Innovative Approaches to Linear Volterra Partial Integro-Differential Equations: A Laplace Residual Power Series Perspective

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Abstract—**This article presents the modified residual power series approach using Laplace transform, the method is used to solve partial integro differential equations. The basic definitions and theorems related to the method are presented and discussed. Moreover, the steps of the method are utilized and applied to solve various examples.**

Index Terms—**Infinite series, Laplace transform, residual power series, Laplace residual power series; integral equation.**

I. Introduction

THE investigation of linear Volterra partial integro-differential equations is an important area in mathematical analysis, with applications spanning various fields such as physics, engineering, and finance. These equations, which incorporate both differential and integral operators, face some challenges in producing exact and approximate solutions. Over the years, numerous methods have been developed to tackle these equations, with the Laplace residual power series method (LRPSM) emerging as a particularly effective approach.

The LRPSM, a hybrid approach merging Laplace transform and residual power series method, has been shown to be a powerful technique to solve fractional system partial differential equations (FSPDEs). This method allows for the simple calculation of solutions in the form of a power series expansion, as demonstrated by 1. Al-Sawalha et al. [1] in their work on FSPDEs using LRPSM. The reliability of the method and its validity in solving the problems have been tested by solving some examples.

The basic concept of the the LRPSM is Laplace transform, which has been extensively discussed in classical texts such as in [2], [3]. These works provide a comprehensive understanding of the Laplace transform, which is basic fin the application of LRPSM.

El-Ajou et al. in [4] further investigate modern analytic methods to solve various kinds of differential equations, highlighting the versatility of analytic techniques in handling complex mathematical problems. Similarly, Trench [5], Boyce, DiPrima, and Meade [6], and Wazwaz [7] have contributed significantly to the understanding of differential equations and integral equations, respectively, laying the groundwork for advanced solution methods like LRPSM.

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The application of LRPSM is extended to solve various types of Volterra partial integral and integro-differential equations, as evidenced by the works of Fahim et al. [8], Gu and Wu [9], and, Prakasam and Arul Joseph [10]. These studies have employed different numerical and analytical methods, showcasing the diversity of approaches in tackling Volterra equations. Khodabin [11], Brunner [12], Avazzadeh et al. [13], and Feldstein and Sopka [14] have also contributed to the field by developing numerical methods for systems of nonlinear integro-parabolic equations of Volterra type and exploring various aspects of Volterra integral equations.

The basic theorems for power series and Laplace transforms, discussed by Heywood [15], [16], Prudnikov and Marichev [17], and others, provide a theoretical foundation for the LRPSM. Chen et al. [18], Dyke [19], Wintner [20], Sherman, Kerr, and González-Parra [21], Pramesti and Solekhudin [22], Davies [23], Pranajaya and Sugiarto [24], ALshemary [25], Cai and Kou [26], McLachlan [27], and Schiff [28] have further expanded on the applications and theoretical aspects of the Laplace transform and its use in solving differential equations [29], [30].

In summary, the Laplace residual power series method stands out as a robust and efficient technique for solving linear Volterra partial integro-differential equations, building upon the foundational work in the Laplace transform and power series methods. The method's ability to handle complex equations efficiently makes it a valuable tool in the arsenal of mathematicians and engineers alike.

II. Preliminaries

In this section, we present some preliminaries about the Laplace transform, power series, and integral equation, including some properties essential to this article. The following definitions and theorems can be found in [31]

Definition 2.1: Let $g(x,t)$ be a piecewise continuous function on $I\times\ [0,\infty)$ and of exponential order δ. Then, the Laplace transform of the function $g(x,t)$ with respect to x, is denoted by $G\left(s,t\right)$ and defined as follows:

$$G\left(s,t\right)=\mathcal{L}[g(x,t)]=\int_0^\infty e^{-sx}g(x,t)dx,\quad s>0. \tag{1}$$

If the integral (1) converges for any value of x, then the Laplace transform of $g(x,t)$ exists; otherwise, it does not.

The expression for the inverse Laplace transform is formulated as:

$$\mathcal{L}^{-1}\left[G\left(s,t\right)\right]=\ g\left(x,t\right)=\frac{1}{2\pi i}\ \int_{c-i\infty}^{c+i\infty}e^{sx}\ G\left(s,t\right)ds,\quad c\in\mathbb{R}. \tag{2}$$

Theorem 2.1: Assume that $g(x,t)$ and $h(x,t)$ are two continuous functions on $I\times\ [0,\infty)$ of exponential order δ, with their Laplace transforms $G(s,t)$, $H(s,t)$ respectively, μ_1 and μ_2 are two constants, then we have the following properties:

i. $\mathcal{L}\ [\mu_1\ g(x,t)+\mu_2 h(x,t)]=\ \mu_1\ \mathcal{L}[g(x,t)] + \mu_2[h(x,t)]=\mu_1\ G(s,t)+\ \mu_2 H(s,t)$.
ii. $\mathcal{L}[x^n g(x,t)]=(-1)^n\frac{\partial^n G(s,t)}{\partial s^n}, n=1,2,\ldots$.
iii. $\lim\limits_{s\to\infty} sG(s,t)=g(0,t)$.
iv. $\mathcal{L}[\ g(x,t)*h(x,t)]=\ \mathcal{L}[g(x,t)]\ \mathcal{L}[h(x,t)] = G(s,t)\ \ H(s,t)$, where $g(x,t)*h(x,t)=\int_0^x g(x-\tau)\,h(x,t)\,dx$.
v. $\mathcal{L}^{-1}[\mu_1 G(s,t)+\mu_2 H(s,t)]=\mu_1 g(x,t)+\mu_2 h(x,t)$.

The Laplace transform of the partial derivatives with respect to t of the function $g(x,t)$, are given by

i. $\mathcal{L}\left[\frac{\partial^n g(x,t)}{\partial x^n}\right]=s^n G(s,t)-\sum_{j=0}^{n-1}s^{n-j-1}\frac{\partial^j g(0,t)}{\partial x^j}$.
ii. $\mathcal{L}\left[\frac{\partial^m g(x,t)}{\partial t^m}\right]=\frac{\partial^m G(s,t)}{\partial t^m}$.

Definition 2.2: An infinite series of the form:

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n=c_0+c_1(x-x_0)+c_2(x-x_0)^2+\ldots, \tag{3}$$

is called a power series about $x=x_0$ where x is a variable and $c_n's$ are constants called the coefficients of the series

Definition 2.3: A series that has the following representation

$$\sum_{n=-\infty}^{\infty} c_n x^n=\sum_{n=1}^{\infty}\frac{c_{-n}}{x^n}+\sum_{n=0}^{\infty}c_n x^n, \tag{4}$$

is called Laurent series about $x = 0$, where x is the variable and $c_n's$ are the coefficients of the series. The series $\sum_{n=0}^{\infty} c_n x^n$ is known as the analytic or the regular part of Laurent series, while $\sum_{n=1}^{\infty}\frac{c_{-n}}{x^n}$ is known as the singular or the principal part of Laurent series.

Theorem 2.2: If there exists a power series representation (expansion) for function $g(x)$ centered at x_0 as follow, $g(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$ has radius of convergence $R>0$, then $g(x)$ is infinitely differentiable in $|x-x_0|<R$, and in this case, the formula provides the coefficients for the series $c_n=\frac{g^{(n)}(x_0)}{n!}$, then $g(x)$ must have the following structure:

$$g(x)=\sum_{n=0}^{\infty}\frac{g^{(n)}(x_0)}{n!}(x-x_0)^n. \tag{5}$$

The series in equation (5) is called the Taylor series of the function $g(x)$ at x_0. For the special case $x_0=0$, the Taylor series becomes called the Maclaurin series.

Theorem 2.3: If $g(x,t)$ is a piecewise continuous function on $I\times\ [0,\infty)$ and of exponential order δ, suppose that $G(s,t)=\mathcal{L}[g(x,t)]$ has a Laurent series representation about $s=0$ as follows:

$$G(s,t)=\frac{c_0(t)}{s}+\sum_{n=1}^{\infty}\frac{c_n(t)}{s^{n+1}},\quad s>0, \tag{6}$$

then the remainder $R_n(s,t)$ of the series (6) satisfies the following condition:

$$R_n(s,t)\le\frac{M(t)}{s^{n+2}},\ x\in I\ \ \&\ \ t\in[a,b).$$

Theorem 2.4: Let $g(x,t)$ be a continuous function on $I\times[0,\infty)$, and assume that $G\ (s,t)=\mathcal{L}[g(x,t)]$ exists and can be expressed in the form of equation (6). If

$$\left|s\ \mathcal{L}\left[\frac{\partial^{(n+1)}g(x,t)}{\partial x^{(n+1)}}\right]\right|\le M(t),\ on\ I\ \times\ [a,b),$$

then the remainder $R_n(s,t)$ of the series (6) satisfies the following condition:

$$R_n(s,t)\le\frac{M(t)}{s^{n+2}},\ x\in I\ \&\ t\in[a,b).$$

Definition 2.4: An integral equation is an equation where the unknown function $g(x,t)$ that needs to be determined appears within an integral. Integral equations are highly valuable mathematical tools in both pure and applied mathematics. A typical form of an integral equation in $g(x,t)$ is of the form:

$$g(x,t)=f(x,t)+\lambda\int_{a(x)}^{b(x)}k(x,\tau)\,g(\tau,t)\,d\tau, \tag{7}$$

where $K(x,\tau)$ is called the kernel of the integral equation (7), and $a(x)$, $b(x)$ are the limits of the integration.

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Definition 2.5: The standard form of Volterra linear integral equations is typically expressed as:

$$h(x,t)g(x,t)=f(x,t)+\lambda\int_a^x k(x,\tau)\,g(\tau,t)\,d\tau, \tag{8}$$

in this equation, the unknown function is $g(x,t)$ which appears linearly under the integral sign. If $h(x,t)=1$, the equation is simplified to:

$$g(x,t)=f(x,t)+\lambda\int_a^x k(x,\tau)\,g(\tau,t)\,d\tau, \tag{9}$$

this equation is referred to as the Volterra integral equation of the second kind. Conversely, if $h(x,t)=0$, the equation becomes:

$$f(x,t)+\lambda\int_a^x k(x,\tau)\,g(\tau,t)\,d\tau=0. \tag{10}$$

Remark 2.1: If $k(x,\tau)=k(x-\tau)$, such that in $(x-\tau)$, $e^{x-\tau},\cdots$,then the equation (8) is called Volterra integral equation of convolution type.

III. LRPS METHOD FOR SOLVING LINEAR VOLTERRA PARTIAL INTEGRO-DIFFERENTIAL EQUATION

This section consists of two parts, the first one includes the steps of the LRPS method for solving partial integro differential equations, and in the other section we solve some illustrative examples to show the simplicity of the method.

A. Steps of LRPS method

To perform the LRPS technique, for solving the linear Volterra partial integro-differential equation of the form:

$$\frac{\partial^n g(x,t)}{\partial x^n}+\frac{\partial^m g(x,t)}{\partial t^m}+h(x,t) =\int_0^x k(x-\tau)\,g(\tau,t)d\tau,\quad n,m=0,1,2,\ldots, \tag{11}$$

where $h(x,t)$ and $k(x)$ are known functions, with the initial conditions:

$$g(0,t)=a_1(t),\frac{\partial g(0,t)}{\partial x}=a_2(t),\frac{\partial^2 g(0,t)}{\partial x^2}=a_3(t),\ \dots,\frac{\partial^{n-1} g(0,t)}{\partial x^{n-1}}=a_n(t),$$

where $a_1,a_2,a_3,\dots,a_n$ are functions of t.

To get the solution by LRPS method, we follow the steps:

Step 1. Starting with the application of the Laplace transform to both sides of equation (11) with respect to x, we obtain the following result:

$$\mathcal{L}\left[\frac{\partial^n g(x,t)}{\partial x^n}\right]+\mathcal{L}\left[\frac{\partial^m g(x,t)}{\partial t^m}\right]+\mathcal{L}\left[h(x,t)\right]=\mathcal{L}\left[\int_0^x k(x-\tau)\,g(\tau,t)d\tau\right]. \tag{12}$$

Step 2. Depending on Laplace transform and convolution theorem, the equation (12), can be rewritten as follows:

$$\begin{aligned}s^nG(s,t)&-s^{n-1}g(0,t)-s^{n-2}\frac{\partial g(0,t)}{\partial x}-\dots\\&-\frac{\partial^{n-1}g(0,t)}{\partial x^{n-1}}+\frac{\partial^m G(s,t)}{\partial t^m}+H(s,t)\\&=K(s)\,G(s,t),\end{aligned} \tag{13}$$

where, $G(s,t)=\mathcal{L}[g(x,t)]$, $H(s,t)=\mathcal{L}[h(x,t)]$, $K(s)=\mathcal{L}[k(x)]$.

Step 3. Multiplying equation (13) by $\frac{1}{s^n}$, and utilizing the initial conditions to simplify equation (13) into the following form:

$$\begin{aligned}G(s,t)=&\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}-\frac{1}{s^n}\frac{\partial^m G(s,t)}{\partial t^m}-\frac{1}{s^n}H(s,t)\\&+\frac{1}{s^n}K(s)\,G(s,t).\end{aligned} \tag{14}$$

Step 4. Examining the solution to equation (14) which takes on the following structure:

$$G(s,\ t)=\sum_{i=0}^{\infty}\frac{c_i(t)}{s^{i+1}},\quad s>0. \tag{15}$$

Step 5. Applying the initial condition by credit Theorem 2.4, we can determine the first n- coefficients of the previous structure, so the equation (15) can be written as follows:

$$G(s,\ t)=\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\sum_{i=n}^{\infty}\frac{c_i(t)}{s^{i+1}},\quad s>0, \tag{16}$$

and the μ-th truncated series of (16) is given by:

$$G_\mu(s,\ t)=\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i(t)}{s^{i+1}},\quad s>0, \tag{17}$$

where, $\mu=n,n+1,\dots$.

Step 6. Evaluating the Laplace residual function from equation (14) and the μth-truncated Laplace residual function independently,

$$\begin{aligned}LRes(s,t)=&G(s,t)-\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\frac{1}{s^n}\frac{\partial^m G(s,t)}{\partial t^m}\\&+\frac{1}{s^n}H(s,t)-\frac{1}{s^n}K(s)\,G(s,t),\end{aligned} \tag{18}$$

and

$$\begin{aligned}LRes_\mu(s,t)=&G_\mu(s,t)-\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}\\&+\frac{1}{s^n}\frac{\partial^m G_\mu(s,t)}{\partial t^m}+\frac{1}{s^n}H(s,t)\\&-\frac{1}{s^n}K(s)\ G_\mu(s,t).\end{aligned} \tag{19}$$

Step 7. Substituting the sum of $G_\mu(s)$ into (17) in place of the term (19) to get:

$$\begin{aligned}LRes_\mu(s,t)=&\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i(t)}{s^{i+1}}-\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}\\&+\frac{1}{s^n}\frac{\partial^m\left(\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i(t)}{s^{i+1}}\right)}{\partial t^m}\\&+\frac{1}{s^n}H(s,t)\\&-\frac{1}{s^n}K(s)\left(\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i(t)}{s^{i+1}}\right).\end{aligned} \tag{20}$$

Step 8. Multiplying both sides of equation (20) by $s^{\mu+1}$, then taking the limit as s approaches infinity, we need the following facts that can be found in [4]

i. $\lim_{\mu\to\infty}LRes_\mu(s)=LRes(s)$, $LRes(s)=0$, for all $s>0$,

ii. $\lim_{\mu\to\infty}s\ LRes(s)=0$, which implies $\lim_{\mu\to\infty}s\ LRes_\mu(s)=0$.

iii. $\lim_{\mu\to\infty}s^{\mu+1}LRes(s)=\lim_{\mu\to\infty}s^{\mu+1}LRes_\mu(s)=0$, $\mu=1,2,3,\cdots$, to obtain:

$$\begin{aligned}&\lim_{\mu\to\infty}s^{\mu+1}\frac{1}{s^n}\frac{\partial^m\left(\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i(t)}{s^{i+1}}\right)}{\partial t^m}\\&-\lim_{\mu\to\infty}s^{\mu+1}\frac{1}{s^n}K(s)\left(\sum_{i=0}^{n-1}\frac{a_{i+1}(t)}{s^{i+1}}+\sum_{i=n}^{\mu}\frac{c_i(t)}{s^{i+1}}\right).\end{aligned} \tag{21}$$

Step 9. Determining the values of the coefficient c_i'$s(t)$ in equation (21) by solving the system in equation (21) for $\mu=1,2,3,\cdots$, recursively.

Step 10. Substituting the calculated values of c_i'$s(t)$ into the truncated series of $G(s,t)$ to derive the approximate solution.

Step 11. Applying the inverse Laplace transform to $G(s,t)$ in order to obtain the approximate solution of $g(x,t)$.

B. Applications

Example 3.1. Consider the partial integro-differential equation of the form:

$$t\frac{\partial g(x,t)}{\partial t}=\frac{\partial^2 g(x,t)}{\partial x^2}+t\sin x+\int_0^x\sin(x-\tau)g(\tau,t)\,d\tau, \tag{22}$$

with the initial conditions: $g(0,t)=0$, $\frac{\partial g(0,t)}{\partial x}=t$.

Solution. Utilizing the Laplace transform on both aspects of the equation with respect to x to obtain:

$$\begin{aligned}\mathcal{L}\left[\frac{t\partial g(x,t)}{\partial t}\right]=&\mathcal{L}\left[\frac{\partial^2 g(x,t)}{\partial x^2}\right]+\mathcal{L}\left[t\sin x\right]\\&+\mathcal{L}\left[\int_0^x\sin(x-\tau)g(\tau,t)\ d\tau\right].\end{aligned} \tag{23}$$

Running Laplace transform

$$t\frac{\partial G(s,t)}{\partial t}=s^{2}G(s,t)-s\ g(0,t)-\frac{\partial g(0,t)}{\partial x}+\frac{t}{s^{2}+1}+\frac{G(s,t)}{s^{2}+1}. \tag{24}$$

Multiplying Equation (24) by $\frac{1}{s^2}$ and utilizing the initial conditions to get the following equation

$$G(s,t)=\frac{t}{s^{2}}\frac{\partial G(s,t)}{\partial t}+\frac{t}{s^{2}}-\frac{t}{s^{2}(s^{2}+1)}-\frac{G(s,t)}{s^{2}(s^{2}+1)}. \tag{25}$$

Assume that:

$$G(s,t)=\sum_{i=0}^{\infty}\frac{c_i(t)}{s^{i+1}},\ s>0. \tag{26}$$

Applying the initial condition, as stated in Theorem 2.4, the Equation (26) can be written as follows:

$$G(s,t)=\frac{t}{s^{2}}+\sum_{i=2}^{\infty}\frac{c_i(t)}{s^{i+1}},\quad s>0, \tag{27}$$

and the μth-truncated series of (27) is given by:

$$G_\mu(s,t)=\frac{t}{s^{2}}+\sum_{i=2}^{\mu}\frac{c_i(t)}{s^{i+1}},\quad s>0. \tag{28}$$

We define the Laplace residual function of equation (25) as follows:

$$LRes(s,t)=G(s,t)-\frac{t}{s^{2}}\frac{\partial G(s,t)}{\partial t}-\frac{t}{s^{2}}+\frac{t}{s^{2}(s^{2}+1)}+\frac{1}{s^{2}(s^{2}+1)}G(s,t), \tag{29}$$

and

$$LRes_\mu(s,t)=G_\mu(s,t)-\frac{t}{s^{2}}\frac{\partial G(s,t)}{\partial t}-\frac{t}{s^{2}}+\frac{t}{s^{2}(s^{2}+1)}+\frac{1}{s^{2}(s^{2}+1)}G_\mu(s,t). \tag{30}$$

To find the second coefficient $c_2(t)$, we define the second truncated series $G_2(s,t)$ as:

$$G_2(s,t)=\frac{t}{s^{2}}+\frac{c_2(t)}{s^{3}},$$

substituting $G_2(s,t)$ in the second Laplace residual function $LRes_2(s,t)$, to get:

$$LRes_2(s,t)=\frac{t}{s^{2}}+\frac{c_2(t)}{s^{3}}-\frac{t}{s^{2}}\frac{\partial}{\partial t}\left(\frac{t}{s^{2}}+\frac{c_2(t)}{s^{3}}\right)-\frac{t}{s^{2}}+\frac{t}{s^{2}(s^{2}+1)}+\frac{1}{s^{2}(s^{2}+1)}\left(\frac{t}{s^{2}}+\frac{c_2(t)}{s^{3}}\right). \tag{31}$$

Multiply both sides by s^3, then we get:

$$s^{3}LRes_2(s,t)=c_2(t)-ts\left(\frac{1}{s^{2}}+\frac{c'_2(t)}{s^{3}}\right)+\frac{ts}{(s^{2}+1)}+\frac{s}{(s^{2}+1)}\left(\frac{t}{s^{2}}+\frac{c_2(t)}{s^{3}}\right). \tag{32}$$

By taking the limit to both sides as $s\to\infty$, we get the value of $c_2(t)$, $c_2(t)=0$. Thus, the second approximation of the solution of Equation (24) $G_2(s,t)=\frac{t}{s^2}$.

Following the same steps to calculate $c_3(t)$, we define the third truncated series $G_3(s,t)$ as:

$$G_3(s,t)=\frac{t}{s^{2}}+\frac{c_3(t)}{s^{4}},$$

substituting $G_3(s,t)$ in the third Laplace residual function $LRes_3(s,t)$, to get:

$$LRes_3(s,t)=\frac{t}{s^{2}}+\frac{c_3(t)}{s^{4}}-\frac{t}{s^{2}}\frac{\partial}{\partial t}\left(\frac{t}{s^{2}}+\frac{c_3(t)}{s^{4}}\right)-\frac{t}{s^{2}}+\frac{t}{s^{2}(s^{2}+1)}+\frac{1}{s^{2}(s^{2}+1)}\left(\frac{t}{s^{2}}+\frac{c_{3(t)}}{s^{4}}\right). \tag{33}$$

Multiplying both sides by s^4, and simplifying the right side of previous equation to get:

$$s^{4}LRes_3(s,t)=c_3(t)-ts\left(\frac{1}{s^{2}}+\frac{c'_3(t)}{s^{4}}\right)+\frac{ts^{2}}{(s^{2}+1)}+\frac{s^{2}}{(s^{2}+1)}\left(\frac{t}{s^{2}}+\frac{c_3(t)}{s^{4}}\right). \tag{34}$$

By taking the limit to both sides as $s\to\infty$, we get the value of $c_3(t)$: $c_3(t)=0$. Thus, the third approximation of the solution of Equation (24) is:

$$G_3(s,t)=\frac{t}{s^{2}},$$

repeating the same steps, we can find the values of the coefficients as follows:

$$c_4(t)=c_5(t)=c_6(t)=\ldots=0.$$

To find the solution $g(x,t)$, of Equation (22), we operate the inverse Laplace transform to $G(s,t)$, to get:

$$g(x,t)=\mathcal{L}^{-1}\left[G(s,t)\right]=\mathcal{L}^{-1}\left[\frac{t}{s^{2}}\right]=xt.$$

In the following figure, Figure 1, we plot the exact solution of Example 3.1.

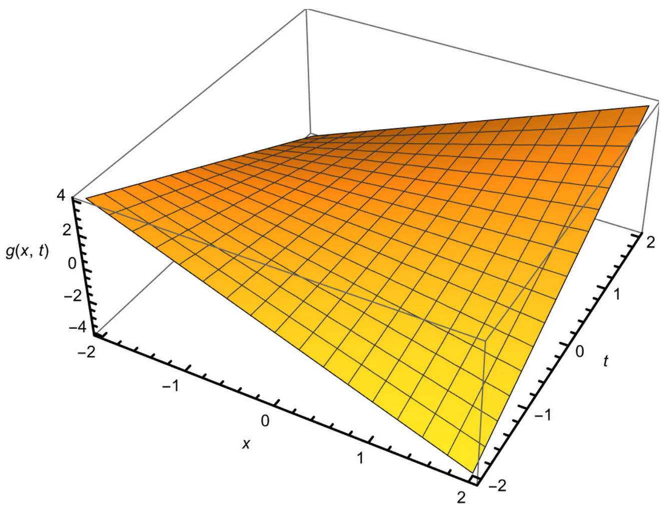

Fig. 1. The graph of solution of Example 3.1

Example 3.2. Consider the partial integro-differential equation of the form:

$$\frac{\partial^2 g(x,t)}{\partial x^2}=\frac{\partial g(x,t)}{\partial t}+2\int_0^x (x-\tau)\ g(\tau,t)\ d\tau-2e^t, \tag{35}$$

with initial conditions: $g(0,t)=e^t$, $\frac{\partial g(0,t)}{\partial x}=0$.

Solution. Utilizing the Laplace transform to both aspects of equation with respect to x to obtain:

$$\mathcal{L}\left[\frac{\partial^2 g(x,t)}{\partial x^2}\right]=\mathcal{L}\left[\frac{\partial g(x,t)}{\partial t}\right] +2\mathcal{L}\left[\int_0^x (x-\tau)\ g(\tau,t)\ d\tau\right]-2\mathcal{L}\left[e^t\right]. \tag{36}$$

Running Laplace transform and using convolution theory of Laplace transform to get:

$$s^2G(s,t)s\ g(0,t)-\frac{\partial g(0,t)}{\partial x}=\frac{\partial G(s,t)}{\partial t} +\frac{2}{s^2}G(s,t)-\frac{2e^t}{s}. \tag{37}$$

Multiplying previous equation by $\frac{1}{s^2}$, and utilizing the initial conditions to simplify Equation (37) into the following form:

$$G(s,t)=\frac{e^t}{s}+\frac{1}{s^2}\frac{\partial G(s,t)}{\partial t}+\frac{2}{s^4}G(s,t)-\frac{2e^t}{s^3}. \tag{38}$$

Assume that:

$$G(s,t)=\sum_{i=0}^{\infty}\frac{c_i(t)}{s^{i+1}},\quad s>0. \tag{39}$$

$$G(s,t)=\frac{e^t}{s}+\sum_{i=2}^{\infty}\frac{c_i(t)}{s^{i+1}},\quad s>0. \tag{40}$$

Applying the initial condition, so the Equation (39) can be written as follows: and the μth-truncated series of (40) is given by:

$$G_\mu(s,t)=\frac{e^t}{s}+\sum_{i=2}^{\mu}\frac{c_i(t)}{s^{i+1}},\quad s>0. \tag{41}$$

We define the Laplace residual function of Equation (38) as follows:

$$LRes\ (s,t)=G(s,t)-\frac{e^t}{s}-\frac{1}{s^2}\frac{\partial G(s,t)}{\partial t}-\frac{2}{s^4}G(s,t)+\frac{2e^t}{s^3}, \tag{42}$$

and

$$LRes_\mu(s,t)=G_\mu(s,t)-\frac{e^t}{s}-\frac{1}{s^2}\frac{\partial G(s,t)}{\partial t} -\frac{2}{s^4}G_\mu(s,t)+\frac{2e^t}{s^3}. \tag{43}$$

To find the second coefficient $c_2(t)$, we define the second truncated series $G_2(s,t)$ as: $G_2(s,t)=\frac{e^t}{s}+\frac{c_2(t)}{s^3}$, substituting $G_2(s,t)$ in the second Laplace residual function $LRes_2(s,t)$, to get:

$$LRes_2(s,t)=\frac{e^t}{s}+\frac{c_2(t)}{s^3}-\frac{e^t}{s}-\frac{1}{s^2}\frac{\partial}{\partial t}\left(\frac{e^t}{s}+\frac{c_2(t)}{s^3}\right) -\frac{2}{s^4}\left(\frac{e^t}{s}+\frac{c_2(t)}{s^3}\right)+\frac{2e^t}{s^3}. \tag{44}$$

Multiplying both sides by s^3, then we get:

$$s^3LRes_2(s,t)=c_2(t)-s\left(\frac{e^t}{s}+\frac{c'_2(t)}{s^3}\right) -\frac{2}{s}\left(\frac{e^t}{s}+\frac{c_2(t)}{s^3}\right)+2e^t. \tag{45}$$

By taking the limit to both sides as $s\to\infty$, we get the value of $c_2(t):c_2(t)=-e^t$.

Thus, the second approximation of the solution of equation (38)

$$G_2(s,t)=\frac{e^t}{s}-\frac{e^t}{s^3}.$$

Following the same steps to calculate $c_3(t)$, we define the third truncated series $G_3(s,t)$ as:

$$G_3(s,t)=\ \frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_3(t)}{s^4},$$

substituting $G_3(s,t)$ in the third Laplace residual function to get

$$LRes_3(s,t)=\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_3(t)}{s^4}-\frac{e^t}{s} -\frac{1}{s^2}\frac{\partial}{\partial t}\left(\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_3(t)}{s^4}\right) -\frac{2}{s^4}\left(\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_3(t)}{s^4}\right)+\frac{2e^t}{s^3}. \tag{46}$$

Multiplying both sides by s^4, and simplifying the right side of previous equation to get:

$$s^4LRes_3(s,t)=c_3(t)+\frac{e^t}{s}-\frac{c'_3(t)}{s^2} -2\left(\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_3(t)}{s^4}\right). \tag{47}$$

By taking the limit to both sides as $s\to\infty$, we get the value of $c_3(t)\ :c_3(t)=0$.

Thus, the third approximation of the solution of Equation (38) is:

$$G_3(s,t)=\frac{e^t}{s}-\frac{e^t}{s^3}.$$

Following the same steps to calculate $c_4(t)$, we define the fourth truncated series $G_4(s,t)$ as:

$$G_4(s,t)=\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_4(t)}{s^5},$$

substituting $G_4(s,t)$ in the fourth Laplace residual function $LRes_4(s)$, to get:

$$LRes_4(s,t)=\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_4(t)}{s^5}-\frac{e^t}{s} -\frac{1}{s^2}\frac{\partial}{\partial t}\left(\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_4(t)}{s^5}\right) -\frac{2}{s^4}\left(\frac{e^t}{s}-\frac{e^t}{s^3}+\frac{c_4(t)}{s^5}\right) +\frac{2e^t}{s^3}. \tag{48}$$

Multiplying both sides by s^5, and simplifying the right side of previous equation to get:

$$s^5LRes_4(s,t)=c_4(t)-e^t-\frac{c'_4(t)}{s^2}-2\left(-\frac{e^t}{s^2}+\frac{c_4(t)}{s^4}\right).$$

By taking the limit to both sides as $s \to \infty$, we get the value of $c_4(t) : c_4(t) = e^t$. Thus, the fourth approximation of the solution of Equation (38)

$$G_4(s,t) = \frac{e^t}{s} - \frac{e^t}{s^3} + \frac{e^t}{s^5}. \tag{49}$$

Similarly

$$c_5(t) = 0,\ c_6(t) = -e^t, \ldots.$$

To find the solution $g(x,t)$, of Equation (35), we operate the inverse Laplace transform to $G(s,t)$, to get:

$$\begin{aligned} g(x,t) &= \mathcal{L}^{-1}\left[G(s,t)\right] \\ &= e^t \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{1}{s^7} + \ldots\right] = e^t \cos x. \end{aligned}$$

In Figure 2, below we sketch the solution of Example 3.2

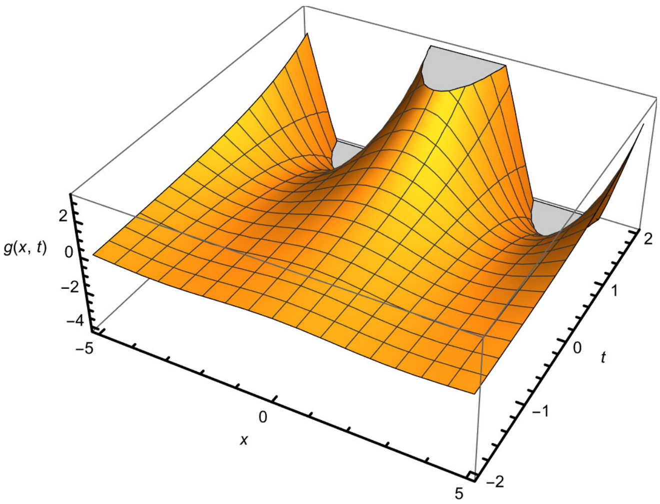

Fig. 2. The graph of solution of Example 3.2

Example 3.3. Consider the partial integro-differential equation of the form:

$$\frac{\partial g(x,t)}{\partial x} + \frac{\partial^3 g(x,t)}{\partial x^3} - \int_0^x \sinh(x-\tau)\frac{\partial^3 g(\tau,t)}{\partial t^3}\, d\tau = 0. \tag{50}$$

With initial conditions: $g(0,t) = 0$, $\frac{\partial g(0,t)}{\partial x} = t$, $\frac{\partial^2 g(0,t)}{\partial x^2} = 0$.

Solution. Utilizing the Laplace transform to both aspects of equation with respect to x to obtain:

$$\begin{aligned} &\mathcal{L}\left[\frac{\partial g(x,t)}{\partial x}\right] + \mathcal{L}\left[\frac{\partial^3 g(x,t)}{\partial x^3}\right] \\ &\quad -\mathcal{L}\left[\int_0^x \sinh(x-\tau)\frac{\partial^3 g(\tau,t)}{\partial t^3}\, d\tau\right] = 0. \end{aligned} \tag{51}$$

Running Laplace transform and convolution theory of Laplace transform to get:

$$\begin{aligned} &sG(s,t) - g(0,t) + s^3 G(s,t) - s^2\, g(0,t) - s\frac{\partial g(0,t)}{\partial x} \\ &\quad - \frac{\partial^2 g(0,t)}{\partial x^2} - \frac{1}{s^2-1}\frac{\partial^3 G(s,t)}{\partial t^3} = 0. \end{aligned} \tag{52}$$

Multiplying previous equation by $\frac{1}{s^3}$, and utilizing the initial conditions to simplify Equation (52) into the following form:

$$G(s,t) = -\frac{1}{s^2}G(s,t) + \frac{t}{s^2} + \frac{1}{s^3(s^2-1)}\frac{\partial^3 G(s,t)}{\partial t^3}. \tag{53}$$

Assume that:

$$G(s,t) = \sum_{i=0}^{\infty} \frac{c_i(t)}{s^{i+1}}, \quad s > 0. \tag{54}$$

Applying the initial condition, so the Equation (54) can be written as follows:

$$G(s,t) = \frac{t}{s^2} + \sum_{i=3}^{\infty} \frac{c_i(t)}{s^{i+1}}, \quad s > 0, \tag{55}$$

and the μth-truncated series of (55) is given by:

$$G_\mu(s,t) = \frac{t}{s^2} + \sum_{i=3}^{\mu} \frac{c_i(t)}{s^{i+1}}, \quad s > 0. \tag{56}$$

We define the Laplace residual function of Equation (53) as follows:

$$\begin{aligned} LRes\ (s,t) &= G(s,t) + \frac{1}{s^2}G(s,t) \\ &\quad - \frac{t}{s^2} - \frac{1}{s^3(s^2-1)}\frac{\partial^3 G(s,t)}{\partial t^3}, \end{aligned} \tag{57}$$

and

$$\begin{aligned} LRes_\mu(s,t) &= G_\mu(s,t)\ + \frac{1}{s^2}G(s,t) \\ &\quad - \frac{t}{s^2} - \frac{1}{s^3(s^2-1)}\frac{\partial^3 G(s,t)}{\partial t^3}. \end{aligned} \tag{58}$$

To find the second coefficient $c_3(t)$, we define the third truncated series $G_3(s,t)$ as: $G_3(s,t) = \frac{t}{s^2} + \frac{c_3(t)}{s^4}$, substituting $G_3(s,t)$ in the third Laplace residual function $LRes_3(s,t)$, to get:

$$\begin{aligned} LRes_3(s,t) &= \frac{t}{s^2} + \frac{c_3(t)}{s^4}\ + \frac{1}{s^2}\left(\frac{t}{s^2} + \frac{c_3(t)}{s^4}\right) \\ &\quad - \frac{t}{s^2} - \frac{1}{s^3(s^2-1)}\left(\frac{c'''_3(t)}{s^4}\right). \end{aligned} \tag{59}$$

Multiply both sides by s^4, then we get:

$$s^4 LRes_3(s,t) = c_3(t) + t + \frac{c_3(t)}{s^2} - \frac{1}{s^3(s^2-1)}\left(c_3'''(t)\right). \tag{60}$$

By taking the limit to both sides as $s \to \infty$, we get the value of $c_3(t)$, $c_3(t) = -t$. Thus, the third approximation of the solution of Equation (53)

$$G_3(s,t) = \frac{t}{s^2} - \frac{t}{s^4}.$$

Following the same steps to calculate $c_4(t)$, we define the fourth truncated series $G_4(s,t)$ as:

$$G_4(s,t) = \frac{t}{s^2} - \frac{t}{s^4} + \frac{c_4(t)}{s^5},$$

substituting $G_4(s,t)$ in the fourth Laplace residual function $LRes_4(s,t)$, to get:

$$\begin{aligned} LRes_4(s,t) &= \frac{t}{s^2} - \frac{t}{s^4} + \frac{c_4(t)}{s^5} + \frac{1}{s^2}\left(\frac{t}{s^2} - \frac{t}{s^4} + \frac{c_4(t)}{s^5}\right) \\ &\quad - \frac{t}{s^2} - \frac{1}{s^3(s^2-1)}\left(\frac{c'''_4(t)}{s^5}\right). \end{aligned} \tag{61}$$

Multiplying both sides by s^4, and simplifying the right side of previous equation to get:

$$s^5 LRes_4(s,t) = c_4(t) + \frac{c_4(t)}{s^2} - \frac{1}{s^3(s^2-1)}\left(c'''_3(t)\right). \tag{62}$$

By taking the limit to both sides as $s \to \infty$, we get the value of $c_4(t)$: $c_4(t) = 0$.

Thus, the fourth approximation of the solution of Equation (53)

$$G_4(s,t) = \frac{t}{s^2} - \frac{t}{s^4}.$$

Following the same steps to calculate $c_5(t)$, we define the fifth truncated series $G_5(s,t)$ as:

$$G_5(s,t) = \frac{t}{s^2} - \frac{t}{s^4} + \frac{c_5(t)}{s^6},$$

substituting $G_5(s,t)$ in the fourth Laplace residual function $LRes_5(s)$, to get:

$$\begin{aligned} LRes_5(s,t) = {} & \frac{t}{s^2} - \frac{t}{s^4} + \frac{c_5(t)}{s^6} + \frac{1}{s^2}\left(\frac{t}{s^2} - \frac{t}{s^4} + \frac{c_5(t)}{s^6}\right) \\ & - \frac{t}{s^2} - \frac{1}{s^3(s^2-1)}\left(\frac{c'''_4(t)}{s^6}\right). \end{aligned} \tag{63}$$

Multiplying both sides by s^6, and simplifying the right side of previous equation to get:

$$s^6 LRes_5(s,t) = c_5(t) - t - \frac{c_5(t)}{s^2} - \frac{1}{s^3(s^2-1)}\left(c'''_5(t)\right). \tag{64}$$

By taking the limit to both sides as $s \to \infty$, we get the value of $c_5(t)$, $c_5(t) = t$. Thus, the fifth approximation of the solution of Equation (53) is:

$$G_5(s,t) = \frac{t}{s^2} - \frac{t}{s^4} + \frac{t}{s^6}.$$

Similarly,

$$c_6(t) = 0, c_7(t) = -t \ldots.$$

To find the solution $g(x,t)$, of Equation (50), we operate the inverse Laplace transform to $G(s,t)$, to get:

$$\begin{aligned} g(x,t) = \mathcal{L}^{-1}[G(s,t)] &= \mathcal{L}^{-1}\left[\frac{t}{s^2} - \frac{t}{s^4} + \frac{t}{s^6} - \frac{t}{s^8} + \ldots\right] \\ &= \left[\frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \ldots\right] \\ &= t\mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \ldots\right] = t\sin x. \end{aligned}$$

In Figure 3, below we sketch the solution of Example 3.3.

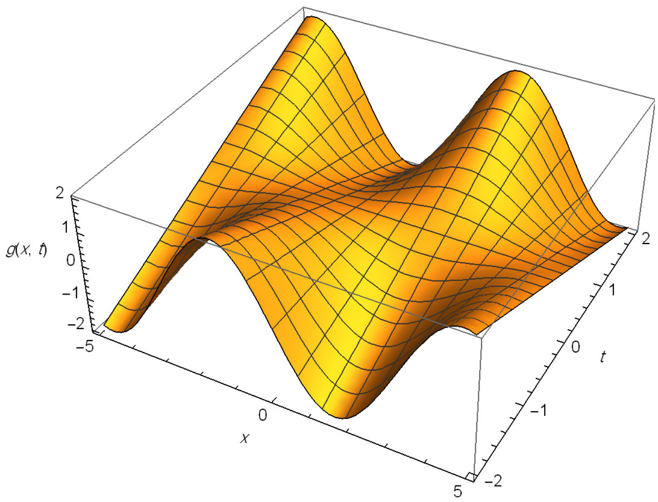

Fig. 3. The graph of solution of Example 3.3

IV. Conclusion

The study validates the LRPSM for linear Volterra partial integro-differential equations. The LRPSM, combines the Laplace transform with a power series, excels in handling complex equations. The research illustrates the steps of LRPSM and its effectiveness through examples, showing its proficiency for accurate approximations. This method is valuable in fields like physics and engineering, with potential for future research in more complex systems [32], [33], [34].

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