Further Study for Individual Consistency Control in Consensus Building

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***Abstract*—In this paper, we examine an individual consistency control in consensus building for group decision-making that was based on a paper of Li, Rodriguez, Martinez, Dong, and Herrera. we point out their important but questionable results and then present our comments. Our results will help researchers understand the structure of individual consistency control in consensus building under fuzzy environments.**

***Index Terms*—Inventory systems, Fuzzy theorems, Consensus building, Individual consistency control**

I. INTRODUCTION

RECENTLY, Li et al. [1] published a paper in a famous journal, IEEE Transactions on Fuzzy Systems, to handle individual consistency control. Up to now, this paper had been cited fifth-two times to indicate this paper is dealt with a hot research topic. We list several related papers in the following: Alexakis and Sarris [2], Tsui and Wen [3], Feizi et al. [4], Roodposhti et al. [5], Pourghahreman and Rajabzadeh Qhatari [6], Andreopoulou et al. [7], Vesyropoulos and Georgiadis [8], Hamzeh et al. [9], and Zhang et al. [10]. After reading Li et al. [1], we find several interesting but dubious outcomes in their paper. Hence, in this article, we will reveal those significant but disputed results and provide our explanation of why those findings are imperative but questionable.

II. REVIEW OF THEIR FORMULAS AND OUR COMMENTS

There are some interesting but unrelated materials in Li et al. [1]. For example, Equations (1-4) of Li et al. [1], that were discussed linguistic preference relations for the 2-tuple linguistic model and numerical scale. However, in their paper, the authors studied linguistic terms on Page 8 left column, Lines 11-12, that are not related to 2-tuple linguistic representation. It reveals that Li et al. [1] mentioned some types of fuzzy data that may be essential in academic society but are not used in their paper.

Furthermore, Equation (4) and Equation (5) of Li et al. [1] look like a pair of twins. It indicates that Li et al. [1] did not prepare their version in a compact form such that main but redundant expressions appeared twice in their paper.

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Moreover, Equations (5-7) of Li et al. [1] are already summarized in Equation (8) of Li et al. [1] such that Equations (5-7) may be deleted to save the precious space of w well-known journal. On the other hand, to provide a detailed explanation for their Equation (8) through Equations (5-7) is useful for ordinary readers to understand the derivation of Equation (8).

Based on our above discussion, we can claim that the formulas in Li et al. [1] are not well-organized. Hence, some improvements will be presented by us to help ordinary readers realize Li et al. [1].

Remark. We are confused by κ with k such that we use δ to replace κ. The authors of Li, Rodriguez, Martinez, Dong, and Herrera should not use two expressions so close to each other that will imply chaos to the standing of their meanings. It is important for help readers absorb Li et al. [1] such that some modifications will be executed by this manuscript.

There are many interesting but questionable expressions in their paper. In the following, we just list several examples. Page 4, Figure 1, of Li et al. [1], we cite that "Individual fuzzy preference relations" that is questionable. Because we refer to their results of "Individual linguistic preference relations" handled by the PIS model to imply "PISs of linguistic terms" for each expert that is a crisp (real) number in the interval, [0,1]. Consequently, they used "fuzzy" which is improper.

We refer to "Algorithm 1. PIS-based clustering algorithm." from Li et al. [1]. We must point out that Li et al. [1] did not provide any reference works for the fuzzy c-means (FCM) algorithm to help readers to understand the meaning of it.

III. RECAP OF THEIR ANALYSIS AND OUR COMMENTS

Based on individual consensus level, two consensus groups, acceptable and unacceptable, denoted as G_A, and G_U, are decided to partition decision makers with

$$CL_k \geq \varepsilon > CL_j, \tag{3.1}$$

for decision-makers $e_k \in G_A$ and $e_j \in G_U$.

Authors wanted to find a decision-maker, say e_δ such that $e_\delta \in G_U$ to indicate the person who is satisfying

$$CL_\delta = \max_{e_j \in G_U} CL_j. \tag{3.2}$$

Authors tried to construct a new pair of acceptable and unacceptable groups, $G_A \bigcup \{e_\delta\}$ and $G_U - \{e_\delta\}$. Each time, shift one decision maker from one group G_U to the other group G_A, until the consensus level (Equation (12) of Li et al. [1]) attains the collective consensus threshold, $\overline{CL}$.

To persuade a person, e_δ to change his/her preference, authors tried to find a moderator, say e_y in two cases A and B.

By PIS-based clustering algorithm, we assume that $e_\delta \in G^h$, a semantic-based cluster, then Case A:

$$G^h \bigcap G_A \neq \phi, \tag{3.3}$$

and Case B:

$$G^h \bigcap G_A = \phi. \tag{3.4}$$

To simplify the expression, we assume the distance between e_α and e_β that

$$d(e_\alpha, e_\beta) = \sum_{i,j=1}^{n} \left| f_{ij}^{\alpha} - f_{ij}^{\beta} \right|. \tag{3.5}$$

For Case A, the moderator e_y satisfies

$$d(e_y, e_\delta) = \min_{e_f \in G^h \bigcap G_A} d(e_f, e_\delta). \tag{3.6}$$

For Case B, the moderator e_y satisfies

$$d(e_y, e_\delta) = \min_{e_f \in G_A} d(e_f, e_\delta). \tag{3.7}$$

We raise our question: If $e_\delta \in G^h$, and $G^h \bigcap G_A \neq \phi$, does

$$\min_{e_f \in G^h \bigcap G_A} d(e_f, e_\delta) = \min_{e_f \in G_A} d(e_f, e_\delta)? \tag{3.8}$$

If our raised problem has a positive answer, then we can merge Case A and Case B into Case B such that directly select e_y, satisfying

$$d(e_y, e_\delta) = \min_{e_f \in G_A} d(e_f, e_\delta), \tag{3.9}$$

without checking whether or not $G^h \bigcap G_A \neq \phi$.

Consequently, if our raised problem has a positive answer, we may save time to overlook Algorithm 1.

We have interesting questions for the direction rule (a)-(e) on Page 7, left column, Lines 11-24, of Li et al. [1].

Authors of Li et al. [1] let $\left(\bar{f}_{ij}^{\delta}\right)$ be the adjusted fuzzy preference relation associated with expert e_δ, then they mentioned that "(a) If $f_{ij}^{\delta} \le f_{ij}^{y} \le f_{ij}^{c}$, then $\bar{f}_{ij}^{\delta} \in \left(f_{ij}^{\delta}, f_{ij}^{y}\right]$."

We predict that the authors of Li et al. [1] wanted

$$\left|\bar{f}_{ij}^{\delta} - f_{ij}^{c}\right| \ge \left|f_{ij}^{y} - f_{ij}^{c}\right|, \tag{3.10}$$

that is, the authors wished that

$$\left|\bar{f}_{ij}^{\delta} - \bar{f}_{ij}^{c}\right| \ge \left|f_{ij}^{y} - \bar{f}_{ij}^{c}\right|, \tag{3.11}$$

owing to $f_{ij}^{\delta} < \bar{f}_{ij}^{\delta} \le f_{ij}^{y} \le f_{ij}^{c} < \bar{f}_{ij}^{c}$.

Li et al. [1] also claimed that "(b) If $f_{ij}^{\delta} \le f_{ij}^{c} \le f_{ij}^{y}$, then $\bar{f}_{ij}^{\delta} \in \left(f_{ij}^{\delta}, f_{ij}^{c}\right]$."

Li et al. [1] did not present any explanation for their assertion of (b) such that the location of the adjusted fuzzy preference relation contains questionable results.

Li et al. [1] also mentioned that "(c) If $f_{ij}^{y} \le f_{ij}^{c} \le f_{ij}^{\delta}$, then $\bar{f}_{ij}^{\delta} \in \left[f_{ij}^{c}, f_{ij}^{\delta}\right)$." Based on the same reason as Case (b), the authors did not inform us how to obtain their Case (c).

Li et al. [1] also noted that "(d) If $f_{ij}^{c} \le f_{ij}^{y} \le f_{ij}^{\delta}$, then $\bar{f}_{ij}^{\delta} \in \left[f_{ij}^{y}, f_{ij}^{\delta}\right)$."

For Case (d), we can expect that the authors wanted

$$\left|\bar{f}_{ij}^{\delta} - \bar{f}_{ij}^{c}\right| \ge \left|f_{ij}^{y} - \bar{f}_{ij}^{c}\right|, \tag{3.12}$$

owing to $f_{ij}^{\delta} > \bar{f}_{ij}^{\delta} \ge f_{ij}^{y} \ge f_{ij}^{c} > \bar{f}_{ij}^{c}$.

At last, the authors concluded that " (e) Otherwise, $\bar{f}_{ij}^{\delta} = f_{ij}^{\delta}$."

The authors only showed us the results for $\left(\bar{f}_{ij}^{\delta}\right)$. However, they did not provide us with the genuine meaning of how to develop those rules.

IV. Review of Their Tables and Our Comments

We rearrange Tables 3-5 of Li et al. [1], according to the value of CL_k.

Expert e_{19} in Table 3 has a rank of 14, but Expert e_{19} in Table 4 has a rank of 1 to indicate that the direction rule (a)-(e) on Page 7, left column, Lines 11-24 of Li et al. [1] must contain severe questionable results.

We have a interesting question for Equation (14) of Li et al. [1] and we provide our explanation in the following.

We quote the first row of Equation (14) of Li et al. [1] in the following, "If $f_{ij,t}^{\delta} \le f_{ij,t}^{y} \le f_{ij,t}^{c}$, then

$$f_{ij,t+1}^{\delta} = \left(f_{ij,t}^{y} - f_{ij,t}^{\delta}\right) / \alpha. \tag{4.1}$$

We recall Page 8, right column, Lines 28-31, of Li et al. [1], the matrix F^1, to imply $F_{23}^1 = 0.667$.

When $\alpha = 8$, we try to find the condition to derive

$$f_{23,2}^{\delta} = \frac{1}{8}\left(0.667 - f_{23,1}^{\delta}\right) < f_{23,1}^{\delta}, \tag{4.2}$$

that is

$$\frac{0.667}{8} \approx 0.08 < f_{23,1}^{\delta}. \tag{4.3}$$

TABLE I
The Comparison of Tables 3-5 of Li et al. [1].

Table 3		Table 4		Table 5	
CL_k	e_k	CL_k	e_k	CL_k	e_k
0.942	e_6	0.970	e_{19}	0.972	e_{19}
0.903	e_{17}	0.938	e_6	0.937	e_6
0.891	e_9	0.912	e_{17}	0.911	e_{17}
0.872	e_5	0.893	e_9	0.892	e_9
0.865	e_7	0.879	e_5	0.878	e_5
0.862	e_8	0.868	e_7	0.869	e_7
0.861	e_{13}	0.865	e_{13}	0.865	e_{13}
0.859	e_{15}	0.859	e_{16}	0.858	e_{16}
0.857	e_1	0.857	e_8	0.857	e_8
0.853	e_4	0.856	e_4	0.857	e_4
0.853	e_{16}	0.855	e_{15}	0.855	e_1
0.806	e_{10}	0.854	e_1	0.854	e_{15}
0.800	e_{11}	0.807	e_{11}	0.820	e_{14}
0.788	e_{19}	0.803	e_{10}	0.805	e_{11}
0.784	e_{14}	0.793	e_{14}	0.804	e_{10}
0.782	e_3	0.789	e_3	0.788	e_3
0.761	e_2	0.761	e_2	0.760	e_2
0.746	e_{20}	0.742	e_{18}	0.742	e_{18}
0.740	e_{18}	0.740	e_{20}	0.741	e_{20}
0.736	e_{12}	0.733	e_{12}	0.734	e_{12}

We check Page 8 for f_{23}^k, with $k = 1,2,...,20$ then s_1 three times, s_2 three times, s_3 two times, s_4 six times, s_5 five times, and s_6 one time, and then based on the above discussion, we know that for $k = 1,2,...,20,$ and $i = 1,2,...6$,

$$NS^k(s_i) \geq 0.157. \tag{4.4}$$

We show that our condition of

$$\frac{0.667}{8} \approx 0.08 < f_{23,1}^{\delta}, \tag{4.5}$$

is satisfied to indicate

$$f_{23,2}^{\delta} < f_{23,1}^{\delta}. \tag{4.6}$$

However, based on Page 7, rule (a), left column, Lines 11-13, of Li et al. [1],

$$\bar{f}_{ij}^{\delta} \in \left(f_{ij}^{\delta}, f_{ij}^{y}\right], \tag{4.7}$$

to imply that

$$f_{23,2}^{\delta} \geq f_{23,1}^{\delta}. \tag{4.8}$$

Therefore, we can assert that Equation (14) of Li et al. [1] that is cited as our Equation (4.1) has severe questions that will be further examined by future studies.

V. A RELATED PROBLEM

To compute the necessity mean for fuzzy numbers, Yoshida et al. [11] published a paper to introduction a sequence approach to define the limit of a point-wise convergent sequence of continuous fuzzy numbers to be the necessity mean for a general fuzzy number. In this note, we provide a simplified method to adopt the version for continuous fuzzy numbers for the necessity mean. After our modification, the sequence approach proposed by Yoshida et al. [11] can be avoided. Our theoretical derivation will be useful for researchers in the future to evaluate the necessity mean for fuzzy numbers.

Owing to the complexity of the real word and rapid changing in human society, fuzzy numbers becomes an appropriate tool to handle the uncertainty in the real word application problems. After researchers developed their objective functions in fuzzy environment, and then they have to use suitable means to defuzzy their models. For continuous fuzzy numbers, $\tilde{a} \in R_C$, the possibility evaluation measure, $M_{\tilde{a}}^{P}$;

the necessity mean, $M_{\tilde{a}}^{N}$ and the credibility mean, $M_{\tilde{a}}^{C}$, were proposed by the following definitions,

$$M_{\tilde{a}}^{P}(I)=\sup_{x\in I}\tilde{a}(x), \quad (5.1)$$

$$M_{\tilde{a}}^{N}(I)=1-\sup_{x\notin I}\tilde{a}(x), \quad (5.2)$$

$$M_{\tilde{a}}^{C}(I)=\frac{1}{2}\left(M_{\tilde{a}}^{P}(I)+M_{\tilde{a}}^{N}\right), \quad (5.3)$$

for $I\in\beta$, where β and I denote the Borel $\sigma-$field and the set of all non-empty bounded closed intervals in the real number, respectively.

For continuous fuzzy numbers, $\tilde{a}\in R_C$, with a compact support and

$$\max_{x}\tilde{a}(x)=1, \quad (5.4)$$

then

$$M^{P}(\tilde{a})=1, \quad (5.5)$$

$$M^{N}(\tilde{a})=1-\alpha, \quad (5.6)$$

and

$$M^{P}(\tilde{a})=1-(\alpha/2). \quad (5.7)$$

Yoshida et al. [11] introduced mean values of a fuzzy number $\tilde{a}$ with an evaluation measure $M_{\tilde{a}}$ as follows,

$$\tilde{E}(\tilde{a})=\int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)g(\tilde{a}_\alpha)d\alpha\Big/\int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)d\alpha \quad (5.8)$$

where

$$g([x,y])=(x+y)/2, \quad (5.9)$$

is the 0.5-weighting function and $\tilde{a}_\alpha$ is the expression of an $\alpha-$cut of a fuzzy number $\tilde{a}$, denoted as $\left[a_\alpha^L, a_\alpha^U\right]$.

Yoshida et al. [11] pointed out that for continuous fuzzy sets, $\tilde{a}\in R_C$, then

$$\tilde{E}^{P}(\tilde{a})=\int_0^1\left(a_\alpha^L+a_\alpha^U\right)d\alpha\Big/2, \quad (5.10)$$

$$\tilde{E}^{N}=\int_0^1(1-\alpha)\left(a_\alpha^L+a_\alpha^U\right)d\alpha, \quad (5.11)$$

and

$$\tilde{E}^{C}(\tilde{a})=\int_0^1(2-\alpha)\left(a_\alpha^L+a_\alpha^U\right)d\alpha\Big/3. \quad (5.12)$$

For fuzzy sets, $\tilde{a}\in R$, Yoshida et al. [11] developed a sequence approach to construct a sequence, $\left\{\tilde{a}^n\right\}_{n=1}^{\infty}$ $(\subset R_C)$, with continuous fuzzy numbers $\tilde{a}^n\in R_C$ which point-wise converges to $\tilde{a}$, and then

$$\tilde{E}(\tilde{a})=\lim_{n\to\infty}\tilde{E}\left(\tilde{a}^n\right). \quad (5.13)$$

They mentioned that if the limit of the sequence of Equation (5.13) is independent of the selection of the sequence $\left\{\tilde{a}^n\right\}_{n=1}^{\infty}\subset R_C$ and then it is assumed to be well-defined.

VI. Our Improvement

We will claim that directly use $1-\alpha$ then to generate a point-wise convergent sequence becomes unnecessary. For discontinuous membership function $\tilde{a}$, to construct a sequence $\left(\tilde{a}^n\right)$ of continuous membership function point-wise converge to it,

$$\lim_{n\to\infty}\tilde{a}^n(x)=\tilde{a}(x), \quad (6.1)$$

for $-\infty<x<\infty$. On the other hand, the necessity mean M^N satisfies the following relationship,

$$\lim_{n\to\infty}M^N\left(\tilde{a}^n\right)\neq M^N(\tilde{a}). \quad (6.2)$$

In the following, we provide our reasoning. Owing to the weighted functions are different for (a) continuous fuzzy number, and (b) discontinuous fuzzy number, such that to obtain different results is a normal phenomenon. Yoshida et al. [11] convert a questionable problem into the motivation of their definition.

Our goal is to construct another sequence $\left(\tilde{b}^n\right)$ of continuous membership function point-wise converge to it, that is,

$$\lim_{n\to\infty}\tilde{b}^n(x)=\tilde{a}(x), \quad (6.3)$$

for $-\infty<x<\infty$.

However, for the necessity mean M^N satisfies the following inequality,

$$\lim_{n\to\infty}M^N\left(\tilde{a}^n\right)\neq\lim_{n\to\infty}M^N\left(\tilde{b}^n\right), \quad (6.4)$$

that is definition of which is related to the continuous sequence.

Yoshida et al. [11] used

$$1-\frac{1}{n}, \quad (6.5)$$

and

$$4+\frac{1}{n}, \quad (6.6)$$

for their $\alpha-$cut.

We try to use $1-\frac{x}{n}$ and $4+\frac{y}{n}$ with $x>0$ and $y>0$, for our $\alpha-$cut.
Hence, our cut have the following expression,

$$\left[\frac{2x\alpha+n-x}{n},\frac{-2y\alpha+4n+y}{n}\right], \quad (6.7)$$

for $0\le\alpha\le 1/2$.

We will compute

$$2\int_0^{1/2}(1-\alpha)\frac{1}{2}\left(\frac{2x\alpha+n-x}{n}+\frac{-2\alpha y+4+y}{n}\right)d\alpha$$
$$=\int_0^{1/2}(1-\alpha)\left(5+\frac{1}{n}A(\alpha)\right)d\alpha, \quad (6.8)$$

where

$$A(\alpha)=2x\alpha-2y\alpha-x+y, \tag{6.9}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{1/2}(1-\alpha)A(\alpha)d\alpha=0. \tag{6.10}$$

We provide a possible explanation for the above inequality,

$$M^N\left(\tilde{a}_\alpha^n\right)=1-\alpha\neq$$

$$M^N\left(\tilde{a}_\alpha\right)=\begin{cases}1, & 0\le\alpha<0.5\\ 1-\alpha, & 0.5\le\alpha\le 1\end{cases}. \tag{6.11}$$

At last, not the least, we point out the result of Yoshida et al. [11] should be revised from

$$\tilde{E}^P(\tilde{a})=\int_0^1 g(\tilde{a}_\alpha)d\alpha, \tag{6.12}$$

to an improved version,

$$\tilde{E}^P(\tilde{a})=\lim_{n\to\infty}\int_0^1 g\left(\tilde{a}_\alpha^n\right)d\alpha, \tag{6.13}$$

where $\left(\tilde{a}^n\right)$ is a sequence of continuous fuzzy number that point-wise converges to $\tilde{a}$.

Moreover, the findings of Yoshida et al. [11] is only valid for continuous fuzzy numbers. For the general case, their findings should be revised to

$$\tilde{E}^N(\tilde{a})=$$

$$\lim_{n\to\infty}\int_0^1 M^N\left(\tilde{a}_\alpha^n\right)g\left(\tilde{a}_\alpha^n\right)d\alpha\Big/\int_0^1 M^N\left(\tilde{a}_\alpha^n\right)d\alpha. \tag{6.14}$$

VII. Related Studies

In this section, we recall several recently published articles that are important to help researchers locate possible examined topics. Liu [12] developed inventory systems with salvage value and shortages with respect to decay products to locate the optimal solution through analytical solution procedure. Referring to integration method for over estimation criterion, Batiha et al. [13] examined a high dimensional linear question to study a time series of parabolic and fractional systems. Suksamran et al. [14] considered solidity examination and diagnostic resolutions for infection related to diffusion and reaction models. For junior-high schools, Side et al. [15] employed time series system to learn video game compulsion. According to illustrated renew mapping, Aung et al. [16] studied programming models of individual proposal performance. For shortages charge under unclear environment, Liu [17] constructed revisions with respect to Newsboy dilemma with a Stochastic setting. Based on multiple dimensional fault improvement, Lubis et al. [18] took analytical commerce for undersized holder. In a dual composition, Manilam and Pochai [19] obtained a mathematical system for coast line progression by a finite stable difference procedure. To realize features pattern recognitions, Wang et al. [20] found out categorization with norm representation. For scrutinizing multiple images, Umilasari et al. [21] got control families with neighborhood resolution. Based on normal mapping system, Feng et al. [22] gained digital figure structure with little resolution. Referring to complicated vehicle data record, Paul et al. [23] acquired marine approaches with figure density. Based on our above literature reviewing, practitioners can locate interesting research trend for their future academic developments.

VIII. Another Related Improvement

In Yang and Chen [24], they tried to solve an open question discussed in Osler [25]. We believe that Yang and Chen [24] already solved the open question discussed in Osler [25]. However, the open question proposed by Osler [25] already have appeared in Sofo [26]. Hence, the original source should not be contributed to Osler [25], and then it should be revised to Sofo [26].

Sofo [26] tried to study three kinds of cubic root or radical root problems. The open question examined by Osler [25] just one of these three directions of examined problems. Hence, we will provide a comprehensive study for Sofo [26] to present a patchwork of Yang and Chen [24].

Sofo [26] tried to solve the following three kinds of problems. In the beginning, Sofo [26] assumed that

$$\left(2+\sqrt{5}\right)^{1/3}+\left(2-\sqrt{5}\right)^{1/3}=U, \tag{8.1}$$

to derive that

$$U^3+3U-4=0, \tag{8.2}$$

and then Sofo [26] demonstrated that

$$U=1, \tag{8.3}$$

is the unique real solution to indicate that the solution procedure of Sofo [26] containing a discussion of complex variables.

For a more general setting, that satisfies

$$\left(p+\sqrt{q}\right)^{1/3}+\left(p-\sqrt{q}\right)^{1/3}=1, \tag{8.4}$$

with p and q are natural numbers, Sofo [26] found that

$$p=3n-1, \tag{8.5}$$

and

$$q=n^2(8n-3), \tag{8.6}$$

for $n=1,2,...$ to indicate that contains infinite pair of natural number solutions.

For the third kind problem of

$$\left(p_1+\sqrt{q_1}\right)^{1/n}+\left(p_1-\sqrt{q_1}\right)^{1/n}=1, \tag{8.7}$$

for $n=3,4,...$, where p and q are positive numbers, Sofo [26] claimed that for a given n, he obtained at least a pair of rational number solutions with

$$p_1=\frac{F_n}{2}+F_{n-1}, \tag{8.8}$$

and

$$q_1=\frac{5}{4}F_n^2, \tag{8.9}$$

where (F_n) are Fibonacci sequence, with

$$F_1=1, \tag{8.10}$$

$$F_2=1, \tag{8.11}$$

and

$$F_{n+1}=F_n+F_{n-1}, \tag{8.12}$$

for $n=2,3,4,...$.

Based on our review for Sofo [26], we will provide a following possible direction for future researchers. To generalize Fibonacci sequence in a more general setting:

$$F_{n+1} = xF_n + yF_{n-1}, \tag{8.13}$$

for $n \geq 2$.

Based on our above discussion, we provide a detailed review of Sofo [26] to present a patchwork for Yang and Chen [24].

IX. Further Discussion for Consistent Test

Recently, Wang et al. [27] examined Ardalan [28], Aull-Hyde [29], and Chu et al. [30] for inventory systems with temporary price discounts. Wang et al. [27] provided an excellent mathematical work to obtain a compact proof for the decreasing assertion of the objective function proposed by Chu et al. [30]. Consequently, the restriction proposed by Chu et al. [30] to assume the domain is the finite union of closed intervals is revised to a natural setting of "closed set" which is a significant revision of Chu et al. [30]. However, there are several questionable results mentioned in Aull-Hyde [29] that still left open. Hence, in this section, we will present a further study for Aull-Hyde [29].

If we consider an Analytic Hierarchy Process problem with three alternatives, A_1, A_2 and A_3, and there are many decision makers, D_k, for $1 \leq k \leq m$. In Saaty [31], each decision maker, say D_k, by pairwise comparison to decide the relative weight for the 1-9 scale then the individual comparison matrix, denoted as $I_k = \lfloor a_{ijk} \rfloor_{3\times 3}$, that satisfies

$$a_{ijk} \in \{9,8,\ldots,1,1/2,\ldots,1/9\}, \tag{9.1}$$

and

$$a_{ijk} a_{jik} = 1. \tag{9.2}$$

On the other hand, in Aull-Hyde [29], she arbitrarily constituted a group of n persons, say P_s, for $1 \leq s \leq n$, with $n = 90$, and then randomly assign a reciprocal comparison matrix, say $M_s = \lfloor b_{ijs} \rfloor_{3\times 3}$ with b_{ijs} is randomly selected from $\{9,8,\ldots,1,1/2,\ldots,1/9\}$.

The group comparison matrix, say $G = \lfloor b_{ij} \rfloor_{3\times 3}$, with

$$b_{ij} = \left(\prod_{s=1}^{n} b_{ijs} \right)^{1/n}. \tag{9.3}$$

From the simulation experiment of Aull-Hyde [29], G will pass the consistency test. If we follow this trend to further extend the group from 90 to 900 and 9000 and then our simulation results will all pass the consistency test.

From our three proposed tests, it demonstrates that if we (a) extend the individual group to a sufficient large group, and (b) randomly assign the individual comparison matrix, then the normalized group priority vector will converge to the uniform weight as follows,

$$(1/3,1/3,1/3)^T, \tag{9.4}$$

since the group comparison matrix pass the consistency test, then the equally weight results will be derived.

It reveals that by the approach proposed by Aull-Hyde [29], then we will derive that every alternative has the same weight. It is not a reasonable conclusion for group decision environment. Hence, we can conclude that something must be wrong in the geometric mean average algorithm proposed by Aull-Hyde [29].

Since $\lim_{n\to\infty} (a_{ijn})^{1/n} = 1$, if a_{ijn} is randomly select.

There is a threshold, say ε (or say R) and a number, say m, if $n \geq m$ such then

$$1 - \varepsilon \leq (a_{ijn})^{1/n} \leq 1 + \varepsilon, \tag{9.5}$$

or

$$1/R \leq (a_{ijn})^{1/n} \leq R. \tag{9.6}$$

Based on our above discussion, we recall the next theory proposed by Aull-Hyde [29].

Theorem 1. (Aull-Hyde [29]) If group comparison matrix with entries between $1/R$ and R, then the group matrix will pass the consistency test, when the group of experts is sufficient large.

If we pick a group of twenty decision makers such that their group 3×3 comparison matrix, say A, is not passed the Saaty's consistent test, since from the data of Aull-Hyde [29], there are 92% simulation trials that passed the consistent test. In other words, there are 8% simulation trials of 1000 experiments such that we have 80 selections. Hence, we assume that there are twenty individual comparison matrices $M_k = \lfloor a_{ijk} \rfloor_{3\times 3}$ for $k = 1,2,\ldots,20$ and $A = \lfloor a_{ij} \rfloor_{3\times 3}$ with the following group decision matrix,

$$a_{ij} = \left(\prod_{k=1}^{20} a_{ijk} \right)^{1/20}. \tag{9.7}$$

Since the group decision matrix, A does not pass the consistency test, by Saaty [31], there must be some improvement for some elements in A. However, if we arbitrarily add another three alternatives, say A_4, A_5 and A_6 and then we construct individual comparison matrix, say $N_k = \lfloor b_{ijk} \rfloor_{6\times 6}$ where $b_{ijk} = a_{ijk}$ for $1 \leq i, j \leq 3$, with $k = 1,2,\ldots,20$, and others b_{ijk} are arbitrarily decided from the 1-9 scales of Saaty [31] that satisfy the reciprocal property of Equation (9.2).

We assume the new group comparison matrix, say $B = \lfloor b_{ij} \rfloor$ with

$$b_{ij} = \left(\prod_{k=1}^{20} b_{ijk} \right)^{1/20}. \tag{9.8}$$

According to Aull-Hyde [29], when group size is twenty, then the group comparison matrix has 100% acceptance. Therefore, researcher will accept the priority vector derived by B then to decide the ranking of A_n for $n = 1,2,\ldots 6$. If we ignore A_n for $n = 4,5,6$, then we can obtain the ranking of A_n for $n = 1,2,3$.

If we carefully examine the above procedure, for different construction of N_k for $k = 1,2,\ldots,20$, there may imply different ranking for of A_n for $n = 1,2,3$.

When generating the individual comparison matrices $A=(a_{ij})$, the a_{ij} entries are randomly selected from the following set $\{1/9,1/8,...,1/2,1/1,1,2,...,8,9\}$. It means that there are 18 numbers where the probability to select "1" is $2/18$ and the probability for other values are $1/18$.
However, researchers assumed that there are 17 possible values for the Saaty's 1-9 scale such that the probability to select "1" is $1/17$.
We may predict that Aull-Hyde [29] on purpose added the probability to select "1" to increase the chance for the synthesized group comparison matrix to have entries close to 1. Therefore, we rerun the same numerical experiment of Aull-Hyde [29] with the selection family of Equation (9.1) that is consistent with the major research trend in Analytic Hierarchy Process.
The claim in Appendix A of Aull-Hyde [29] should not be derived by the Hospital rule. It should be owing to the fact that the selection set $\{1/9,1/8,...,1/2,1/1,1,2,...,8,9\}$ of Equation (9.1) having the geometric mean equals to 1.
For example, if we select the entries, a_{ij}, for the individual comparison matrix, not followed the 1-9 scale rule but from the following set, $\{2\}$. It means that we always assume that $a_{ij}\equiv 2$, then we compute the limit problem

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n}a_{ijk}\right)^{1/n}=\lim_{n\to\infty}\left(\prod_{k=1}^{n}2\right)^{1/n}=\lim_{n\to\infty}\left(2^n\right)^{1/n}=2. \tag{9.9}$$

Equation (9.9) indicates that the Hospital rule cannot be applied for this problem. In the next, we provide an improvement for the proof for Appendix A of Aull-Hyde [29].
By the weak law of large number, we know that $\frac{1}{n}\left(\sum_{k=1}^{n}\ln a_{ijk}\right)$ will converge to its mean of $\frac{1}{18}\sum_{k=1}^{9}\left(\ln k+\ln\frac{1}{k}\right)$. We derive that

$$\frac{1}{18}\sum_{k=1}^{9}\left(\ln k+\ln\frac{1}{k}\right)=\frac{1}{18}\sum_{k=1}^{9}\left(\ln k-\ln k\right)=0, \tag{9.10}$$

From

$$\ln\left(\lim_{n\to\infty}\left(\prod_{k=1}^{n}a_{ijk}\right)^{1/n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln a_{ijk}=0, \tag{9.11}$$

we obtain that

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n}a_{ijk}\right)^{1/n}=1, \tag{9.12}$$

as proposed by Aull-Hyde [29].

X. Conclusion

In this paper, we recall the important but doubtful findings of Li et al. [1] regarding their formulas expressions, theoretical analysis, and numerical examples, and then we raise our comments to point out their vital but questionable derivations. Our comments will help practitioners face the problems in individual consistency control with fuzzy information. In the direction of future research, we list the following four papers: Baek et al. [32], Dong and Herrera-Viedma [33], Parajuli et al. [34], and Rahman et al. [35] that are worthy to mention to pay attention to them.

References

[1] C. Li, R. M. Rodriguez, L. Martinez, Y. Dong, F. Herrera, "Consensus building with individual consistency control in group decision making," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 2, 2019, pp. 319-332.

[2] D. D. Alexakis, A. Sarris, "Integrated GIS and remote sensing analysis for landfill sitting in Western Crete, Greece," *Environmental Earth Sciences*, vol. 72, no. 2, 2014, pp. 467–482.

[3] C. W. Tsui, U. P. Wen, "A hybrid multiple criteria group decision-making approach for green supplier selection in the TFT-LCD industry," *Mathematical Problems in Engineering*, vol. 2014, Article ID 709872, 2014, 13 pages.

[4] F. Feizi, A. KarbalaeiRamezanali, E. Mansouri, "Calcic iron skarn prospective mapping based on fuzzy AHP method, a case study in Varan area, Markazi province, Iran," *Geosciences Journal*, vol. 21, no. 1, 2017, pp. 123–136.

[5] M. S. Roodposhti, S. Rahimi, M. J. Beglou, "PROMETHEE II and fuzzy AHP: An enhanced GIS-based landslide susceptibility mapping," *Natural Hazards*, vol. 73, no. 1, 2014, pp. 77–95.

[6] N. Pourghahreman, A. Rajabzadeh Qhatari, "Supplier selection in an agent-based pharmaceutical supply chain: An application of TOPSIS and PROMETHEE II," *Uncertain Supply Chain Management*, vol. 3, no. 3, 2015, pp. 231–240.

[7] Z. Andreopoulou, C. Koliouska, C. Lemonakis, C. Zopounidis, "National Forest Parks development through Internet technologies for economic perspectives," *Operational Research*, vol. 15, no. 3, 2015, pp. 395–421.

[8] N. Vesyropoulos, C. K. Georgiadis, "QoS-Based Filters in Web Service Compositions: Utilizing Multi-Criteria Decision Analysis Methods," *Journal of Multi-Criteria Decision Analysis*, vol. 22, no. 5-6, 2015, pp. 279–292.

[9] M. Hamzeh, R. Ali Abbaspour, R. Davalou, "Raster-based outranking method: a new approach for municipal solid waste landfill (MSW) siting," *Environmental Science and Pollution Research*, vol. 22, no. 16, 2015, pp. 12511–12524.

[10] L. Zhang, J. Yu, B. K. Sovacool, J. Ren, "Measuring energy security performance within China: Toward an inter-provincial perspective," *Energy*, vol. 125, 2017, pp. 825–836.

[11] Y. Yoshida, M. Yasuda, J. Nakagami, M. Kurano, "A new evaluation of mean value for fuzzy numbers and its applications to American put option under uncertainty," *Fuzzy Sets and Systems*, vol. 157, 2006, pp. 2614-2626.

[12] Jinyuan Liu, "Inventory Models of Decaying Items with Shortages and Salvage Value," IAENG International Journal of Applied Mathematics, vol. 54, no. 2, pp308-316, 2024

[13] Iqbal M. Batiha, Amal Benguesmia, Taki-Eddine Oussaeif, Iqbal H. Jebril, Adel Ouannas, and Shaher Momani, "Study of a Superlinear Problem for a Time Fractional Parabolic Equation Under Integral Over-determination Condition," IAENG International Journal of Applied Mathematics, vol. 54, no. 2, pp187-193, 2024

[14] Jeerawan Suksamran, Somkid Amornsamankul, and Yongwimon Lenbury, "Reaction-Diffusion-Integral System Modeling SARS-CoV-2 Infection-Induced versus Vaccine-Induced Immunity: Analytical Solutions and Stability Analysis," IAENG International Journal of Applied Mathematics, vol. 54, no. 2, pp223-231, 2024

[15] Syafruddin Side, Miftahul Jannah, Abdul Saman, Muhammad Isbar Pratama, and Wahidah Sanusi, "Optimal Control and Analysis of the SEIRS Model on the Problem of Online Game Addiction : A Case Study Among Class VIII Students of the State Junior High Schools in Makassar City," IAENG International Journal of Applied Mathematics, vol. 54, no. 2, pp232-237, 2024

[16] Soe Thandar Aung, Lynn Htet Aung, Nobuo Funabiki, Shigo Yamaguchi, Yan Watequlis Syaifudin, and Wen-Chung Kao, "An Implementation of Web-based Personal Platform for Programming Learning Assistant System with Instance File Update Function," Engineering Letters, vol. 32, no. 2, pp226-243, 2024

[17] Jinyuan Liu, "Improvement for a Stochastic Newsboy Problem with Fuzzy Shortages Cost," Engineering Letters, vol. 32, no. 2, pp305-314, 2024

[18] Anita Rizky Lubis, Nelva Meyriani Br Ginting, Sri Fajar Ayu, Rezzy Eko Caraka, Yunho Kim, Prana Ugiana Gio, and Bens Pardamean, "Business Predictive Analytics of Smallholder Indonesian Maize using Vector Error Correction," Engineering Letters, vol. 32, no. 2, pp429-435, 2024

[19] Surasak Manilam, and Nopparat Pochai, "An Unconditionally Stable Explicit Finite Difference Method for a Non-Dimensional Mathematical Model of Shoreline Evolution with a Twin Groins Structure," Engineering Letters, vol. 32, no. 2, pp296-304, 2024

[20] Jing Wang, Xiao Xie, Li Zhang, Xusheng Chen, Hongzhou Yue, and Huaping Guo, "Generalized Representation-based Classification by Lp-norm for Face Recognition," IAENG International Journal of Computer Science, vol. 51, no. 2, pp104-114, 2024

[21] Ting-Chen Yang, and Yu-Chuan Chen, "Solution for Open Questions in Yen (2021) Osler (2001) and Caliskan (2020, 2022)," IAENG International Journal of Applied Mathematics, vol. 53, no.1, pp48-57, 2023

[22] Xinling Feng, Cuiqing Zhu, and Zixia Ge, "Research on Low Resolution Digital Image Reconstruction Method Based on Rational Function Model," IAENG International Journal of Computer Science, vol. 51, no. 2, pp75-82, 2024

[23] Raj Kumar Paul, Ankit Vishwakarma, and Saravanan Chandran, "An Aquatic Image Compression Scheme based on Optimized Deep Convolutional Autoencoder," IAENG International Journal of Computer Science, vol. 51, no. 2, pp83-90, 2024

[24] Ting-Chen Yang, and Yu-Chuan Chen, "Solution for Open Questions in Yen (2021) Osler (2001) and Caliskan (2020, 2022)," IAENG International Journal of Applied Mathematics, vol. 53, no.1, pp48-57, 2023

[25] T. J. Osler, "An easy look at the cubic formula," *Mathematics and Computer Education*, vol. 36, 2002, pp. 287-290.

[26] A. Sofo, "Generalization of a radical identity," *The Mathematical Gazette*, vol. 83, 1999, pp. 274-276.

[27] Chao-Min Wang, Yu-Lan Wang, and Ming-Li Chen, "Discussion for Consistency Test in Analytic Hierarchy Process," IAENG International Journal of Applied Mathematics, vol. 53, no.3, pp1123-1131, 2023

[28] A. Ardajan, "Optimal ordering policies in response to a sale," *IIE Transactions*, vol. 20, 1988, pp. 292-294.

[29] R. L. Aull-Hyde, "Evaluation of supplier-restricted purchasing options under temporary price discounts," *IIE Transactions*, vol. 24, 1992, pp. 184-186.

[30] P. Chu, P. S. Chen, T. Niu, "Note on supplier-restricted order quantity under temporary price discounts," *Mathematical Methods of Operations Research*, vol. 58, 2003, pp. 141-147.

[31] T.L. Saaty, *The analytic hierarchy process*, New York: McGraw-Hill, 1980.

[32] C. Y. Baek, K. H. Park, K. Tahara, Y. Y. Chun, “Data quality assessment of the uncertainty analysis applied to the greenhouse gas emissions of a dairy cow system,” *Sustainability*, vol. 9, no. 10, 2017.

[33] Y. Dong, E. Herrera-Viedma, “Consistency-Driven Automatic Methodology to Set Interval Numerical Scales of 2-Tuple Linguistic Term Sets and Its Use in the Linguistic GDM with Preference Relation,” *IEEE Transactions on Cybernetics*, vol. 45, no. 4, 2015, pp. 780–792.

[34] R. Parajuli, M. T. Knudsen, T. Dalgaard, “Multi-criteria assessment of yellow, green, and woody biomasses: Prescreening of potential biomasses as feedstocks for biorefineries,” *Biofuels, Bioproducts and Biorefining*, vol. 9, no. 5, 2015, pp. 545–566.

[35] M. A. Rahman, L. Jaumann, N. Lerche, “Selection of the Best Inland Waterway Structure: A Multicriteria Decision Analysis Approach,” *Water Resources Management*, vol. 29, no. 8, 2015, pp. 2733–2749.

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