The Law of Importation for Quantum Implications

Lei Du, Yingying Xu, Haifeng Song, Songsong Dai

***Abstract*—In classical logic, the law of importation $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$ is a tautology, and it has been extensively studied in fuzzy logics. This paper explores the law of importation in orthomodular quantum logic. Our investigation reveals that the law of importation does not hold for five quantum implications within the context of orthomodular quantum logic. Furthermore, we examine six other quantum implication functions in detail.**

***Index Terms*—Quantum logic, orthomodular lattice, quantum implication, law of importation.**

I. Introduction

In 1936, Birkhoff and von Neumann [1] proposed the quantum logic as a logic of quantum mechanics. It is currently defined as orthomodular lattices [2]. With the rapid development of quantum computation, Ying [3], [4], [5] studied the automata theory of computation based on orthomodular lattices. This theory can be seen as a logical approach to quantum computation [6], [7], [8], [9].

The law of importation, given by the equality

$$(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r) \tag{1}$$

is an important property of implication operators. In classical logic, it is a tautology. In the framework of fuzzy logic, it has been studied extensively for various fuzzy implication operators [10], [11], [12]. Mas et al.[13], [14] studied the law of importation for discrete implications and several uninorm derived implications. Additionally, Massanet and Torrens [15] examined the relationship between the law of importation and the exchange principle on fuzzy implications. Massanet et al [16], [17], [18] studied the law of importation with fixed a fixed t-norm (or uninorm) for fuzzy implications. Li and Qin [19] investigated the characterization of a class of fuzzy implications satisfying the law of importation with respect to uninorms with continuous underlying operators. Li et al. [20] considered the stability of the law of importation for (S,N)-implications. Furthermore, Wang et al. [21], [22] introduced the derivations on fuzzy implication algebras, and Zhu et al. [23], [24], [25] studied implicative derivations on residuated lattices. In the side of quantum logic, we have a corresponding problem: does Eq.1 hold for some quantum implication operators?

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In this paper, we consider the law of implication in the setting of quantum logic. Furthermore, we consider the case that the multiplication & replaces $\wedge$ in Eq. (1), i.e.,

$$(p\&q) \rightarrow r \equiv p \rightarrow (q \rightarrow r). \tag{2}$$

We also consider the case that $p = q$ in Eq. (1), i.e.,

$$p \rightarrow r \equiv p \rightarrow (p \rightarrow r), \tag{3}$$

owing to the property that $p \wedge p = p$ in an orthomodular lattice. Eq.(3) is called the derived iterative Boolean law.

Moreover, we consider the distributivity of quantum implications, i.e.,

$$p \wedge q \rightarrow r = (p \rightarrow r) \vee (q \rightarrow r), \tag{4}$$

$$p \vee q \rightarrow r = (p \rightarrow r) \wedge (q \rightarrow r), \tag{5}$$

$$p \rightarrow (q \wedge r) = (p \rightarrow q) \wedge (p \rightarrow r), \tag{6}$$

$$p \rightarrow (q \vee r) = (p \rightarrow q) \vee (p \rightarrow r). \tag{7}$$

This article is structured as follows. In Section 2, some preliminaries concerning orthomodular lattices are given. In Section 3, we study the quantum implication functions Eqs.(1-7) in orthomodular quantum logic. In Section 4, concluding remarks are given.

II. Orthomodular lattice

For the sake of readability, this section gives some preliminaries concerning orthomodular lattice, and the details are referred to refs. [26], [27].

An orthocomplemented lattice L is a lattice with an orthocomplement $\perp: L \rightarrow L$ satisfying: $\forall p, q \in L$,

(i). $p^{\perp\perp} = p$;
(ii). $p \wedge p^{\perp} = 0, p \vee p^{\perp} = 1$;
(iii). $p \leq q \quad \Rightarrow \quad q^{\perp} \leq p^{\perp}$

An orthomodular lattice is an orthocomplemented lattice satisfying the orthomodular law:

$$p \geq q \quad \Rightarrow \quad p \wedge (p^{\perp} \vee q) = q, \quad \forall p, q \in L. \tag{8}$$

Eq.(8) also can be represented as follows

$$p \vee (p^{\perp} \wedge (p \vee q)) = p \vee q, \quad \forall p, q \in L. \tag{9}$$

By using the above conditions (i-iii), Eq.(9) can be equivalently stated as:

$$p^{\perp} \wedge (p \vee (p^{\perp} \wedge q^{\perp})) = p^{\perp} \wedge q^{\perp}, \quad \forall p, q \in L. \tag{10}$$

In an orthomodular lattice, a reasonable implication connective is required to satisfy the Birkhoff-von Neumann condition [1]:

$$p \leq q \text{ if and only if } p \rightarrow q = 1, \quad \forall p, q \in L. \tag{11}$$

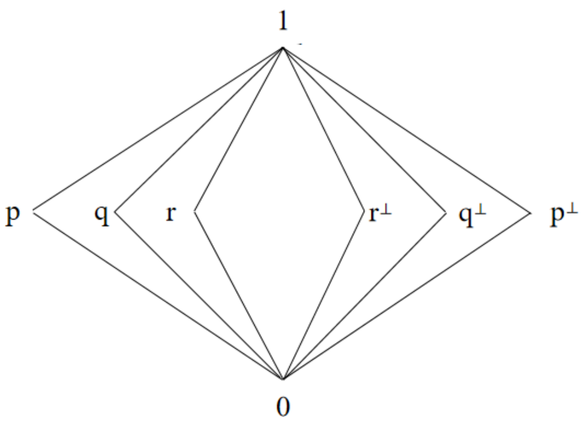

Fig. 1. Chinese lantern.

Indeed, it has been verified [28], [29] that there are exactly five implications satisfying the Birkhoff-von Neumann condition:

(Sasaki): $p \to_1 q = p^\perp \vee (p \wedge q)$;
(Dishkant): $p \to_2 q = q \vee (p^\perp \wedge q^\perp)$;
(relevance): $p \to_3 q = (p^\perp \wedge q) \vee (p \wedge q) \vee (p^\perp \wedge q^\perp)$;
(non-tollens): $p \to_4 q = (p^\perp \wedge q) \vee (p \wedge q) \vee ((p^\perp \vee q) \wedge q^\perp)$;
(Kalmbach): $p \to_5 q = (p^\perp \wedge q) \vee (p^\perp \wedge q^\perp) \vee (p \wedge (p^\perp \vee q))$.

In classical logic, these five implications are equivalent to "material implication" $\to_0$, i.e., $p \to_0 q = p^\perp \vee q$. Note that $\to_0$ does not satisfy the Birkhoff-von Neumann condition. The multiplication & is defined as $p\&q = q \wedge (p \vee q^\perp)$. Note that $p\&q = p \wedge q$ in classical logic.

III. Main results

A. $(p \wedge q) \to r \equiv p \to (q \to r)$ in orthomodular lattice

Theorem 1. *There exists an orthomodular lattice, such that none of the above implications $\to_i$ ($1 \leq i \leq 5$) satisfies Eq.(1).*

Proof: Consider the orthomodular lattice visualized by Fig. 1.

For $\to_1$, we have

$$\begin{aligned} p \wedge q \to_1 r &= (p \wedge q)^\perp \vee (p \wedge q \wedge r) \\ &= (0)^\perp \vee (0) \\ &= 1. \end{aligned}$$

and

$$\begin{aligned} p \to_1 (q \to_1 r) &= p \to_1 \big(q^\perp \vee (q \wedge r)\big) \\ &= p \to_1 q^\perp \\ &= p^\perp \vee (p \wedge q^\perp) \\ &= p^\perp. \end{aligned}$$

Thus $p \wedge q \to_1 r \neq p \to_1 (q \to_1 r)$.

For $\to_2$, we have

$$\begin{aligned} p \wedge q \to_2 r &= r \vee \big((p \wedge q)^\perp \wedge r^\perp\big) \\ &= r \vee \big((0)^\perp \wedge r^\perp\big) \\ &= r \vee \big(1 \wedge r^\perp\big) \\ &= r \vee r^\perp \\ &= 1. \end{aligned}$$

and

$$\begin{aligned} p \to_2 (q \to_2 r) &= p \to_2 \big(r \vee (q^\perp \wedge r^\perp)\big) \\ &= p \to_2 r \\ &= r \vee \big(p^\perp \wedge (r^\perp)\big) \\ &= r. \end{aligned}$$

Thus $p \wedge q \to_2 r \neq p \to_2 (q \to_2 r)$.

For $\to_3$, we have

$$\begin{aligned} & p \wedge q \to_3 r \\ =\ & \big((p \wedge q)^\perp \wedge r\big) \vee \big(p \wedge q \wedge r\big) \vee \big((p \wedge q)^\perp \wedge r^\perp\big) \\ =\ & r \vee 0 \vee r^\perp \\ =\ & 1. \end{aligned}$$

and

$$\begin{aligned} & p \to_3 (q \to_3 r) \\ =\ & p \to_3 \big((q^\perp \wedge r) \vee (q \wedge r) \vee (q^\perp \wedge r^\perp)\big) \\ =\ & p \to_3 0 \\ =\ & (p^\perp \wedge 0) \vee (p \wedge 0) \vee (p^\perp \wedge 0^\perp) \\ =\ & 0 \vee 0 \vee p^\perp \\ =\ & p^\perp. \end{aligned}$$

Thus $p \wedge q \to_3 r \neq p \to_3 (q \to_3 r)$.

For $\to_4$, we have

$$\begin{aligned} & p \wedge q \to_4 r \\ =\ & \big((p \wedge q)^\perp \wedge r\big) \vee \big(p \wedge q \wedge r\big) \vee \big(((p \wedge q)^\perp \vee r) \wedge r^\perp\big) \\ =\ & r \vee 0 \vee \big((1 \vee r) \wedge r^\perp\big) \\ =\ & r \vee r^\perp \\ =\ & 1. \end{aligned}$$

and

$$\begin{aligned} & p \to_4 (q \to_4 r) \\ =\ & p \to_4 \big((q^\perp \wedge r) \vee (q \wedge r) \vee ((q^\perp \vee r) \wedge r^\perp)\big) \\ =\ & p \to_4 r^\perp \\ =\ & (p^\perp \wedge r^\perp) \vee (p \wedge (r^\perp)^\perp) \vee ((p^\perp \vee r^\perp) \wedge (r^\perp)^\perp) \\ =\ & 0 \vee 0 \vee r \\ =\ & r. \end{aligned}$$

Thus $p \wedge q \to_4 r \neq p \to_4 (q \to_4 r)$.

For $\to_5$, we have

$$\begin{aligned} & p \wedge q \to_5 r \\ =\ & \big((p \wedge q)^\perp \wedge r\big) \vee \big((p \wedge q)^\perp \wedge r^\perp\big) \\ & \vee \big((p \wedge q) \wedge ((p \wedge q)^\perp \vee r)\big) \\ =\ & r \vee r^\perp \vee \big(p \wedge q\big) \\ =\ & 1. \end{aligned}$$

and

$$\begin{aligned} & p \to_5 (q \to_5 r) \\ =\ & p \to_5 \big((q^\perp \wedge r) \vee (q^\perp \wedge r^\perp) \vee (q \wedge (q^\perp \vee r))\big) \\ =\ & p \to_5 q \\ =\ & (p^\perp \wedge q) \vee (p^\perp \wedge q^\perp) \vee (p \wedge (p^\perp \vee q)) \\ =\ & 0 \vee 0 \vee (p \wedge 1) \\ =\ & p. \end{aligned}$$

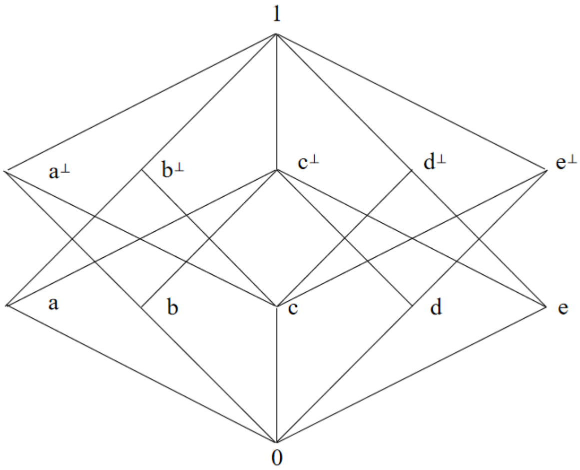

Fig. 2. Greechie lattice $\mathscr{G}_{12}$.

Thus $p \wedge q \rightarrow_5 r \neq p \rightarrow_5 (q \rightarrow_5 r)$. ■

From above theorem, we know that all the relatively reasonable five implication operators in quantum logic do not satisfy the law of importation. In their book [26] (see page 167), the authors presented a list of critical logical truths that are violated, among which is the Eq. (1). This equation's proof, however, was not provided. Here, a detailed proof is included.

B. $(p\&q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$ *in orthomodular lattice*

Theorem 2. *There exists an orthomodular lattice , such that none of the above implications* $\rightarrow_i$ *(*$1 \leq i \leq 5$*) satisfies Eq.(2)*

Proof: Consider the orthomodular lattice (Greechie lattice $\mathscr{G}_{12}$) represented in Fig. 2.

We have $a\&e = e \wedge (a \vee e^{\perp}) = e$ and $a^{\perp}\&b^{\perp} = b^{\perp} \wedge (a^{\perp} \vee (b^{\perp})^{\perp}) = b^{\perp} \wedge a^{\perp} = c$

For $\rightarrow_1$, we have

$$\begin{aligned} a\&e \rightarrow_1 d &= e \rightarrow_1 d \\ &= e^{\perp} \vee (e \wedge d) \\ &= e^{\perp}. \end{aligned}$$

and

$$\begin{aligned} a \rightarrow_1 (e \rightarrow_1 d) &= a \rightarrow_1 e^{\perp} \\ &= a^{\perp} \vee (a \wedge e^{\perp}) \\ &= a^{\perp}. \end{aligned}$$

Thus $a\&e \rightarrow_1 d \neq a \rightarrow_1 (e \rightarrow_1 d)$.

For $\rightarrow_2$, we have

$$\begin{aligned} a^{\perp}\&b^{\perp} \rightarrow_2 e^{\perp} &= c \rightarrow_2 e^{\perp} \\ & 1. \end{aligned}$$

and

$$\begin{aligned} & a^{\perp} \rightarrow_2 (b^{\perp} \rightarrow_2 e^{\perp}) \\ = \; & a^{\perp} \rightarrow_2 (e^{\perp} \vee ((b^{\perp})^{\perp} \wedge (e^{\perp})^{\perp}) \\ = \; & a^{\perp} \rightarrow_2 (e^{\perp} \vee 0) \\ = \; & a^{\perp} \rightarrow_2 e^{\perp} \\ = \; & e^{\perp} \vee ((a^{\perp})^{\perp} \wedge (e^{\perp})^{\perp}) \\ = \; & e^{\perp} \vee (a \wedge e) \\ = \; & e^{\perp} \vee 0 \\ = \; & e^{\perp}. \end{aligned}$$

Thus $a^{\perp}\&b^{\perp} \rightarrow_2 e^{\perp} \neq a^{\perp} \rightarrow_2 (b^{\perp} \rightarrow_2 e^{\perp})$.

For $\rightarrow_3$, we have

$$\begin{aligned} a\&e \rightarrow_3 d &= e \rightarrow_3 d \\ &= (e^{\perp} \wedge d) \vee (e \wedge d) \vee (e^{\perp} \wedge d^{\perp}) \\ &= d \vee 0 \vee c \\ &= e^{\perp}. \end{aligned}$$

and

$$\begin{aligned} a \rightarrow_3 (e \rightarrow_3 d) &= a \rightarrow_3 e^{\perp} \\ &= (a^{\perp} \wedge e^{\perp}) \vee (a \wedge e^{\perp}) \vee (a^{\perp} \wedge (e^{\perp})^{\perp}) \\ &= 0 \vee 0 \vee 0 \\ &= 0. \end{aligned}$$

Thus $a\&e \rightarrow_3 d \neq a \rightarrow_3 (e \rightarrow_3 d)$.

For $\rightarrow_4$, we have

$$\begin{aligned} a\&e \rightarrow_4 d &= e \rightarrow_4 d \\ &= (e^{\perp} \wedge d) \vee (e \wedge d) \vee ((e^{\perp} \vee d) \wedge d^{\perp}) \\ &= d \vee 0 \vee (e^{\perp} \wedge d^{\perp}) \\ &= d \vee 0 \vee c \\ &= e^{\perp}. \end{aligned}$$

and

$$\begin{aligned} a \rightarrow_4 (e \rightarrow_4 d) &= a \rightarrow_4 e^{\perp} \\ &= (a^{\perp} \wedge e^{\perp}) \vee (a \wedge e^{\perp}) \\ &\quad \vee ((a^{\perp} \vee e^{\perp}) \wedge (e^{\perp})^{\perp}) \\ &= 0 \vee 0 \vee (1 \wedge e) \\ &= e. \end{aligned}$$

Thus $a\&e \rightarrow_4 d \neq a \rightarrow_4 (e \rightarrow_4 d)$.

For $\rightarrow_5$, we have

$$\begin{aligned} & a\&e \rightarrow_5 d \\ = \; & e \rightarrow_5 d \\ = \; & (e^{\perp} \wedge d) \vee (e^{\perp} \wedge d^{\perp}) \vee (e \wedge (e^{\perp} \vee d)) \\ = \; & d \vee c \vee (e \wedge e^{\perp}) \\ = \; & d \vee c \vee 0 \\ = \; & e^{\perp}. \end{aligned}$$

and

$$\begin{aligned} & a \rightarrow_5 (e \rightarrow_5 d) \\ = \; & a \rightarrow_5 e^{\perp} \\ = \; & (a^{\perp} \wedge e^{\perp}) \vee (a \wedge (e^{\perp})^{\perp}) \vee (a \wedge (a^{\perp} \vee e^{\perp})) \\ = \; & 0 \vee 0 \vee (a \wedge 1) \\ = \; & a. \end{aligned}$$

Thus $a\&e \to_5 d \neq a \to_5 (e \to_5 d)$. ■

From above theorem, we know that all five relatively reasonable implication operators in quantum logic do not satisfy $(p\&q) \to r = p \to (q \to r)$.

C. $p \to r \equiv p \to (p \to r)$ in orthomodular lattice

Theorem 3. *For any orthomodular lattice, the implications $\to_i$, $i \in \{1, 2\}$, satisfies Eq.(3).*

Proof: For $\to_1$, we have

$$\begin{aligned} & p \to_1 (p \to_1 r) \\ = \;& p \to_1 \big(p^\perp \vee (p \wedge r)\big) \\ = \;& p^\perp \vee \Big(p \wedge \big(p^\perp \vee (p \wedge r)\big)\Big) \\ = \;& p^\perp \vee (p \wedge r) \quad \text{(by the orthomodular law Eq.9)} \\ = \;& p \to_1 r. \end{aligned}$$

Thus $p \to_1 (p \to_1 r) = p \to_1 r$.

For $\to_2$, we have

$$\begin{aligned} & p \to_2 (p \to_2 r) \\ = \;& p \to_2 \big(r \vee (p^\perp \wedge r^\perp)\big) \\ = \;& \big(r \vee (p^\perp \wedge r^\perp)\big) \vee \Big(p^\perp \wedge \big(r \vee (p^\perp \wedge r^\perp)\big)^\perp\Big). \end{aligned}$$

Since

$$\begin{aligned} & p^\perp \wedge \big(r \vee (p^\perp \wedge r^\perp)\big)^\perp \\ = \;& p^\perp \wedge \big(r^\perp \wedge ((p^\perp)^\perp \vee (r^\perp)^\perp)\big) \\ = \;& p^\perp \wedge \big(r^\perp \wedge (p \vee r)\big) \\ = \;& (p^\perp \wedge r^\perp) \wedge (p \vee r) \\ = \;& (p \vee r)^\perp \wedge (p \vee r) \\ = \;& 0. \end{aligned}$$

Then $p \to_2 (p \to_2 r) = \big(r \vee (p^\perp \wedge r^\perp)\big) \vee 0 = r \vee (p^\perp \wedge r^\perp) = p \to_2 r$. ■

Theorem 4. *There exists an orthomodular lattice, such that none of the implications $\to_i$, $i \in \{3, 4, 5\}$, satisfies Eq.(3).*

Proof: Consider the orthomodular lattice visualized by Fig. 1.

For $\to_3$, we have

$$\begin{aligned} p \to_3 q &= (p^\perp \wedge q) \vee (p \wedge q) \vee (p^\perp \wedge q^\perp) \\ &= 0 \vee 0 \vee 0 \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} p \to_3 (p \to_3 q) &= a \to_3 0 \\ &= (p^\perp \wedge 0) \vee (p \wedge 0) \vee (p^\perp \wedge 0^\perp) \\ &= p^\perp. \end{aligned}$$

Thus $p \to_3 (p \to_3 q) \neq p \to_3 q$.

For $\to_4$, we have

$$\begin{aligned} p \to_4 q &= (p^\perp \wedge q) \vee (p \wedge q) \vee ((p^\perp \vee q) \wedge q^\perp) \\ &= 0 \vee 0 \vee (1 \wedge q^\perp) \\ &= q^\perp. \end{aligned}$$

and

$$\begin{aligned} & p \to_4 (p \to_4 q) \\ = \;& p \to_4 q^\perp \\ = \;& (p^\perp \wedge q^\perp) \vee (p \wedge q^\perp) \vee ((p^\perp \vee q^\perp) \wedge (q^\perp)^\perp) \\ = \;& 0 \vee 0 \vee (1 \wedge q) \\ = \;& q. \end{aligned}$$

Thus $p \to_4 (p \to_4 q) \neq p \to_4 q$.

For $\to_5$, we have

$$\begin{aligned} p \to_5 q &= (p^\perp \wedge q) \vee (p^\perp \wedge q^\perp) \vee (p \wedge (p^\perp \vee q)) \\ &= 0 \vee 0 \vee (p \wedge 1) \\ &= p. \end{aligned}$$

and

$$\begin{aligned} p \to_5 (p \to_5 q) &= p \to_5 p \\ &= 1. \end{aligned}$$

Thus $p \to_5 (p \to_5 q) \neq p \to_5 q$. ■

From above theorems, we know that $\to_1$ and $\to_2$ satisfy the equality $p \to r \equiv p \to (p \to r)$, but $\to_3$, $\to_4$ and $\to_5$ do not satisfy this equality.

We have confirmed that the equivalence $p \to r \equiv p \to (p \to r)$ holds for both Sasaki and Dishkant implications, whereas $(p \wedge q) \to r \equiv p \to (q \to r)$ does not. It's important to note that $p \to r \equiv p \to (p \to r)$ is a specific instance of $(p \wedge q) \to r \equiv p \to (q \to r)$ when $p = q$.

If we substitute $r = p \to r$, then $r \to (p \to r)$ can be rewritten as $(p \to r) \to (p \to (p \to r))$. It also should be noted that the expression $r \to (p \to r)$ is violated for both Sasaki and Dishkant implications [26], whereas $(p \to r) \to (p \to (p \to r))$ is not.

D. $p \wedge q \to r = (p \to r) \vee (q \to r)$ in orthomodular lattice

Theorem 5. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{1, 4, 5\}$ satisfies Eq. (4).*

Proof: Consider the orthomodular lattice (Greechie lattice $\mathscr{G}_{12}$) represented in Fig. 2.

For $\to_1$, we have

$$\begin{aligned} a^\perp \wedge d^\perp \to_1 c^\perp &= c \to_1 c^\perp \\ &= c^\perp \vee (c \wedge c^\perp) \\ &= c^\perp \vee 0 \\ &= c^\perp \end{aligned}$$

and

$$\begin{aligned} & (a^\perp \to_1 c^\perp) \vee (d^\perp \to_1 c^\perp) \\ = \;& (a \vee (a^\perp \wedge c^\perp) \vee (d \vee (d^\perp \wedge c^\perp) \\ = \;& (a \vee b) \vee (d \vee e) \\ = \;& 0 \vee 0 \\ = \;& 0. \end{aligned}$$

Thus $a^\perp \wedge d^\perp \to_1 c^\perp \neq (a^\perp \to_1 c^\perp) \vee (d^\perp \to_1 c^\perp)$.

For $\to_4$, we have

$$\begin{aligned} a \wedge e \to_4 d &= 0 \to_4 d \\ &= 1 \end{aligned}$$

and

$$\begin{aligned}&(a \to_4 d) \vee (e \to_4 d)\\ =\ & \Big((a^\perp \wedge d) \vee (a \wedge d) \vee ((a^\perp \vee d) \wedge d^\perp)\Big)\\ & \vee\Big((e^\perp \wedge d) \vee (e \wedge d) \vee ((e^\perp \vee d) \wedge d^\perp)\Big)\\ =\ & \Big(0 \vee 0 \vee (1 \wedge d^\perp)\Big) \vee \Big(0 \vee 0 \vee (1 \wedge d^\perp)\Big)\\ =\ & d^\perp \vee d^\perp\\ =\ & d^\perp.\end{aligned}$$

Thus $a \wedge e \to_4 d \neq (a \to_4 d) \vee (e \to_4 d)$.

For $\to_5$, we have

$$\begin{aligned}a \wedge e \to_5 d &= 0 \to_5 d\\ &= 1\end{aligned}$$

and

$$\begin{aligned}&(a \to_5 d) \vee (e \to_5 d)\\ =\ & \Big((a \wedge d) \vee (a^\perp \wedge d^\perp) \vee (a \wedge (a^\perp \vee d))\Big)\\ & \vee\Big((e \wedge d) \vee (e^\perp \wedge d^\perp) \vee (e \wedge (e^\perp \vee d))\Big)\\ =\ & \Big(0 \vee c \vee (a \wedge 1\Big) \vee \Big(0 \vee c \vee (e \wedge e^\perp)\Big)\\ =\ & b^\perp \vee c\\ =\ & b^\perp.\end{aligned}$$

Thus $a \wedge e \to_5 d \neq (a \to_5 d) \vee (e \to_5 d)$. ■

Theorem 6. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{2,3\}$ satisfies Eq. (4).*

Proof: Consider the orthomodular lattice visualized by Fig. 1.

For $\to_2$, from the proof of Theorem 1, we have $p \wedge q \to_2 r = 1$. Moreover,

$$\begin{aligned}&(p \to_2 r) \vee (q \to_2 r)\\ =\ & (r \vee (q^\perp \wedge r^\perp) \vee (r \vee (p^\perp \wedge r^\perp)\\ =\ & (r \vee 0) \vee (r \vee 0)\\ =\ & r \vee r\\ =\ & r.\end{aligned}$$

Thus $p \wedge q \to_2 r \neq (p \to_2 r) \vee (q \to_2 r)$.

For $\to_3$, from the proof of Theorem 1, we have $p \wedge q \to_3 r = 1$. Moreover,

$$\begin{aligned}&(p \to_3 r) \vee (q \to_3 r)\\ =\ & \Big((p^\perp \wedge r) \vee (p \wedge r) \vee (p^\perp \wedge r^\perp)\Big)\\ & \vee\Big((q^\perp \wedge r) \vee (q \wedge r) \vee (q^\perp \wedge r^\perp)\Big)\\ =\ & (0 \vee 0 \vee 0) \vee (0 \vee 0 \vee 0)\\ =\ & 0.\end{aligned}$$

Thus $p \wedge q \to_3 r \neq (p \to_3 r) \vee (q \to_3 r)$. ■

From above two theorems, we know that all five relatively reasonable implication operators in quantum logic do not satisfy $p \wedge q \to r = (p \to r) \vee (q \to r)$.

E. $p \vee q \to r = (p \to r) \wedge (q \to r)$ in orthomodular lattice

Theorem 7. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{1,2\}$ satisfies Eq. (5).*

Proof: Consider the orthomodular lattice (Greechie lattice $\mathscr{G}_{12}$) represented in Fig. 2.

For $\to_1$, we have

$$\begin{aligned}a \vee b \to_1 d &= c^\perp \to_1 d\\ &= c \vee (c^\perp \wedge d)\\ &= c \vee d\\ &= e^\perp\end{aligned}$$

and

$$\begin{aligned}&(a \to_1 d) \wedge (b \to_1 d)\\ =\ & (a^\perp \vee (a \wedge d)) \wedge (b^\perp \vee (b \wedge d))\\ =\ & (a^\perp \vee 0) \wedge (b^\perp \vee 0)\\ =\ & a^\perp \wedge b^\perp\\ =\ & c.\end{aligned}$$

Thus $a \vee b \to_1 d \neq (a \to_1 d) \wedge (b \to_1 d)$.

For $\to_2$, we have

$$\begin{aligned}a \vee c \to_2 e &= b^\perp \to_2 e\\ &= e \vee (b \wedge e^\perp)\\ &= e \vee 0\\ &= e\end{aligned}$$

and

$$\begin{aligned}&(a \to_2 e) \wedge (c \to_2 e)\\ =\ & (e \vee (a^\perp \wedge e^\perp)) \wedge (e \vee (c^\perp \wedge e^\perp))\\ =\ & (e \vee c) \wedge (e \vee d)\\ =\ & d^\perp \wedge c^\perp\\ =\ & 1.\end{aligned}$$

Thus $a \vee b \to_2 d \neq (a \to_2 d) \wedge (b \to_2 d)$. ■

Theorem 8. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{3,4,5\}$ satisfies Eq. (5).*

Proof: Consider the orthomodular lattice visualized by Fig. 1.

For $\to_3$, we have

$$\begin{aligned}p \vee q \to_3 r &= 1 \to_3 r\\ &= (0 \wedge r) \vee (1 \wedge r) \vee (0 \wedge r^\perp)\\ &= 0 \vee r \vee 0\\ &= r\end{aligned}$$

and

$$\begin{aligned}&(p \to_3 r) \wedge (q \to_3 r)\\ =\ & (p^\perp \wedge r) \vee (p \wedge r) \vee (p^\perp \wedge r^\perp)\\ & \wedge(q^\perp \wedge r) \vee (q \wedge r) \vee (q^\perp \wedge r^\perp)\\ =\ & (0 \vee 0 \vee 0) \wedge (0 \vee 0 \vee 0)\\ =\ & 0.\end{aligned}$$

Thus $a \vee b \to_3 d \neq (a \to_3 d) \wedge (b \to_3 d)$.

For $\to_4$, we have

$$\begin{aligned} & p \vee q \to_4 r \\ = \;& 1 \to_4 r \\ = \;& (0 \wedge r) \vee (1 \wedge r) \vee (r^{\perp} \wedge (0 \vee r)) \\ = \;& 0 \vee r \vee 0 \\ = \;& r \end{aligned}$$

and

$$\begin{aligned} & (p \to_4 r) \wedge (q \to_4 r) \\ = \;& (p^{\perp} \wedge r) \vee (p \wedge r) \vee (r^{\perp} \wedge (p^{\perp} \vee r)) \\ & \wedge (q^{\perp} \wedge r) \vee (q \wedge r) \vee (r^{\perp} \wedge (q^{\perp} \vee r)) \\ = \;& (0 \vee 0 \vee r^{\perp}) \wedge (0 \vee 0 \vee r^{\perp}) \\ = \;& r^{\perp} \wedge r^{\perp} \\ = \;& r^{\perp}. \end{aligned}$$

Thus $a \vee b \to_4 d \neq (a \to_4 d) \wedge (b \to_4 d)$.

For $\to_5$, we have

$$\begin{aligned} & p \vee q \to_5 r \\ = \;& 1 \to_5 r \\ = \;& (0 \wedge r) \vee (0 \wedge r^{\perp}) \vee (1 \wedge (0 \vee r)) \\ = \;& 0 \vee 0 \vee r \\ = \;& r \end{aligned}$$

and

$$\begin{aligned} & (p \to_5 r) \wedge (q \to_5 r) \\ = \;& (p^{\perp} \wedge r) \vee (p^{\perp} \wedge r^{\perp}) \vee (p \wedge (p^{\perp} \vee r)) \\ & \wedge (q^{\perp} \wedge r) \vee (q^{\perp} \wedge r^{\perp}) \vee (q \wedge (q^{\perp} \vee r)) \\ = \;& (0 \vee 0 \vee p) \wedge (0 \vee 0 \vee q) \\ = \;& p \wedge q \\ = \;& 0. \end{aligned}$$

Thus $a \vee b \to_5 d \neq (a \to_5 d) \wedge (b \to_5 d)$. ■

From above two theorems, we know that all five relatively reasonable implication operators in quantum logic do not satisfy $p \vee q \to r = (p \to r) \wedge (q \to r)$.

F. $p \to (q \wedge r) = (p \to q) \wedge (p \to r)$ in orthomodular lattice

Theorem 9. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{2, 3, 4\}$ satisfies Eq. (6).*

Proof: Consider the orthomodular lattice visualized by Fig. 1.

For $\to_2$, we have

$$\begin{aligned} & p \to_2 (q \wedge r) \\ = \;& p \to_2 0 \\ = \;& 0 \vee (p^{\perp} \wedge 1) \\ = \;& 0 \vee p^{\perp} \\ = \;& p^{\perp} \end{aligned}$$

and

$$\begin{aligned} & (p \to_2 q) \wedge (p \to_2 r) \\ = \;& (q \vee (p^{\perp} \wedge q^{\perp})) \wedge (r \vee (p^{\perp} \wedge r^{\perp})) \\ = \;& (q \vee 0) \wedge (r \vee 0) \\ = \;& q \wedge r \\ = \;& 0. \end{aligned}$$

Thus $p \to_2 (q \wedge r) \neq (p \to_2 q) \wedge (p \to_2 r)$.

For $\to_3$, we have

$$\begin{aligned} & p \to_3 (q \wedge r) \\ = \;& p \to_3 0 \\ = \;& (p^{\perp} \wedge 0) \vee (p \wedge 0) \vee (p^{\perp} \wedge 0^{\perp}) \\ = \;& 0 \vee 0 \vee p^{\perp} \\ = \;& p^{\perp} \end{aligned}$$

and

$$\begin{aligned} & (p \to_3 q) \wedge (p \to_3 r) \\ = \;& \Big((p^{\perp} \wedge q) \vee (p \wedge q) \vee (p^{\perp} \wedge q^{\perp})\Big) \\ & \wedge \Big((p^{\perp} \wedge r) \vee (p \wedge r) \vee (p^{\perp} \wedge r^{\perp})\Big) \\ = \;& (0 \vee 0 \vee 0) \wedge (0 \vee 0 \vee 0) \\ = \;& 0. \end{aligned}$$

Thus $p \to_3 (q \wedge r) \neq (p \to_3 q) \wedge (p \to_3 r)$.

For $\to_4$, we have

$$\begin{aligned} & p \to_4 (q \wedge r) \\ = \;& p \to_4 0 \\ = \;& (p^{\perp} \wedge 0) \vee (p \wedge 0) \vee ((p^{\perp} \vee 0) \wedge 0^{\perp}) \\ = \;& 0 \vee 0 \vee (p^{\perp} \wedge 1) \\ = \;& p^{\perp} \end{aligned}$$

and

$$\begin{aligned} & (p \to_4 q) \wedge (p \to_4 r) \\ = \;& \Big((p^{\perp} \wedge q) \vee (p \wedge q) \vee ((p^{\perp} \vee q) \wedge q^{\perp})\Big) \\ & \wedge \Big((p^{\perp} \wedge r) \vee (p \wedge r) \vee ((p^{\perp} \vee r) \wedge r^{\perp})\Big) \\ = \;& (0 \vee 0 \vee q^{\perp}) \wedge (0 \vee 0 \vee r^{\perp}) \\ = \;& 0. \end{aligned}$$

Thus $p \to_4 (q \wedge r) \neq (p \to_4 q) \wedge (p \to_4 r)$. ■

Theorem 10. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{1, 5\}$ satisfies Eq. (6).*

Proof: Consider the orthomodular lattice (Greechie lattice $\mathscr{G}_{12}$) represented in Fig. 2.

For $\to_1$, we have

$$\begin{aligned} c^{\perp} \to_1 (d \wedge e) &= c^{\perp} \to_1 0 \\ &= c \vee (c^{\perp} \wedge 0) \\ &= c \vee 0 \\ &= c \end{aligned}$$

and

$$\begin{aligned} & (c^{\perp} \to_1 d) \wedge (c^{\perp} \to_1 e) \\ = \;& \Big(c \vee (c^{\perp} \wedge d)\Big) \wedge \Big(c \vee (c^{\perp} \wedge e)\Big) \\ = \;& (a \vee d) \wedge (a \vee e) \\ = \;& c^{\perp} \wedge c^{\perp} \\ = \;& c^{\perp}. \end{aligned}$$

Thus $p \to_1 (q \wedge r) \neq (p \to_1 q) \wedge (p \to_1 r)$.

For $\to_5$, we have

$$\begin{aligned} & a^{\perp} \to_5 (c \wedge e) \\ = \;& a^{\perp} \to_5 0 \\ = \;& (a \wedge 0) \vee (a \wedge 0^{\perp}) \vee (a^{\perp} \wedge (a \vee 0)) \\ = \;& 0 \vee a \vee (a^{\perp} \wedge a) \\ = \;& 0 \vee a \vee 0 \\ = \;& a \end{aligned}$$

and

$$\begin{aligned} & (a^{\perp} \to_5 c) \wedge (a^{\perp} \to_5 e) \\ = \;& \Big((a^{\perp} \wedge c) \vee (a^{\perp} \wedge c^{\perp}) \vee (a \wedge (a^{\perp} \vee c))\Big) \\ & \wedge\Big((a^{\perp} \wedge e) \vee (a^{\perp} \wedge e^{\perp}) \vee (a \wedge (a^{\perp} \vee e))\Big) \\ = \;& (c \vee b \vee (a \wedge c)) \wedge (e \vee c \vee (a \wedge 0)) \\ = \;& (c \vee b \vee 0) \wedge (e \vee c \vee 0) \\ = \;& a^{\perp} \wedge d^{\perp} \\ = \;& c. \end{aligned}$$

Thus $p \to_5 (q \wedge r) \neq (p \to_5 q) \wedge (p \to_5 r)$. ■

From above two theorems, we know that all five relatively reasonable implication operators in quantum logic do not satisfy $p \to (q \wedge r) = (p \to q) \wedge (p \to r)$.

G. $p \to (q \vee r) = (p \to q) \vee (p \to r)$ in orthomodular lattice

Theorem 11. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{1, 3, 5\}$ satisfies Eq. (7).*

Proof: Consider the orthomodular lattice visualized by Fig. 1.

For $\to_1$, we have

$$\begin{aligned} p \to_1 (q \vee r) &= p \to_1 1 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} & (p \to_1 q) \vee (p \to_1 r) \\ = \;& (p^{\perp} \vee (p \wedge q)) \wedge (p^{\perp} \vee (p \wedge r)) \\ = \;& p^{\perp} \vee p^{\perp} \\ = \;& p^{\perp}. \end{aligned}$$

Thus $p \to_1 (q \vee r) \neq (p \to_1 q) \vee (p \to_1 r)$.

For $\to_3$, we have

$$\begin{aligned} & p \to_3 (q \vee r) \\ = \;& p \to_3 1 \\ = \;& 1 \end{aligned}$$

and

$$\begin{aligned} & (p \to_3 q) \wedge (p \to_3 r) \\ = \;& \Big((p^{\perp} \wedge q) \vee (p \wedge q) \vee (p^{\perp} \wedge q^{\perp})\Big) \\ & \vee\Big((p^{\perp} \wedge r) \vee (p \wedge r) \vee (p^{\perp} \wedge r^{\perp})\Big) \\ = \;& (0 \vee 0 \vee 0) \vee (0 \vee 0 \vee 0) \\ = \;& 0. \end{aligned}$$

Thus $p \to_3 (q \vee r) \neq (p \to_3 q) \vee (p \to_3 r)$.

For $\to_5$, we have

$$\begin{aligned} p \to (q \vee r) &= p \to_5 1 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} & (p \to_5 q) \vee (p \to_5 r) \\ = \;& \Big((p^{\perp} \wedge q) \vee (p^{\perp} \wedge q^{\perp}) \vee (p \wedge (p^{\perp} \vee q))\Big) \\ & \vee\Big((p^{\perp} \wedge q) \vee (p^{\perp} \wedge q^{\perp}) \vee (p \wedge (p^{\perp} \vee q))\Big) \\ = \;& (0 \vee 0 \vee p) \vee (0 \vee 0 \vee p) \\ = \;& p. \end{aligned}$$

Thus $p \to_5 (q \vee r) \neq (p \to_5 q) \vee (p \to_5 r)$. ■

Theorem 12. *There exists an orthomodular lattice such that none of implications $\to_i$, $i \in \{2, 4\}$ satisfies Eq. (7).*

Proof: Consider the orthomodular lattice (Greechie lattice $\mathscr{G}_{12}$) represented in Fig. 2.

For $\to_2$, we have

$$\begin{aligned} & a^{\perp} \to_2 (c \vee e) \\ = \;& a^{\perp} \to_2 d \\ = \;& d \vee (a \wedge d^{\perp}) \\ = \;& d \vee 0 \\ = \;& d \end{aligned}$$

and

$$\begin{aligned} & (a^{\perp} \to_2 c) \vee (a^{\perp} \to_2 e) \\ = \;& \Big(c \vee (a \wedge c^{\perp})\Big) \\ & \wedge\Big(e \vee (a \wedge e^{\perp})\Big) \\ = \;& (c \vee a) \wedge (e \vee 0) \\ = \;& b^{\perp} \wedge e \\ = \;& 0. \end{aligned}$$

Thus $p \to_2 (q \vee r) \neq (p \to_2 q) \vee (p \to_2 r)$.

For $\to_4$, we have

$$\begin{aligned} & a^{\perp} \to_4 (c \vee e) \\ = \;& a^{\perp} \to_4 d \\ = \;& (a \wedge d) \vee (a^{\perp} \wedge d) \vee ((a \vee d) \wedge d^{\perp}) \\ = \;& 0 \vee 0 \vee (c^{\perp} \wedge d^{\perp}) \\ = \;& 0 \vee e \\ = \;& e \end{aligned}$$

and

$$\begin{aligned} & (a^{\perp} \to_4 c) \vee (a^{\perp} \to_4 e) \\ = \;& \Big((a \wedge c) \vee (a^{\perp} \wedge c) \vee ((a \vee c) \wedge c^{\perp})\Big) \\ & \wedge\Big((a \wedge e) \vee (a^{\perp} \wedge e) \vee ((a \vee e) \wedge e^{\perp})\Big) \\ = \;& (0 \vee c \vee (b^{\perp} \wedge c^{\perp})) \vee (0 \vee 0 \vee (c^{\perp} \wedge e^{\perp})) \\ = \;& c \vee a \vee d \\ = \;& b^{\perp} \vee d \\ = \;& 1. \end{aligned}$$

Thus $p \to_4 (q \vee r) \neq (p \to_4 q) \vee (p \to_4 r)$.

TABLE I
SEVEN FUNCTIONAL EQUATIONS IN QUANTUM LOGIC

	$\to_1$	$\to_2$	$\to_3$	$\to_4$	$\to_5$
$p \wedge q \to r \equiv p \to (q \to r)$	×	×	×	×	×
$p \& q \to r \equiv p \to (q \to r)$	×	×	×	×	×
$p \to r \equiv p \to (p \to r)$	√	√	×	×	×
$p \wedge q \to r \equiv (p \to r) \vee (q \to r)$	×	×	×	×	×
$p \vee q \to r \equiv (p \to r) \wedge (q \to r)$	×	×	×	×	×
$p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r)$	×	×	×	×	×
$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$	×	×	×	×	×

■

From above two theorems, we know that all five relatively reasonable implication operators in quantum logic do not satisfy $p \to (q \vee r) = (p \to q) \vee (p \to r)$.

IV. Concluding Remarks

In this paper, our investigation focuses on seven quantum implication functions under five reasonable implication operators. Our main results are summarized as follows, and are also illustrated in Table I.

(i) We prove that all the five relatively reasonable implication operators in quantum logic do not satisfy the law of importation, as demonstrated by Theorem 1.
(ii) We show that none of the five relatively reasonable implication operators in quantum logic satisfy the Eq. (2), as proven by Theorem 2.
(iii) We observe that Sasaki implication $\to_1$ and Dishkant implication $\to_2$ adhere to the derived iterative Boolean law, whereas relevance implication $\to_3$, non-tollens implication $\to_4$, and Kalmbach implication $\to_5$ do not, as confirmed by Theorems 3 and 4.
(iv) We prove that all the five relatively reasonable implication operators in quantum logic do not satisfy the the distributivity of implications, as demonstrated by Theorems 5-12.

It is important to note that our study exclusively examines three specific implication functions with respect to quantum implication operators. However, a more comprehensive examination of other quantum implication functions would be both necessary and interesting for future research.

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