An Effective Algorithm for Gracefulness on the Cartesian Product of Cayley Digraphs

R Thamizharasi and Ismail Naci Cangul

***Abstract*— Labeled digraphs are frequently used to denote real-world issues because of their unique features. They were employed in modelling communication, interconnection networks, parallel computer architecture, field of bioinformatics and issues with DNA sequencing. Graceful labelings are proven to exist for a number of infinite classes of graphs. One can treat the present work as assistance to pick a suitable combinatorial model of large directed secured networks.**

Index Terms— **Cartesian product, Cayley digraph, Digraph labelling, Edge odd graceful, Super edge graceful**

I. INTRODUCTION

As an operation of graph theory, the Cartesian product has been broadly used to design large scale networks from small ones. This product of digraphs has headed to the discovery of many properties of digraphs. There are several kinds of larger digraphs like Cartesian product of Cayley digraphs with other digraphs, and these classes have real-world applications in interconnection networks. These graphs not only serve as attractive theoretical representations of practical issues, but they also offer precise algebraic answers to issues relating to group theory.

Rosa [1] formulated β-valuation for a graph. Later, Golomb [2] described such a formula as graceful, which is today a commonly used phrase. Several publications have created modified vertex and edge labels in graceful graph analogues. Edge-graceful graphs are a notion that was introduced by Lo [3]. Thereafter volume of papers published on edge graceful labeling with some modifications.

Mitchem and Simoson [4] defined super edge-graceful labelings and demonstrated that trees which are super edge graceful also admit edge graceful under certain conditions. Lee et al. [5] considered the above labeling for Eulerian graphs. Further Lee and Yong-Song Ho[6] disclosed that odd order trees with three allow super edge-graceful labelling technique. Edge odd graceful is the one labeling technique among the significant labelings which deals its edge domain with odd numbers. This technique was first suggested by Solairaju and Chitra [7] who explored it for ladders, paths and odd cycles.

Thamizharasi and Rajeswari [8], [9], [10], [11] implemented several labelling techniques on Cayley digraphs and its line digraphs. Li Wang, Jingwen Li, Lijing Zhang [12] detailly mentioned about total labelings. Salat Arti and Sharma Amit, [13] visited H graph with Palindromic labelling. Thamizharasi and Rajeswari [14] analysed cordial labelings on Cay (G,S) $\times \overrightarrow{K_2}$. Sathish K and Pratap H[15] investigated the graphs for interconnection networks. Here we go with the super edge graceful and edge odd graceful labelling on the complex graph called Cartesian product of Cayley digraph Cay (G,S) and the cycle $\overrightarrow{K_2}$ i.e., Cay (G,S) $\times \overrightarrow{K_2}$.

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II. PRELIMINARIES

A. Definition

A graph with direction G (V,E) of m vertices and n arcs permits super edge graceful labelling, when a bijection g occurred on E *to* $\left\{\pm 1, \pm 2, \dots, \pm \frac{n-1}{2}\right\}$, if n is an odd number, and on E *to* $\left\{\pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\}$ if n is an even number in a way that the resultant node labeling g* got by $g^*(v_i) = \sum g(e_{ij})$ the summation is over the outward arcs of v_i, $1 \le i \le m$ results a bijection on V to $\left\{0, \pm 1, \pm 2, \dots, \pm \frac{m-1}{2}\right\}$ if p is an odd number and on V to $\left\{\pm 1, \pm 2, \dots, \pm \frac{m}{2}\right\}$ if m is an even number.

B. Definition

The digraph G (m, n) is said to be edge -odd- graceful when a bijection f: E(G) → {1 ,3, 5, …, 2n-1} occurred in a way that every node is assumed the sum of its outward arc labels under mod 2q, then the resultant node labels are different.

III. GRACEFULNESS OF THE CARTESIAN PRODUCT OF CAYLEY DIGRAPHS

A. Super Edge Graceful labeling

1) Theorem

The Cartesian product Cay (G, S) $\times \overrightarrow{K_2}$ permits super edge graceful labeling.

Proof:

Assume a Cayley digraph with the generating set S Cay (G, S) contains p nodes which also has m generators. Every vertex has m indegree and m outdegree. Totally the Cayley digraph has $mp_l = q$ arcs. Now multiply Cay (G, S) with $\overrightarrow{K_2}$ by the Cartesian product. From the definition of the Cartesian product of digraphs, we have resultant digraph Cay (G, S) × $\overrightarrow{K_2}$ has $2p_l$ vertices. First p_l vertices have m+1 outgoing and m incoming arcs. Another p_l vertices have m outgoing and m+1 entering arcs. Hence Cay (G, S) × $\overrightarrow{K_2}$ has $2p_l$ vertices and mp_l + (m+1) p_l = 2q + p_l arcs. Let r = 2q + p_l and n = $2p_l$.

To establish that the digraph got from the Cartesian product, Cay (G,S) × $\overrightarrow{K_2}$ of $2p_l$ nodes and r outgoing arcs is super edge graceful, we must prove there occurs an bijection f defined on the set of arcs to $\left\{0, \pm 1, \pm 2, \dots, \pm \frac{r-1}{2}\right\}$ if r is an odd number and set of edges to $\left\{\pm 1, \pm 2, \dots, \pm \frac{r}{2}\right\}$ if r is an even number in a way that the resultant mapping f* given by $f^*(v_i) = \sum f(e_{ij}) \pmod{2r}$ taken over all outgoing arcs of v_i, $1 \le i \le 2p_l$ where e_{ij} is j^{th} outward arc of i^{th} vertex is a bijection defined from the set of nodes to $\{\pm 1, \pm 2, \dots, \pm p_l\}$. Though Cay *(G,S)* × $\overrightarrow{K_2}$ has $2p_l$ nodes which is always even, the number of nodes of Cay (G,S) also plays an important role to define function on the edge set. We prove this statement in four different cases with respect to the number of nodes of Cayley digraph and the number of generators of the digraph.

(i) m is an odd number and p_l is an even number

We know that the first p_l vertices have m+1 outdegree and other p_l vertices have m outdegree. Since the p_l is even, r also even. Now define the bijective function f defined on the arcs set of Cay (G,S) × $\overrightarrow{K_2}$ to the set $\left\{\pm 1, \pm 2, \dots, \pm \frac{r}{2}\right\}$ as follows.

For, $1 \le i \le p_l$ *and* $1 \le j \le m+1$

$$f(e_{ij}) = \begin{cases} \frac{r}{2} - i - \frac{p_l(j-1)}{2} + 1 & for\ j = 1,3,5,\dots,m \\ -\left[\frac{r}{2} - i - \frac{p_l(j-2)}{2} + 1\right] & for\ j = 24,6,\dots,m-1 \end{cases}$$

For *j = m + 1*,

$$f(e_{ij}) = \begin{cases} -\left[\frac{r}{2} - \frac{p_l(j-2)}{2} + 1 - 2i\right] & \text{for } 1 \le i \le \frac{p_l}{2} \\ -\left[\frac{r}{2} - \frac{p_l(j-2)}{2} + 2 + p_l - 2i\right] & \text{for } \frac{p_l}{2} + 1 \le i \le p_l \end{cases}$$

For $p_l + 1 \le i \le 2p_l$ and $1 \le j \le m$

$$f(e_{ij}) = \begin{cases} \frac{q}{2} - \frac{p_l(j-1)}{2} + 1 + p_l - i & \text{for } j = 1, 3, 5, \dots, m-2 \\ -\left[\frac{q}{2} - \frac{p_l(j-2)}{2} + 1 + p_l - i\right] & \text{for } j = 2, 4, 6, \dots m-3 \end{cases}$$

For j = m -1,

$$f(e_{ij}) = -\left[\frac{q}{2} - \frac{p_l(j-2)}{2} + i - 2p\right] \text{ for } p_l + 1 \le i \le 2p_l$$

For j = m,

$$f(e_{ij}) = \begin{cases} i - p_l & \text{for } p_l + 1 \le i \le \frac{3p_l}{2} \\ -[2p_l + 1 - i] & \text{for } \frac{3p_l}{2} + 1 \le i \le 2p_l \end{cases}$$

The resultant function for $1 \le I \le p_l$ is

$$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$$

For $1 \le i \le \frac{p_l}{2}$,

$$f^*(v_i) = \frac{r}{2} - i - p_l(0) + 1 - \left[\frac{r}{2} - i - p_l(0) + 1\right] + \cdots$$

$$+ \left[\frac{r}{2} - i - \frac{p_l(m-1)}{2} + 1\right] - \left[\frac{r}{2} - 2i - \frac{p_l(m-1)}{2} + 1\right]$$

$$= -i + 1 - 1 + 2i$$

$$= i$$

For $\frac{p_l}{2} + 1 \le i \le p_l$

$$f^*(v_i) = \frac{r}{2} - i - p_l(0) + 1 - \left[\frac{r}{2} - i - p_l(0) + 1\right] + \dots +$$

$$\left[\frac{r}{2} - i - \frac{p_l(m-1)}{2} + 1\right] - \left[\frac{r}{2} - \frac{p_l(m-1)}{2} - 2i + p_l + 2\right]$$

$$= -i + 2i - p_l - 2$$

$$= i - p_l - 1$$

The induced function for $p_l+1 \le i \le 2p_l$ is

$$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$$

For, $p_l + 1 \le i \le \frac{3p_l}{2}$

$f^*(v_i) = \frac{q}{2} - i - p_l(0) + p_l + 1 - \left[\frac{q}{2} - i - p_l(0) + p_l + 1\right] + \cdots + \left[\frac{q}{2} - i - \frac{p_l(m-3)}{2} + p_l + 1\right] - \left[\frac{q}{2} - \frac{p_l(m-3)}{2} + i - 2p_l\right] + i - p_l$

$= p_l + 1 - i - i + 2p_l + i - p_l$

$= 2p_l + 1 - i$

For $\frac{3p_l}{2} + 1 \leq i \leq 2p_l$

$f^*(v_i) = \frac{q}{2} - i - p_l(0) + p_l + 1 - \left[\frac{q}{2} - i - p_l(0) + p_l + 1\right] + \cdots + \left[\frac{q}{2} - i - \frac{p_l(m-3)}{2} + p_l + 1\right] - \left[\frac{q}{2} - \frac{p_l(m-3)}{2} + i - 2p_l\right] - [2p_l + 1 - i]$

$= p_l + 1 - i - i + 2p_l - 2p_l - 1 + i$

$= p_l - i$

$$f^*(v_i) = \begin{cases} i & \text{for } 1 \leq i \leq \frac{p_l}{2} \\ i - p_l - 1 & \text{for } \frac{p_l}{2} + 1 \leq i \leq p_l \\ 2p_l + 1 - i & \text{for } p_l + 1 \leq i \leq \frac{3p_l}{2} \\ p_l - i & \text{for } \frac{3p_l}{2} + 1 \leq i \leq 2p_l \end{cases}$$

From the above equation we observed that the values of $f^*(v_i)$ are distinct for every i and they belongs to the set $\{\pm 1, \pm 2, \ldots, \pm p_l\}$. Also noted that no two sums are equal for different values of i. Therefore f* is clearly a bijective from the vertex set of Cay (G,S) × $\overrightarrow{K_2}$ to $\{\pm 1, \pm 2, \ldots, \pm p_l\}$. Hence Cay (G,S) × $\overrightarrow{K_2}$ admits super edge graceful labeling when m is an odd number and p is an even number.

(ii) m and p_l are even

Since p_l is even, r also even. Now define the bijective function f defined from the arc set of Cay (G,S) × $\overrightarrow{K_2}$ to the set $\{\pm 1, \pm 2, \ldots, \pm \frac{r}{2}\}$ as mentioned below.

For, $1 \leq i \leq p_l$ and $1 \leq j \leq m + 1$

$$f(e_{ij}) = \begin{cases} \frac{r}{2} - i - \frac{p_l(j-1)}{2} + 1 & \text{for } j = 1,3,5,\ldots,m-1 \\ -\left[\frac{r}{2} - i - \frac{p_l(j-2)}{2} + 1\right] & \text{for } j = 2,4,6,\ldots,m-2 \end{cases}$$

For, j = m,

$$f(e_{ij}) = -\left[\frac{r}{2} - \frac{p_l(j-2)}{2} + i - p_l\right]$$

For j = m + 1,

$$f(e_{ij}) = \begin{cases} i & \text{for } 1 \leq i \leq \frac{p_l}{2} \\ -[p_l - i + 1] & \text{for } \frac{p_l}{2} + 1 \leq i \leq p_l \end{cases}$$

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$f(e_{ij}) = \begin{cases} \frac{r-q}{2} - \frac{p_l(j-1)}{2} + 1 + p_l - i & \text{for } j = 1,3,\ldots,m-1 \\ -\left[\frac{r-q}{2} - \frac{p_l(j-2)}{2} + 1 + p_l - i\right] & \text{for } j = 24,\ldots m-2 \end{cases}$$

For j = m,

$$f(e_{ij}) = \begin{cases} -\left[\frac{r-q}{2} - \frac{p_l(j-2)}{2} + 2p_l - 2i + 1\right] & \text{for } p + 1 \leq i \leq p_l + \frac{p_l}{2} \\ -\left[\frac{r-q}{2} - \frac{p_l(j-2)}{2} + 3p_l + 2 - 2i\right] & \text{for } \frac{3p_l}{2} + 1 \leq i \leq 2p_l \end{cases}$$

Then the induced function for $1 \leq i \leq p_l$ is

$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$

For $1 \leq i \leq \frac{p_l}{2}$,

$$f^*(v_i) = \frac{r}{2} - i - p_l(0) + 1 - \left[\frac{r}{2} - i - p_l(0) + 1\right] + \cdots + \left[\frac{r}{2} - i - \frac{p_l(m-2)}{2} + 1\right] - \left[\frac{r}{2} - \frac{p_l(m-2)}{2} + i - p_l\right] + i$$

$= -i + 1 - i + p_l + i$

$= 1 + p_l - i$

For $\frac{p_l}{2} + 1 \leq i \leq p_l$

$$f^*(v_i) = \frac{r}{2} - i - p_l(0) + 1 - \left[\frac{r}{2} - i - p_l(0) + 1\right] + \cdots +$$

$$\left[\frac{r}{2} - i - \frac{p_l(m-2)}{2} + 1\right] - \left[\frac{r}{2} - \frac{p_l(m-1)}{2} + i - p_l\right] - [p_l - i + 1]$$

$$= -i + 1 - i + p_l - p_l + i - 1$$

$$= -i$$

The induced function for $p_l+1 \le i \le 2p_l$ is

$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$

For $p_l + 1 \le i \le \frac{3p_l}{2}$

$$f^*(v_i) = \frac{r-q}{2} - i - p_l(0) + p_l + 1 - \left[\frac{r-q}{2} - i - p_l(0) + p_l + 1\right] + \cdots + \left[\frac{r-q}{2} - i - \frac{p_l(m-2)}{2} + p_l + 1\right] - \left[\frac{r-q}{2} - \frac{p_l(m-2)}{2} - 2i + 1\right]$$

$$= p_l + 1 - i - 2p_l + 2i - 1$$

$$= i - p_l$$

For $\frac{3p_l}{2} + 1 \le i \le 2p_l$

$$f^*(v_i) = \frac{r-q}{2} - i - p_l(0) + p_l + 1 - \left[\frac{r-q}{2} - i - p_l(0) + p_l + 1\right] + \cdots + \left[\frac{r-q}{2} - i - \frac{p_l(m-2)}{2} + p_l + 1\right] - \left[\frac{r-q}{2} - \frac{p_l(m-2)}{2} + 3p_l + 2 - 2i\right]$$

$$= p_l + 1 - i - 3p_l - 2 + 2i$$

$$= i - 1 - 2p_l$$

$$f^*(v_i) = \begin{cases} p_l + 1 - i & \text{for } 1 \le i \le \frac{p_l}{2} \\ -i & \text{for } \frac{p_l}{2} + 1 \le i \le p_l \\ i - p_l & \text{for } p_l + 1 \le i \le \frac{3p}{2} \\ i - 1 - 2p_l & \text{for } \frac{3p_l}{2} + 1 \le i \le 2p_l \end{cases}$$

From the above equation we observed that the values of $f^*(v_i)$ are distinct for every i and they belongs to the set $\{\pm 1, \pm 2, \ldots, \pm p_l\}$. Also noted that no two sums are equal for different values of i. Therefore f* is clearly a bijective defined on the nodes set of Cay (G,S) $\times \overrightarrow{K_2}$ to $\{\pm 1, \pm 2, \ldots, \pm p_l\}$. Therefore Cay (G,S) $\times \overrightarrow{K_2}$ admits super edge graceful labeling if m and p_l both are even numbers.

(iii) m and p_l are odd

Since the p_l is an odd number, r also odd. Now define the bijective function f defined on the set of arcs of Cay (G,S) $\times \overrightarrow{K_2}$ to $\left\{0, \pm 1, \pm 2, \ldots, \pm \frac{r-1}{2}\right\}$ as mentioned below.

For, $1 \le i \le p_l$ and $1 \le j \le m+1$

$$f(e_{ij}) = \begin{cases} \frac{r-1}{2} - i - \frac{p_l(j-1)}{2} + 1 & \text{for } j = 1,3,5,\ldots,m \\ -\left[\frac{r-1}{2} - i - \frac{p_l(j-2)}{2} + 1\right] & \text{for } j = 2,4,\ldots,m-1 \end{cases}$$

For $j = m+1$,

$$f(e_{ij}) = \begin{cases} -\left[\frac{r-1}{2} - \frac{p_l(j-2)}{2} + 1 - 2i\right] & \text{for } 1 \le i \le \frac{p_l-1}{2} \\ -\left[\frac{r-1}{2} - \frac{p_l(j-2)}{2} + 1 + p_l - 2i\right] & \text{for } \frac{p_l+1}{2} \le i \le p_l - 1 \\ -\left[\frac{r-1}{2} - i + 1 - \frac{jp_l}{2}\right] & \text{for } i = p_l \end{cases}$$

For $p_l + 1 \le i \le 2p_l$ and $1 \le j \le m$

$$f(e_{ij}) = \begin{cases} \frac{q-1}{2} - \frac{p_l(j-1)}{2} + 1 + p_l - i & \text{for } j = 1,3,5,\ldots,m-2 \\ -\left[\frac{q-1}{2} - \frac{p_l(j-2)}{2} + 1 + p_l - i\right] & \text{for } j = 2,4,\ldots m-3 \end{cases}$$

For $j = m-1$ and $p_l + 1 \le i \le 2p_l$

$$f(e_{ij}) = -\left[\frac{q-1}{2} - \frac{p_l(j-2)}{2} + i - 2p_l + 1\right]$$

For $j = m$,

$$f(e_{ij}) = \begin{cases} i - p_l & \text{for } p_l + 1 \le i \le \frac{3p_l-1}{2} \\ -[2p_l - i] & \text{for } \frac{3p_l+1}{2} \le i \le 2p_l \end{cases}$$

Then the resultant function for $1 \le i \le p_l$ is

$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$

For $1 \le i \le \frac{p_l-1}{2}$,

$f^*(v_i) = \frac{r-1}{2} - i - p_l(0) + 1 - \left[\frac{r-1}{2} - i - p_l(0) + 1\right] + \cdots$

$+ \left[\frac{r-1}{2} - i - \frac{p_l(m-1)}{2} + 1\right] - \left[\frac{r-1}{2} - 2i - \frac{p_l(m-1)}{2} + 1\right]$

$= -i + 1 - 1 + 2i$

$= i$

For $\frac{p_l+1}{2} \le i \le p_l - 1$

$f^*(v_i) = \frac{r-1}{2} - i - p_l(0) + 1 - \left[\frac{r-1}{2} - i - p_l(0) + 1\right]$

$+ \cdots +$

$\left[\frac{r-1}{2} - i - \frac{p_l(m-1)}{2} + 1\right] - \left[\frac{r-1}{2} - \frac{p_l(m-1)}{2} - 2i + p_l + 1\right]$

$= -i + 1 + 2i - p_l - 1$

$= i - p_l$

For $i = p_l$,

$f^*(v_i) = \frac{r-1}{2} - i - p_l(0) + 1 - \left[\frac{r-1}{2} - i - p_l(0) + 1\right] + \cdots + \left[\frac{r-1}{2} - i - \frac{p_l(m-1)}{2} + 1\right] - \left[\frac{r-1}{2} - \frac{p_l(m+1)}{2} - i + 1\right]$

$= -\frac{p_l(m-1)}{2} + 1 - i + i - 1 + \frac{p_l(m+1)}{2}$

$= p_l$

The induced function for $p_l+1 \le i \le 2p_l$ is
$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$
For $p_l + 1 \le i \le \frac{3p_l-1}{2}$

$f^*(v_i) = \frac{q-1}{2} - i - p_l(0) + p_l + 1 - \left[\frac{q-1}{2} - i - p_l(0) + p_l + 1\right] + \cdots + \left[\frac{q-1}{2} - I - \frac{p_l(m-3)}{2} + p_l + 1\right]$

$-\left[\frac{q-1}{2} - \frac{p_l(m-3)}{2} + i - 2p_l + 1\right] + i - p_l$

$= p_l + 1 - i - i + 2p_l - 1 + i - p_l$

$= 2p_l - i$

For $\frac{3p_l+1}{2} \le i \le 2p_l$

$f^*(v_i) = \frac{q-1}{2} - i - p_l(0) + p_l + 1 - \left[\frac{q-1}{2} - i - p_l(0) + p_l + 1\right] + \cdots + \left[\frac{q-1}{2} - i - \frac{p_l(m-3)}{2} + p_l + 1\right]$

$-\left[\frac{q-1}{2} - \frac{p_l(m-3)}{2} + i - 2p_l + 1\right] - [2p_l - i]$

$= p_l + 1 - i - i + 2p_l - 1 - 2p_l + i$

$= p_l - i$

$$f^*(v_i) = \begin{cases} i & \text{for } 1 \le i \le \frac{p_l-1}{2} \\ i - p_l & \text{for } \frac{p_l+1}{2} \le i \le p_l - 1 \\ p_l & \text{for } i = p_l \\ 2p_l - i & \text{for } p_l + 1 \le i \le \frac{3p_l-1}{2} \\ p_l - i & \text{for } \frac{3p_l+1}{2} \le i \le 2p_l \end{cases}$$

From the above equation we observed that the values of $f^*(v_i)$ are distinct for every i and they belong to the set $\{\pm 1, \pm 2, \ldots, \pm p\}$. Also noted that no two sums are equal for different values of i. Therefore f* is clearly a bijective on the nodes set of Cay (G,S) × $\overrightarrow{K_2}$ to $\{\pm 1, \pm 2, \ldots, \pm p\}$. Hence Cay (G,S) × $\overrightarrow{K_2}$ admits the labelling technique called super edge graceful if numbers m and p are odd.

(iv) m is even number and p_l is odd number

Since p is odd, r is also an odd number. Now define a bijective function f defined on the set of arcs of Cay (G,S) × $\overrightarrow{K_2}$ to $\{0, \pm 1, \pm 2, \ldots, \pm \frac{r-1}{2}\}$ as mentioned below.

For, $1 \le i \le p_l$ and $1 \le j \le m + 1$

$$f(e_{ij}) = \begin{cases} \frac{r-1}{2} - i - \frac{p_l(j-1)}{2} + 1 \\ \quad \text{for } j = 1, 3, 5, \ldots, m-1 \\ -\left[\frac{r-1}{2} - i - \frac{p_l(j-2)}{2} + 1\right] \\ \quad \text{for } j = 2, 4, 6, \ldots, \quad m-2 \end{cases}$$

For j = m,

$$f(e_{ij}) = -\left[\frac{r-1}{2} - \frac{p_l(j-2)}{2} + i - p_l\right]$$

For **j = m + 1**,

$$f(e_{ij}) = \begin{cases} i & \text{for } 1 \le i \le \frac{p_l - 1}{2} \\ -[p_l - i + 1] & \text{for } \frac{p_l + 1}{2} \le i \le p_l \end{cases}$$

For $p_l + 1 \le i \le 2p_l$ and $1 \le j \le m$

$$f(e_{ij}) = \begin{cases} \frac{r-q-1}{2} - \frac{p_l(j-1)}{2} + 1 + p_l - i & \text{for } j = 1,3,5,\dots,m-1 \\ -\left[\frac{r-q-1}{2} - \frac{p_l(j-2)}{2} + 1 + p_l - i\right] & \text{for } j = 2,4,\dots,m-2 \end{cases}$$

For j = m,

$$f(e_{ij}) = \begin{cases} -\left[\frac{r-q-1}{2} - \frac{p_l(j-2)}{2} + 2p_l - 2i + 1\right] & \text{for } p_l + 1 \le i \le \frac{3p_l - 1}{2} \\ -\left[\frac{r-q-1}{2} - \frac{p_l(j-2)}{2} + 3p_l + 1 - 2i\right] & \text{for } \frac{3p_l + 1}{2} \le i \le 2p_l - 1 \\ 0 & \text{for } i = 2p_l \end{cases}$$

Then the induced function for $1 \le i \le p_l$ is

$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$

For $1 \le i \le \frac{p_l - 1}{2}$,

$$f^*(v_i) = \frac{r-1}{2} - i - p_l(0) + 1 - \left[\frac{r-1}{2} - i - p_l(0) + 1\right] + \cdots + \left[\frac{r-1}{2} - i - \frac{p_l(m-2)}{2} + 1\right] - \left[\frac{r-1}{2} - \frac{p_l(m-2)}{2} + i - p_l\right] + i$$

$= -i + 1 - i + p_l + i$

$= 1 + p_l - i$

For $\frac{p_l+1}{2} \le i \le p_l$

$$f^*(v_i) = \frac{r-1}{2} - i - p_l(0) + 1$$

$$-\left[\frac{r-1}{2} - i - p_l(0) + 1\right] + \cdots +$$

$$\left[\frac{r-1}{2} - i - \frac{p_l(m-2)}{2} + 1\right]$$

$$-\left[\frac{r-1}{2} - \frac{p_l(m-2)}{2} + i - p_l\right] - [p_l - i + 1]$$

$= -i + 1 - i + p_l - p_l + i - 1$

$= -i$

The induced function for $p_l + 1 \le i \le 2p_l$ is

$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$

For $p_l + 1 \le i \le \frac{3p_l - 1}{2}$

$$f^*(v_i) = \frac{r-q-1}{2} - i - p_l(0) + p_l + 1 -$$

$$\left[\frac{r-q-1}{2} - i - p_l(0) + p_l + 1\right] +$$

$$\dots + \left[\frac{r-q-1}{2} - i - \frac{p_l(m-2)}{2} + p_l + 1\right]$$

$$-\left[\frac{r-q-1}{2} - \frac{p_l(m-2)}{2} + 2p_l - 2i + 1\right]$$

$= p_l + 1 - i - 2p_l + 2i - 1$

$= i - p_l$

For $\frac{3p_l+1}{2} \le i \le 2p_l - 1$

$$f^*(v_i) = \frac{r-q-1}{2} - i - p_l(0) + p_l + 1 - \left[\frac{r-q-1}{2} - i - p_l(0) + p_l + 1\right] + \dots + \left[\frac{r-q-1}{2} - i - \frac{p_l(m-2)}{2} + p_l + 1\right]$$

$$-\left[\frac{r-q-1}{2} - \frac{p_l(m-2)}{2} + 3p_l + 1 - 2i\right]$$

$= p_l + 1 - i - 3p_l - 1 + 2i$

$= i - 2p_l$

For $i = 2p_l$,

$$f^*(v_i) = \frac{r-q-1}{2} - i - p_l(0) + p_l + 1 -$$

$$\left[\frac{r-q-1}{2} - i - p_l(0) + p_l + 1\right] +$$

$$\dots + \left[\frac{r-q-1}{2} - i - \frac{p_l(m-2)}{2} + p_l + 1\right] + 0$$

$$= \frac{r-q-1}{2} - 2p_l - \frac{p_l(m-2)}{2} + p_l + 1$$

$$= \frac{2q+p_l-q-1}{2} - \frac{p_l(m-2)}{2} - p_l + 1$$

$$= \frac{q+p_l-1}{2} - \frac{p_l(m-2)}{2} - p_l + 1$$

$$= \frac{mp_l+p_l-1-p_l\,(m-2)}{2} - p_l + 1$$

$$= \frac{p_l+1}{2}$$

$$f^*(v_i) = \begin{cases} p_l+1-i & \text{for } 1 \le i \le \frac{p_l-1}{2} \\ -i & \text{for } \frac{p_l+1}{2} \le i \le p_l \\ i-p_l & \text{for } p_l+1 \le i \le \frac{3p_l-1}{2} \\ i-2p_l & \text{for } \frac{3p_l+1}{2} \le i \le 2p_l-1 \\ \frac{p_l+1}{2} & \text{for } i = 2p_l \end{cases}$$

From the above equation we observed that the values of $f^*(v_i)$ are distinct for every i and they belongs to the set $\{\pm 1, \pm 2, \dots, \pm p_l\}$. Also noted that no two sums are equal for different values of i. Therefore f* is clearly a bijective on the nodes set of Cay (G,S) $\times \overrightarrow{K_2}$ to $\{\pm 1, \pm 2, \dots, \pm p_l\}$. Hence Cay (G,S) $\times \overrightarrow{K_2}$ admits super edge graceful labeling if the numbers m is an even number and p_l is odd.

Therefore Cay (G,S) $\times \overrightarrow{K_2}$ permits super edge graceful labeling.

2) Algorithm

Input:

The digraph product Cay(G,S) X K_2 with $2p_l$ vertices such that first p_l vertices with m+1 outgoing arcs and another p_l vertices with m outgoing arcs

Step: 1
Denote the vertex set $V = \{v_1, v_2, \dots v_{2p_l}\}$
Step: 2
Denote the arc set E= $\{e_{ij}\}$ where $1 \le i \le 2p_l$ and $1 \le j \le m+1$
Step: 3
Check whether p_l is odd or even.
If p_l is even, go to step 4
If p_l is odd, go to step 5
Step: 4
Check whether m is even or odd
If m is odd go to step 6
If m is even go to step 7
Step: 5
Check whether m is even or odd
If m is odd go to step 8
If m is even go to step 9
Step: 6
Define f for all arcs as defined in case (i) of the above theorem.
Go to step 10
Step: 7
Define f for all arcs as defined in case (ii) of the above theorem.
Go to step 10
Step: 8
Define f for all arcs as defined in case (iii) of the above theorem.
Go to step 10
Step: 9
Define f for all arcs as defined in case (iv) of the above theorem.
Go to step 10
Step: 10
The induced function for $1 \le i \le 2p_l$ is
$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$

Output: Cay (G,S) $\times \overrightarrow{K_2}$ with super edge graceful labelling.

B. Edge odd graceful labeling

1) Theorem

The Cartesian product Cay (G, S) $\times \overrightarrow{K_2}$ permits edge -odd-graceful labeling.

Proof:

Assume a Cayley digraph with the generating set S. Then the digraph Cay (G, S) contains p nodes which also has m generators. Every vertex has m indegree and m outdegree. Totally the Cayley digraph has $mp_l = q$ arcs. Now multiply Cay (G, S) with $\overrightarrow{K_2}$ by the Cartesian product. After the Cartesian product of digraphs, we have resultant digraph Cay (G, S) $\times \overrightarrow{K_2}$ has $2p_l$ vertices. The first p_l vertices have m+1 outgoing and m incoming arcs whereas the another p_l vertices have m outgoing and m+1 entering arcs. Hence Cay (G, S) $\times \overrightarrow{K_2}$ has $2p_l$ vertices and $mp_l + (m+1)\,p_l = 2q + p_l$ arcs. Let $r = 2q + p_l$ and $n = 2p_l$.

To establish Cay{(G,S)} $\times \overrightarrow{K_2}$ of $2p_l$ nodes and r outgoing arcs is edge odd graceful we must demonstrate that there occurs a bijection f on the arcs set to {1,3,5,…,2r-1} such that the outcomes of the resultant mapping f*, given by f*(vi) = $\sum f(e_{ij})$ (mod 2r) summation taken up all outward arcs of v_i, $1 \le i \le 2p_l$ are different where e_{ij} is j^{th} outward arc of i^{th} vertex. Now define f: E → $\{1,3,5,\dots,2r-1\}$ as

$$f(e_{ij}) = 2(j-1)n + 2i - 1$$

For, $1 \le i \le 2p_l$ *and* $1 \le j \le m+1$

For $1 \le i \le p_l$ and $1 \le j \le m+1$,

$$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$$

$$= 2i - 1 + 2n + 2i - 1 + 4n + 2i - 1$$

$$+ \cdots + 2(m+1-1)n + 2i - 1$$

$$= 2i(m+1) - 1(m+1) + 2n(1+2+\cdots+m)$$

$$=2i(m+1) - (m+1) + nm(m+1) \quad (1)$$

The above equation gives diverse outputs for each i, $1 \le i \le p_l$ under the modulus 2r.

For, $p_l + 1 \le i \le 2p_l$ *and* $1 \le j \le m$,

$$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$$

$$= 2i - 1 + 2n + 2i - 1 + 4n + 2i - 1$$

$$+ \ldots + 2(m-1)n + 2i - 1$$

$$= 2i(m) - 1(m) + 2n(1+2+\cdots+(m-1))$$

$$=2mi - m + nm(m-1) \quad (2)$$

The above equation gives diverse values for all i, $p_l + 1 \le i \le 2p_l$ under mod 2r.

Let s and t be two integers such that $1 \le s \le p_l$ and $p_l + 1 \le t \le 2p_l$.

From Equations (1) and (2), $f^*(v_s) = f^*(v_t)$ for any s and t, $1 \le t \le p_l$ and $p_l + 1 \le s \le 2p_l$

$$2s(m+1) - (m+1) + nm(m+1) = 2mt - m + nm(m-1)$$

$$2s(m+1) - 1 + 2mn = 2mt$$

For m =1, we have $4s - 1 + 4p_l = 2t$

This is a contradiction to the fact that $p_l + 1 \le t \le 2p_l$. Therefore $f^*(v_s) \ne f^*(v_t)$ if $s \ne t$. $f^*(v_i)$ gives diverse values for each node $1 \le i \le 2p_l$. Therefore Cay (G,S) $\times \overrightarrow{K_2}$ admits edge -odd- graceful labeling.

2) Algorithm

Input:

The digraph product Cay(G,S) X K_2 with $2p_l$ vertices such that first p_l vertices with m+1 outgoing arcs and another p_l vertices with m outgoing arcs

Step: 1
Denote the vertex set $V = \{v_1, v_2, \ldots v_{2p_l}\}$
Step: 2
Denote the arc set E= $\{e_{ij}\}$ where $1 \le i \le 2p_l$ and $1 \le j \le m+1$
Step: 3
Define f : $E \to \{1,3,5,\ldots,2r-1\}$ as
For, $1 \le i \le 2p_l$ and $1 \le j \le m+1$
$f(e_{ij}) = 2(j-1)n + 2i - 1$
Step: 4
Define $f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$

Output: Cay (G,S) $\times \overrightarrow{K_2}$ with edge odd graceful labelling.

3) Example

Consider the symmetric group S_3 with generating set $\{(1,2)(1,2,3)\}$. The following figure 1 shows (Cay S_3) $\times \overrightarrow{K_2}$ with its edge odd graceful labeling.

IV. COROLLARY

We established that the Cayley digraph permits super edge graceful and edge odd graceful labeling. So, we can make a proposition that

"If the Cayley- digraph permits super edge graceful labelling, then the Cartesian product Cay (G,S) $\times \overrightarrow{K_2}$ permits the labelling techniques called edge odd graceful and super edge graceful".

V. CONCLUSION

We have shown that the Cartesian product of Cayley digraph with K_2 permits super edge graceful labelling and edge odd graceful labeling. Cayley digraphs have the greatest potential applications in various fields when we looking for larger networks with symmetric property. This Cartesian product with graceful labeling technology makes it easier for researchers to select the best combinatorial model with security.

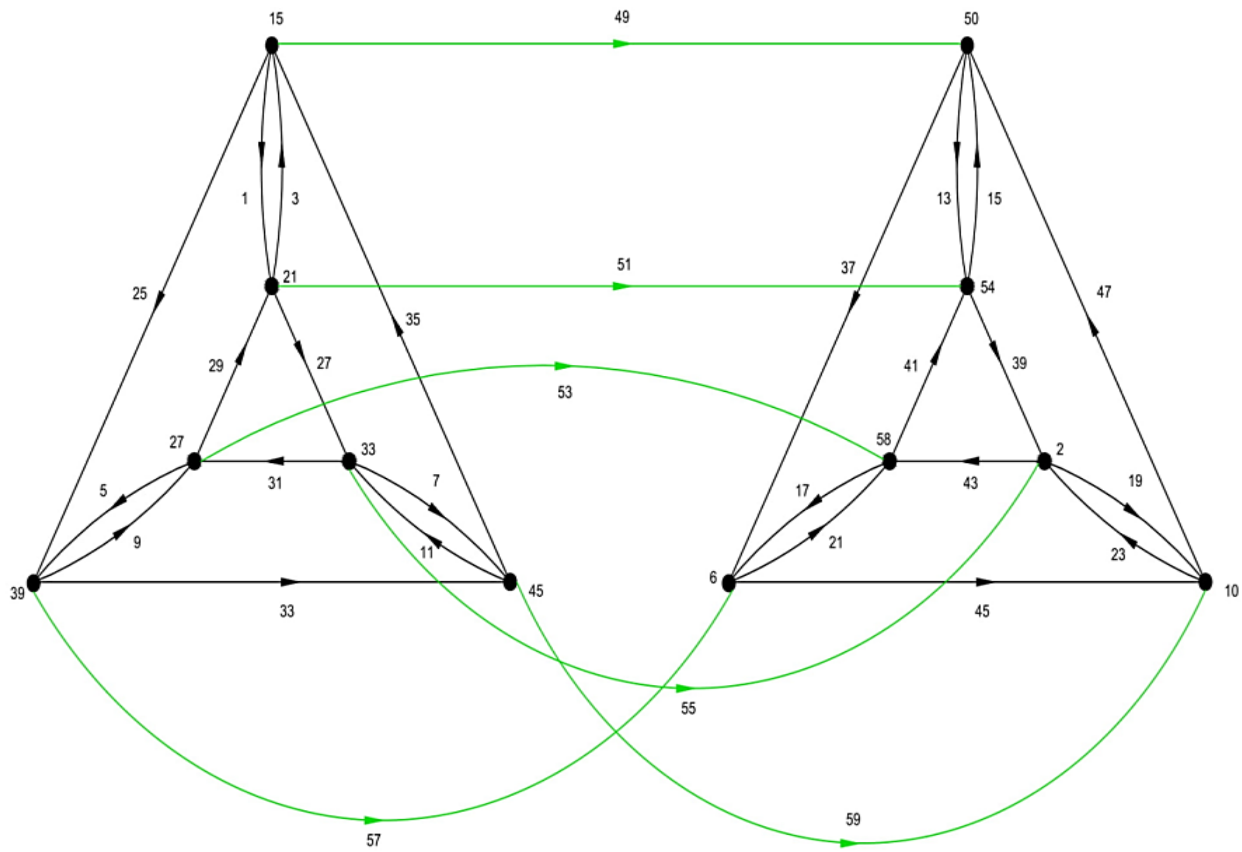

Figure 1 Edge odd graceful labeling of (Cay S_3) $\times$ $\overrightarrow{K_2}$

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