Essential Intuitionistic Fuzzy Ideals in Semigroups

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***Abstract*—In this paper, we give the concepts of essential intuitionistic fuzzy ideals in semigroups. We proved properties and relationships between essential intuitionistic fuzzy ideals and essential ideals in semigroups. We extend to 0-essential intuitionistic fuzzy ideals in semigroups. Finally, we study properties of essential intuitionistic fuzzy $(m, n)-$ideals.**

***Index Terms*—essential fuzzy ideals, essential intuitionistic fuzzy ideals, intuitionistic fuzzy ideals, 0-essential intuitionistic fuzzy ideals.**

I. Introduction

THE CONCEPTS of fuzzy sets was published by L. A. Zadeh in 1965 [1]. In 1986 K. T. Atanassov [2], gave the concept of intuitionistic fuzzy sets such that both a degree of membership and a degree of non-membership are real numbers between 0 and 1. The concept has been applied in different fields, such as logic programming, decision-making, problem and medical diagnosis, etc. Later, in 2002 K. H. Kim and Y. B. Jun [3] discussed the concept of intuitionistic fuzzy ideals in semigroups. Many authors used the theory of intuitionistic fuzzy sets go applications, see, e.g., [4], [5], [6], [7]. In 1971 U. Medhi et al. [8] discussed the Essential fuzzy ideals of ring. In 2013, U. Medhi and H.K. Saikia [9] studied the concept of T-fuzzy essential ideals and the properties of T-fuzzy essential ideals. In 2017 S. Wani and K. Pawar [10] extended the concept of essential ideals in semigroups to ternary semiring and studied essential ideals in ternary semiring. In 2020, S. Baupradist et al. [11] studied essential ideals and essential fuzzy ideals in semigroups. Together with 0-essential ideals and 0-essential fuzzy ideals in semigroups.

In 2022 T. Gaketem et al. [12] studied essential bi-ideals and fuzzy essential bi-ideals in semigroups. Moreover, T. Gaketem and A. Iampan [13], [14] used knowledge of essential ideals in semigroups to study essential ideals in UP-algebra. In the same year P. Khamrot and T. Gaketem, [15] studied essential ideals in a bipolar fuzzy set. In 2023, R. Rittichuai et al. [16] studied essential ideals and essential fuzzy ideals in ternary semigroups. Recently, N. Kaewmanee and T. Gaketem [17] studied essential hyperideals and fuzzy hyperideals in hypersemigroups.

In this paper, we study essential intuitionistic fuzzy ideals in semigroup and investigate some basic properties of essential intuitionistic fuzzy ideals. Moreover, we characterize essential intuitionistic fuzzy ideals and 0-essential intuitionistic fuzzy ideals of semigroups. And together essential intuitionistic fuzzy $(m, n)-$ideals.

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II. Preliminaries

In this section, we review the concept's basic definitions and the theorem used to prove all results in the next section.

A non-empty subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called a *subsemigroup* of $\mathcal{S}$ if $\mathcal{I}^2 \subseteq \mathcal{I}$. A non-empty subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called a *left (right)* ideal of $\mathcal{S}$ if $\mathcal{S}\mathcal{I} \subseteq \mathcal{I}$ ($\mathcal{I}\mathcal{S} \subseteq \mathcal{I}$). An *ideal* $\mathcal{I}$ of a semigroup $\mathcal{S}$ is a non-empty subset which is both a left ideal and a right ideal of $\mathcal{S}$. An *essential ideal* $\mathcal{I}$ of a semigroup $\mathcal{S}$ if $\mathcal{I}$ is an ideal of $\mathcal{S}$ and $\mathcal{I} \cap \mathfrak{f} \neq \emptyset$ for every ideal $\mathfrak{f}$ of $\mathcal{S}$.

A subsemigroup $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be an (m, n)-*ideal* of $\mathcal{S}$ if $\mathcal{I}^m S \mathcal{I}^n \subseteq \mathcal{I}$ for any $m, n \in \mathbb{N}$.

A subsemigroup $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be an $(m, 0)$-*ideal* of $\mathcal{S}$ if $\mathcal{I}^m S \subseteq \mathcal{I}$ for any $m \in \mathbb{N}$.

A subsemigroup $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be an $(0, n)$-*ideal* of $\mathcal{S}$ if $S\mathcal{I}^n \subseteq \mathcal{I}$ for any $n \in \mathbb{N}$.

For a non-empty subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ and $m, n \in \mathbb{N}$, we denote the

$[\mathcal{I}](m, n) = \bigcup_{r=1}^{m+n} \mathcal{I}^r \cap \mathcal{I}^m \mathcal{S} \mathcal{I}^n$ is the principal (m, n)-ideal by $\mathcal{I}$,

$[\mathcal{I}](m, 0) = \bigcup_{r=1}^{m} \mathcal{I}^r \cap \mathcal{I}^m \mathcal{S}$ is the principal $(m, 0)$-ideal by $\mathcal{I}$

and

$[\mathcal{I}](0, n) = \bigcup_{r=1}^{n} \mathcal{I}^r \cap S\mathcal{I}^n$ is the principal $(0, n)$-ideal by $\mathcal{I}$,

i.e., the smallest (m, n)-ideal, the smallest $(m, 0)$-ideal and the smallest $(0, n)$-ideal of $\mathcal{S}$ containing $\mathcal{S}$, respectively.

Lemma 2.1. *[18] Let $\mathcal{S}$ be a semigroup and m, n positive integers, $[u]_{(m,n)}$ the principal (m, n)-ideal generated by the element u. Then*

(1) $([u]_{(m,0)})^m \mathcal{S} = u^m \mathcal{S}$.
(2) $\mathcal{S}([u]_{(0,n)})^n = \mathcal{S} u^n$.
(3) $([u]_{(m,0)})^m \mathcal{S}([u]_{(0,n)})^n = u^m \mathcal{S} u^n$.

We see that for any $\delta_1, \delta_2 \in [0, 1]$, we have

$$\delta_1 \vee \delta_2 = \max\{\delta_1, \delta_2\} \quad \text{and} \quad \delta_1 \wedge \delta_2 = \min\{\delta_1, \delta_2\}.$$

A fuzzy set δ of a non-empty set $\mathcal{T}$ is function from $\mathcal{T}$ into unit closed interval $[0, 1]$ of real numbers, i.e., $\delta : \mathcal{T} \to [0, 1]$.

For any two fuzzy sets δ and ϑ of a non-empty set $\mathcal{T}$, define $\geq, =, \wedge$ and $\vee$ as follows:

(1) $\delta \geq \vartheta \Leftrightarrow \delta(\mathfrak{e}) \geq \vartheta(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{T}$,
(2) $\delta = \vartheta \Leftrightarrow \delta \geq \vartheta$ and $\vartheta \geq \delta$,
(3) $(\delta \wedge \vartheta)(\mathfrak{e}) = \min\{\delta(\mathfrak{e}), \vartheta(\mathfrak{e})\} = \delta(\mathfrak{e}) \wedge \vartheta(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{T}$,

(4) $(\delta \vee \vartheta)(\mathfrak{e}) = \max\{\delta(\mathfrak{e}), \vartheta(\mathfrak{e})\} = \delta(\mathfrak{e}) \vee \vartheta(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{T}$.
For the symbol $\delta \leq \vartheta$, we mean $\vartheta \geq \delta$.

The following definitions are types of fuzzy subsemigroups on semigroups.

Definition 2.2. *[20] A fuzzy set δ of a semigroup $\mathcal{S}$ is said to be a fuzzy ideal of $\mathcal{S}$ if $\eta(\mathfrak{u}\mathfrak{v}) \geq \eta(\mathfrak{u}) \vee \eta(\mathfrak{v})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.*

Definition 2.3. *A fuzzy set δ of an ordered semigroups $\mathcal{S}$ is said to be an fuzzy (m,n)-ideal of $\mathcal{S}$ if*

$$\delta(u_1u_2\cdots u_m z v_1 v_2 \cdots v_n) \geq \delta(u_1) \wedge \delta(u_2) \wedge ... \wedge \delta(u_m) \wedge \delta_{\mathfrak{f}}(v_1) \wedge \delta(v_2) \wedge ... \wedge \delta(v_n)$$

for all $u_1, u_2, ..., u_m, v_1, v_2, ..., v_n, z \in \mathcal{S}$ and $m, n \in \mathbb{N}$.

Definition 2.4. *[11] An* essential fuzzy ideal *δ of a semigroup $\mathcal{S}$ if δ is a nonzero fuzzy ideal of $\mathcal{S}$ and $\delta \wedge \vartheta \neq 0$ for every nonzero fuzzy ideal ϑ of $\mathcal{S}$.*

Definition 2.5. *[19] An* essential fuzzy *(m,n)-ideal δ of a semigroup $\mathcal{S}$ if δ is a nonzero fuzzy (m,n)-ideal of $\mathcal{S}$ and $\delta \wedge \vartheta \neq 0$ for every nonzero fuzzy (m,n)-ideal ϑ of $\mathcal{S}$.*

Now, we review the intuitionistic fuzzy set.

Definition 2.6. *[2] An **intuitionsic fuzzy set** (IF set) $\mathfrak{f}$ in a non-empty set $\mathcal{T}$ is an object having the form*

$$\mathfrak{f} := \{(\mathfrak{e}, \delta(\mathfrak{e}), \vartheta(\mathfrak{e})) \mid \mathfrak{e} \in \mathcal{T}\},$$

where $\delta : \mathcal{T} \to [0,1]$ and $\vartheta : \mathcal{T} \to [0,1]$ denote the degree of membership and the degree of non-membership respectively, and $0 \leq \delta(\mathfrak{e}) + \vartheta(\mathfrak{e}) \leq 1$ for all $\mathfrak{e} \in \mathcal{T}$.

Remark 2.7. *For the sake of simplicity we shall use the symbol $\mathfrak{f} = (\delta, \vartheta)$ for the IF set $\mathfrak{f} = \{(\mathfrak{e}, \delta(\mathfrak{e}), \vartheta(\mathfrak{e})) \mid \mathfrak{e} \in \mathcal{T}\}$.*

For IF sets $\mathfrak{f}_1 = (\delta_{\mathfrak{f}_1}, \vartheta_{\mathfrak{f}_1})$ and $\mathfrak{f}_2 = (\delta_{\mathfrak{f}_2}, \vartheta_{\mathfrak{f}_2})$ of $\mathcal{T}$, defined the relation as follows:

(1) $\mathfrak{f}_1 \subseteq \mathfrak{f}_2$ if and only if $\delta_{\mathfrak{f}_1}(\mathfrak{e}) \leq \delta_{\mathfrak{f}_2}(\mathfrak{e})$ and $\vartheta_{\mathfrak{f}_1}(\mathfrak{e}) \geq \vartheta_{J_2}(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{T}$,
(2) $\mathfrak{f}_1 = \mathfrak{f}_2$ if and only if $\mathfrak{f}_1 \subseteq \mathfrak{f}_2$ and $\mathfrak{f}_2 \subseteq \mathfrak{f}_1$,
(3) $\mathfrak{f}_1 \cap \mathfrak{f}_2 = \{(\delta_{\mathfrak{f}_1} \wedge \delta_{\mathfrak{f}_2})(\mathfrak{e}), (\vartheta_{\mathfrak{f}_1} \vee \vartheta_{\mathfrak{f}_2})(\mathfrak{e}) \mid \mathfrak{e} \in \mathcal{T}\}$,
(4) $\mathfrak{f}_1 \cup \mathfrak{f}_2 = \{(\delta_{\mathfrak{f}_1} \vee \delta_{\mathfrak{f}_2})(\mathfrak{e}), (\vartheta_{\mathfrak{f}_1} \wedge \vartheta_{\mathfrak{f}_2})(\mathfrak{e}) \mid \mathfrak{e} \in \mathcal{T}\}$.

For $\mathfrak{e} \in \mathcal{T}$, define $F_{\mathfrak{e}} = \{(\mathfrak{y}, \mathfrak{z}) \in \mathcal{T} \times \mathcal{T} \mid \mathfrak{e} = \dagger\ddagger\}$.

For IF sets $\mathfrak{f}_1 = (\delta_{\mathfrak{f}_1}, \vartheta_{\mathfrak{f}_1})$ and $\mathfrak{f}_2 = (\delta_{\mathfrak{f}_2}, \vartheta_{\mathfrak{f}_2})$, defined the products
$\mathfrak{f}_1 \circ \mathfrak{f}_2 = (\delta_{\mathfrak{f}_1} \circ \delta_{\mathfrak{f}_2}, \vartheta_{\mathfrak{f}_1} \circ \vartheta_{\mathfrak{f}_2})$ as follows:

$$(\delta_{\mathfrak{f}_1} \circ \delta_{\mathfrak{f}_2})(\mathfrak{e}) = \begin{cases} \bigvee_{(\mathfrak{y},\mathfrak{z}) \in F_{\mathfrak{e}}} \{\delta_{\mathfrak{f}_1}(\mathfrak{y}) \wedge \delta_{\mathfrak{f}_2}(\mathfrak{z})\} & \text{if} \quad F_{\mathfrak{e}} \neq \emptyset, \\ 0 & \text{if otherwise.} \end{cases}$$

and

$$(\vartheta_{\mathfrak{f}_1} \circ \vartheta_{\mathfrak{f}_2})(\mathfrak{e}) = \begin{cases} \bigwedge_{(\mathfrak{y},\mathfrak{z}) \in F_{\mathfrak{e}}} \{\vartheta_{\mathfrak{f}_1}(\mathfrak{y}) \vee \vartheta_{\mathfrak{f}_2}(\mathfrak{z})\} & \text{if} \quad F_{\mathfrak{e}} \neq \emptyset, \\ 0 & \text{if otherwise.} \end{cases}$$

Definition 2.8. *[3] Let $\mathfrak{f}$ be a non-empty set of a semigroup $\mathcal{S}$. An **intuitionistic characteristic function** $\chi_{\mathfrak{f}} = (\delta_{\chi_{\mathfrak{f}}}, \vartheta_{\chi_{\mathfrak{f}}})$*

$$\delta_{\chi_{\mathfrak{f}}}(\mathfrak{e}) := \begin{cases} 1 & \mathfrak{e} \in \mathfrak{f}, \\ 0 & \mathfrak{e} \notin \mathfrak{f}, \end{cases}$$

and

$$\vartheta_{\chi_{\mathfrak{f}}}(\mathfrak{e}) := \begin{cases} 0 & \mathfrak{e} \in \mathfrak{f}, \\ 1 & \mathfrak{e} \notin \mathfrak{f}. \end{cases}$$

Remark 2.9. *For the sake of simplicity we shall use the symbol $\chi_{\mathfrak{f}} = (\delta_{\chi_{\mathfrak{f}}}, \vartheta_{\chi_{\mathfrak{f}}})$ for the IF set $\chi_{\mathfrak{f}} := \{(\mathfrak{e}, \delta_{\chi_{\mathfrak{f}}}(\mathfrak{e}), \vartheta_{\chi_{\mathfrak{f}}}(\mathfrak{e})) \mid \mathfrak{e} \in \mathcal{S}\}$.*

Definition 2.10. *[3] An IF set $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ on a semigroup $\mathcal{S}$ is called an* IF ideal *on $\mathcal{S}$ if $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \delta_{\mathfrak{f}}(\mathfrak{e}) \vee \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \leq \vartheta_{\mathfrak{f}}(\mathfrak{e}) \wedge \vartheta_{\mathfrak{f}}(\mathfrak{f})$, for all $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$.*

The following theorems are true.

Theorem 2.11. *[3] Let $\mathcal{J}$ be a nonempty subset of semigroup $\mathcal{S}$. Then $\mathcal{J}$ is a subsemigroup (ideal, bi-ideal) of $\mathcal{S}$ if and only if intuitionistic characteristic function $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an IF subsemigrup (ideal, bi-ideal) of $\mathcal{S}$.*

Theorem 2.12. *[3] Let $\mathfrak{f}$ and $\mathcal{J}$ be subsets of a non-empty set $\mathcal{S}$. Then the following holds.*

(1) $\delta_{\chi_{\mathfrak{f} \cap \chi_{\mathcal{J}}}} = \delta_{\chi_{\mathfrak{f}}} \wedge \delta_{\chi_{\mathcal{J}}}$.
(2) $\vartheta_{\chi_{\mathfrak{f} \cup \chi_{\mathcal{J}}}} = \vartheta_{\chi_{\mathfrak{f}}} \vee \vartheta_{\chi_{\mathcal{J}}}$.
(3) $\delta_{\chi_{\mathfrak{f}}} \circ \delta_{\chi_{\mathcal{J}}} = \delta_{\chi_{\mathfrak{L}\mathcal{J}}}$.
(4) $\vartheta_{\chi_{\mathfrak{f}}} \circ \vartheta_{\chi_{\mathcal{J}}} = \vartheta_{\chi_{\mathfrak{L}\mathcal{J}}}$.

Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be an IF set of a non-empty of $\mathcal{S}$. Then the *support* of $\mathfrak{f}$ instead of $\text{supp}(\mathfrak{f}) = \{\mathfrak{e} \in \mathcal{S} \mid \delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0 \,\text{and}\, \vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0\}$ for all $\mathfrak{e} \in \mathcal{S}$.

Theorem 2.13. *Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be a nonzero IF set of a semigroup $\mathcal{S}$. Then $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of $\mathcal{S}$ if and only if* supp($\mathfrak{f}$) *is an ideal of $\mathcal{S}$.*

Proof: Suppose that $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of $\mathcal{S}$ and let $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$ with $\mathfrak{e}, \mathfrak{f} \in \text{supp}(\mathfrak{f})$ Then $\delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\delta_{\mathfrak{f}}(\mathfrak{f}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\vartheta_{\mathfrak{f}}(\mathfrak{f}) \neq 0$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of $\mathcal{S}$ we have $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \delta_{\mathfrak{f}}(\mathfrak{e}) \vee \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \leq \vartheta_{\mathfrak{f}}(\mathfrak{e}) \wedge \vartheta_{\mathfrak{f}}(\mathfrak{f})$. Thus, $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \neq 0$. It implies that $\mathfrak{e}\mathfrak{f} \in \text{supp}(\mathfrak{f})$. Hence $\text{supp}(\mathfrak{f})$ is an ideal of $\mathcal{S}$.

Conversely, suppose that $\text{supp}(\mathfrak{f})$ is an ideal of $\mathcal{S}$ and let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is not an IF ideal of $\mathcal{S}$. Then there exist $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$ such that $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) < \delta_{\mathfrak{f}}(\mathfrak{e}) \wedge \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) > \vartheta_{\mathfrak{f}}(\mathfrak{e}) \wedge \vartheta_{\mathfrak{f}}(\mathfrak{f})$. Since $\text{supp}(\mathfrak{f})$ is an ideal of $\mathcal{S}$ we have $\mathfrak{e}\mathfrak{f} \in \text{supp}(\mathfrak{f})$. Thus, $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \neq 0$.

If $\mathfrak{e} \in \text{supp}(\mathfrak{f})$ or $\mathfrak{f} \in \text{supp}(\mathfrak{f})$, then $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \delta_{\mathfrak{f}}(\mathfrak{e}) \wedge \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) > \vartheta_{\mathfrak{f}}(\mathfrak{e}) \vee \vartheta_{\mathfrak{f}}(\mathfrak{f})$. It is a contradiction.

If $\mathfrak{e} \notin \text{supp}(\mathfrak{f})$ and $\mathfrak{f} \notin \text{supp}(\mathfrak{f})$, then $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \delta_{\mathfrak{f}}(\mathfrak{e}) \wedge \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) > \vartheta_{\mathfrak{f}}(\mathfrak{e}) \vee \vartheta_{\mathfrak{f}}(\mathfrak{f})$. It is a contradiction.

Thus, $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \delta_{\mathfrak{f}}(\mathfrak{e}) \vee \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \leq \vartheta_{\mathfrak{f}}(\mathfrak{e}) \wedge \vartheta_{\mathfrak{f}}(\mathfrak{f})$. Hence $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of $\mathcal{S}$. ■

III. ESSENTIAL INTUITIONISTIC FUZZY IDEALS IN A SEMIGROUP

In this section, we give a definition of essential intuitionistic fuzzy ideals and study the properties of essential intuitionistic fuzzy ideals in semigroups.

Definition 3.1. *An* essential IF ideal *$\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ of a semigroup $\mathcal{S}$ if $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a nonzero IF ideal of $\mathcal{S}$ and $\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}} \neq 0$ and $\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}} \neq 0$ for every nonzero IF ideal $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ of $\mathcal{S}$.*

Theorem 3.2. *Let $\mathcal{J}$ be an ideal of a semigroup $\mathcal{S}$. Then $\mathcal{J}$ is an essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an essential IF ideal of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{J}$ is an essential ideal of $\mathcal{S}$ and let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be a nonzero IF ideal of $\mathcal{S}$. Then by Theorem

2.13 $\text{supp}(\mathfrak{f})$ is an ideal of $\mathcal{S}$. Since $\mathcal{J}$ is an essential ideal of $\mathcal{S}$ we have $\mathcal{J}$ is an ideal of $\mathcal{S}$. Thus $\mathcal{J} \cap \text{supp}(\mathfrak{f}) \neq \emptyset$. So there exists $\mathfrak{e} \in \mathcal{J} \cap \text{supp}(\mathfrak{f})$. Since $\mathcal{J}$ is an ideal of $\mathcal{S}$ we have $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an IF ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a nonzero IF ideal of $\mathcal{S}$ we have $(\delta_{\chi_{\mathcal{J}}} \wedge \delta_{\mathfrak{f}})(\mathfrak{e}) \neq 0$ and $(\vartheta_{\chi_{\mathcal{J}}} \vee \vartheta_{\mathfrak{f}})(\mathfrak{e}) \neq 0$ Thus, $\delta_{\chi_{\mathcal{J}}} \wedge \delta_{\mathfrak{f}} \neq 0$ and $\vartheta_{\chi_{\mathcal{J}}} \vee \vartheta_{\mathfrak{f}} \neq 0$. Therefore, $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an essential IF ideal of $\mathcal{S}$.

Conversely, assume that $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an essential IF ideal of $\mathcal{S}$ and let $\mathcal{K}$ be an ideal of $\mathcal{S}$. Then $\chi_{\mathcal{K}} = (\delta_{\chi_{\mathcal{K}}}, \vartheta_{\chi_{\mathcal{K}}})$ is a nonzero IF ideal of $\mathcal{S}$. Since $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an essential IF ideal of $\mathcal{S}$ we have $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an IF ideal of $\mathcal{S}$. Thus, $\delta_{\chi_{\mathcal{J}}} \wedge \delta_{\chi_{\mathcal{K}}} \neq 0$ and $\vartheta_{\chi_{\mathcal{J}}} \vee \vartheta_{\chi_{\mathcal{K}}} \neq 0$ So by Theorem 2.12, $\delta_{\chi_{\mathcal{J}\cap\mathcal{K}}} \neq 0$ and $\vartheta_{\chi_{\mathcal{J}\cup\mathcal{K}}} \neq 0$. Hence, $\mathcal{J} \cap \mathcal{K} \neq \emptyset$. Therefore, $\mathcal{J}$ is an essential ideal of $\mathcal{S}$. ∎

Theorem 3.3. *Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be a nonzero IF ideal of a semigroup $\mathcal{S}$. Then $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an essential IF ideal of $\mathcal{S}$ if and only if $\text{supp}(\mathfrak{f})$ is an essential ideal of $\mathcal{S}$.*

Proof: Assume that $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an essential IF ideal of $\mathcal{S}$ and let $\mathcal{L}$ be an ideal of $\mathcal{S}$. Then by Theorem 2.11, $\chi_{\mathcal{L}} = (\delta_{\chi_{\mathcal{L}}}, \vartheta_{\chi_{\mathcal{L}}})$ is an IF ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an essential IF ideal of $\mathcal{S}$ we have $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of $\mathcal{S}$. Thus, $\delta_{\mathfrak{f}} \wedge \delta_{\chi_{\mathcal{L}}} \neq 0$ and $\vartheta_{\mathfrak{f}} \vee \vartheta_{\chi_{\mathcal{L}}} \neq 0$. So, there exists $\mathfrak{e} \in \mathcal{S}$ such that $(\delta_{\mathfrak{f}} \wedge \delta_{\chi_{\mathcal{L}}})(\mathfrak{e}) \neq 0$ and $(\vartheta_{\mathfrak{f}} \vee \vartheta_{\chi_{\mathcal{L}}})(\mathfrak{e}) \neq 0$. It implies that $\delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\delta_{\chi_{\mathcal{L}}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\vartheta_{\chi_{\mathcal{L}}}(\mathfrak{e}) \neq 0$. Hence, $\mathfrak{e} \in \text{supp}(\mathfrak{f}) \cap \mathcal{L}$ so $\text{supp}(\mathfrak{f}) \cap \mathcal{L} \neq \emptyset$. Therefore, $\text{supp}(\mathfrak{f})$ is an essential ideal of $\mathcal{S}$.

Conversely, assume that $\text{supp}(\mathfrak{f})$ is an essential ideal of $\mathcal{S}$ and let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be a nonzero IF ideal of $\mathcal{S}$. Then by Thoerem 2.13 $\text{supp}(\mathcal{J})$ is an ideal of $\mathcal{S}$. Since $\text{supp}(\mathfrak{f})$ is an essential ideal of $\mathcal{S}$ we have $\text{supp}(\mathfrak{f})$ is an ideal of $\mathcal{S}$. Thus, $\text{supp}(\mathfrak{f}) \cap \text{supp}(\mathcal{J}) \neq \emptyset$. So, there exists $\mathfrak{e} \in \text{supp}(\mathfrak{f}) \cap \text{supp}(\mathcal{J})$. It implies that $\delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\delta_{\mathcal{J}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\vartheta_{\mathcal{J}}(\mathfrak{e}) \neq 0$. for all $\mathfrak{e} \in \mathcal{S}$. Hence, $(\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}})(\mathfrak{e}) \neq 0$ and $(\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}})(\mathfrak{e}) \neq 0$ for all $\mathfrak{e} \in \mathcal{S}$. Therefore, $\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}} \neq 0$ and $\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}}$. We conclude that $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an essential IF ideal of $\mathcal{S}$. ∎

Theorem 3.4. *Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be an essential IF ideal of a semigroup $\mathcal{S}$. If $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is an IF ideal of $\mathcal{S}$ such that $\delta_{\mathfrak{f}} \leq \delta_{\mathcal{J}}$ and $\vartheta_{\mathfrak{f}} \geq \vartheta_{\mathcal{J}}$, then $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is also an essential IF ideal of $\mathcal{S}$.*

Proof: Let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is an IF ideal of $\mathcal{S}$ such that $\delta_{\mathfrak{f}} \leq \delta_{\mathcal{J}}$ and $\vartheta_{\mathfrak{f}} \geq \vartheta_{\mathcal{J}}$ and
let $\mathcal{K} = (\delta_{\mathcal{K}}, \vartheta_{\mathcal{K}})$ be any IF ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an essential IF ideal of $\mathcal{S}$ we have $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of $\mathcal{S}$. Thus, $\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{K}} \neq 0$ and $\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{K}} \neq 0$. So, $\delta_{\mathcal{J}} \wedge \delta_{\mathcal{K}} \neq 0$ and $\vartheta_{\mathcal{J}} \vee \vartheta_{\mathcal{K}} \neq 0$. Hence, $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is an essential IF ideal of $\mathcal{S}$. ∎

Theorem 3.5. *Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be essential IF ideals of a semigroup $\mathcal{S}$. Then $\mathfrak{f} \cup \mathcal{J}$ and $\mathfrak{f} \cap \mathcal{J}$ are essential IF ideals of $\mathcal{S}$.*

Proof: By Theorem 3.4, we have $\mathfrak{f} \cup \mathcal{J}$ is an essential IF ideal of $\mathcal{S}$.

Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ are essential IF ideals of $\mathcal{S}$ we have
$\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ are IF ideals of $\mathcal{S}$. Thus $\mathfrak{f} \cap \mathcal{J}$ is an IF ideal of $\mathcal{S}$.
Let $\mathcal{K} = (\delta_{\mathcal{K}}, \vartheta_{\mathcal{K}})$ be an IF ideal of $\mathcal{S}$. Then $\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{K}} \neq 0$ and $\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{K}} \neq 0$. Thus there exists $\mathfrak{e} \in \mathcal{S}$ such that $(\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{K}})(\mathfrak{e}) \neq 0$ and $(\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{K}})(\mathfrak{e}) \neq 0$. So $\delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\delta_{\mathcal{K}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathcal{K}}(\mathfrak{e}) \neq 0$. Since $\delta_{\mathcal{K}} \neq 0$ and $\vartheta_{\mathcal{K}} \neq 0$ and let $\mathfrak{f} \in \mathcal{S}$ such that $\delta_{\mathcal{J}}(\mathfrak{f}) \neq 0$ and $\vartheta_{\mathcal{J}}(\mathfrak{f}) \neq 0$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ are IF ideals of $\mathcal{S}$ we have $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \delta_{\mathfrak{f}}(\mathfrak{e}) \vee \delta_{\mathfrak{f}}(\mathfrak{f}) \geq 0$, $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \leq \vartheta_{\mathfrak{f}}(\mathfrak{e}) \wedge \vartheta_{\mathfrak{f}}(\mathfrak{f}) \leq 0$ and $\delta_{\mathcal{J}}(\mathfrak{e}\mathfrak{f}) \geq \delta_{\mathcal{J}}(\mathfrak{e}) \wedge \delta_{\mathcal{J}}(\mathfrak{f}) \geq 0$, $\vartheta_{\mathcal{J}}(\mathfrak{e}\mathfrak{f}) \leq \vartheta_{\mathcal{J}}(\mathfrak{e}) \vee \vartheta_{\mathcal{J}}(\mathfrak{f}) \leq 0$ Thus, $(\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}})(\mathfrak{e}\mathfrak{f}) = \delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \wedge \delta_{\mathcal{J}}(\mathfrak{e}\mathfrak{f}) \neq 0$ and $(\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}})(\mathfrak{e}\mathfrak{f}) = \vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \vee \vartheta_{\mathcal{J}}(\mathfrak{e}\mathfrak{f}) \neq 0$. Since $\mathcal{K} = (\delta_{\mathcal{K}}, \vartheta_{\mathcal{K}})$ is an IF ideal of $\mathcal{S}$ and $\delta_{\mathcal{K}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathcal{K}}(\mathfrak{e}) \neq 0$ we have $\delta_{\mathcal{K}}(\mathfrak{e}\mathfrak{f}) \neq 0$ and $\vartheta_{\mathcal{K}}(\mathfrak{e}\mathfrak{f}) \neq 0$ for all $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$. Thus, $((\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}}) \wedge \delta_{\mathcal{K}})(\mathfrak{e}\mathfrak{f}) \neq 0$ and $((\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}}) \vee \vartheta_{\mathcal{K}})(\mathfrak{e}\mathfrak{f}) \neq 0$. Hence $((\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}}) \wedge \delta_{\mathcal{K}}) \neq 0$ and $((\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}}) \vee \vartheta_{\mathcal{K}}) \neq 0$. Therefore, $\mathfrak{f} \cap \mathcal{J}$ is an essential IF ideal of $\mathcal{S}$. ∎

Next, we defined minimal, maximal, prime, and semiprime essential intuitionistic fuzzy ideals and studied the properties of minimal, prime, and semiprime essential intuitionistic fuzzy ideals in semigroups.

Definition 3.6. *[11] An essential ideal $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every essential ideal of $\mathcal{L}$ of $\mathcal{S}$ such that $\mathcal{L} \subseteq \mathcal{I}$, we have $\mathcal{L} = \mathcal{I}$,*
(2) *a* maximal *if for every essential ideal of $\mathcal{L}$ of $\mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{L}$, we have $\mathcal{L} = \mathcal{I}$,*
(3) *a* prime *if $\mathfrak{e}\mathfrak{f} \in \mathcal{I}$ implies $\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{f} \in \mathcal{I}$,*
(4) *a* semiprime *if $\mathfrak{e}^2 \in \mathcal{I}$ implies $\mathfrak{e} \in \mathcal{I}$, for all $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$.*

Example 3.7. *[11] Let $\mathcal{S}$ be a semigroup with zero. Then $\{0\}$ is a unique minimal essential ideal of $\mathcal{S}$, since $\{0\}$ is an eseential ideal of $\mathcal{S}$.*

Definition 3.8. *An essential IF ideal $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every essential IF ideal of $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ of $\mathcal{S}$ such that $\delta_{\mathcal{J}} \leq \delta_{\mathfrak{f}}$ and $\vartheta_{\mathcal{J}} \geq \vartheta_{\mathfrak{f}}$, we have* $\text{supp}(\mathfrak{f}) = \text{supp}(\mathcal{J})$,
(2) *a* maximal *if for every essential IF ideal of $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ of $\mathcal{S}$ such that $\delta_{\mathcal{J}} \geq \delta_{\mathfrak{f}}$ and $\vartheta_{\mathcal{J}} \leq \vartheta_{\mathfrak{f}}$, we have* $\text{supp}(\mathfrak{f}) = \text{supp}(\mathcal{J})$,
(3) *a* prime *if $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \leq \delta_{\mathfrak{f}}(\mathfrak{e}) \vee \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \vartheta_{\mathfrak{f}}(\mathfrak{e}) \wedge \vartheta_{\mathfrak{f}}(\mathfrak{f})$* ,
(4) *a* semiprime *if $\delta_{\mathfrak{f}}(\mathfrak{e}^2) \leq \delta_{\mathfrak{f}}(\mathfrak{e})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}^2) \geq \vartheta_{\mathfrak{f}}(\mathfrak{e})$, for all $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$.*

Theorem 3.9. *Let $\mathcal{I}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds.*

(1) *$\mathcal{I}$ is a minimal essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{J}}})$ is a minimal essential IF ideal of $\mathcal{S}$,*
(2) *$\mathcal{I}$ is a maximal essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{J}}})$ is a maximal essential IF ideal of $\mathcal{S}$,*
(3) *$\mathcal{I}$ is a prime essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a prime essential IF ideal of $\mathcal{S}$,*
(4) *$\mathcal{I}$ is a semiprime essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a semiprime essential IF ideal of $\mathcal{S}$.*

Proof:

(1) Suppose that $\mathcal{I}$ is a minimal essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. By Theorem 3.2, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is an essential IF ideal of $\mathcal{S}$. Let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be an essential IF ideal of $\mathcal{S}$ such that $\delta_{\mathcal{J}} \leq \delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\mathcal{J}} \geq \vartheta_{\chi_{\mathcal{I}}}$. Then $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}})$. Thus, $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}}) = \mathcal{I}$. Hence, $\text{supp}(\mathcal{J}) \subseteq$

$\mathcal{I}$. Since $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is an essential IF ideal of $\mathcal{S}$ we have $\text{supp}(\mathcal{J})$ is an essential ideal of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{J}) = \mathcal{I} = \text{supp}(\chi_{\mathcal{I}})$. Hence, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a minimal essential IF ideal of $\mathcal{S}$.

Conversely, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a minimal essential IF ideal of $\mathcal{S}$ and let $\mathcal{L}$ be an essential ideal of $\mathcal{S}$ such that $\mathcal{L} \subseteq \mathcal{I}$. Then $\mathcal{L}$ is an ideal of $\mathcal{S}$. Thus by Theorem 3.2, $\chi_{\mathcal{L}} = (\delta_{\chi_{\mathcal{L}}}, \vartheta_{\chi_{\mathcal{L}}})$ is an essential IF ideal of $\mathcal{S}$ such that $\delta_{\chi_{\mathcal{L}}} \leq \delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\chi_{\mathcal{L}}} \geq \vartheta_{\chi_{\mathcal{I}}}$. So, $\mathcal{L} = \text{supp}(\chi_{\mathcal{L}}) \subseteq \text{supp}(\chi_{\mathcal{I}}) = \mathcal{I}$. By assumption, we have $\mathcal{L} = \mathcal{I}$. Therefore, $\mathcal{I}$ is a minimal essential ideal of $\mathcal{S}$.

(2) Suppose that $\mathcal{I}$ is a maximal essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. By Theorem 3.2, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is an essential IF ideal of $\mathcal{S}$. Let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be an essential IF ideal of $\mathcal{S}$ such that $\delta_{\mathcal{J}} \geq \delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\mathcal{J}} \geq \vartheta_{\chi_{\mathcal{I}}}$. Then $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}})$. Thus, $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}}) = \mathcal{I}$. Hence, $\text{supp}(\mathcal{J}) \subseteq \mathcal{I}$. Since $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is an essential IF ideal of $\mathcal{S}$ we have $\text{supp}(\mathcal{J})$ is an essential ideal of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{J}) = \mathcal{I} = \text{supp}(\chi_{\mathcal{I}})$. Hence, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a maximal essential IF ideal of $\mathcal{S}$.

Conversely, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a maxiimal essential IF ideal of $\mathcal{S}$ and let $\mathcal{L}$ be an essential ideal of $\mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{L}$. Then $\mathcal{J}$ is an ideal of $\mathcal{S}$. Thus by Theorem 3.2, $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is an essential IF ideal of $\mathcal{S}$ such that $\delta_{\chi_{\mathcal{I}}} \leq \delta_{\chi_{\mathcal{L}}}$ and $\vartheta_{\chi_{\mathcal{I}}} \geq \vartheta_{\chi_{\mathcal{L}}}$. So, $\mathcal{I} = \text{supp}(\chi_{\mathcal{I}}) \subseteq \text{supp}(\chi_{\mathcal{L}}) = \mathcal{L}$. By assumption, we have $\mathcal{J} = \mathcal{I}$. Therefore, $\mathcal{I}$ is a maximal essential ideal of $\mathcal{S}$.

(3) Suppose that $\mathcal{I}$ is a prime essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Thus by Theorem 3.2 $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is an essential IF ideal of $\mathcal{S}$. Let $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$. If $\mathfrak{ef} \in \mathcal{I}$, then $\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{f} \in \mathcal{I}$. Thus, $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \vee \delta_{\chi_{\mathcal{I}}}(\mathfrak{f}) = 1 \geq \delta_{\chi_{\mathcal{I}}}(\mathfrak{ef})$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \wedge \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{f}) = 0 \leq \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{ef})$.

If $\mathfrak{ef} \notin \mathcal{I}$, then $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \vee \delta_{\chi_{\mathcal{I}}}(\mathfrak{f}) \geq \delta_{\chi_{\mathcal{I}}}(\mathfrak{ef})$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \wedge \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{f}) \leq \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{ef})$.

Thus, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a prime essential IF ideal of $\mathcal{S}$.

Conversely, suppose that $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a prime essential IF ideal of $\mathcal{S}$. Then $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is an essential IF ideal. Thus by Theorem 3.2, $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Let $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$. If $\mathfrak{ef} \in \mathcal{I}$, then $\delta_{\chi_{\mathcal{I}}}(\mathfrak{ef}) = 1$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{ef}) = 0$. By assumption, $\delta_{\chi_{\mathcal{I}}}(\mathfrak{ef}) \leq \delta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \vee \delta_{\chi_{\mathcal{I}}}(\mathfrak{f})$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{ef}) \geq \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \wedge \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{f})$. Thus, $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \vee \delta_{\chi_{\mathcal{I}}}(\mathfrak{f}) = 1$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \wedge \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{f}) = 0$ so, $\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{f} \in \mathcal{I}$. Hence, $\mathcal{I}$ is a prime essential ideal of $\mathcal{S}$.

(4) Suppose that $\mathcal{I}$ is a semiprime essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Thus by Theorem 3.2, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is an essential IF ideal of $\mathcal{S}$. Let $\mathfrak{e} \in \mathcal{S}$.

If $\mathfrak{e}^2 \in \mathcal{I}$, then $\mathfrak{e} \in \mathcal{I}$ Thus $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}) = \delta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) = 1$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}) = \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) = 0$.

Hence $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) \leq \delta_{\chi_{\mathcal{I}}}(\mathfrak{e})$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) \geq \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e})$.

If $\mathfrak{e}^2 \notin \mathcal{I}$, then $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) = 0 \leq \delta_{\chi_{\mathcal{I}}}(\mathfrak{e})$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) = 1 \geq \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e})$.

Thus, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a semiprime essential IF ideal of $\mathcal{S}$.

Conversely, suppose that $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a semiprime essential IF ideal of $\mathcal{S}$. Then $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is an essential IF ideal of $\mathcal{S}$. Thus by Theorem 3.2, $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Let $\mathfrak{e} \in \mathcal{S}$ with $\mathfrak{e}^2 \in \mathcal{I}$. Then $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) = 1$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) = 0$. By assumption, $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) \leq \delta_{\chi_{\mathcal{I}}}(\mathfrak{e})$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}^2) \geq \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e})$. Thus, $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}) = 1$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}) = 0$ so $\mathfrak{e} \in \mathcal{I}$. Hence, $\mathcal{I}$ is a semiprime essential ideal of $\mathcal{S}$.

■

IV. 0-Essential Intuitionistic Fuzzy Ideals in Semigroups

In this section, we let $\mathcal{S}$ be a semigroup with zero. begin we review the definition 0-essential ideal of $\mathcal{S}$ as follows:

Definition 4.1. *[11] A nonzero ideal $\mathcal{I}$ of a semigroup with zero $\mathcal{S}$ is called a* 0-essential ideal *of $\mathcal{S}$ if $\mathcal{I} \cap \mathcal{J} \neq \{0\}$ for every nonzero ideal of $\mathcal{J}$ of $\mathcal{S}$.*

Example 4.2. *[11] Let $(\mathbb{Z}_{12}, +)$ be semigroup. Then $\{0, 2, 4, 6, 8, 10\}$ and $\mathbb{Z}_{12}$ are 0-essential ideal of $\mathbb{Z}_{12}$.*

Definition 4.3. *An IF ideal $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ of a semigroup with zero $\mathcal{S}$ is called a* nontrivial IF ideal *of $\mathcal{S}$ if there exists a nonzero element $\mathfrak{e} \in \mathcal{S}$ such that $\delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$.*

Next, we define 0-essential intuitionistic fuzzy ideals and study the properties of 0-essential intuitionistic fuzzy ideals in semigroups.

Definition 4.4. *A* 0-essential IF ideal *$\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ of a semigroup with zero $\mathcal{S}$ if $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a nonzero IF ideal of $\mathcal{S}$ and supp$(\mathfrak{f} \wedge \mathcal{J}) \neq \{0\}$ for every nonzero IF ideal $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ of $\mathcal{S}$.*

Theorem 4.5. *Let $\mathcal{I}$ be a nonzero ideal of a semigroup with zero $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a 0-essential IF ideal of $\mathcal{S}$.*

Proof: Suppose that $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$ and let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be a nontrivial IF ideal of $\mathcal{S}$. Then by Theorem 2.13, $\text{supp}(\mathfrak{f})$ is a nonzero ideal of $\mathcal{S}$. Since $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$ we have $\mathcal{I}$ is a nonzero ideal of $\mathcal{S}$. Thus $\mathcal{I} \cap \text{supp}(\mathfrak{f}) \neq \{0\}$. So there exists $\mathfrak{e} \in \mathcal{I} \cap \text{supp}(\mathfrak{f})$. Since $\mathcal{I}$ is a nonzero ideal of $\mathcal{S}$ we have $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a nonzero IF ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ we have $\text{supp}(\chi_{\mathcal{I}} \wedge \mathfrak{f})(\mathfrak{e}) \neq 0$. Thus, $\delta_{\chi_{\mathcal{I}}} \wedge \delta_{\mathfrak{f}} \neq 0$ and $\vartheta_{\chi_{\mathcal{I}}} \vee \vartheta_{\mathfrak{f}} \neq 0$. Therefore, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a 0-essential IF ideal of $\mathcal{S}$.

Conversely, assume that $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a 0-essential IF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be a nonzero ideal of $\mathcal{S}$. Then $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is a nonzero IF ideal of $\mathcal{S}$. Sicne $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a 0-essential IF ideal of $\mathcal{S}$ we have $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a nontrivial IF ideal of $\mathcal{S}$. Thus, $\text{supp}(\chi_{\mathcal{I}} \wedge \chi_{\mathcal{J}}) \neq \{0\}$. So by Theorem 2.12, $\delta_{\chi_{\mathcal{I} \cap \mathcal{J}}} \neq 0$ and $\vartheta_{\chi_{\mathcal{I} \cup \mathcal{J}}} \neq 0$. Hence, $\mathcal{I} \cap \mathcal{J} \neq \{0\}$. Therefore $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. ■

Theorem 4.6. *Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be a nonzero IF ideal of a semigroup with zero $\mathcal{S}$. Then $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a 0-essential IF ideal of $\mathcal{S}$ if and only if* $\text{supp}(\mathfrak{f})$ *is a 0-essential ideal of $\mathcal{S}$.*

Proof: Assume that $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a 0-essential IF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be a nontrivial ideal of $\mathcal{S}$. Then by Theorem 2.11, $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is a nonzero IF ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a 0-essential IF ideal of $\mathcal{S}$ we have $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a nonzero IF ideal of $\mathcal{S}$. Thus $\delta_{\mathfrak{f}} \wedge \delta_{\chi_{\mathcal{J}}} \neq 0$ and $\vartheta_{\mathfrak{f}} \vee \vartheta_{\chi_{\mathcal{J}}} \neq 0$. So there exists a nonzero element $\mathfrak{e} \in \mathcal{S}$ such that $(\delta_{\mathfrak{f}} \wedge \delta_{\chi_{\mathcal{J}}})(\mathfrak{e}) \neq 0$ and $(\vartheta_{\mathfrak{f}} \vee \vartheta_{\chi_{\mathcal{J}}})(\mathfrak{e}) \neq 0$. It implies that $\delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\delta_{\chi_{\mathcal{J}}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\vartheta_{\chi_{\mathcal{J}}}(\mathfrak{e}) \neq 0$. Hence,

$\mathfrak{e} \in \text{supp}(\mathfrak{f}) \cap \mathcal{J}$ so $\text{supp}(\mathfrak{f}) \cap \mathcal{J} \neq \{0\}$. Therefore $\text{supp}(\mathfrak{f})$ is a 0-essential ideal of $\mathcal{S}$.

Conversely, assume that $\text{supp}(\mathfrak{f})$ is a 0-essential ideal of $\mathcal{S}$ and let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be a nonzero IF ideal of $\mathcal{S}$. Then by Theorem 2.13 $\text{supp}(\mathcal{J})$ is a nontrivial zero ideal of $\mathcal{S}$. Since $\text{supp}(\mathfrak{f})$ is a 0-essential ideal of $\mathcal{S}$ we have $\text{supp}(\mathfrak{f})$ is a nonzero ideal of $\mathcal{S}$. Thus $\text{supp}(\mathfrak{f}) \cap \text{supp}(\mathcal{J}) \neq \{0\}$. So there exists $\mathfrak{e} \in \text{supp}(\mathfrak{f}) \cap \text{supp}(\mathcal{J})$, this implies that $(\delta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $(\delta_{\mathcal{J}}(\mathfrak{e}) \neq 0$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}) \neq 0$, $\vartheta_{\mathcal{J}}(\mathfrak{e}) \neq 0$ for all $\mathfrak{e} \in \mathcal{S}$. Hence, $(\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}})(\mathfrak{e}) \neq 0$ and $(\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}})(\mathfrak{e}) \neq 0$ for all $\mathfrak{e} \in \mathcal{S}$. Therefore, $\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}} \neq 0$ and $\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}} \neq 0$. We conclude that $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a 0-essential IF ideal of $\mathcal{S}$. ■

Theorem 4.7. *Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ be a 0-essential IF ideal of a semigroup $\mathcal{S}$. If $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is an IF ideal of $\mathcal{S}$ such that $\delta_{\mathfrak{f}} \leq \delta_{\mathcal{J}}$ and $\vartheta_{\mathfrak{f}} \geq \vartheta_{\mathcal{J}}$, then $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is also a 0-essential IF ideal of $\mathcal{S}$.*

Proof: Let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is an IF ideal of $\mathcal{S}$ such that $\delta_{\mathfrak{f}} \leq \delta_{\mathcal{J}}$ and $\vartheta_{\mathfrak{f}} \geq \vartheta_{\mathcal{J}}$ and
let $\mathcal{K} = (\delta_{\mathcal{K}}, \vartheta_{\mathcal{K}})$ be any nonzero IF ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is a 0-essential IF ideal of $\mathcal{S}$ we have $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of $\mathcal{S}$. Thus $\text{supp}(\mathcal{J} \wedge \mathcal{K}) \neq 0$. So $\delta_{\mathcal{J}} \wedge \delta_{\mathcal{K}} \neq 0$ and $\vartheta_{\mathcal{J}} \vee \vartheta_{\mathcal{K}} \neq 0$. Hence $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is a 0-essential IF ideal of $\mathcal{S}$. ■

Theorem 4.8. *Let $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be 0-essential IF ideals of a semigroup $\mathcal{S}$. Then $\mathfrak{f} \cup \mathcal{J}$ and $\mathfrak{f} \cap \mathcal{J}$ are 0-essential IF ideals of $\mathcal{S}$.*

Proof: By Theorem 4.7, we have $\mathfrak{f} \cup \mathcal{J}$ is a 0-essential IF ideal of $\mathcal{S}$.

Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ are 0-essential IF ideals of $\mathcal{S}$ we have
$\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ are IF ideals of $\mathcal{S}$. Thus $\mathfrak{f} \cap \mathcal{J}$ is an IF ideal of $\mathcal{S}$.
Let $\mathcal{K} = (\delta_{\mathcal{K}}, \vartheta_{\mathcal{K}})$ be a nontrival IF ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ is an IF ideal of S we have $\text{supp}(\mathfrak{f})$ is an ideal of $\mathcal{S}$. Then $\text{supp}(\mathfrak{f} \wedge \mathcal{K}) = \{0\}$. Thus there exists $\mathfrak{e} \in \mathcal{S}$ such that $(\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{K}})(\mathfrak{e}) \neq 0$ and $(\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{K}})(\mathfrak{e}) \neq 0$. Since $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is a 0-essential IF ideal of $\mathcal{S}$ we have $\text{supp}(\mathcal{J})$ is a 0-essential IF ideal of $\mathcal{S}$. Thus $\text{supp}(\mathcal{J} \wedge \mathcal{K}) \neq \{0\}$. So there exists a nonzero element $\mathfrak{f} \in \text{supp}(\mathcal{J} \wedge \mathcal{K})(\mathfrak{f}) \neq 0$ implies $\delta_{\mathcal{J}}(\mathfrak{f}) \neq 0$ and $\vartheta_{\mathcal{J}}(\mathfrak{f}) \neq 0$. Since $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ and $\mathcal{K} = (\delta_{\mathcal{K}}, \vartheta_{\mathcal{K}})$ are IF ideals of $\mathcal{S}$ we have $\delta_{\mathfrak{f}}(\mathfrak{f}) \geq \delta_{\mathfrak{f}}(\mathfrak{e})$, $\delta_{\mathcal{K}}(\mathfrak{f}) \geq \delta_{\mathcal{K}}(\mathfrak{e})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{f}) \leq \vartheta_{\mathfrak{f}}(\mathfrak{e})$, $\vartheta_{\mathcal{K}}(\mathfrak{f}) \leq \vartheta_{\mathcal{K}}(\mathfrak{e})$. So $((\delta_{\mathfrak{f}} \wedge \delta_{\mathcal{J}}) \wedge \delta_{\mathcal{K}})(\mathfrak{f}) \neq 0$ and $((\vartheta_{\mathfrak{f}} \vee \vartheta_{\mathcal{J}}) \vee \vartheta_{\mathcal{K}})(\mathfrak{f}) \neq 0$. Thus $\text{supp}((\mathfrak{f} \wedge \mathcal{J}) \wedge \mathcal{K}) \neq \{0\}$. Therefore $\mathfrak{f} \cap \mathcal{J}$ is a 0-essential IF ideals of $\mathcal{S}$. ■

Definition 4.9. *[11] A 0-essential ideal $\mathcal{I}$ of a semigroup with zero $\mathcal{S}$ is called*

(1) *a* minimal *if for every 0-essential ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$, we have $\mathcal{J} = \mathcal{I}$,*
(2) *a* maximal *if for every 0-essential ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{J}$, we have $\mathcal{J} = \mathcal{I}$,*
(3) *a* prime *if $\mathfrak{e}\mathfrak{f} \in \mathcal{I}$ implies $\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{f} \in \mathcal{I}$,*
(4) *a* semiprime *if $\mathfrak{e}^2 \in \mathcal{I}$ implies $\mathfrak{e} \in \mathcal{I}$, for all $\rceil, \{ \in \mathcal{S}$.*

Example 4.10. *Let $(\mathbb{Z}_{12}, +)$ be a semigroup with zero. Then $\{0, 2, 4, 6, 8, 10\}$ is a minimal 0-essential ideal of S.*

Definition 4.11. *A 0-essential IF ideal $\mathfrak{f} = (\delta_{\mathfrak{f}}, \vartheta_{\mathfrak{f}})$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every 0-essential IF ideal of $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ of $\mathcal{S}$ such that $\delta_{\mathfrak{f}} \leq \delta_{\mathcal{J}}$ and $\vartheta_{\mathfrak{f}} \geq \vartheta_{\mathcal{J}}$, we have* $\text{supp}(\mathfrak{f}) = \text{supp}(\mathcal{J})$,
(2) *a* maximal *if for every 0-essential IF ideal of $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ of $\mathcal{S}$ such that $\delta_{\mathcal{J}} \geq \delta_{\mathfrak{f}}$ and $\vartheta_{\mathcal{J}} \leq \vartheta_{\mathfrak{f}}$, we have* $\text{supp}(\mathfrak{f}) = \text{supp}(\mathcal{J})$,
(3) *a* prime *if $\delta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \leq \delta_{\mathfrak{f}}(\mathfrak{e}) \vee \delta_{\mathfrak{f}}(\mathfrak{f})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}\mathfrak{f}) \geq \vartheta_{\mathfrak{f}}(\mathfrak{e}) \wedge \vartheta_{\mathfrak{f}}(\mathfrak{f})$,*
(4) *a* semiprime *if $\delta_{\mathfrak{f}}(\mathfrak{e}^2) \leq \delta_{\mathfrak{f}}(\mathfrak{e})$ and $\vartheta_{\mathfrak{f}}(\mathfrak{e}^2) \geq \vartheta_{\mathfrak{f}}(\mathfrak{e})$, for all $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$.*

Theorem 4.12. *Let $\mathcal{I}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds.*

(1) *$\mathcal{I}$ is a minimal 0-essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{J}}})$ is a minimal 0-essential ideal IF ideal of $\mathcal{S}$,*
(2) *$\mathcal{I}$ is a prime 0-essential ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a prime 0-essential ideal IF ideal of $\mathcal{S}$,*
(3) *$\mathcal{I}$ is a semiprime 0-essential ideal of S if and only if $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a semiprime essential IF ideal of $\mathcal{S}$.*

Proof:

(1) Suppose that $\mathcal{I}$ is a minimal 0-essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. By Theorem 4.5, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a 0-essential IF ideal of $\mathcal{S}$. Let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be a 0-essential IF ideal of $\mathcal{S}$ such that $\delta_{\mathcal{J}} \leq \delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\mathcal{J}} \geq \vartheta_{\chi_{\mathcal{I}}}$. Then $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}})$. Then $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}}) = \mathcal{I}$. Thus $\text{supp}(\mathcal{J}) \subseteq \mathcal{I}$. Since $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is a 0-essential IF ideal of $\mathcal{S}$ we have $\text{supp}(\mathcal{J})$ is a 0-essential ideal of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{J}) = \mathcal{I} = \text{supp}(\chi_{\mathcal{I}})$. Hence, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a minimal 0-essential IF ideal of $\mathcal{S}$.
Conversely, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a minimal 0-essential IF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be an essential ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$. Then $\mathcal{B}$ is an ideal of $\mathcal{S}$. Thus by Theorem 4.5, $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is a 0-essential IF ideal of $\mathcal{S}$ such that $\delta_{\chi_{\mathcal{J}}} \geq \delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\chi_{\mathcal{J}}} \leq \vartheta_{\chi_{\mathcal{I}}}$. So $\mathcal{J} = \text{supp}(\chi_{\mathcal{J}}) \subseteq \text{supp}(\chi_{\mathcal{I}}) = \mathcal{I}$. By assumption, we have $\mathcal{J} = \mathcal{I}$. Therefore $\mathcal{I}$ is a minimal 0-essential ideal of $\mathcal{S}$.
(2) Suppose that $\mathcal{I}$ is a maximal 0-essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. By Theorem 3.2, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a 0-essential IF ideal of $\mathcal{S}$.
Let $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ be a 0-essential IF ideal of $\mathcal{S}$ such that $\delta_{\mathcal{J}} \geq \delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\mathcal{J}} \geq \vartheta_{\chi_{\mathcal{I}}}$. Then $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}})$. Thus, $\text{supp}(\mathcal{J}) \subseteq \text{supp}(\chi_{\mathcal{I}}) = \mathcal{I}$. Hence, $\text{supp}(\mathcal{J}) \subseteq \mathcal{I}$. Since $\mathcal{J} = (\delta_{\mathcal{J}}, \vartheta_{\mathcal{J}})$ is a 0-essential IF ideal of $\mathcal{S}$ we have $\text{supp}(\mathcal{J})$ is a 0-essential ideal of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{J}) = \mathcal{I} = \text{supp}(\chi_{\mathcal{I}})$. Hence, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a maximal 0-essential IF ideal of $\mathcal{S}$.
Conversely, $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a maximal 0-essential IF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be a 0-essential ideal of $\mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{J}$. Then $\mathcal{J}$ is an ideal of $\mathcal{S}$. Thus by Theorem 3.2, $\chi_{\mathcal{J}} = (\delta_{\chi_{\mathcal{J}}}, \vartheta_{\chi_{\mathcal{J}}})$ is a 0-essential IF ideal of $\mathcal{S}$ such that $\delta_{\chi_{\mathcal{J}}} \geq \delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\chi_{\mathcal{J}}} \leq \vartheta_{\chi_{\mathcal{I}}}$. So, $\mathcal{I} = \text{supp}(\chi_{\mathcal{I}}) \subseteq \text{supp}(\chi_{\mathcal{J}}) = \mathcal{J}$. By assumption, we have $\mathcal{J} = \mathcal{I}$. Therefore, $\mathcal{I}$ is a maximal 0-essential ideal of $\mathcal{S}$.
(3) Suppose that $\mathcal{I}$ is a prime 0-essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Thus by Theorem 4.5 $\chi_{\mathcal{I}} = (\delta_{\chi_{\mathcal{I}}}, \vartheta_{\chi_{\mathcal{I}}})$ is a 0-essential IF ideal of $\mathcal{S}$. Let $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$. If $\mathfrak{e}\mathfrak{f} \in \mathcal{I}$, then $\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{f} \in \mathcal{I}$. Thus $\delta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \vee \delta_{\chi_{\mathcal{I}}}(\mathfrak{f}) = 1 \geq \delta_{\chi_{\mathcal{I}}}(\mathfrak{e}\mathfrak{f})$ and $\vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}) \wedge \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{f}) = 0 \leq \vartheta_{\chi_{\mathcal{I}}}(\mathfrak{e}\mathfrak{f})$.

If $\mathfrak{ef} \notin \mathcal{I}$, then $\delta_{\chi_\mathcal{I}}(\mathfrak{e}) \vee \delta_{\chi_\mathcal{I}}(\mathfrak{f}) \geq \delta_{\chi_\mathcal{I}}(\mathfrak{ef})$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}) \wedge \vartheta_{\chi_\mathcal{I}}(\mathfrak{f}) \leq \vartheta_{\chi_\mathcal{I}}(\mathfrak{ef})$.
Thus $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is a prime 0-essential IF ideal of $\mathcal{S}$.
Conversely, suppose that $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is a prime 0-essential IF ideal of $\mathcal{S}$. Then $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is an essential IF ideal. Thus by Theorem 4.5, $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Let $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$. If $\mathfrak{ef} \in \mathcal{I}$, then $\delta_{\chi_\mathcal{I}}(\mathfrak{ef}) = 1$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{ef}) = 0$. By assumption, $\delta_{\chi_\mathcal{I}}(\mathfrak{ef}) \leq \delta_{\chi_\mathcal{I}}(\mathfrak{e}) \vee \delta_{\chi_\mathcal{I}}(\mathfrak{f})$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{ef}) \geq \vartheta_{\chi_\mathcal{I}}(\mathfrak{e}) \wedge \vartheta_{\chi_\mathcal{I}}(\mathfrak{f})$. Thus $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}) \wedge \vartheta_{\chi_\mathcal{I}}(\mathfrak{f}) = 1$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}) \wedge \vartheta_{\chi_\mathcal{I}}(\mathfrak{f}) = 0$ so $\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{f} \in \mathcal{I}$. Hence $\mathcal{I}$ is a prime 0-essential ideal of $\mathcal{S}$.

(4) Suppose that $\mathcal{I}$ is a semiprime 0-essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Thus by Theorem 4.5, $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is a 0-essential IF ideal of $\mathcal{S}$. Let $\mathfrak{e} \in \mathcal{S}$.
If $\mathfrak{e}^2 \in \mathcal{I}$, then $\mathfrak{e} \in \mathcal{I}$ Thus $\delta_{\chi_\mathcal{I}}(\mathfrak{e}) = \delta_{\chi_\mathcal{I}}(\mathfrak{e}^2) = 1$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}) = \vartheta_{\chi_\mathcal{I}}(\mathfrak{e}^2) = 0$.
Hence $\delta_{\chi_\mathcal{I}}(\mathfrak{e}^2) \leq \delta_{\chi_\mathcal{I}}(\mathfrak{e})$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}^2) \geq \vartheta_{\chi_\mathcal{I}}(\mathfrak{e})$.
If $\mathfrak{e}^2 \notin \mathcal{I}$, then $\delta_{\chi_\mathcal{I}}(\mathfrak{e}^2) = 0 \leq \delta_{\chi_\mathcal{I}}(\mathfrak{e})$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}^2) = 1 \geq \vartheta_{\chi_\mathcal{I}}(\mathfrak{e})$.
Thus $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is a semiprime 0-essential IF ideal of $\mathcal{S}$.
Conversely, suppose that $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is a semiprime 0-essential IF ideal of $\mathcal{S}$. Then $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is a 0-essential IF ideal of $\mathcal{S}$. Thus by Theorem 4.5, $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Let $\mathfrak{e} \in \mathcal{S}$ with $\mathfrak{e}^2 \in \mathcal{I}$. Then $\delta_{\chi_\mathcal{I}}(\mathfrak{e}^2) = 1$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}^2) = 0$. By assumption, $\delta_{\chi_\mathcal{I}}(\mathfrak{e}^2) \leq \delta_{\chi_\mathcal{I}}(\mathfrak{e})$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}^2) \geq \vartheta_{\chi_\mathcal{I}}(\mathfrak{e}^2)$. Thus $\delta_{\chi_\mathcal{I}}(\mathfrak{e}) = 1$ and $\vartheta_{\chi_\mathcal{I}}(\mathfrak{e}) = 0$ so $\mathfrak{e} \in \mathcal{I}$. Hence $\mathcal{I}$ is a semiprime 0-essential ideal of $\mathcal{S}$.

■

5. Essential (m, n)-ideals and essential intuitionistic fuzzy (m, n)-ideals

In this section, we will study concepts of essential IF (m, n)-ideals in a semigroup and properties of those.

Before, we defined the definition of essential (m, n)-ideals and intuitionistic fuzzy (m, n)-ideals in semigroups.

Definition 5.1. *[19] An* essential (m, n)-ideal $\mathcal{I}$ *of a semigroup* $\mathcal{S}$ *if* $\mathcal{I}$ *is an* (m, n)*-ideal of* $\mathcal{S}$ *and* $\mathcal{I} \cap \mathcal{J} \neq \emptyset$ *for every* (m, n)*-ideal* $\mathcal{J}$ *of* $\mathcal{S}$.

Definition 5.2. *An IF set* $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ *of a semigroups* $\mathcal{S}$ *is said to be an IF* (m, n)*-ideal of* $\mathcal{S}$ *if*

$$\delta_\mathfrak{f}(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \geq \delta_\mathfrak{f}(u_1) \wedge \delta_\mathfrak{f}(u_2) \wedge \ldots \wedge \delta_\mathfrak{f}(u_m) \wedge \delta_\mathfrak{f}(v_1) \wedge \delta_\mathfrak{f}(v_2) \wedge \ldots \wedge \delta_\mathfrak{f}(v_n)$$

and

$$\vartheta_\mathfrak{f}(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \leq \vartheta_\mathfrak{f}(u_1) \vee \vartheta_\mathfrak{f}(u_2) \vee \ldots \vee \vartheta_\mathfrak{f}(u_m) \vee \vartheta_\mathfrak{f}(v_1) \vee \vartheta_\mathfrak{f}(v_2) \vee \ldots \vee \vartheta_\mathfrak{f}(v_n).$$

for all $u_1, u_2, ..., u_m, v_1, v_2, ..., v_n, z \in \mathcal{S}$ *and* $m, n \in \mathbb{N}$.

Example 5.3. *(1) We have that a semigroup* $\mathcal{S}$ *is an essential IF* (m, n)*-ideal of* S *for all* $m, n \in \mathbb{N}$.

(2) Let $\mathcal{S}$ *be a semigroup with zero. Then every ideal of* $\mathcal{S}$ *is an essential* (m, n)*-ideal of* $\mathcal{S}$ *for all* $m, n \in \mathbb{N}$.

Next, we give definition essential IF (m, n)-ideals in a semigroup and prove some properties of it.

Definition 5.4. *An* essential IF (m, n)-ideal $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ *of a semigroup* $\mathcal{S}$ *if* $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ *is a nonzero fuzzy* (m, n)*-ideal of* $\mathcal{S}$ *and* $\delta_\mathfrak{f} \wedge \delta_\mathcal{J} \neq 0$, $\vartheta_\mathfrak{f} \vee \vartheta_\mathcal{J} \neq 0$ *for every nonzero IF* (m, n)*-ideal* $\mathcal{J} = (\delta_\mathcal{J}, \vartheta_\mathcal{J})$ *of* $\mathcal{S}$.

Theorem 5.5. *Let* $\mathcal{I}$ *be an* (m, n)*-ideal of a semigroup* $\mathcal{S}$. *Then* $\mathcal{I}$ *is an essential* (m, n)*-ideal of* $\mathcal{S}$ *if and only if* $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ *is an essential fuzzy* (m, n)*-ideal of* $\mathcal{S}$.

Proof: Suppose that $\mathcal{I}$ is an essential (m, n)-ideal of $\mathcal{S}$ and let $\mathcal{J} = (\delta_\mathcal{J}, \vartheta_\mathcal{J})$ be any nonzero IF (m, n)-ideal of $\mathcal{S}$. Then $\mathrm{supp}(\mathcal{J})$ is an (m, n)-ideal of $\mathcal{S}$. By assumption we have $\mathcal{I}$ is an (m, n)-ideal of $\mathcal{S}$. Then $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is an (m, n)-ideal of $\mathcal{S}$. Thus $\mathcal{I} \cap \mathrm{supp}(\mathcal{J}) \neq \emptyset$. So there exist $u_1, u_2, ..., u_m, v_1, v_2, ..., v_n, z \in I \cap \mathrm{supp}(\mathcal{J})$. It implies that

$$(\delta_{\chi_\mathcal{I}} \wedge \delta_\mathcal{J})(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0$$

and

$$(\vartheta_{\chi_\mathcal{I}} \vee \vartheta_\mathcal{J})(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0.$$

Hence, $\delta_{\chi_\mathcal{I}} \wedge \delta_\mathcal{J} \neq 0$ and $\vartheta_{\chi_\mathcal{I}} \vee \vartheta_\mathcal{J} \neq 0$. Therefore $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is an essential IF (m, n)-ideal of $\mathcal{S}$.

Conversely, assume that $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is an essential IF (m, n)-ideal of $\mathcal{S}$ and let $\mathcal{J}$ be any (m, n)-ideal of $\mathcal{S}$. Then $\chi_\mathcal{J} = (\delta_{\chi_\mathcal{J}}, \vartheta_{\chi_\mathcal{J}})$ is a nonzero IF (m, n)-ideal of $\mathcal{S}$. Since $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is an essential IF (m, n)-ideal of $\mathcal{S}$ we have $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is an IF (m, n)-ideal of S. Thus, $\delta_{\chi_{\mathcal{I} \cap \mathcal{J}}} \neq 0$ and $\vartheta_{\chi_{\mathcal{I} \cup \mathcal{J}}} \neq 0$. Hence, $\mathcal{I} \cap \mathcal{J} \neq \emptyset$. Therefore $\mathcal{I}$ is an essential (m, n)-ideal of $\mathcal{S}$. ■

Theorem 5.6. *Let* $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ *be a nonzero IF* (m, n)*-ideal of a semigroup* $\mathcal{S}$. *Then* $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ *is an essential IF* (m, n)*-ideal of* $\mathcal{S}$ *if and only if* $\sup(\mathfrak{f})$ *is an essential* (m, n)*-ideal of* $\mathcal{S}$.

Proof: Assume that $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ is an essential IF (m, n)-ideal of $\mathcal{S}$. Then $\sup(\mathfrak{f})$ is an (m, n)-ideal of $\mathcal{S}$. Let $\mathcal{I}$ be any (m, n)-ideal of $\mathcal{S}$. Then by Theorem 5.5, $\chi_\mathcal{I} = (\delta_{\chi_\mathcal{I}}, \vartheta_{\chi_\mathcal{I}})$ is an (m, n)-ideal of $\mathcal{S}$. Since $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ is an essential IF (m, n)-ideal of $\mathcal{S}$, we have $\mathfrak{f} = (\delta_\mathfrak{f}, \vartheta_\mathfrak{f})$ is an IF (m, n)-ideal of $\mathcal{S}$. Thus, $\delta_\mathfrak{f} \wedge \delta_{\chi_\mathcal{I}} \neq \emptyset$ and $\vartheta_\mathfrak{f} \wedge \vartheta_{\chi_\mathcal{I}} \neq \emptyset$. So there exist $u_1, u_2, ..., u_m, v_1, v_2, ..., v_n, z \in \mathcal{S}$ such that

$$(\delta_\mathfrak{f} \wedge \delta_{\chi_\mathcal{I}})(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0$$

and

$$(\vartheta_\mathfrak{f} \vee \vartheta_{\chi_\mathcal{I}})(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0.$$

It implies that

$$\delta_\mathfrak{f}(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0$$
$$\vartheta_\mathfrak{f}(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0$$

and

$$\delta_{\chi_\mathcal{I}}(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0,$$

$$\vartheta_{\chi_\mathcal{I}}(u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n) \neq 0.$$

Hence, $u_1 u_2 \cdots u_m z v_1 v_2 \cdots v_n \in \sup(\mathfrak{f}) \cap \mathcal{I}$ so, $\sup(\mathfrak{f}) \cap \mathcal{I} \neq \emptyset$. It implies that $\sup(\mathfrak{f})$ is an essential (m, n)-ideal of $\mathcal{S}$.

Conversely, assume that $\sup(\mathfrak{f})$ is an essential (m, n)-ideal of $\mathcal{S}$ and let $\mathcal{J} = (\delta_\mathcal{J}, \vartheta_\mathcal{J})$ be a nonzero IF (m, n)-ideal of $\mathcal{S}$. Then $\sup(\mathcal{J})$ is an (m, n)-ideal of $\mathcal{S}$. Thus, $\sup(\mathfrak{f}) \cap \sup(\mathcal{J}) \neq \emptyset$. So there exist $u_1 u_2 \ldots u_m z v_1 v_2 \ldots v_n \in \sup(\mathfrak{f}) \cap \sup(\mathcal{J})$. This implies that

$$\delta_{\mathfrak{f}}(u_1u_2\cdots u_mzv_1v_2\cdots v_n)\neq 0,$$

$$\vartheta_{\mathfrak{f}}(u_1u_2\cdots u_mzv_1v_2\cdots v_n)\neq 0$$

and

$$\delta_{\mathcal{J}}(u_1u_2\cdots u_mzv_1v_2\cdots v_n)\neq 0,$$

$$\vartheta_{\mathcal{J}}(u_1u_2\cdots u_mzv_1v_2\cdots v_n)\neq 0$$

for all $u_1,u_2,\ldots,u_m.z.v_1.v_2.\ldots,v_n\in\mathcal{S}$.

Hence, $(\delta_{\mathfrak{f}}\wedge\delta_{\mathcal{J}})(u_1u_2\cdots u_mzv_1v_2\cdots v_n)\neq 0$ and $(\vartheta_{\mathfrak{f}}\vee\vartheta_{\mathcal{J}})(u_1u_2\cdots u_mzv_1v_2\cdots v_n)\neq 0$ for all $u_1,u_2,\ldots,u_m.z.v_1.v_2.\ldots,v_n\in\mathcal{S}$. Therefore, $\delta_{\mathfrak{f}}\wedge\delta_{\mathcal{J}}\neq 0$ and $\vartheta_{\mathfrak{f}}\vee\vartheta_{\mathcal{J}}\neq 0$. We conclude that $\mathfrak{f}=(\delta_{\mathfrak{f}},\vartheta_{\mathfrak{f}})$ is an essential IF (m,n)-ideal of $\mathcal{S}$. ∎

Definition 5.7. *An essential IF (m,n)-ideal $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every essential (m,n)-ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J}\subseteq\mathcal{I}$, we have $\mathcal{J}=\mathcal{I}$,*

(2) *a* maximal *if for every essential (m,n)-ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{I}\subseteq\mathcal{J}$, we have $\mathcal{J}=\mathcal{I}$,*

Definition 5.8. *An essential IF (m,n)-ideal $\mathfrak{f}=(\delta_{\mathfrak{f}},\vartheta_{\mathfrak{f}})$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every essential IF (m,n)-ideal of $\mathcal{J}=(\delta_{\mathcal{J}},\vartheta_{\mathcal{J}})$ of $\mathcal{S}$ such that $\delta_{\mathcal{J}}\leq\delta_{\mathfrak{f}}$ and $\vartheta_{\mathcal{J}}\geq\vartheta_{\mathfrak{f}}$, we have* $\mathrm{supp}(\mathfrak{f})=\mathrm{supp}(\mathcal{J})$,

(2) *a* maximal *if for every essential IF (m,n)-ideal of $\mathcal{J}=(\delta_{\mathcal{J}},\vartheta_{\mathcal{J}})$ of $\mathcal{S}$ such that $\delta_{\mathcal{J}}\geq\delta_{\mathfrak{f}}$ and $\vartheta_{\mathcal{J}}\leq\vartheta_{\mathfrak{f}}$, we have* $\mathrm{supp}(\mathfrak{f})=\mathrm{supp}(\mathcal{J})$,

Theorem 5.9. *Let $\mathcal{I}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds.*

(1) *$\mathcal{I}$ is a minimal essential (m,n)-ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{J}}})$ is a minimal essential IF (m,n)-ideal of $\mathcal{S}$,*

(2) *$\mathcal{I}$ is a maximal essential (m,n)-ideal of $\mathcal{S}$ if and only if $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{J}}})$ is a maximal essential IF (m,n)-ideal of $\mathcal{S}$,*

Proof:

(1) Suppose that $\mathcal{I}$ is a minimal essential (m,n)-ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. By Theorem 5.5, $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{I}}})$ is an essential IF (m,n)-ideal of $\mathcal{S}$. Let $\mathcal{J}=(\delta_{\mathcal{J}},\vartheta_{\mathcal{J}})$ be an essential IF (m,n)-ideal of $\mathcal{S}$ such that $\delta_{\mathcal{J}}\leq\delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\mathcal{J}}\geq\vartheta_{\chi_{\mathcal{I}}}$. Then $\mathrm{supp}(\mathcal{J})\subseteq\mathrm{supp}(\chi_{\mathcal{I}})$. Thus, $\mathrm{supp}(\mathcal{J})\subseteq\mathrm{supp}(\chi_{\mathcal{I}})=\mathcal{I}$. Hence, $\mathrm{supp}(\mathcal{J})\subseteq\mathcal{I}$. Since $\mathcal{J}=(\delta_{\mathcal{J}},\vartheta_{\mathcal{J}})$ is an essential IF (m,n)-ideal of $\mathcal{S}$ we have $\mathrm{supp}(\mathcal{J})$ is an essential (m,n)-ideal of $\mathcal{S}$. By assumption, $\mathrm{supp}(\mathcal{J})=\mathcal{I}=\mathrm{supp}(\chi_{\mathcal{I}})$. Hence, $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{I}}})$ is a minimal essential IF (m,n)-ideal of $\mathcal{S}$.

Conversely, $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{I}}})$ is a minimal essential IF (m,n)-ideal of $\mathcal{S}$ and let $\mathcal{J}$ be an essential $(m,n)-$ideal of $\mathcal{S}$ such that $\mathcal{J}\subseteq\mathcal{I}$. Then $\mathcal{J}$ is an (m,n)-ideal of $\mathcal{S}$. Thus by Theorem 5.5, $\chi_{\mathcal{J}}=(\delta_{\chi_{\mathcal{J}}},\vartheta_{\chi_{\mathcal{J}}})$ is an essential IF (m,n)-ideal of $\mathcal{S}$ such that $\delta_{\chi_{\mathcal{J}}}\geq\delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\chi_{\mathcal{J}}}\leq\vartheta_{\chi_{\mathcal{I}}}$. So, $\mathcal{J}=\mathrm{supp}(\chi_{\mathcal{J}})\subseteq\mathrm{supp}(\chi_{\mathcal{I}})=\mathcal{I}$. Hence, $\mathcal{J}=\mathcal{I}$. By assumption, we have $\mathcal{J}\subseteq\mathcal{I}$. Therefore, $\mathcal{I}$ is a minimal essential (m,n)-ideal of $\mathcal{S}$.

(2) Suppose that $\mathcal{I}$ is a maximal essential (m,n)-ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential (m,n)-ideal of $\mathcal{S}$. By Theorem 5.5, $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{I}}})$ is an essential IF (m,n)-ideal of $\mathcal{S}$. Let $\mathcal{J}=(\delta_{\mathcal{J}},\vartheta_{\mathcal{J}})$ be an essential IF (m,n)-ideal of $\mathcal{S}$ such that $\delta_{\mathcal{J}}\geq\delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\mathcal{J}}\geq\vartheta_{\chi_{\mathcal{I}}}$. Then $\mathrm{supp}(\mathcal{J})\subseteq\mathrm{supp}(\chi_{\mathcal{I}})$. Thus, $\mathrm{supp}(\mathcal{J})\subseteq\mathrm{supp}(\chi_{\mathcal{I}})=\mathcal{I}$. Hence, $\mathrm{supp}(\mathcal{J})\subseteq\mathcal{I}$. Since $\mathcal{J}=(\delta_{\mathcal{J}},\vartheta_{\mathcal{J}})$ is an essential IF (m,n)-ideal of $\mathcal{S}$ we have $\mathrm{supp}(\mathcal{J})$ is an essential (m,n)-ideal of $\mathcal{S}$. By assumption, $\mathrm{supp}(\mathcal{J})=\mathcal{I}=\mathrm{supp}(\chi_{\mathcal{I}})$. Hence, $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{I}}})$ is a maximal essential IF (m,n)-ideal of $\mathcal{S}$.

Conversely, $\chi_{\mathcal{I}}=(\delta_{\chi_{\mathcal{I}}},\vartheta_{\chi_{\mathcal{I}}})$ is a maximal essential IF (m,n)-ideal of $\mathcal{S}$ and let $\mathcal{J}$ be an essential ideal of $\mathcal{S}$ such that $\mathcal{I}\subseteq\mathcal{J}$. Then $\mathcal{J}$ is an (m,n)-ideal of $\mathcal{S}$. Thus by Theorem 5.5, $\chi_{\mathcal{J}}=(\delta_{\chi_{\mathcal{J}}},\vartheta_{\chi_{\mathcal{J}}})$ is an essential IF (m,n)-ideal of $\mathcal{S}$ such that $\delta_{\chi_{\mathcal{J}}}\geq\delta_{\chi_{\mathcal{I}}}$ and $\vartheta_{\chi_{\mathcal{J}}}\leq\vartheta_{\chi_{\mathcal{I}}}$. So, $\mathcal{I}=\mathrm{supp}(\chi_{\mathcal{I}})\subseteq\mathrm{supp}(\chi_{\mathcal{J}})=\mathcal{J}$. Hence, $\mathcal{I}\subseteq\mathcal{J}$. By assumption, we have $\mathcal{J}=\mathcal{I}$. Therefore, $\mathcal{I}$ is a maximal essential (m,n)-ideal of $\mathcal{S}$.

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VI. Conclusion

In Section III, we define intuitionistic fuzzy ideals in semigroups. We present that the union and intersection of essential intuitionistic fuzzy ideals are also essential intuitionistic fuzzy ideals of semigroups. Moreover, we prove some relationship between essential ideals and essential intuitionistic fuzzy ideals. In Section IV, we define 0-essential intuitionistic fuzzy ideals in semigroups with zero. In Section V, we define essential intuitionistic fuzzy (m,n)-ideals in semigroups. In the future work, we can discuss essential i-ideals and essential fuzzy i-ideals in n-ary semigroups and algebraic systems.

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