Maximizing Deviations Method in Intuitionistic Fuzzy Setting

Jinyuan Liu

Abstract—**We studied a paper to apply a method of maximizing deviations to multiple attribute decision-makings under intuitionistic fuzzy environment that have found several new methods and theorems for maximum problem under some specific conditions with insufficient information environment. We showed that the Lagrange multiplication method used by the paper can be replaced by our simplify approach with the Cauchy Schwarz inequality. The purpose of this paper is fourfold. First, the iteration method for the problem within the range of weights is well developed and with appropriate explanation if the weight vector of attributes is bounded. Second, if the weight vector of attributes is completely unknown, we could directly and swiftly derive the weight by the Cauchy–Schwarz inequality such that the complicated approach by the Lagrange multiplication method becomes redundant. Third, we prove the results of score function and rank for one-norm will not be preserved in the two-norm. Fourth, the same numerical examples are examined again and have different outcome to demonstrate our findings is superior to the previously published results.**

Index Terms—**Maximizing deviation method, Intuitionistic fuzzy sets, Intuitionistic fuzzy weighted averaging operator, Multiple attribute decision making**

I. Introduction

In this paper, we developed two solution approaches to revise the questionable results of Wei [1] for (a) attribute weights are partially known and (b) attribute weights are completely unknown under one norm, for maximum deviation problem under intuitionistic fuzzy environment. There are various multiple attribute decision-making problems in which the information cannot be collected and evaluated precisely in an exact quantitative way but may be expressed in a fuzzy numbers to describe the data. At the same time, researchers are no longer able to use the well-developed analytical approach to determine the optimal solution under fuzzy environment. Hence, many different procedures for computing a compromise solution under some specific situation have been proposed.

Atanassov [2] initially developed the concept of intuitionistic fuzzy set which is the generalization of the concept of the fuzzy set. Xu [3] developed several arithmetic aggregation operators which are intuitionistic fuzzy weighted averaging operator, the intuitionistic fuzzy ordered weighted averaging operator and the intuitionistic fuzzy hybrid aggregation operator. These operators functioned with respect to their own characteristic in order to reduce the fuzzy problem to a classical mathematical optimization problem. Xu [4] proposed a method regard to the intuitionistic fuzzy multiple attribute decision making with the information about attribute weights which is incompletely known or completely unknown. Consequently, Wei [1] revised Xu [4] by using the normalized Hamming distance in Atanassov [2] and Herrera and Herrera-Viedma [5] for the maximizing deviation of accuracy and score functions among alternatives to convert a fuzzy problem to a non-linear programming model. However, Wei [1] did not explain how he would solve the maximization model with incompletely known weights.

The essence of this problem is not non-linear actually and the result of the answer through this method was wrong all of these will be discussed in this paper. On the other hand, he considered another so-called "non-linear programming model (M-2)" with completely unknown weights. Moreover, he did not show how to solve this linear programming problem when the information is partially known in (M-1). Instead, Wei's approach becomes very lengthy and complicated under the completely unknown situation. According to the method for solving maximum-minimum problem in the context of "Operations Research", mostly, objective functions are fulfilling under many different restrictions to fulfill the demand. But for multiple decision problems, many of their restriction are confined to one-norm for weights which is only one constraint $\sum_{j=1}^{n} w_j = 1$ with respect to the weights when information is completely unknown.

The benefit for this phenomenon makes this kind of maximum and minimum problem linear and simpler. We can directly solve the maximum deviation problem with one-norm that will simplify the solution procedure in Wei [1]. Wei [1] also exploited the maximum deviation problem under the restriction $\sum_{j=1}^{n} w_j^2 = 1$ of two-norm to solve the same problem but without any explanation for the benefit and property on this issue. We will prove that two-norm has different results for ranking with one-norm. The purpose of this paper is to provide a patchwork to simplify and enhance the solution structure of Wei [1].

Several related papers of Cheng [7], Wang [8], Wu and Chen [9], and Yang [10] were worthy to mention to indicate the hot spot of research trend.

II. Review of Their Approach

We directly adopt the same notation and assumptions as Wei [1]. The family of a discrete set of m alternatives is denoted as

$$A = \{A_1, \ldots, A_m\}, \tag{2.1}$$

and the family of a discrete set of n attributes is denoted as

Manuscript received January 14, 2024; revised July 2, 2024.

Jinyuan Liu is an Associate Professor in the School of General Studies, Weifang University of Science and Technology, Weifang 262799, China (e-mail: shgljy@126.com).

$$G = \{G_1, \ldots, G_n\}. \quad (2.2)$$

The information about attribute weights is allowed to be completely unknown or incompletely known.

An intuitionistic set is expressed as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \quad (2.3)$$

which is denoted by a membership mapping $\mu_A: X \to [0,1]$, $\mu_A(x)$ points out the scale of connection of an element x to the set A and $v_A(x)$ denotes the scale of non-relationship of an element x to the set A.

An intuitionistic fuzzy decision matrix is expressed as

$$\widetilde{R} = (\widetilde{r}_{ij})_{m \times n} = ((\mu_{ij}, \nu_{ij}))_{m \times n}, \quad (2.4)$$

where μ_{ij} denotes the scale that the alternative A_i satisfies the attribute G_j given by the decision maker, and v_{ij} denotes the scale that the alternative A_i dissatisfies the attribute G_j given by the decision maker, under the normalization restriction as

$$\mu_{ij} + \nu_{ij} \le 1. \quad (2.5)$$

The scale of undecided of the element x to intuitionistic fuzzy set A is expressed as

$$\pi_{ij} = 1 - (\mu_{ij} + \nu_{ij}), \quad (2.6)$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, n$.

We define a simplified expression as

$$\tilde{a} = (\mu, \nu), \quad (2.7)$$

to be an intuitionistic number.

There are several methods to de-fuzzy an intuitionistic fuzzy set to a real number. We recall the approach of Zadeh [6] as follows. The score function,

$$S(\tilde{a}) = \mu - v, \quad (2.8)$$

with $-1 \le S(\tilde{a}) \le 1$, and the accuracy function,

$$H(\tilde{a}) = \mu + v, \quad (2.9)$$

with $0 \le H(\tilde{a}) \le 1$. As follows, we also explain several useful basic terminologies with respect to intuitionistic fuzzy sets.

If $\widetilde{R} = (\widetilde{r}_{ij})_{m \times n} = ((\mu_{ij}, \nu_{ij}))_{m \times n}$ is an intuitionistic fuzzy decision matrix for alternatives related to attributes, and then for the synthesized attribute value for alternative A_i, Xu [3] assumed the intuitionistic fuzzy weighted averaging operator as follows, for $i = 1,2, \ldots, m$,

$$\widetilde{r_i} = (\mu_i, v_i)$$

$$= \mathrm{IFWA}_w(\widetilde{r_{i1}}, \widetilde{r_{i2}}, \ldots, \widetilde{r_{in}})$$

$$= \left(1 - \prod_{j=1}^{n}(1 - \mu_{ij})^{w_j}, \prod_{j=1}^{n} v_{ij}{}^{w_j}\right), \quad (2.10)$$

where

$$\tilde{r}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}), \quad (2.11)$$

is the vector of attribute values corresponding to the alternative A_i, under the condition $\sum_{j=1}^{n} w_j = 1$ for one-norm and $\sum_{j=1}^{n} w_j^2 = 1$ for two-norm where $w = \{w_1, \ldots, w_n\}$ is the weight vector of attributes, where $w_j \ge 0$ for $j = 1, \ldots, n$.

If $\tilde{a}_1 = (\mu_1, \nu_1)$ and $\tilde{a}_2 = (\mu_2, \nu_2)$ are two intuitionistic fuzzy numbers, then the normalized Hamming distance between $\tilde{a}_1 = (\mu_1, \nu_1)$ and $\tilde{a}_2 = (\mu_2, \nu_2)$ is assumed in the following,

$$d(\tilde{a}_1, \tilde{a}_2) = \frac{1}{2}(|\mu_1 - \mu_2| + |\nu_1 - \nu_2|). \quad (2.12)$$

Wei [1] applied the maximum deviation method to compute the differences between the performance values of two alternatives. For the attribute G_j, the deviation of alternative A_i to all the other alternatives is defined as follows:

$$D_{ij}(w) = \sum_{k=1}^{m} d(\tilde{r}_{ij}, \tilde{r}_{kj}) w_j, \quad (2.13)$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ is the renormalized Hamming distance between $\widetilde{r}_{ij} = (\mu_{ij}, \nu_{ij})$, and $\widetilde{r}_{kj} = (\mu_{kj}, \nu_{kj})$. Wei [1] assumed $D_j(w)$ to indicate the deviation value of all alternatives to other alternatives for the attribute $G_j \in G$ that is assumed as

$$D_j(w) = \sum_{i=1}^{m} D_{ij}(w)$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{m} d(\widetilde{r_{ij}}, \widetilde{r_{kj}}) w_j, \quad (2.14)$$

for $j = 1, \ldots, n$. When the weight vector of attributes was incompletely known, that is there are lower bound and upper bound for weights, Wei [1] tried to choose the weight vector to maximize deviation values such that he constructed the following problem, denoted as (M-1) model,

$$\max D(w) = \sum_{j=1}^{n} D_j(w)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} d(\widetilde{r_{ij}}, \widetilde{r_{kj}}) w_j, \quad (2.15)$$

subject to $w_j \geq 0$ for $j = 1, \ldots, n$, $w \in H$, $\sum_{j=1}^{n} w_j = 1$, and satisfies the following restrictions,

$$a_j \leq w_j \leq b_j, \tag{2.16}$$

where b_j is the upper bound and a_j is the lower bound for the attribute weights that was derived by reliable sources or experts. For the situation about the completely unknown information, the main objective function in (M-2) model is the same as (M-1) except the relative weights of w_j is completely unknown in (M-2), that is, there is no restriction on attribute weights as Equation (2.16).

There are two most common possible norms for (M-2) model: two-norm and one-norm, that is, $\sum_{j=1}^{n} w_j^2 = 1$, and $\sum_{j=1}^{n} w_j = 1$, respectively.

Wei [1] applied the Lagrange multiplication method to solve the maximization problem (M-2) of Equation (2.15) and then he normalized the weights w_j^* for $j = 1, \ldots, n$ to imply the whole sum being one so that

$$w_j^* = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right)}{\sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right)}, \tag{2.17}$$

under the restriction of one-norm, $\sum_{j=1}^{n} w_j = 1$ and

$$w_j^* = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right)}{\sqrt{\sum_{j=1}^{n} \left[\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right)\right]^2}}, \tag{2.18}$$

under the restriction of two-norm, $\sum_{j=1}^{n} w_j^2 = 1$.

III. Our Improvement

For the (M-1) model, Wei tried to apply a numerical example to demonstrate his developed method for partially known information case. However, in the numerical example, we will show that Wei [1] did not obtain the optimal solution. Moreover, he also did not explain why the optimal solution for the (M-2) model of two-norm could be directly derived by re-normalization of that of 1-norm. In this paper, we will point that Wei's approach for (M-2) model with two-norm is corrected but his approach for (M-2) model with one-norm is false. We will find an improvement method for (M-1) model and a revised approach for (M-2) model with one-norm. In our research, there are only one model under 3 cases. We will define cases (C-1), (C-2) and (C-3) as below.

For the (M-1) model, Wei tried to apply a numerical example to demonstrate his developed method for partially known information case. However, in the numerical example, we will show that Wei [1] did not obtain the optimal solution. Moreover, he also did not explain why the optimal solution for the (M-2) model of two-norm could be directly derived by re-normalization of that of 1-norm. In this paper, we will point that Wei's approach for (M-2) model with two-norm is corrected but his approach for (M-2) model with one-norm is false. We will find an improvement method for (M-1) model and a revised approach for (M-2) model with one-norm. In our research, there are only one model under 3 cases. We will define three cases: (I) (Case-1), (II) (Case-2), and (III) (Case-3) as below.

(I) (Case-1) case

By the same example, we will demonstrate Wei [1] could not find optimal solution for (M-1) case. We consider the maximization problem in (M-1) model where the lower bound and upper bound for attribute weights are known from experts or decision makers. We may abstractly express the constraints in set space H to say that

$$a_j \leq w_j \leq b_j, \tag{3.1}$$

where a_j is lower bound and b_j is upper bound for $j = 1, \ldots, n$. Based on the range for each weight, there are lower bounds for w_j, that must be satisfied by each weight. At the moment, we overlook the upper bound of weights. The remainder, that is $1 - \sum_{j=1}^{n} a_j$, will be distributed to the weights individually depending on the objective function. Due to the problem of the maximization, after we satisfy the lower bound constraint, comparing the coefficient in Equation (2.15), then the best policy is to fulfill the need for weight with the highest coefficient as much as possible in the maximum problem and should be less than upper bound to obey the restrictions such that we consider the upper bound to imply that the weight with the most great coefficient has to be satisfied by the value $b_j - a_j$ in order to make the value of maximum function most for the (M-1) model in Wei [1]. This work for weights has to be done one by one till the condition of $\sum_{j=1}^{n} w_j = 1$ is satisfied.

For completeness, we organize and summarize our approach about the maximization problem for the attribute weights with lower bound and upper bound in the following algorithm.

Our algorithm

Step 1. Satisfy the lower bound constraint for each criterion.

Step 2. Calculate the difference between 1 and the summation of all lower bounds.

Step 3. Compare the coefficients in the objective function and select the highest one to satisfy the upper bound and keep on doing this procedure for the second highest coefficient and so on under the constraint of upper bound and lower bound.

Step 4. The maximum value for objective will to be achieved when there is no remaining weight can be assigned.

(II) (Case-2) case

This is the case for the improvement of (M-2) model with respect to two-norm in Wei [1]. This case is the situation that the knowledge with respect to criteria weights is entirely unknown and with a constraint $\sum_{j=1}^{n} w_j^2 = 1$. The acquiring for w_j is also can be hold by the Cauchy-Schwarz inequality theory.

In light of the Cauchy-Schwarz inequality, we will use this inequality to deal with the following maximum problem,

$$\max \quad \sum_{j=1}^{n} \alpha_j w_j , \tag{3.2}$$

under two-norm, $\sum_{j=1}^{n} w_j^2 = 1$. From Schwarz inequality, we derive that

$$\sum_{j=1}^{n} \alpha_j w_j = \langle \alpha, w \rangle \le \|\alpha\| \, \|w\|, \tag{3.3}$$

where

$$\alpha_j = \sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right), \tag{3.4}$$

with $\alpha = (\alpha_1, ..., \alpha_n)$, $w = (w_1, ..., w_n)$ and $\langle \alpha, w \rangle$ is the inner product. From the inner product $\langle \alpha, w \rangle = \|\alpha\| \, \|w\|$ if and only if $w = \lambda \alpha$ for some positive number λ. Under the restriction that w is a normalized vector with $\sum_{j=1}^{n} w_j^2 = 1$, then $\lambda^2 \sum_{j=1}^{n} \alpha_j^2 = 1$ when $w_j = \lambda \alpha_j$. It yields that

$$\lambda = 1 \Big/ \sqrt{\sum_{j=1}^{n} \alpha_j^2} , \tag{3.5}$$

and

$$w_j^* = \alpha_j \Big/ \sqrt{\sum_{j=1}^{n} \alpha_j^2}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) \Big/ \sqrt{\sum_{j=1}^{n} \left(\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) \right)^2} . \tag{3.6}$$

(III) (Case-3) case

We will provide an improvement of (M-1) model with respect to one-norm in Wei [1]. The maximum problem is expressed as

$$\max \sum_{j=1}^{n} \alpha_j w_j \tag{3.7}$$

under one-norm, $\sum_{j=1}^{n} w_j = 1$. We assume that $\alpha_s = \max \{\alpha_j : 1 \le j \le n\}$, and then we obtain that

$$\sum_{j=1}^{n} \alpha_j w_j \le \sum_{j=1}^{n} \alpha_s w_j$$

$$= \alpha_s = \alpha_s (w_s = 1), \tag{3.8}$$

such that we obtain the optimal solution with $w_s = 1$ and other $w_j = 0$.

IV. Illustrative Example

To illustrate our approach and compare with Wei [1], we scrutinize the same example in Herrera and Herrera-Viedma [5] and Wei [1] with the following problem. Let $A_i (i = 1, ..., 5)$ be five possible alternatives to invest money and $G_j (j = 1, ..., 4)$ be four different attributes. A decision maker will decide the relationship between the above four attributes and five possible alternatives with respect to intuitionistic fuzzy data. The evaluation results are denoted by the next matrix. .

$$\tilde{R} = \begin{bmatrix} (0.5, 0.4) & (0.6, 0.3) & (0.3, 0.6) & (0.2, 0.7) \\ (0.7, 0.3) & (0.7, 0.2) & (0.7, 0.2) & (0.4, 0.5) \\ (0.6, 0.4) & (0.5, 0.4) & (0.5, 0.3) & (0.6, 0.3) \\ (0.8, 0.1) & (0.6, 0.3) & (0.3, 0.4) & (0.2, 0.6) \\ (0.6, 0.2) & (0.4, 0.3) & (0.7, 0.1) & (0.5, 0.3) \end{bmatrix}. \tag{4.1}$$

(1) For (Case-1) case

The attribute weights are partially known, such that the lower bound and upper bound related to weights ~~is~~ given by experts are listed below,

$$H = \{0.15 \le w_1 \le 0.2, 0.16 \le w_2 \le 0.18,$$
$$0.3 \le w_3 \le 0.35, 0.3 \le w_4 \le 0.45\}. \tag{4.2}$$

Utilize the example for model (M-1) in Wei [1] to establish the following maximum objective programming model:

$$\max \quad D(w) = 1.7 w_1 + 1.4 w_2$$
$$+ 2.7 w_3 + 3.1 w_4 , \tag{4.3}$$

subject to for $j = 1,...,4$, $w_j \geq 0$, $w \in H$ and $\sum_{j=1}^{4} w_j = 1$.

We may rewrite the objective function for illustrative example in Wei [1] as

$$\max \quad D(w) = a + 1.7x_1 + 1.4x_2 + 2.7x_3 + 3.1x_4, \quad (4.4)$$

with an abbreviation of a, with

$$a = 1.7(0.15) + 1.4(0.16) + 2.7(0.3) + 3.1(0.3) = 2.219, \quad (4.5)$$

to satisfy the lower bound restriction and $x_j = w_j - a_j$ for $j = 1,...,4$,

$$\sum_{j=1}^{4} x_j = 0.09, \quad (4.6)$$

with

$$H = \{0 \leq x_1 \leq 0.05, 0 \leq x_2 \leq 0.02, 0 \leq x_3 \leq 0.05, 0 \leq x_4 \leq 0.15\}. \quad (4.7)$$

After we satisfy the lower bound constraint initially, comparing the coefficient in equation (4.4), the best policy is to fulfill x_4 as much as possible and do not violate the upper bound restriction such that we imply that $x_4 = 0.09$, and $x_1 = x_2 = x_3 = 0$. Hence, we will choose $w_1 = 0.15$, $w_2 = 0.16$, $w_3 = 0.3$ and $w_4 = 0.3 + 0.09 = 0.39$ with our maximum value $D(w) = 2.498$. If we recall the optimal solution of Wei [1] to quote his maximum solution as $w_1 = 0.2$, $w_2 = 0.18$, $w_3 = 0.32$ and $w_4 = 0.3$ with his maximum value $D(w) = 2.386$, it demonstrates that our approach is a simple and effective approach to obtain the maximum value. Next, we have the overall intuitionistic fuzzy set for alternatives:

$$\tilde{r}_1 = (0.36, 0.54), \quad (4.8)$$

$$\tilde{r}_2 = (0.61, 0.30), \quad (4.9)$$

$$\tilde{r}_3 = (0.56, 0.33), \quad (4.10)$$

$$\tilde{r}_4 = (0.44, 0.36), \quad (4.11)$$

and

$$\tilde{r}_5 = (0.57, 0.20). \quad (4.12)$$

The scores $S(\tilde{r}_i)$ for $\tilde{r}_i$, $i = 1,2,...,5$ are

$$S(\tilde{r}_1) = -0.18, \quad (4.13)$$

$$S(\tilde{r}_2) = 0.31, \quad (4.14)$$

$$S(\tilde{r}_3) = 0.23, \quad (4.15)$$

$$S(\tilde{r}_4) = 0.08, \quad (4.16)$$

and

$$S(\tilde{r}_5) = 0.37. \quad (4.18)$$

We quote the scores of Wei [1] as follows,

$$S(\tilde{r}_1) = -0.13, \quad (4.19)$$

$$S(\tilde{r}_2) = 0.35, \quad (4.20)$$

$$S(\tilde{r}_3) = 0.22, \quad (4.21)$$

$$S(\tilde{r}_4) = 0.16, \quad (4.22)$$

and

$$S(\tilde{r}_5) = 0.39, \quad (4.23)$$

to indicate that based on improper weights of attribute, Wei [1] overestimated the scores for most alternatives. Fortunately, the questionable of scores for alternatives in Wei [1] did not influence the rank for alternatives such that

$$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1, \quad (4.24)$$

holds in our derivation and Wei [1].

The rank for all alternatives in accordance with the scores $S(\tilde{r}_i)$ of the overall intuitionistic fuzzy preference values $\tilde{r}_i (i = 1,...m)$ in our approach is

$$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1, \quad (4.27)$$

and the most desirable alternative is A_2 but not A_5 for (M-1) and (M-2) model in Wei [6].

V. The Second Model with Two-norm

In this section, we examine for the (C-2) case. The objective function is

$$\max \quad D(w) = 1.7x_1 + 1.4x_2 + 2.7x_3 + 3.1x_4, \quad (5.1)$$

such that $\sum_{j=1}^{4} w_j^2 = 1$.

From equation (3.5), we found

$$\sum_{i=1}^{4} \alpha_i^4 = 21.75, \quad (5.2)$$

and

$$\sqrt{\sum_{i=1}^{4}\alpha_i^2} = 4.66, \tag{5.3}$$

such that we derived $w_1 = 0.36$, $w_2 = 0.30$, $w_3 = 0.58$ and $w_4 = 0.66$, and then the intuitionistic fuzzy set for alternatives are

$$\tilde{r}_1 = (0.59, 0.29), \tag{5.4}$$

$$\tilde{r}_2 = (0.84, 0.10), \tag{5.5}$$

$$\tilde{r}_3 = (0.79, 0.12), \tag{5.6}$$

$$\tilde{r}_4 = (0.70, 0.13), \tag{5.7}$$

and

$$\tilde{r}_5 = (0.81, 0.05). \tag{5.8}$$

The scores $S(\tilde{r}_i)$ for $\tilde{r}_i$, $i = 1,2,...,5$ we derived that

$$S(\tilde{r}_1) = 0.30, \tag{5.9}$$

$$S(\tilde{r}_2) = 0.74, \tag{5.10}$$

$$S(\tilde{r}_3) = 0.67, \tag{5.11}$$

$$S(\tilde{r}_4) = 0.57, \tag{5.12}$$

and

$$S(\tilde{r}_5) = 0.76, \tag{5.13}$$

such that the same rank as case (C-1) is derived.

VI. The Second Model with One-norm

In this section, we examine for the (C-3) case with one-norm.

The objective function is

$$\max \quad D(w) = 1.7x_1 + 1.4x_2 + 2.7x_3 + 3.1x_4, \tag{6.1}$$

such that $\sum_{j=1}^{4} w_j = 1$.

We found the maximum under $w_1 = w_2 = w_3 = 0$ and $w_4 = 1$ with maximum value 3.1. Next, we quote the weights of attributes obtained by Wei [1] as $w_1 = 0.1910$, $w_2 = 0.1573$, $w_3 = 0.3034$ and $w_4 = 0.3483$, and then $D(w) = 2.44$ to point out that Wei [1] did derive the maximum value. The intuitionistic fuzzy set for alternatives are derived as follows,

$$\tilde{r}_1 = (0.2, 0.7), \tag{6.1}$$

$$\tilde{r}_2 = (0.4, 0.5), \tag{6.2}$$

$$\tilde{r}_3 = (0.6, 0.3), \tag{6.3}$$

$$\tilde{r}_4 = (0.2, 0.6), \tag{6.4}$$

and

$$\tilde{r}_5 = (0.5, 0.3). \tag{6.5}$$

For he scores $S(\tilde{r}_i)$ for $\tilde{r}_i$, $i = 1,2,...,5$, we derived in the following,

$$S(\tilde{r}_1) = -0.5, \tag{6.6}$$

$$S(\tilde{r}_2) = -0.1, \tag{6.7}$$

$$S(\tilde{r}_3) = 0.3, \tag{6.8}$$

$$S(\tilde{r}_4) = -0.4, \tag{6.9}$$

and

$$S(\tilde{r}_5) = 0.2, \tag{6.10}$$

and then the rank for alternatives is obtained

$$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1. \tag{6.11}$$

If we quote the rank derived by Wei [1], based on questionable weight of attributes, it yielded that

$$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1, \tag{6.12}$$

to reveal that questionable weight of attributes of Wei [1] will mislead the decision maker to adopt the second best alternative. All above examples for cases have different outcome and ranking. This result can be brought to decision maker to put in the options for all possible solutions of strategy.

VII. A Related Problem

We study a lately published article of Wu [11] in which Glock et al. [12] was fully examined. We are aware of in Wu [11] that mentioned Glock et al. [12] had been cited by nine papers. However, we have run a comprehensive study to find that there are twenty-two articles that have referred to Glock et al. [12] in their articles. We execute a detailed reviewing of those related articles, to know that eighteen of them: Shrivastava and Gorantiwar [13], Jaaron and Backhouse [14], Al Masud et al. [15], Maity [16], Ghosh et al. [17], Kumar et al. [18], Kazemi et al. [19, 20], Soni and Joshi [21], Kazemi et al. [22, 23], Kurdhi et al. [24], Soni et al. [25], Shekarian et al. [26], Karmakar et al. [27], and Goyal et al. [28], just talked about Glock et al. [12] in their opening section without offering any consideration of Glock et al. [12]. While the other four papers, Andriolo et al. [29] is a reviewing paper to evaluate more than two hundred articles during the past one hundred years with respect to inventory models initiated by Harris [30]. Kim and Glock [31] only mentioned Glock et al. [12] in the section of possible direction for future study. Mahapatra et al. [32] examined inventory systems with promotional effort, learning effect, linear demand, deteriorated items, and finite planning horizon. they developed three inventory models: crisp, fuzzy, and fuzzy-learning that is an extension of Glock et al. [12]. However, Mahapatra et al. [32] did not consider the open question proposed by Glock et al. [12]. Öztürk [33] considered two production inventory models where the input parameters are assumed to be fuzzy numbers. The production time is crisp for the first model and is a fuzzy number for the second model. He applied the graded mean integration representation method to defuzzy his fuzzy objective function to a crisp problem. Therefore, it can be assumed that until now, researchers did not provide a response for the open problem raised by Glock et al. [12] to examine whether or not the optimal replenishment policy should be operated when all order quantities are the same for each replenishment cycle. Consequently, Wu [11] is the first paper to provide a positive answer for the open problem raised by Glock et al. [12]. Moreover, Wu [11] provided an optimal solution in a closed format for her new inventory system with fuzzy demand. According to the above discussion, we offer a detailed literature reviewing to support the originality of Wu [11].

On the other hand, for completeness, we offer a patch work for the derivation in Wu [11].

We call that Wu [11] extended the domain from discrete natural numbers $\{1,2,\ldots\}$ to the positive real number $\{x>0\}$ to convert the objective function, F(n) to F(x) as follows,

$$F(x) = xA + \frac{Dh}{2x} + \frac{xA}{3D}(\Delta_2 - \Delta_1), \tag{7.1}$$

for $x > 0$. Referring to Equation (7.1), we show that

$$\frac{dF(x)}{dx} = A - \frac{Dh}{2x^2} + \frac{A}{3D}(\Delta_2 - \Delta_1). \tag{7.2}$$

According to Equation (7.2), the derivation of Wu [11] for the second derivative can be implied as follows,

$$\frac{d^2F(x)}{dx^2} = \frac{Dh}{x^3}. \tag{7.3}$$

Consequently, we provide a complete derivation to show that the objective function, F(x), is convex up for $x > 0$.

VIII. A Related Problem

In this section, we will consider a brief discussion for Chen and Klein [34] for deciding the group order through synthesized fuzzy operators. We assume that x is an interval fuzzy value, denoted as $x=[t_x, 1-f_x]$, under the condition $0 \le t_x \le 1-f_x \le 1$.

Based on the fuzzy value, there are three information: (i) the truth membership part, denoted as t_x, (ii) the false membership part, denoted as f_x, and (iii) the unknown part, represented as $1-t_x-f_x$. Traditionally, researchers assumed the empty vague set, denoted as $[0,1]$, with $t_x=0$, and $f_x=0$.

Chen and Klein [34] defined the score function, denoted as S, with

$$S(x) = t_x - f_x, \tag{8.1}$$

such that the score function is derived as $S(x) \in [-1,1]$.

Chen and Klein [34] defined the degree of similarity between the vague values x and y,

$$M(x,y) = 1 - \frac{|S(x) - S(y)|}{2}. \tag{8.2}$$

If x is a fuzzy value with $x=[t_x, 1-f_x]$, the weighted score function S_w is defined as follows,

$$S_w(x) = at_x + bf_x + c(1-t_x-f_x), \tag{8.3}$$

where c indicates the weight of the unknown part, b indicates the weight of the false membership part, and a indicates the weight of the truth membership part, under the restriction, $a \ge c \ge 0 \ge b$, and then we know the function value of the weighted score function, is derived as $S_w(x) \in [b,a]$.

Chen and Klein [34] defined the weighted similarity measure between the interval fuzzy values x and y, denoted as M_w

$$M_w(x,y) = 1 - \frac{|S_w(x) - S_w(y)|}{a-b}. \tag{8.4}$$

We suppose that $A_1, \ldots, A_m$ is a group of interval fuzzy values defined in the universe of discourse U. We assume that u_i, and u_j are elements (members) in the universe of discourse U.

The similarity measure between two elements u_i, and u_j in fuzzy value A_k, where $k=1,\ldots,m$, is defined as follows,

$$S_e(u_i, u_j) = \frac{1}{m}\sum_{k=1}^{m} \Phi_k, \tag{8.5}$$

where Φ_k is an abbreviation that is defined as follows,

$$\Phi_k = M_w\left(A_k(u_i), A_k(u_j)\right), \tag{8.6}$$

$$A_k(u_i) = \left(t_{A_k}(u_i), 1 - f_{A_k}(u_i)\right), \tag{8.7}$$

and

$$A_k(u_j) = \left(t_{A_k}(u_j), 1 - f_{A_k}(u_j)\right). \tag{8.8}$$

In this following, we will consider numerical example for our previous discussion. Based on behavior analysis problems mentioned in Chen and Klein [34], we will provide an improvement for their example. According to our above discussion, we can answer the following two problems proposed by Chen and Klein [34].

Problem A: According to what scale two groups A and B could work together?

Problem B: By what scale the numbers u_2 and u_3 in the universe of discourse that could be arranged in the same group.

Based on the following evaluation,

$$M_w(A,B) = 0.4625, \tag{8.9}$$

to show that the percentage that two groups A and B can be worked together is about 46%. On the other hand, based on the estimation of the following,

$$S_e(u_2, u_3) = 0.7333, \tag{8.10}$$

to derive the percentage that two members u_2 and u_3 in the universe of discourse could be classified into the same group is about 73%. Our derivations of Equations (8.9) and (8.10) contained more information than the original interval fuzzy values. Consequently, our approach with respect to behavior analysis is more elastic than the results proposed by Chen and Klein [34].

IX. Further Discussion

If the universe of discourse, $X = \{x_j : j = 1,\ldots,n\}$, Szmidt and Kacprzyk [35] defined the normalized Hamming distance between two IFSs A and B as follows:

$$d(A,B) = \frac{1}{2n}\sum_{j=1}^{n}\Big(\left|\mu_A(x_j) - \mu_B(x_j)\right| + \left|v_A(x_j) - v_B(x_j)\right| + \left|\pi_A(x_j) - \pi_B(x_j)\right|\Big). \quad (9.1)$$

Xu and Yager [36] derived that

$$\left|\pi_A(x_j) - \pi_B(x_j)\right| = \left|1 - \mu_A(x_j) - v_A(x_j) - \left(1 - \mu_B(x_j) - v_B(x_j)\right)\right| = \left|\left(\mu_A(x_j) - \mu_B(x_j)\right) - \left(v_A(x_j) - v_B(x_j)\right)\right|, \quad (9.2)$$

and then they rewrote Equation (9.1) as

$$d(A,B) = \frac{1}{2n}\sum_{j=1}^{n}\Big(\left|\mu_A(x_j) - \mu_B(x_j)\right| + \left|v_A(x_j) - v_B(x_j)\right| + \left|\left(\mu_A(x_j) - \mu_B(x_j)\right) - \left(v_A(x_j) - v_B(x_j)\right)\right|\Big). \quad (9.3)$$

When the universe of discourse is a singleton, $X = \{x\}$, Grzegorzewski [37] defined a distance measure for two IFSs A and B that was based on the Hausdorff metric as follows:

$$d_1(A,B) = \max\left\{\left|\mu_A(x) - \mu_B(x)\right|, \left|v_A(x) - v_B(x)\right|\right\}. \quad (9.4)$$

Xu and Yager [36] used the well known relation in the following between two real numbers:

$$\max\{x,y\} = \frac{1}{2}\left(x + y + |x - y|\right), \quad (9.5)$$

to derive that

$$d_1(\alpha_1,\alpha_2) = \frac{1}{2}\Big(\left|\mu_{\alpha 1} - \mu_{\alpha 2}\right| + \left|v_{\alpha 1} - v_{\alpha 2}\right| + \Big|\left|\mu_{\alpha 1} - \mu_{\alpha 2}\right| - \left|v_{\alpha 1} - v_{\alpha 2}\right|\Big|\Big), \quad (9.6)$$

where $\alpha_1 = (\mu_{\alpha 1}, v_{\alpha 1}, \pi_{\alpha 1})$ and $\alpha_2 = (\mu_{\alpha 2}, v_{\alpha 2}, \pi_{\alpha 2})$ are two IFSs with the universe of discourse is a singleton. Xu and Yager [36] used

$$\left|(\mu_{\alpha 1} - \mu_{\alpha 2}) - (v_{\alpha 1} - v_{\alpha 2})\right| \geq \Big|\left|\mu_{\alpha 1} - \mu_{\alpha 2}\right| - \left|v_{\alpha 1} - v_{\alpha 2}\right|\Big|, \quad (9.7)$$

with Equations (9.3) and (9.6) to obtain that

$$d(\alpha_1,\alpha_2) \geq d_1(\alpha_1,\alpha_2). \quad (9.8)$$

X. Our Revisions of Their Method

Xu and Yager [36] had developed a new similarity measure that not only patch the shortcoming of Szmidt and Kacprzyk [35] but also applied to group decision making under IFS and IVIFS environment. However, there are some questionable results contained in their derivations. Hence, we will provide an improvement. The result of Equation (9.2) proposed by Xu and Yager [36] should be revised as

$$\left|\pi_A(x_j) - \pi_B(x_j)\right| = \left|1 - \mu_A(x_j) - v_A(x_j) - \left(1 - \mu_B(x_j) - v_B(x_j)\right)\right| = \left|\left(\mu_A(x_j) - \mu_B(x_j)\right) + \left(v_A(x_j) - v_B(x_j)\right)\right|. \quad (10.1)$$

Consequently, Equation (9.3) proposed by Xu and Yager [36] also modified as

$$d(\alpha_1,\alpha_2) = \frac{1}{2}\Big(\left|\mu_{\alpha 1} - \mu_{\alpha 2}\right| + \left|v_{\alpha 1} - v_{\alpha 2}\right| + \left|(\mu_{\alpha 1} - \mu_{\alpha 2}) + (v_{\alpha 1} - v_{\alpha 2})\right|\Big), \quad (10.2)$$

where $\alpha_1 = (\mu_{\alpha 1}, v_{\alpha 1}, \pi_{\alpha 1})$ and $\alpha_2 = (\mu_{\alpha 2}, v_{\alpha 2}, \pi_{\alpha 2})$ are two IFSs with the universe of discourse is a singleton. From the triangle inequality, we know that

$$\left|(\mu_{\alpha 1} - \mu_{\alpha 2}) + (v_{\alpha 1} - v_{\alpha 2})\right| \geq \Big|\left|\mu_{\alpha 1} - \mu_{\alpha 2}\right| - \left|v_{\alpha 1} - v_{\alpha 2}\right|\Big|. \quad (10.3)$$

If we combine the findings of Equations (9.6), (10.2) and (10.3) to yield that

$$d(\alpha_1,\alpha_2) \geq d_1(\alpha_1,\alpha_2). \quad (10.4)$$

XI. Our Novel Proof

In this section, we provide our approach. For two real numbers, x and y, we will derive that

$$\frac{1}{2}\{|x| + |y| + |x + y|\} = \begin{cases} |x| + |y|, & \text{if } xy \geq 0 \\ \max\{|x|,|y|\}, & \text{if } xy < 0 \end{cases}. \quad (11.1)$$

If $xy \geq 0$, we know that

$$|x + y| = |x| + |y|. \quad (11.2)$$

On the other hand, if $xy < 0$, it follows that

$$|x + y| = \max\{|x|,|y|\} - \min\{|x|,|y|\}. \quad (11.3)$$

Moreover, we know that

$$|x| + |y| = \max\{|x|,|y|\} + \min\{|x|,|y|\}. \quad (11.4)$$

From the above discussion of Equations (11.2-11.4), we prove the assertion of Equation (11.1) is verified.

From Equation (10.2), we obtain that

$$d(\alpha_1,\alpha_2) = \frac{1}{2}\{|x| + |y| + |x + y|\}, \quad (11.5)$$

where $x = \mu_{\alpha 1} - \mu_{\alpha 2}$ and $y = v_{\alpha 1} - v_{\alpha 2}$.

By Equation (9.4), it yields that

$$d_1(\alpha_1,\alpha_2) = \max\{|x|,|y|\}, \quad (11.6)$$

where $x = \mu_{\alpha 1} - \mu_{\alpha 2}$ and $y = v_{\alpha 1} - v_{\alpha 2}$.

Here, we combine the results of Equations (11.1), (11.5) and (11.6) to imply that

$$d(\alpha_1,\alpha_2) \geq d_1(\alpha_1,\alpha_2). \quad (11.7)$$

We summarize our findings in the next theorem.

Theorem 1. For two IFSs, $\alpha_1 = (\mu_{\alpha 1}, v_{\alpha 1}, \pi_{\alpha 1})$ and $\alpha_2 = (\mu_{\alpha 2}, v_{\alpha 2}, \pi_{\alpha 2})$ with the universe of discourse is a singleton, we show that
(a) If $(\mu_{\alpha 1} - \mu_{\alpha 2})(v_{\alpha 1} - v_{\alpha 2}) \geq 0$, then

$$d(\alpha_1, \alpha_2) \geq d_1(\alpha_1, \alpha_2). \quad (11.8)$$

(b) If $(\mu_{\alpha 1} - \mu_{\alpha 2})(v_{\alpha 1} - v_{\alpha 2}) < 0$, then

$$d(\alpha_1, \alpha_2) > d_1(\alpha_1, \alpha_2). \quad (11.9)$$

We first revise Xu and Yager [36] to compare two similarity measures. We also find the sufficient and necessary conditions to guarantee that these two similarity measures are equal. Our results may help decision-makers realize and apply the similarity measures of Xu and Yager [36].

XII. Discussion of Interval Neutrosophic Numbers

Ye [38] considered interval neutrosophic numbers to deal with two boundary points, such that their interval neutrosophic numbers can be treated as the interval crisp number in a different formation. Ye [38] also studied single-valued neutrosophic numbers. Ye [38] have examined (i) Idempotency, and (ii) Monotonicity, because Ye [38] did not consider the hesitant fuzzy sets, such that there is no problems with respect to (i) Idempotency, and (ii) Monotonicity.
For an element of interval neutrosophic numbers, denoted as B, and then

$$B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle, x \in X\}, \quad (12.1)$$

where $T_B(x)$ is the truth membership mapping, $I_B(x)$ is the indeterminacy membership mapping, and $F_B(x)$ is the falsity membership mapping.
For elements in a group of interval neutrosophic numbers, denoted as A_t, with

$$A_t = \langle (T_t^L, T_t^U), (I_t^L, I_t^U), (F_t^L, F_t^U)\rangle, \quad (12.2)$$

with $t \in \Omega$, the ideal element, denoted as A_{id}, is constructed as follows,

$$A_{id} = \langle (T_{id}^L, T_{id}^U), (I_{id}^L, I_{id}^U), (F_{id}^L, F_{id}^U)\rangle, \quad (12.3)$$

where

$$T_{id}^L = \max\{T_t^L, t \in \Omega\}, \quad (12.4)$$

$$T_{id}^U = \max\{T_t^U, t \in \Omega\}, \quad (12.5)$$

$$I_{id}^L = \min\{I_t^L, t \in \Omega\}, \quad (12.6)$$

$$I_{id}^U = \min\{I_t^U, t \in \Omega\}, \quad (12.7)$$

$$F_{id}^L = \min\{F_t^L, t \in \Omega\}, \quad (12.8)$$

and

$$F_{id}^U = \min\{F_t^U, t \in \Omega\}. \quad (12.9)$$

For two interval neutrosophic numbers A_i and A_j, Ye [38] assumed that $A_i \cdot A_j$ in the following,

$$A_i \cdot A_j = T_i^L T_j^L + T_i^U T_j^U + I_i^L I_j^L + I_i^U I_j^U + F_i^L F_j^L + F_i^U F_j^U, \quad (12.10)$$

Ye [38] defined the projection measure between A_i and A_j as follows,

$$\text{Proj}_{A_j}(A_t) = A_i \cdot A_j / \|A_j\|, \quad (12.11)$$

where $A_i \cdot A_j$ is defined in Equation (12.10) and

$$\|A_j\| = \left((T_j^L)^2 + (T_j^U)^2 + (I_j^L)^2 (I_j^U)^2 + (F_j^L)^2 + (F_j^U)^2 \right)^{1/2}. \quad (12.12)$$

Consequently, Ye [38] compare two elements in a group of interval neutrosophic numbers, denoted as A_i and A_j, if

$$\text{Proj}_{A_{id}}(A_i) > \text{Proj}_{A_{id}}(A_j), \quad (12.13)$$

then Ye [38] defined that

$$A_i \succ A_j. \quad (12.14)$$

Ye [38] applied two methods to result in two different ranking:

$$A_1 \succ A_3 \succ A_2 \succ A_4 \succ A_5, \quad (12.15)$$

and

$$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5, \quad (12.15)$$

and then Ye [38] mentioned that Applying two methods to decide the ranking of alternatives may be different. It also happened in Liu and Wang [39], Ye [40], and Peng et al. [41]. Ye [38] claimed that the difference of two ranking approach by various measuring methods is unavoidable and then Ye [38] mentioned that by his novel procedure with respect to credibility of criteria.
Ye [42] introduced the Hamming and Euclidean distance between interval neutrosophic sets which is the maximum norm in the common sense.
If there is only one decision maker who decided

$$w = (0.17, 0.18, 0.25, 0.2, 0.2), \quad (12.16)$$

and then he selected

$$c = (1, 0.7, 0.6, 0.7, 0.6). \quad (12.17)$$

Based on Equations (12.16) and (12.17), and the following normalization process, we derive that

$$v_k = w_k c_k / \textstyle\sum_{t=1}^{5} w_t c_t, \quad (12.18)$$

, such that a new weight $v = (v_1, v_2, v_3, v_4, v_5)$ was created as follows,

$$v_1 = 0.240793, \quad (12.19)$$

$$v_2 = 0.178470, \quad (12.20)$$

$$v_3 = 0.212465, \quad (12.21)$$

$$v_4 = 0.198300, \quad (12.22)$$

and

$$v_5 = 0.169972. \quad (12.23)$$

XIII. Our Improvement

We must point out that the definition of Equations (12.10-12.12) are correct but we can provide a simplification. We recall the comparison for two interval neutrosophic numbers, A_i and A_j in a group of interval neutrosophic numbers, and then Ye [38] suggested to compute the following,

$$\text{Proj}_{A_{id}}(A_i) = A_i \cdot A_{id} / \|A_{id}\|, \quad (13.1)$$

and

$$\text{Proj}_{A_{id}}(A_j) = A_j \cdot A_{id} / \|A_{id}\|. \quad (13.2)$$

If we observe Equations (13.1) and (13.2), and then they have a common denominator. Therefore, we can provide a simplification for Ye [38] to define $A_i \succ A_j$ if and only if

$$A_i \cdot A_{id} > A_{id} \cdot A_j. \quad (13.3)$$

Moreover, the construction of $\text{Proj}_{A_j}(A_t)$ is unnecessary to decide which interval neutrosophic number is the optimal choice among elements in the group of interval neutrosophic numbers.
In the past, there is a severe challenge about pattern recognition problems. We may classify them into two kinds of problems:
(i) A researcher selected a measure, denoted as Msr, and derive that

$$\text{Msr}(A_1, A_{id}) > Msr(A_2, A_{id}). \quad (13.4)$$

The researcher added another interval neutrosophic number, denoted as A_3, and then he obtained that

$$\mathrm{Msr}(A_2, A_{id}) > Msr(A_1, A_{id}) \geq \mathrm{Msr}(A_3, A_{id}), \quad (13.5)$$

such that after adding A_3, the most favorable interval neutrosophic number is changed from A_1 to A_2.

(ii) A researcher selected two measures, denoted as Msr1, and Msr2, and then the researcher obtained that

$$\mathrm{Msr1}(A_1, A_{id}) > Msr1(A_2, A_{id}), \quad (13.6)$$

and

$$\mathrm{Msr2}(A_2, A_{id}) > Msr2(A_1, A_{id}). \quad (13.7)$$

Consequently, to decide the group of interval neutrosophic numbers, and the proposed measure to calculate the similarity to the ideal element become a predesigned duty.

We observe that $\mathrm{Proj}_{A_{id}}(A_1) = 1.1803$ and $\mathrm{Proj}_{A_{id}}(A_3) = 1.1810$ to indicate that those two values are too close together such that different aggregation method or different weight may imply different rankings.

We compare w of Equation (12.16) and v of Equations (12.19-23) to find that

$$v_1 > w_1, \quad (13.8)$$

and

$$v_k < w_k, \quad (13.9)$$

for $k = 2,3,4,5$.

We check the value of c by Equation (12.17) to find that

$$c_1 \geq 1, \quad (13.10)$$

and

$$c_k < 1, \quad (13.11)$$

for $k = 2,3,4,5$.

We observe that $c_2 = c_4$, and $c_3 = c_5$ and then we compute that

$$\frac{v_2}{w_2} = \frac{w_2 c_2 / \sum_{t=1}^{5} w_t c_t}{w_2} = \frac{c_2}{\sum_{t=1}^{5} w_t c_t} = \frac{c_4}{\sum_{t=1}^{5} w_t c_t} = \frac{w_4 c_4 / \sum_{t=1}^{5} w_t c_t}{w_4} = \frac{v_4}{w_4}. \quad (13.12)$$

By the same approach, we can prove that

$$\frac{v_3}{w_3} = \frac{v_5}{w_5}. \quad (13.13)$$

Based on the findings of Equations (13.12) and (13.13), we derive the following theorem.

Theorem 1. If $c_i = c_j$, and then $\frac{v_i}{w_i} = \frac{v_j}{w_j}$.

In the following, we proposed our novel ranking procedure. We will directly apply the Hamming distance between a interval neutrosophic number and the ideal element. We proposed the following two numerical examples to illustrate our approach.

For the first numerical example, we assume that

$$A_1 = \langle [0.5,0.7], [0.3,0.4], [0.5,0.6] \rangle, \quad (13.14)$$

and

$$A_2 = \langle [0.4,0.5], [0.3,0.6], [0.3,0.4] \rangle. \quad (13.15)$$

According to Equations (13.14) and (13.15), we derive the ideal element,

$$A_{id} = \langle [0.5,0.7], [0.3,0.4], [0.3,0.4] \rangle. \quad (13.16)$$

We compute the distance to find the Hamming distance that

$$\mathrm{Dist}(A_1, A_{id}) = 0.4 < Dist(A_2, A_{id}) = 0.5 \quad (13.17)$$

such that we claim that

$$A_1 \succ A_2. \quad (13.18)$$

For the second numerical example, we assume that

$$B_1 = \langle [0.3,0.6], [0.3,0.4], [0.3,0.4] \rangle, \quad (13.19)$$

and

$$B_2 = \langle [0.5,0.7], [0.3,0.5], [0.3,0.5] \rangle. \quad (13.20)$$

According to Equations (13.19) and (13.20), we derive the ideal element,

$$B_{id} = \langle [0.5,0.7], [0.3,0.4], [0.3,0.4] \rangle. \quad (13.21)$$

We compute the distance to find the Hamming distance that

$$\mathrm{Dist}(B_1, B_{id}) = 0.3 > \mathrm{Dist}(B_2, B_{id}) = 0.2 \quad (13.22)$$

such that we claim that

$$B_2 \succ B_1. \quad (13.23)$$

XIV. Directions for Future Research

To point out the current research, we cited some articles published in 2024 in the following. Hao et al. [43] developed an extended automatic regression system by fusing nonlinearity and linearity. Referring to restoring mileage and time, Yang et al. [44] constructed a train optimal model with several components of electrical circulation. Ouyang et al. [45] considered nonlinear exchanged fuzzy models to deal with stipulated reaction performance. According to periodic feature fusion and time and space estimation, Wang et al. [46] examined transit stream estimation over a short period. Berot et al. [47] studied a network with a selection of variables and parameters under several investigational information. With K-mean adjacent classification and the grasshopper discrete procedure, Qi et al. [48] developed a model to decide the best characteristic choice. Applying geographical and biological optimal processes, Hou et al. [49] constructed a multiple-echelon examination system to locate exterior imperfections. Kakarlapudi et al. [50] considered an iterative algorithm to deal with complicated systems by the valley of attraction. Using intuitionistic interval-value fuzzy sets, Rakhmawati et al. [51] examined the multiple graphs to locate the best route. We add four recently published articles, Dutta and Banik [52, 53], Dutta and Borah [54], and Chen and Tang [55], which used the maximizing deviations approach in their decision-making problems under an intuitionistic fuzzy environment. Based on our above citation, practitioners can locate research directions that are interesting.

XV. Conclusion

We have studied the solution procedure of Wei [1] for the maximum deviation problem and its influence on multiple attribute decision-making in an intuitionistic fuzzy environment. For attribute weights with one norm under (a) lower and upper bound and (b) completely unknown, we provided an improved solution method to derive the optimal solution. From the same numerical example, we point out that questionable results of Wei [1] sometimes cannot obtain the best alternative. Our findings will be valuable in future applications for decision-making problems under an intuitionistic fuzzy environment.

In Wu [11], a study by Glock et al. [12] for an inventory system with fuzzy demand under the condition of equal ordering quantity for each replenishment cycle was conducted. Glock et al. [12] claimed their inventory system is restrict to ordering lots with equal sizes. In possible directions for future research, Glock et al. [12] raised an open question that practitioners may develop inventory systems with unequal order quantities. Wu [11] was motivated by the open question stated above and constructed an inventory model with fuzzy demand without the restriction of equal ordering quantity for each replenishment cycle. In this paper, we provide two improvements for Wu [11] to help researchers realize her important contribution to inventory systems under fuzzy environments.

We also locate the necessary and sufficient criteria to assurance that the discussed two similarity measures are equal which was proposed by Xu and Yager [36]. Our results will help practitioners understand and use the similarity measures of Xu and Yager [36].

At last, not the least, we examine Ye [38] and Ye [42] to revive their projection measure and c

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Jinyuan Liu received his Master degree from School of Mathematics and Systems Science, Shandong University of Science and Technology in 2009. He currently is an Associate Professor at the School of General Studies of Weifang University of Science and Technology. The main research directions are Pattern Recognition, Lanchester's Model, Fuzzy Set Theorem, Isolate Points, Analytical Hierarchy Process, and Inventory Models.