vertices u_i are coloured using 1 to n colours as they are adjacent to the each other. The vertices v_n and v_{n-1} are coloured by $n + 2$ and $n + 4$ respectively.Then the vertices of the path are coloured by $1, 3, 5$ consecutively. This colouring format shows that $rn(T_{n,n}) \leq \Delta + 1$. Hence $T_{n,n}$ is $\Delta + 1$. □

Theorem 6.4. The radio chromatic number of n-level sibling tree ST_r is $\Delta + 1$, where $r \geq 3$.

Proof: For any integer r, the complete binary tree T_r of height r is the basic structure of a sibling tree that is obtained by adding edges between the left and right children of the same parent. The sibling vertices are labelled as follows: The root vertex has label-1 and the children of vertices x are labelled as $2x$ and $2x+1$. It implies that the root vertex is at level 0. The r-level sibling tree is denoted as ST_r, which has $(2^{r+1}–1)$ vertices and $3(2^r - 1)$ edges.

Sibling tree is a type of tree and by observation 6.1, and colouring pattern the radio chromatic number is $\Delta + 1$. The following example shows the colouring pattern of sibling tree. □

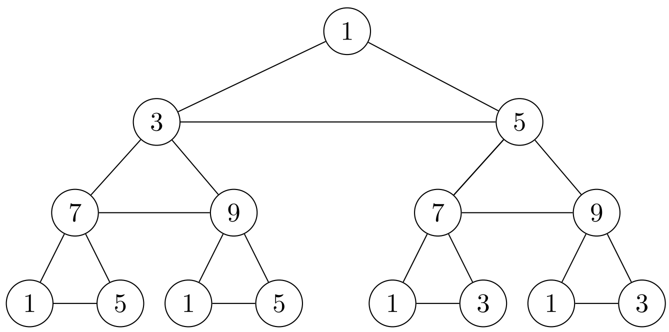

Figure 6: Radio colouring of ST_3

Theorem 6.5. The radio chromatic number Mycielski of path is is $\Delta + 1$, where $n \geq 3$. □

Theorem 6.6. The radio chromatic number Mycielski of cycle is is $\Delta + 1$, where $n \geq 3$. □

7 COMPARISION OF CHROMATIC PARAMETERS

Based on the results obtained from my research work the inequalities are derived in this section. Radio colouring and distance two(L(2,1)) colouring are applied in frequency assignment problem, and this comparison shows that radio number is more applicable compared with the span of the graph. The number of vertices of all graphs under consideration are greater than or equal to three.

Observation 7.1. If S_k is a sunlet graph of order $k \geq 3$, then $rn(S_k) < \lambda(S_k)$.

Observation 7.2. If $L(S_k)$ is a line graph of sunlet graph of order $k \geq 3$, then $rn(L(S_k)) < \lambda(L(S_k))$.

Observation 7.3. If $M(S_k)$ is a middle graph of sunlet graph of order $k \geq 5$, then $rn(M(S_k)) < \lambda(M(S_k))$.

Observation 7.4. If $T(S_k)$ is a total graph of sunlet graph of order $k \geq 7$, then $rn(T(S_k)) \leq \lambda(T(S_k))$.

Observation 7.5. If $S(S_k)$ is a subdivision graph of sunlet graph of order $k \geq 3$, then $rn(S(S_k) = \lambda(S(S_k))$.

Observation 7.6. If $B_{k,k}$ is a bistar graph of order k, then $rn(B_{k,k}) = \lambda(B_{k,k})$.

Observation 7.7. If $B^2_{k,k}$ is a square graph of bistar graph of order $k \geq 3$, then $rn(B^2_{k,k}) < \lambda(B^2_{k,k})$.

Observation 7.8. If $D_2(\mathrm{B}_{k,k})$ is a shadow graph of bistar graph of order $k \geq 3$, then $rn(D_2(B_{k,k})) < \lambda(D_2(B_{k,k}))$.

Observation 7.9. If $S'(B_{k,k})$ is a splitting graph of bistar graph of order $k \geq 4$, then $rn(S'(B_{k,k})) < \lambda(S'(B_{k,k}))$.

Observation 7.10. If $DS(B_{k,k})$ is a degree splitting graph of bistar graph of order $k \geq 3$, then $rn(DS(B_{k,k})) = \lambda(DS(B_{k,k}))$.

Observation 7.11. If Pc_k is a pencil graph of order $k \geq 7$, then $rn(Pc_k) < \lambda(Pc_k)$.

Observation 7.12. If $L(Pc_k)$ is a line graph of pencil graph of order $k \geq 6$, then $rn(L(Pc_k)) \leq \lambda(L(Pc_k))$.

Chromatic parameters interpreted above are according to their order, size and maximum degree of graph. It is observed that for the graphs discussed above the radio number is less than or equal to span which provides a better bound for frequency assignment problem.

8 CONCLUSION

Several issues related to the design of radio networks, as well as other problems in telecommunications, may be formulated as graph colouring problems. If this problem is modeled by graphs, transmitters correspond to vertices, distances to edges and assigning frequency slots corresponds to colouring the vertices so that adjacent vertices and vertices at distance two have different colours. In a similar way we may deal with the task of assigning channels to the radio base stations. Particular instances of these problems are very often NP-complete optimization problems and finding an optimal solution is a computationally hard task. In this article we analyze the span number of sunlet, bistar and pencil graph families and compare the span and radio number of some graphs.

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