Synchronous Controller of Drive-Response Neural Networks of n Reaction-Diffusion Systems of FitzHugh-Nagumo type

Phan Van Long Em

Abstract—**This work studies the identical synchronization of drive-response neural networks of n reaction-diffusion systems of FitzHugh-Nagumo type with arbitrary topological structures. We propose a nonlinear adaptive controller in order to achieve the desired synchronization of these two networks of the same node dynamics. This work also shows the simulation results to verify if the proposed method is effective.**

Index Terms—**drive-response neural networks, identical synchronization, synchronous controller, reaction-diffusion system of FitzHugh-Nagumo.**

I. Introduction

SYNCHRONIZATION is known as one of the most important topics of dynamical systems that have been investigated in recent years. Especially, it has been interesting in complex networks. As everyone knows, complex networks are various in nature and human society such as transportation networks, biological networks, social relationship networks, neuronal networks, etc [5], [11], [12], [13], [14], [21], and have been studied in many domains, especially in neuroscience, and applied mathematics, [12], [14]. There are many researchs reflecting the synchronization of complex networks, and their application in the real world [1], [2], [7], [15], [16], [18], [19], [20]. The meaning of *synchronization* is to the same behavior at the same time [5]. Its principle can be seen in many different applications, also in social production and human activities [12], [14]. Many studies on the synchronization of complex networks have been achieved. Especially, internal synchronization in a complex network is studied in many papers, for example, [1], [2], [7], [15], [16], [18], [19], [20]. In reality, synchronization can also be seen between two networks presenting two different groups of cells with arbitrary network structures. In this paper, the identical synchronization of two complex networks of the same reaction-diffusion systems is studied. Specifically, each node is presented by one reaction-diffusion system of FitzHugh-Nagumo type (see Figure 1). In Figure 1, the left graph is called the drive network, and the right one is called the response network. They may have different structures and connections which means it is difficult to synchronize them. That is a reason why in this paper we would like to design a nonlinear controller so that the identical synchronization of two complex networks with the same nodes and different topological structures is realized.

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As everyone knows, there are many kinds of synchronization [5], and the identical one is studied the most. Specifically, the identical synchronization of a network of n reaction-diffusion systems of Hindmarsh-Rose type is studied in [15], [16]. There are also a lot of works studying the identical synchronization of n dynamical systems in complete networks such as [7], [18], [19], [20]. However, there are not any studies on the identical synchronization of two neural networks of n reaction-diffusion systems of FitzHugh-Nagumo model. In other words, this paper studies the problem of identical synchronization for drive-response neural networks of n reaction-diffusion systems of FitzHugh-Nagumo type. In this work, we consider the drive-response neural networks with arbitrary topological structures and designs an appropriate adaptive controller in order to get the identical synchronization of these two networks. The proof is based on Lyapunov stability theory and LaSalle's invariance principle. This paper also shows the numerical results to verify if the proposed method is effective.

This paper is organized as follows. In Section 2, the definition of the identical synchronization between the drive-response complex networks of n reaction-diffusion systems and some preliminary knowledge are given. By using the Lyapunov theory and LaSalle's invariance principle, we design some schemes to construct the response networks to achieve the desired synchronization. In Section 3, we display two numerical examples to see if the proposed method in Section 2 is effective. Finally, conclusions are displayed in Section 4.

II. Identical synchronization between two complex networks of n reaction-diffusion systems of FitzHugh-Nagumo type

In 1952, Hodgkin and Huxley introduced a four-dimensional mathematical system that could approximate many properties of neural membrane potential [7], [8], [10]. Based on this system, a lot of scientists published simpler models describing the neuron voltage dynamics. In 1962, FitzHugh and Nagumo introduced a new simpler model called FitzHugh-Nagumo model [7], [1], [2], [3]. This system was known as a simplified two-dimensional model from Hodgkin-Huxley's famous model [11]. Although this system is simpler, it has a lot of extraordinary analytical results and retains the energizing properties and biological significance of cells. Specifically, this model consists of two equations in the two variables u and v. The first variable is the fast one. It is excitatory and represents the transmembrane voltage. The second one is the slow recovery variable presenting some

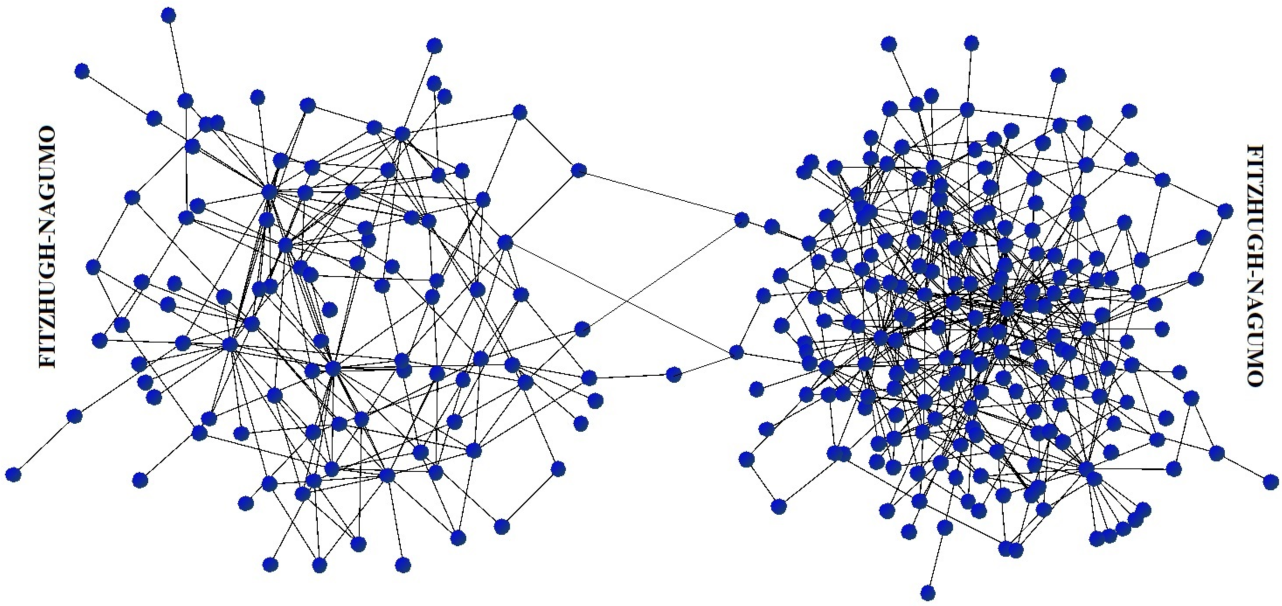

Fig. 1. An example of the drive-response networks of reaction-diffusion systems of FitzHugh-Nagumo type. Each node is represented by a reaction-diffusion system of FitzHugh-Nagumo type.

physical quantities, such as the electrical conductivity of ion currents across the membrane. The ordinary differential equations of FitzHugh-Nagumo type are given by [1], [2], [3]:

$$\begin{cases} \varepsilon \dfrac{du}{dt} = \varepsilon u_t = f(u) - v + I, \\ \\ \dfrac{dv}{dt} = v_t = au - bv + c, \end{cases} \tag{1}$$

where a, b, c are constants (a, b are strictly positive), $0 < \varepsilon < 1$, $f(u) = -u^3 + 3u$, I presents the external current, and t presents the time.

However, the system (1) is not strong enough in order to describe the propagation of action potential along the axon of the cell. To solve this problem, the cable equation is investigated in this work. It means the Laplace operator can be added to the first equation of the system (1) in order to obtain the model that can describe the propagation of action potential. This mathematical system is obtained from a circuit model of the membrane and its intracellular and extracellular space to provide a quantitative description of current flow and voltage change both within and between neurons. It allows us to understand how cells function quantitatively and qualitatively. Hence, the reaction-diffusion system of FitzHugh-Nagumo type (FHN) is considered as follows:

$$\begin{cases} \varepsilon u_t = f(u) - v + I + d\Delta u, \\ \\ v_t = au - bv + c, \end{cases} \tag{2}$$

where $u = u(x,t)$, $v = v(x,t)$, $(x,t) \in \Omega \times \mathbb{R}^+$, d is a positive constant, Δu is the Laplace operator of u, $\Omega \subset \mathbb{R}^N$ is a regular bounded open set and with Neumann zero flux boundary conditions, and N is a positive integer. This model allows the appearance of many patterns and relevant phenomena in physiology. This model consists of two nonlinear partial differential equations. The first one presents the action potential and the second one introduces the recovery variable describing some physical quantities, such as the electrical conductivity of ion currents across the membrane. Besides, the first equation is similar to the cable equation. It presents the distribution of the membrane potential along the axon of a single cell [8], [11]. Hereafter, system (2) is considered as a neural model, and a network of n coupled systems (2) is constructed as follows:

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i + I_i + d_i \Delta u_i + \sum\limits_{j=1}^{n} c_{ij} h(u_i, u_j), \\ v_{it} = au_i - bv_i + c, \\ i = 1, 2, ..., n, \end{cases} \tag{3}$$

where $(u_i, v_i), I_i, d_i, i = 1, 2, ..., n$ is defined as (2). The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n \times n}$, defined by: $c_{ij} > 0$ if neuron ith and jth are coupled, $c_{ij} = 0$ if neuron ith and jth are not coupled; $c_{ii} = \sum\limits_{j=1, j \neq i}^{n} c_{ij}$. This matrix also presents the network topology. The function h presents the coupling function describing the type of connection between cell ith and jth. It is known that neurons connect through synapses, then it leads to two types of connections between cells such as chemical connections and electrical ones.

If the neurons connect through the chemical synapse, then the coupling function is nonlinear [1], [2], [7] and is given by the following formula:

$$h(u_i, u_j) = -(u_i - V_{syn}) g_{syn} \frac{1}{1 + \exp(-\lambda(u_j - \theta_{syn}))}, j \neq i, \tag{4}$$

where g_{syn} represents a positive number that is called the coupling strength; V_{syn} introduces the reversal potential, and its value must be larger than $u_i(x,t)$, for all $i = 1, 2, ..., n$, $x \in \Omega, t \geq 0$ since synapses are supposed excitatory; θ_{syn} represents the threshold value that is reached by every action potential; λ is a positive number that could be big enough to approach the Heaviside function [6], [7].

If the neurons connect through the electrical synapse, then the coupling function is linear [7], [15] and is given by the following formula:

$$h(u_i, u_j) = -g_{syn}(u_i - u_j),\ j \neq i. \tag{5}$$

The network (3) is considered a drive network, and the following system is the corresponding response network:

$$\begin{cases} \varepsilon \overline{u}_{it} = f(\overline{u}_i) - \overline{v}_i + I_i + d_i \Delta \overline{u}_i + \sum\limits_{j=1}^{n} \overline{c}_{ij} \overline{h}(\overline{u}_j) + w_i, \\ \overline{v}_{it} = a\overline{u}_i - b\overline{v}_i + c, \\ i = 1, 2, ..., n, \end{cases} \tag{6}$$

where $(\overline{u}_i, \overline{v}_i), i = 1, 2, ..., n$ is defined by (2). The coefficients $\overline{c}_{ij}\,(i, j = 1, 2, ..., n)$ and the function $\overline{h}$ are defined as c_{ij} and h in network (3), respectively; $w_i, i = 1, 2, ..., n$ is the synchronous controller to be designed.

We can see from the model of drive-response neural networks (3) and (6) that the structures of these two networks can be different. For example, all cells in (3) are coupled by chemical synapses, and all cells in (6) are coupled by electrical synapses. Then, it is difficult to synchronize them because of their different structures. Therefore, we need to design the adaptive controllers to observe our desired synchronization.

Before going to the main results, we need to see some following remarks that help to prove our desired results.

Remark 1. *The function f satisfies the following condition:*

$$|f(u_i) - f(u_j)| \leq \alpha |u_i - u_j|,\ i, j = 1, 2, ..., n, \tag{7}$$

where u_i, u_j present the transmembrane voltages, and α is a positive number.

Proof: For all $u_i, u_j, i, j = 1, 2, ..., n$, we have:

$$\begin{aligned} f(u_i) - f(u_j) &= -u_i^3 + 3u_i + u_j^3 - 3u_j \\ &= (u_i - u_j)\left[3 - (u_i - u_j)^2 - u_i u_j\right]. \end{aligned}$$

Since $u_i, u_j, i, j = 1, 2, ..., n$ are bounded in [3], then we can find a positive constant α such that:

$$|f(u_i) - f(u_j)| \leq \alpha |u_i - u_j|,\ i, j = 1, 2, ..., n.$$

■

Remark 2. *The function h defined by* (4) *satisfies the following condition:*

$$|h(u_i, u_k) - h(u_j, u_l)| \leq \beta |u_i - u_j|,\ i, j, k, l = 1, 2, ..., n, \tag{8}$$

where u_i, u_j, u_k, u_l present the transmembrane voltages, and β is a positive number.

Proof: For all $u_i, u_j, i, j = 1, 2, ..., n$, we have:

$$\begin{aligned} &|h(u_i, u_k) - h(u_j, u_l)| = \\ &\quad \left| g_{syn}(u_i - V_{syn}) \frac{1}{1 + \exp(-\lambda(u_k - \theta_{syn}))} \right. \\ &\quad \left. - g_{syn}(u_j - V_{syn}) \frac{1}{1 + \exp(-\lambda(u_l - \theta_{syn}))} \right|, \end{aligned}$$

where $k \neq i, l \neq j$.

Since $u_k, u_l, k, l = 1, 2, ..., n$ are bounded in [3], then we can find a positive constant K such that:

$$-K \leq \frac{1}{1 + \exp(-\lambda(u_k - \theta_{syn}))} \leq K,$$

and

$$-K \leq \frac{1}{1 + \exp(-\lambda(u_l - \theta_{syn}))} \leq K.$$

Besides that $u_i - V_{syn} < 0;\ u_j - V_{syn} < 0$, for all $u_i, u_j, i, j = 1, 2, ..., n$, since we only consider the rapid chemical excitatory synapses [6], [7].

Thus

$$\begin{aligned} &|h(u_i, u_k) - h(u_j, u_l)| \leq \\ &\qquad |K g_{syn}(u_i - V_{syn}) + K g_{syn}(u_j - V_{syn})| \\ &\qquad \leq K g_{syn} |u_i + u_j - 2V_{syn}|. \end{aligned}$$

Moreover, $u_i, u_j, i, j = 1, 2, ..., n$ are bounded in [3], then we can find a positive constant β such that:

$$|h(u_i, u_k) - h(u_j, u_l)| \leq \beta |u_i - u_j|.$$

■

Remark 3. *It is easy to see that the function h defined by* (5) *satisfies the following condition:*

$$|h(u_i, u_k) - h(u_j, u_l)| \leq \beta |u_i - u_j|,\ i, j, k, l = 1, 2, ..., n, \tag{9}$$

where u_i, u_j, u_k, u_l present the transmembrane voltages, and β is a positive number.

Remark 4. *The function $\overline{h}$ in network* (6) *is defined as h in network* (3)*, then it also verifies the conditions in Remark 2 and Remark 3.*

Definition 1 (see [1])**.** *Let $S_i = (u_i, v_i),\ i = 1, 2, ..., n$ and $S = (S_1, S_2, ..., S_n)$ be a network. We say that S is identically synchronous if*

$$\lim_{t \to +\infty} \sum_{i=1}^{n-1} \left(\|u_i - u_{i+1}\|_{L^2(\Omega)} + \|v_i - v_{i+1}\|_{L^2(\Omega)} \right) = 0,$$

where $L^2(\Omega)$ is function space on Ω defined using a natural generalization of the 2-norm for finite-dimensional vector spaces.

By applying Definition 1 to this study, we let the node error of complete synchronization between two systems (3) and (6) be $e_i = \overline{u}_i - u_i, \overline{e}_i = \overline{v}_i - v_i,\ i = 1, 2, ..., n$.

If there is a controller w_i such that the Definition 1 satisfies, it means:

$$\lim_{t \to +\infty} \sum_{i=1}^{n} \left(\|e_i\|_{L^2(\Omega)} + \|\overline{e}_i\|_{L^2(\Omega)} \right) = 0,$$

then the networks (3) and (6) are said to be identical synchronization.

To get the identical synchronization of networks (3) and (6), the controller w_i is designed as follows:

$$w_i = \varepsilon u_{it} - f(u_i) + v_i - I_i - d_i \Delta u_i - \sum_{j=1}^{n} \overline{c}_{ij} \overline{h}(u_i, u_j) - k_i e_i, \tag{10}$$

with the updated rules defined as follows:

$$k_{it} = r_i e_i^2, \tag{11}$$

where r_i is an arbitrary positive constant, for $i = 1, 2, ..., n$.

Under the action of the controller, the error dynamic equations of the system are described as:

$$\begin{aligned}\varepsilon e_{it} &= \varepsilon(\overline{u}_{it} - u_{it}) \\ &= f(\overline{u}_i) - \overline{v}_i + I_i + d_i\Delta\overline{u}_i + \sum_{j=1}^{n}\overline{c}_{ij}\overline{h}(\overline{u}_i,\overline{u}_j) \\ &\quad -f(u_i) + v_i - I_i - d_i\Delta u_i - \sum_{j=1}^{n}\overline{c}_{ij}\overline{h}(u_i,u_j) - k_ie_i \\ &= f(\overline{u}_i) - f(u_i) - (\overline{v}_i - v_i) + d_i\Delta(\overline{u}_i - u_i) \\ &\quad + \sum_{j=1}^{n}\overline{c}_{ij}(\overline{h}(\overline{u}_i,\overline{u}_j) - \overline{h}(u_i,u_j)) - k_ie_i,\end{aligned} \tag{12}$$

and

$$\overline{e}_{it} = \overline{v}_{it} - v_{it} = a(\overline{u}_i - u_i) - b(\overline{v}_i - v_i), \tag{13}$$

for $i = 1, 2, ..., n$.

Next, we investigate the identical synchronization problem of networks (3) and (6). The main result is given by the following theorem.

Theorem 1. *The drive-response neural networks* (3) *and* (6) *can achieve identical synchronization under the adaptive controller* (10) *and updated rule* (11).

Proof: We construct the Lyapunov function as follows:

$$V(x,t) = \frac{1}{2}\sum_{i=1}^{n}\int_{\Omega}\left[a\varepsilon e_i^2 + \overline{e}_i^2 + \frac{a}{r_i}(k_i - \overline{k})^2\right]dx, \tag{14}$$

where $\overline{k}$ is a positive constant to be determined.

Calculating the time derivative of $V(x,t)$ along the error systems (12) and (13), we get:

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} &= \sum_{i=1}^{n}\int_{\Omega}\left[a\varepsilon e_ie_{it} + \overline{e}_i\overline{e}_{it} + \frac{a}{r_i}(k_i - \overline{k})k_{it}\right]dx \\ &= \sum_{i=1}^{n}\int_{\Omega}\Big[ae_i\left(f(\overline{u}_i) - f(u_i) - (\overline{v}_i - v_i) + d_i\Delta(\overline{u}_i - u_i)\right. \\ &\quad \left. + \sum_{j=1}^{n}\overline{c}_{ij}(\overline{h}(\overline{u}_i,\overline{u}_j) - \overline{h}(u_i,u_j)) - k_ie_i\right) \\ &\quad + \overline{e}_i\left(a(\overline{u}_i - u_i) - b(\overline{v}_i - v_i)\right) + \frac{a}{r_i}(k_i - \overline{k})k_{it}\Big]dx.\end{aligned} \tag{15}$$

By using the Green formula and Neumann zero flux boundary conditions, (15) becomes:

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} &\le \sum_{i=1}^{n}\int_{\Omega}\Big[ae_i\left(f(\overline{u}_i) - f(u_i)\right) \\ &\quad + ae_i\sum_{j=1}^{n}\overline{c}_{ij}(\overline{h}(\overline{u}_i,\overline{u}_j) - \overline{h}(u_i,u_j)) \\ &\quad - k_iae_i^2 - b\overline{e}_i^2 + k_iae_i^2 - a\overline{k}e_i^2\Big]dx \\ &\le \sum_{i=1}^{n}\int_{\Omega}\Big[ae_i\left(f(\overline{u}_i) - f(u_i)\right) \\ &\quad + ae_i\sum_{j=1}^{n}\overline{c}_{ij}(\overline{h}(\overline{u}_i,\overline{u}_j) - \overline{h}(u_i,u_j)) \\ &\quad - b\overline{e}_i^2 - a\overline{k}e_i^2\Big]dx.\end{aligned} \tag{16}$$

By using Remarks 1-4, it is easy to obtain:

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} &\le \\ &\le \sum_{i=1}^{n}\int_{\Omega}\left[a\alpha e_i^2 - b\overline{e}_i^2 - a\overline{k}e_i^2 + \sum_{j=1}^{n}a\beta\left|\overline{c}_{ij}\right|\left|e_i\right|\left|e_j\right|\right]dx.\end{aligned} \tag{17}$$

Besides that, we can see:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}a\beta\left|\overline{c}_{ij}\right|\left|e_i\right|\left|e_j\right| \\ &= a\beta\sum_{i=1}^{n}\left(\left|e_i\right|\sum_{j=1}^{n}\left|\overline{c}_{ij}\right|\left|e_j\right|\right) \\ &\le a\beta\left(\sum_{i=1}^{n}e_i^2.\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|\overline{c}_{ij}\right|\left|e_j\right|\right)^2\right)^{\frac{1}{2}} \\ &\le a\beta\left(\sum_{i=1}^{n}e_i^2.\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\overline{c}_{ij}^2.\sum_{j=1}^{n}e_j^2\right)\right)^{\frac{1}{2}} \\ &\le a\beta\left(\left(\sum_{i=1}^{n}e_i^2\right)^2.\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\overline{c}_{ij}^2\right)\right)^{\frac{1}{2}} \\ &\le a\beta\left(\left(\sum_{i=1}^{n}e_i^2\right)^2.n^2\max_{1\le i,j\le n}\overline{c}_{ij}^2\right)^{\frac{1}{2}} \\ &\le a\beta n\max_{1\le i,j\le n}\left|\overline{c}_{ij}\right|\sum_{i=1}^{n}e_i^2.\end{aligned} \tag{18}$$

Combining (17) and (18) yields:

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} &\le \\ &\le \sum_{i=1}^{n}\int_{\Omega}\left[a\alpha e_i^2 - b\overline{e}_i^2 - a\overline{k}e_i^2 + a\beta n\max_{1\le i,j\le n}\left|\overline{c}_{ij}\right|e_i^2\right]dx \\ &\le \sum_{i=1}^{n}\int_{\Omega}\left[(a\alpha - a\overline{k} + a\beta n\max_{1\le i,j\le n}\left|\overline{c}_{ij}\right|)e_i^2 - b\overline{e}_i^2\right]dx \\ &\le \sum_{i=1}^{n}\int_{\Omega}\left[-(a\overline{k} - a\alpha - a\beta n\max_{1\le i,j\le n}\left|\overline{c}_{ij}\right|)e_i^2 - b\overline{e}_i^2\right]dx.\end{aligned} \tag{19}$$

Take $\overline{k} > \alpha + \beta n\max_{1\le i,j\le n}\left|\overline{c}_{ij}\right|$, then (19) can be estimated as:

$$\frac{\partial V(x,t)}{\partial t} \le -\gamma\sum_{i=1}^{n}\int_{\Omega}\left[\frac{1}{2}(a\varepsilon e_i^2 + \overline{e}_i^2)\right]dx, \tag{20}$$

where

$$\gamma = \min\left\{\frac{2(\overline{k} - \alpha - \beta n\max_{1\le i,j\le n}\left|\overline{c}_{ij}\right|)}{\varepsilon}; 2b\right\}.$$

It can be found from (20) that $0 \le V(x,t) \le V(x,0)$, this together with (14) signifies $V(x,t)$ is bounded. Based on Lyapunov stability theory and LaSalle's invariance principle [4], we have:

$$\lim_{t\to+\infty}\sum_{i=1}^{n}\left(\left\|e_i\right\|_{L^2(\Omega)} + \left\|\overline{e}_i\right\|_{L^2(\Omega)}\right) = 0.$$

Then it follows from Definition 1 that the drive-response networks (3) and (6) achieve identical synchronization. The proof is completed. ■

III. Illustrative Numerical Examples

This section concretely shows two examples of drive and response networks to demonstrate the effectiveness of the proposed method in the previous section.

The numerical results are realized to observe the identical synchronization of 2 networks of reaction-diffusion systems of FHN with different network topologies. The integration of systems is realized by using C++ and the results are represented by Gnuplot.

Some parameters are fixed as [1], [2], [6], [7], [15], [16]:

$$f(u) = -u^3 + 3u, a = 1, b = 0.001, c = 0, \varepsilon = 0.1,$$

$$I_i = 0,\ d_i = 0.05,\ i = 1, 2, ..., n,$$

$$[0; T] \times \Omega = [0; 500] \times [0; 100] \times [0; 100],$$

$$\lambda = 10,\ V_{syn} = 2,\ \theta_{syn} = -0,25.$$

A. Example 1

Consider a drive network of 3 nodes that has the structure called a ring network with unidirectionally linear coupling as follows:

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + I_1 + d_1 \Delta u_1 - g_{syn}(u_1 - u_2), \\ v_{1t} = a u_1 - b v_1 + c, \\ \varepsilon u_{2t} = f(u_2) - v_2 + I_2 + d_2 \Delta u_2 - g_{syn}(u_2 - u_3), \\ v_{2t} = a u_2 - b v_2 + c, \\ \varepsilon u_{3t} = f(u_3) - v_3 + I_3 + d_3 \Delta u_3 - g_{syn}(u_3 - u_1), \\ v_{3t} = a u_3 - b v_3 + c, \end{cases} \tag{21}$$

and the response network of 3 nodes that has the structure called a complete (full) network with linear coupling is described by:

$$\begin{cases} \varepsilon \overline{u}_{1t} = f(\overline{u}_1) - \overline{v}_1 + I_1 + d_1 \Delta \overline{u}_1 \\ \qquad - g_{syn}(\overline{u}_1 - \overline{u}_2) - g_{syn}(\overline{u}_1 - \overline{u}_3) + w_1, \\ \overline{v}_{1t} = a \overline{u}_1 - b \overline{v}_1 + c, \\ \varepsilon \overline{u}_{2t} = f(\overline{u}_2) - \overline{v}_2 + I_2 + d_2 \Delta \overline{u}_2 \\ \qquad - g_{syn}(\overline{u}_2 - \overline{u}_3) - g_{syn}(\overline{u}_2 - \overline{u}_1) + w_2, \\ \overline{v}_{2t} = a \overline{u}_2 - b \overline{v}_2 + c, \\ \varepsilon \overline{u}_{3t} = f(\overline{u}_3) - \overline{v}_3 + I_3 + d_3 \Delta \overline{u}_3 \\ \qquad - g_{syn}(\overline{u}_3 - \overline{u}_1) - g_{syn}(\overline{u}_3 - \overline{u}_2) + w_3, \\ \overline{v}_{3t} = a \overline{u}_3 - b \overline{v}_3 + c, \end{cases} \tag{22}$$

with the adaptive controllers defined as:

$$\begin{cases} w_1 = \varepsilon u_{1t} - f(u_1) + v_1 - I_1 - d_1 \Delta u_1 \\ \qquad + g_{syn}(u_1 - u_2) + g_{syn}(u_1 - u_3) - k_1(\overline{u}_1 - u_1), \\ w_2 = \varepsilon u_{2t} - f(u_2) + v_2 - I_2 - d_2 \Delta u_2 \\ \qquad + g_{syn}(u_2 - u_1) + g_{syn}(u_2 - u_3) - k_2(\overline{u}_2 - u_2), \\ w_3 = \varepsilon u_{3t} - f(u_3) + v_3 - I_3 - d_3 \Delta u_3 \\ \qquad + g_{syn}(u_3 - u_1) + g_{syn}(u_3 - u_2) - k_3(\overline{u}_3 - u_3), \end{cases} \tag{23}$$

where

$$\begin{cases} k_{1t} = r_1(\overline{u}_1 - u_1)^2, \\ k_{2t} = r_2(\overline{u}_2 - u_2)^2, \\ k_{3t} = r_3(\overline{u}_3 - u_3)^2. \end{cases} \tag{24}$$

Notice that the adaptive controllers (23) and the updated rules (24) are defined as (10) and (11), respectively.

Figure 2 below illustrates the synchronization errors of drive-response networks (21) and (22) without the adaptive controllers (23) and the updated rules (24). Figure 2(a), 2(b), 2(c) represent the synchronization errors of the coupled solutions, respectively:

$$(u_1(x_1, x_2, t), \overline{u}_1(x_1, x_2, t)),\ (u_2(x_1, x_2, t), \overline{u}_2(x_1, x_2, t)),$$
$$\text{and}\ (u_3(x_1, x_2, t), \overline{u}_3(x_1, x_2, t)),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 0.1$. This figure shows that the synchronization errors do not reach zero, it means the drive-response networks can not achieve identical synchronization.

Figure 3 below illustrates the synchronization errors of drive-response networks (21) and (22) with the adaptive controllers (23) and the updated rules (24). The simulations show that this adaptive scheme is effective and we can get:

$$\lim_{t \to +\infty} \sum_{i=1}^{3} \left(\|e_i\|_{L^2(\Omega)} + \|\overline{e}_i\|_{L^2(\Omega)} \right) = 0,$$

where $e_i = \overline{u}_i - u_i, \overline{e}_i = \overline{v}_i - v_i$, $i = 1, 2, 3$. Specifically, Figure 3(a), 3(b), 3(c) represent the synchronization errors of the coupled solutions, respectively:

$$(u_1(x_1, x_2, t), \overline{u}_1(x_1, x_2, t)),\ (u_2(x_1, x_2, t), \overline{u}_2(x_1, x_2, t)),$$
$$\text{and}\ (u_3(x_1, x_2, t), \overline{u}_3(x_1, x_2, t)),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 0.01, r_1 = 0.1, r_2 = 0.2, r_3 = 0.3$. Here, we even take the coupling strength smaller than before, this figure shows that the synchronization errors reach zero, it means:

$$u_1(x_1, x_2, t) \approx \overline{u}_1(x_1, x_2, t),\ u_2(x_1, x_2, t) \approx \overline{u}_2(x_1, x_2, t)$$
$$\text{and}\ u_3(x_1, x_2, t) \approx \overline{u}_3(x_1, x_2, t),$$

for all $(x_1, x_2) \in \Omega$.

Figure 4(a), 4(b), 4(c) represent the solutions $u_i(x_1, x_2, 499), i = 1, 2, 3$, of the drive network (21), and Figure 4(d), 4(e), 4(f) perform the solutions $\overline{u}_i(x_1, x_2, 499)$, $i = 1, 2, 3$, of the response network (22). We can see that the neuron ith of network (21) and the neuron ith of network (22) have the same shape ($i = 1, 2, 3$), i.e., the synchronization is performed.

B. Example 2

Consider a drive network of 2 nodes that has the structure called a complete (full) network with linear coupling as follows:

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + I_1 + d_1 \Delta u_1 - g_{syn}(u_1 - u_2), \\ v_{1t} = a u_1 - b v_1 + c, \\ \varepsilon u_{2t} = f(u_2) - v_2 + I_2 + d_2 \Delta u_2 - g_{syn}(u_2 - u_1), \\ v_{2t} = a u_2 - b v_2 + c, \end{cases} \tag{25}$$

and the response network of 2 nodes that has the structure called a chain network with nonlinear coupling is described by:

$$\begin{cases} \varepsilon \overline{u}_{1t} = f(\overline{u}_1) - \overline{v}_1 + I_1 + d_1 \Delta \overline{u}_1 \\ \qquad - g_{syn}(\overline{u}_1 - V_{syn}) \dfrac{1}{1 + \exp(-\lambda(\overline{u}_2 - \theta_{syn}))} + w_1, \\ \overline{v}_{1t} = a \overline{u}_1 - b \overline{v}_1 + c, \\ \varepsilon \overline{u}_{2t} = f(\overline{u}_2) - \overline{v}_2 + I_2 + d_2 \Delta \overline{u}_2 + w_2, \\ \overline{v}_{2t} = a \overline{u}_2 - b \overline{v}_2 + c, \end{cases} \tag{26}$$

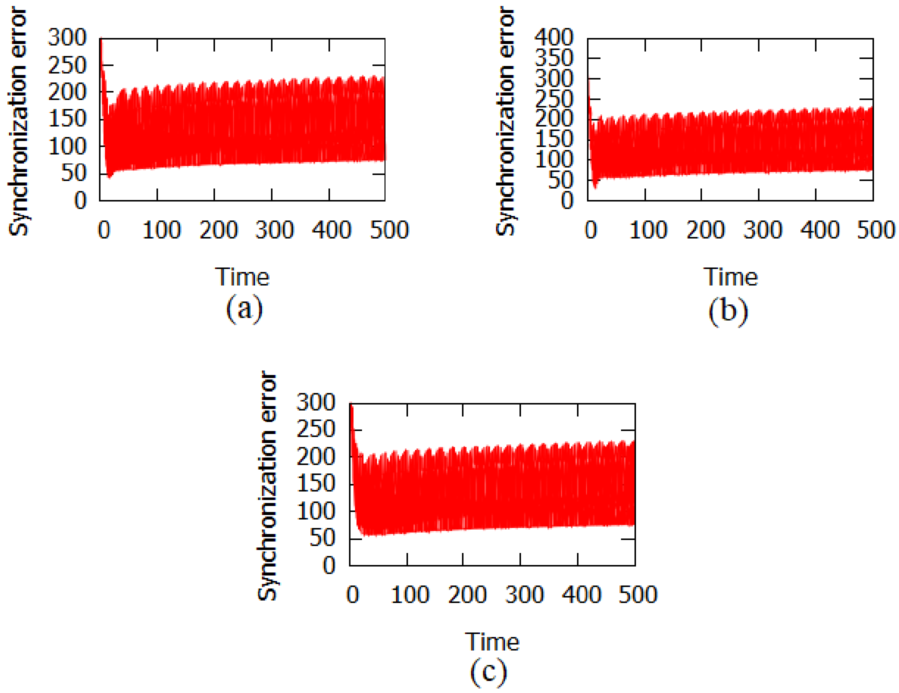

Fig. 2. Synchronization errors of the drive-response networks (21) and (22) without controllers. We can see that the synchronization errors do not reach zero which means there is not the synchronization.

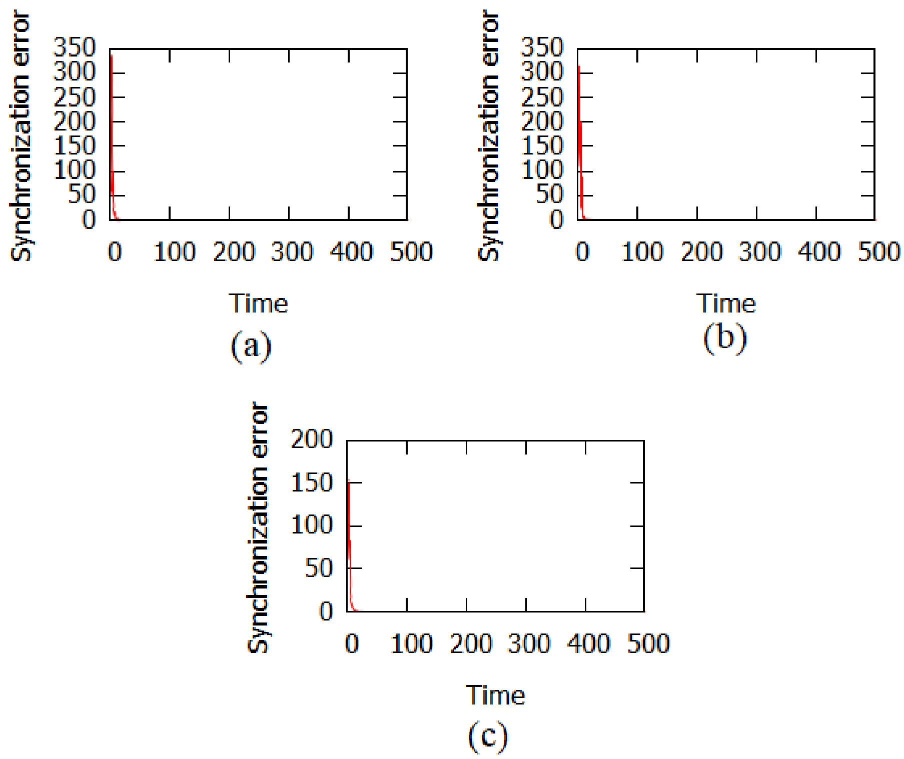

Fig. 3. Synchronization errors of the drive-response networks (21) and (22) with controllers. We can see that the synchronization errors asymptotically reaches zero which means the synchronization occurs.

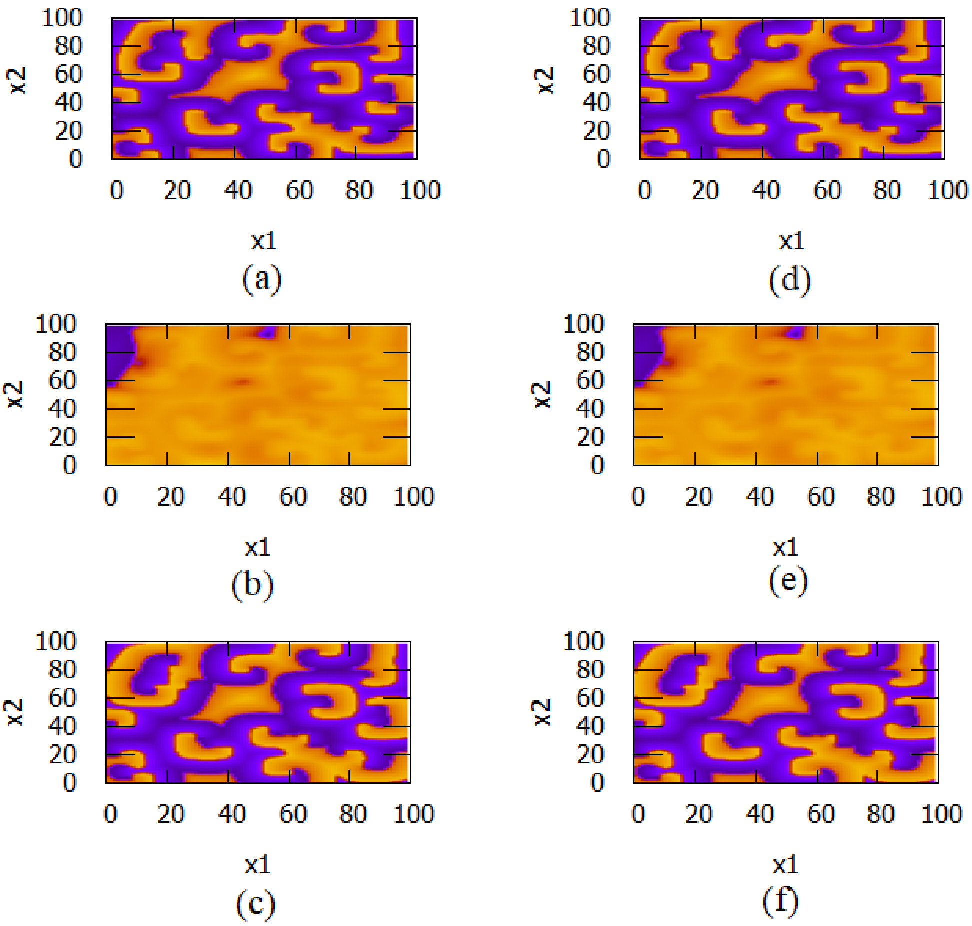

Fig. 4. Synchronization patterns of the drive-response networks (21) and (22) with controllers. Figure 4(a), 4(b), 4(c) represent the solutions $u_i(x_1, x_2, 499), i = 1, 2, 3,$ of the drive network (21), and Figure 4(d), 4(e), 4(f) perform the solutions $\overline{u}_i(x_1, x_2, 499),\ i = 1, 2, 3,$ of the response network (22). We can see that the patterns of the second column have the same shape with the patterns of the first column, respectively. In other words, the response network (22) synchronizes with the drive network (21).

with the adaptive controllers defined as:

$$\begin{cases} w_1 = \varepsilon u_{1t} - f(u_1) + v_1 - I_1 - d_1 \Delta u_1 \\ \qquad +g_{syn}(u_1 - V_{syn}) \dfrac{1}{1 + \exp(-\lambda(u_2 - \theta_{syn}))} \\ \qquad -k_1(\overline{u}_1 - u_1), \\ w_2 = \varepsilon u_{2t} - f(u_2) + v_2 - I_2 - d_2 \Delta u_2 - k_2(\overline{u}_2 - u_2), \end{cases} \tag{27}$$

where

$$\begin{cases} k_{1t} = r_1(\overline{u}_1 - u_1)^2, \\ k_{2t} = r_2(\overline{u}_2 - u_2)^2. \end{cases} \tag{28}$$

Notice that the adaptive controllers (27) and the updated rules (28) are defined as (10) and (11), respectively.

Figure 5 below illustrates the synchronization errors of drive-response networks (25) and (26) without the adaptive controllers (27) and the updated rules (28). Figure 5(a), 5(b) represent the synchronization errors of the coupled solutions, respectively:

$$(u_1(x_1, x_2, t), \overline{u}_1(x_1, x_2, t)),$$

and

$$(u_2(x_1, x_2, t), \overline{u}_2(x_1, x_2, t)),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 0.1$. This figure shows that the synchronization errors do not reach zero, which means the drive-response networks can not achieve identical synchronization.

Figure 6 below illustrates the synchronization errors of drive-response networks (25) and (26) with the adaptive controllers (27) and the updated rules (28). The simulations show that this adaptive scheme is effective and we can get:

$$\lim_{t \to +\infty} \sum_{i=1}^{2} \left(\|e_i\|_{L^2(\Omega)} + \|\overline{e}_i\|_{L^2(\Omega)} \right) = 0,$$

where $e_i = \overline{u}_i - u_i, \overline{e}_i = \overline{v}_i - v_i,\ i = 1, 2$. Specifically, Figure 6(a), 6(b) represent the synchronization errors of the coupled solutions, respectively:

$$(u_1(x_1, x_2, t), \overline{u}_1(x_1, x_2, t)),$$

and

$$(u_2(x_1, x_2, t), \overline{u}_2(x_1, x_2, t)),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 0.01, r_1 = 0.1, r_2 = 0.2$. Here, we even take the coupling strength smaller than before, this figure shows that the synchronization errors reach zero, it means:

$$u_1(x_1, x_2, t) \approx \overline{u}_1(x_1, x_2, t),$$

and

$$u_2(x_1, x_2, t) \approx \overline{u}_2(x_1, x_2, t),$$

for all $(x_1, x_2) \in \Omega$. In other words, the controllers that we built effectively work.

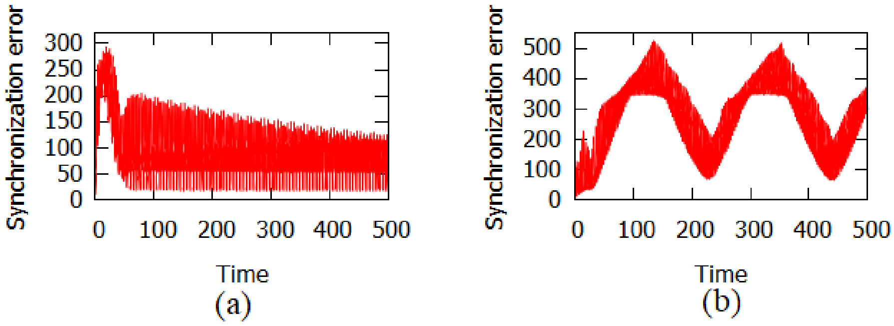

Fig. 5. Synchronization errors of the drive-response networks (25) and (26) without controllers. We can see that the synchronization errors do not reach zero which means there is not the synchronization.

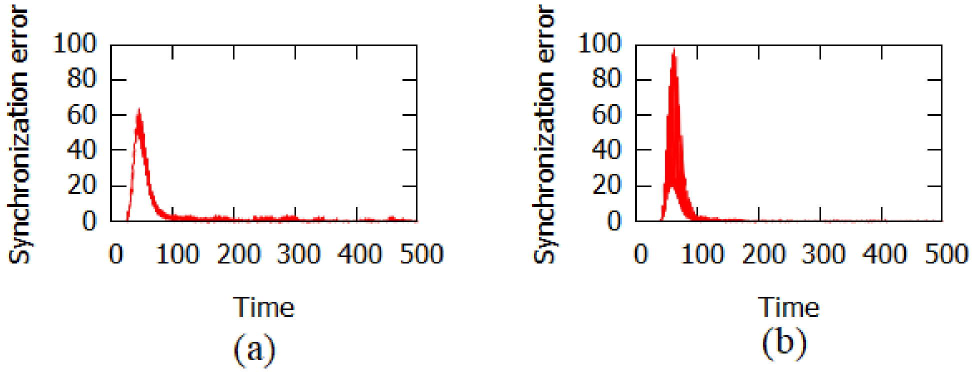

Fig. 6. Synchronization errors of the drive-response networks (25) and (26) with controllers. We can see that the synchronization errors asymptotically reaches zero which means the synchronization occurs.

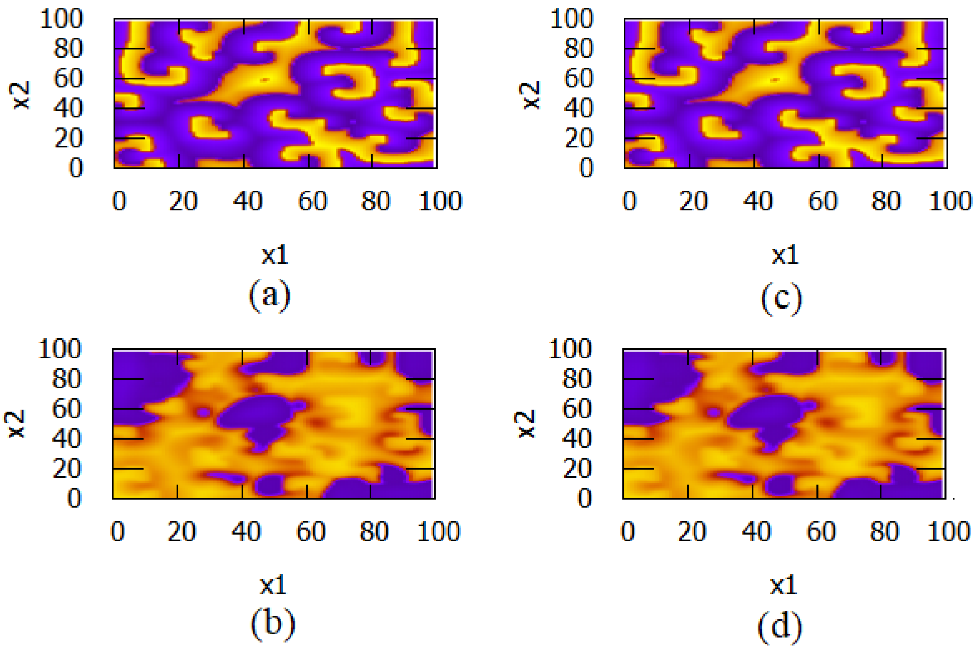

Fig. 7. Synchronization patterns of the drive-response networks (25) and (26) with controllers. Figure 7(a), 7(b) represent the solutions $u_i(x_1, x_2, 499)$, $i = 1, 2$, of the drive network (25), and Figure 7(c), 7(d) perform the solutions $\overline{u}_i(x_1, x_2, 499)$, $i = 1, 2$, of the response network (26). We can see that the patterns of the second column have the same shape with the patterns of the first column, respectively. In other words, the response network (25) synchronizes with the drive network (26).

Figure 7(a), 7(b) represent the solutions $u_i(x_1, x_2, 499), i = 1, 2,$ of the drive network (25) and Figure 7(c), 7(d) perform the solutions $\overline{u}_i(x_1, x_2, 499)$, $i = 1, 2$, of the response network (26). We can see that the neuron ith of network (25) and the neuron ith of network (26) have the same shape ($i = 1, 2$), i.e., the synchronization is performed.

Remark 5. Notice that the synchronization above is the identical synchronization between two networks of cells. It means the ith cell of the first network will have the same pattern as the ith cell of the second network. In other words, one network will copy the behavior of the other. Meanwhile, the behavior of cells in the same network can be different (see Figure 4 and Figure 7). However, if we want all cells of both networks to have the same pattern, we need to increase the value of the coupling strength such that it is larger than the necessary threshold value [6], [7]. Then all cells in both networks will have the same pattern (see Figure 9). In Fig. 9, we take $g_{syn} = 2$, this value is big enough to get the same behavior for all cells of networks (25) and (26).

Clearly, Figure 8(a), 8(b), 8(c), and 8(d) represent the synchronization errors of the coupled solutions, respectively:

$$\begin{aligned} &\left(u_1(x_1, x_2, t), \overline{u}_1(x_1, x_2, t)\right), \\ &\left(u_2(x_1, x_2, t), \overline{u}_2(x_1, x_2, t)\right), \\ &\left(u_1(x_1, x_2, t), u_2(x_1, x_2, t)\right), \end{aligned}$$

and

$$\left(\overline{u}_1(x_1, x_2, t), \overline{u}_2(x_1, x_2, t)\right),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 2, r_1 = 0.1, r_2 = 0.2$. This figure shows that the synchronization errors reach zero, which means:

$$\begin{aligned} u_1(x_1, x_2, t) &\approx \overline{u}_1(x_1, x_2, t), \\ u_2(x_1, x_2, t) &\approx \overline{u}_2(x_1, x_2, t) \\ u_1(x_1, x_2, t) &\approx u_2(x_1, x_2, t), \end{aligned}$$

and

$$\overline{u}_1(x_1, x_2, t) \approx \overline{u}_2(x_1, x_2, t),$$

for all $(x_1, x_2) \in \Omega$.

Figure 9(a), 9(b) represent the solutions $u_i(x_1, x_2, 499), i = 1, 2,$ of the drive network (25), and Figure 9(c), 9(d) perform the solutions $\overline{u}_i(x_1, x_2, 499)$, $i = 1, 2$, of the response network (26). We can see that they have the same shape, i.e., the synchronization is performed for all cells of both networks.

IV. Conclusion

This paper investigates the identical synchronization of drive-response neural networks of n reaction-diffusion systems of FitzHugh-Nagumo type with arbitrary topological structures. We design the nonlinear adaptive controllers and construct a suitable Lyapunov function so that the desired synchronization is achieved. The numerical results showed the effectiveness of the proposed method. The synchronous controllers are complex and actually necessary for identical synchronization in this work. How to simplify the controllers and investigate the drive-response networks of n different reaction-diffusion systems will be a problem to be studied in the future.

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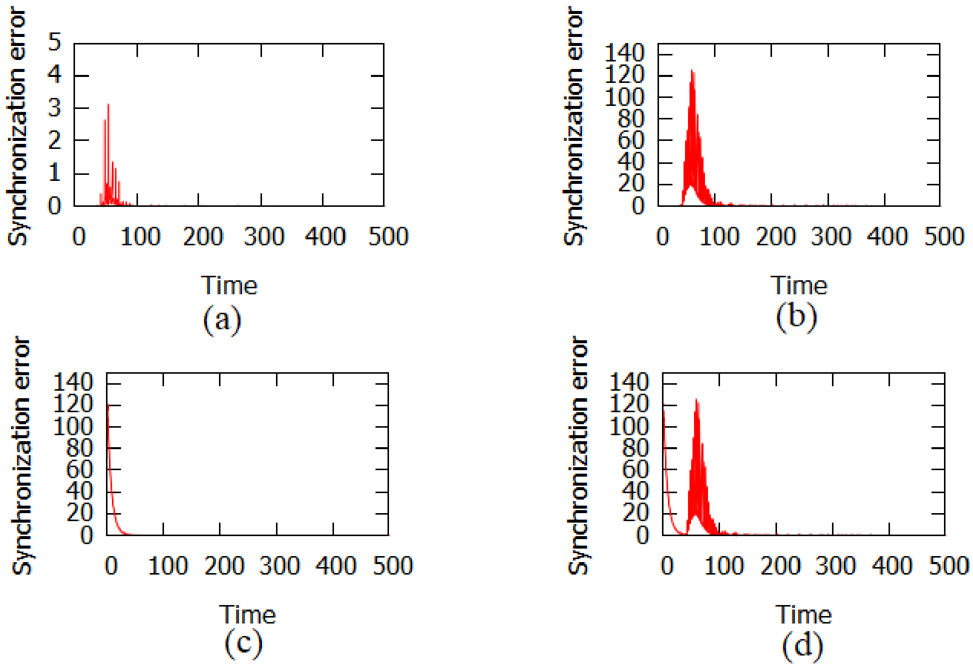

Fig. 8. Synchronization errors of the drive-response networks (25) and (26) with controllers and big enough coupling strength. We can see that the synchronization errors asymptotically reaches zero which means the synchronization occurs.

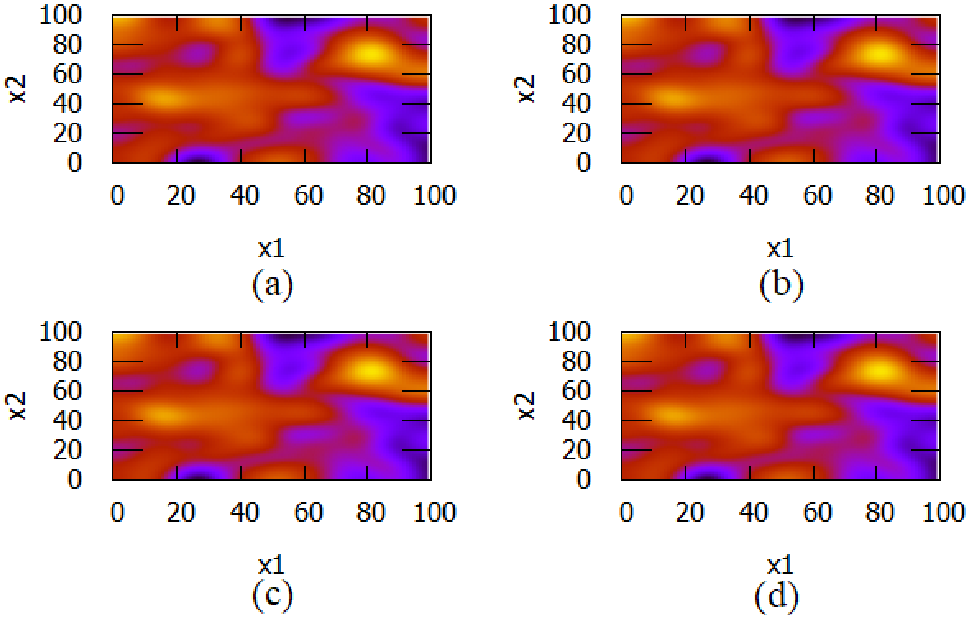

Fig. 9. Synchronization patterns of the drive-response networks (25) and (26) with controllers and large enough coupling strength. Figure 9(a), 9(b) represent the solutions $u_i(x_1, x_2, 499), i = 1, 2$, of the drive network (25), and Figure 7(c), 7(d) perform the solutions $\overline{u}_i(x_1, x_2, 499)$, $i = 1, 2$, of the response network (26). We can see that all the patterns of two networks have the same shape. In other words, all the nodes of the response network (25) and the drive network (26) identically synchronize with a large enough value of coupling strength.