Analysis of Relative Weights for Attributes under Intuitionistic Fuzzy Setting Environments

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***Abstract*—We provide an analytic examination for a new proposed relative weights to find the inherent property of this new relative weight. Our findings will help researchers in the future to apply this new relative weight and develop their own new relative weights to face the complex and changing real world.**

***Index Terms*—Relative weights, Intuitionistic fuzzy sets, Entropy measures, Analytic Examination**

I. Introduction

Chen and Li [1] listed twenty-one entropy measures to convert an intuitionistic fuzzy set to a crisp value. They first used four entropy measures to demonstrate that different entropy measures will imply different findings for relative weights for attributes. Next, they developed a new normalization method to convert the total sum of relative weights being equal to one. However, they did not provide a motivation for their new normalization method. The purpose of this paper is to present an explanation for the new normalization method proposed by Chen and Li [1]. Our derivation reveals that Chen and Li [1] implicitly increased the weight for those stronger weights (that is, before normalization, those weights are greater than the mean of all weights). On the other hand, they implicitly decreased the weight for those weaker weights (that is, before normalization, those weights are smaller than the mean of all weights).

II. A Brief Review of Previous Results

Among twenty-one entropy measures mentioned in Chen and Li [1], they applied the following four entropy measures to convert intuitionistic fuzzy sets into crisp values. The first entropy measure is defined as follows,

$$E_{BB}(A)=\sum_{i=1}^{n}\left(1-\mu_A(x_i)-v_A(x_i)\right)\Delta_i\,, \quad (2.1)$$

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which is proposed by Burillo and Bustince [2] and Hung and Yang [3], where

$$\Delta_i=\sin\left(\pi\left(\mu_A(x_i)+v_A(x_i)\right)/2\right), \quad (2.2)$$

is a abbreviation developed by this paper to simplify the expression.

The second entropy measure is defined as follows,

$$E_{SK}(A)=\frac{\sum_{i=1}^{n}\left[\left((\Delta+\Phi)/4\right)+\left(\max\{\Delta,\Phi\}/2\right)\right]}{\sum_{i=1}^{n}\left[\left((\Gamma+\mathrm{H})/4\right)+\left(\max\{\Gamma,\mathrm{H}\}/2\right)\right]}, \quad (2.3)$$

where the following abbreviations are used,

$$\Delta=\mu_{near}(x_i)-\mu_A(x_i), \quad (2.4)$$

$$\Phi=v_{near}(x_i)-v_A(x_i), \quad (2.5)$$

$$\Gamma=\mu_{farr}(x_i)-\mu_A(x_i), \quad (2.6)$$

and

$$\mathrm{H}=v_{farr}(x_i)-v_A(x_i), \quad (2.7)$$

which is a combination of entropy measures of Szmidt and Kacprzyk [4] and Wang and Xia [5].

The third entropy measure is defined as follows,

$$E_{ZL}(A)=1-\left|\mu_A(x_i)-v_A(x_i)\right|, \quad (2.8)$$

which is proposed by Zeng and Li [6].

The fourth entropy measure is defined as follows, for $\alpha > 1$ and $\alpha \neq 1$,

$$E_{HY}^{\alpha}(A)=\left(1-\left(\mu_A^{\alpha}+v_A^{\alpha}+\pi_A^{\alpha}\right)\right)/(\alpha-1), \quad (2.9)$$

When $\alpha = 1$,

$$E_{HY}^{\alpha}(A)=\mu_A\log\mu_A-v_A\log v_A-\pi_A\log\pi_A\,, \quad (2.10)$$

which is proposed by Hung and Yang [3].

If there are m alternative measures P_i, for $i=1,2,...,m$, and n attributes and then the intuitionistic fuzzy decision matrix, denoted as D, in the following Table 1.

Chen and Li [1] applied entropy measures of Equations (2.1), (2.3), (2.8), (2.9) and (2.10) to convert an intuitionistic fuzzy numbers to a crisp number in the next Table 2.

They adjusted the maximum value of each column being one as follows,

$$h_{ij} = \frac{E_{i1}}{\max(E_{i1})}, \frac{E_{i2}}{\max(E_{i2})}, \ldots, \frac{E_{ij}}{\max(E_{ij})}. \quad (2.11)$$

They used h_{ij} to denote the adjusted values to imply the following adjusted decision matrix in the following Table 3.

They defined relative weights for attributes x_j, for $j = 1,2,\ldots,n$ as

$$w_j = (1 - a_j)/(n - T), \quad (2.12)$$

with an abbreviation, that is denoted as

$$a_j = \sum_{i=1}^{m} h_{ij}, \quad (2.13)$$

for $j = 1,2,\ldots,n$, and

$$T = \sum_{j=1}^{n} a_j. \quad (2.14)$$

However, Chen and Li [1] did not inform us the motivation why they construct relative weights by Equation (2.12).

The most important goal of our article is to present a detailed examination for Equation (2.12) to find out relations between (a) traditional relative weights, and (b) new relative weights proposed by Chen and Li [1].

III. Our Improvement

In the following, we will point out two questionable results in Chen and Li [1]. First, we recall the definition of h_{ij} should be revised from Equation (2.11) to a revised expression,

$$h_{ij} = E_{ij} / \max\{E_{\alpha j} : \alpha = 1,2,\ldots,m\}, \quad (3.1)$$

for $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.

We must point out that Chen and Li [1] did not provide their reason why did they define w_j as Equation (2.12).

We recall the popular definition for relative weight, denoted as W^j,

$$W^j = a_j / T, \quad (3.2)$$

for $j = 1,2,\ldots,n$, and $T = \sum_{j=1}^{n} a_j$ to keep the total sum of W^j, for $j = 1,2,\ldots,n$, equals to one.

We run a comparison between w_j, and W^j. It is trivial that $\sum_{j=1}^{n} W^j = 1$ and we compute that

$$\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} (a_j - 1)/(T - n)$$

$$= \sum_{j=1}^{n} (a_j - 1) \Big/ (T - n),$$

$$= \left[\left(\sum_{j=1}^{n} a_j \right) - \sum_{j=1}^{n} 1 \right] \Big/ (T - n)$$

$$= (T - n)/(T - n) = 1. \quad (3.3)$$

Based on above computation, we can generalize the above definition as

$$W_k = (a_k - \varepsilon)/(T - n\varepsilon), \quad (3.4)$$

for $k = 1,2,\ldots,n$, where ε satisfies the following condition,

$$\varepsilon < \min\{a_k : k = 1,2,\ldots,n\}, \quad (3.5)$$

to guarantee the positivity of W_k. For example, we can assume that

$$\varepsilon = [1 + \min\{a_k : k = 1,2,\ldots,n\}]/2, \quad (3.6)$$

owing to $a_k > 1$, for $k = 1,2,\ldots,n$.

Based on the same numerical example of Chen and Li [1], with $n = 6$, $a_1 = 2.88$, $a_2 = 3.09$, $a_3 = 2.39$, $a_4 = 1.69$, $a_5 = 3.09$ and $a_6 = 2.39$, we compare (a) the relative weights w_j of Chan and Li [1], and (b) the commonly used relative weights W^j. Hence, we consider

$$W_k(\alpha) = (a_k - \alpha)/(T - n\alpha), \quad (3.7)$$

for $k = 1,2,\ldots,n$, and $\alpha = 0, 0.1, 0.2, \ldots, 1$, such that

$$W_k(0) = W^k, \tag{3.8}$$

for $k = 1,2,...,n$ as the traditional relative weights. On the other hand,

$$W_k(1) = w_k, \tag{3.9}$$

for $k = 1,2,...,n$ as the relative weights proposed by Chen and Li [1]. We express our findings in the following Table 4.

From Table 4, researchers may predict that $W_1(\alpha)$, $W_2(\alpha)$ and $W_5(\alpha)$ are increasing function with respect to the variable of α, and on the other hand, $W_3(\alpha)$, $W_4(\alpha)$ and $W_6(\alpha)$ are decreasing function with respect to the variable of α.

Consequently, for a numerical example point of view, those practitioners will predict the following Lemmas based on data in the above Table 4.

Lemma 1. If $W_k(0.1) > W_k(0)$, then $W_k(\alpha)$ is increasing function with respect to the variable of α.

Lemma 2. If $W_k(0.1) < W_k(0)$, then $W_k(\alpha)$ is decreasing function with respect to the variable of α.

Lemma 3. If $W_k(\alpha)$ is increasing function with respect to the variable of α. Moreover, under the condition of $W_i(0) > W_k(0)$, then $W_i(\alpha)$ is also increasing with α.

Lemma 4. If $W_k(\alpha)$ is decreasing with respect to the variable of α. On the other hand, under the condition of $W_i(0) < W_k(0)$, then $W_i(\alpha)$ is also decreasing with α.

However, on the other hand, from an analytical point of view, we will take the derivative with respect to α, such that we obtain that

$$W_j(\alpha) = (a_j - \alpha) \Big/ \left[\left(\sum_{i=1}^{n} a_i\right) - n\alpha\right], \tag{3.10}$$

for $j = 1,2,...,n$, since $a_j > 1$, and

$$0 \le \alpha \le 1 < \min\{a_i : i = 1,2,...,n\}. \tag{3.11}$$

We compute that

$$\frac{d}{d\alpha} W_j(\alpha) = \frac{n\left(a_j - \left(\sum_{i=1}^{n} a_i \Big/ n\right)\right)}{\left[\left(\sum_{i=1}^{n} a_i\right) - n\alpha\right]^2}. \tag{3.12}$$

From Equation (3.11), we find that $W_j(\alpha)$ is a monatomic function of α, depending on the sign of $a_j - \left(\sum_{i=1}^{n} a_i \Big/ n\right)$.

Therefore, we derive the following theorem.

Theorem 1.

(I) If $a_j > \sum_{i=1}^{n} a_i \Big/ n$, then $W_j(\alpha)$ is an increasing function with the variable of α.

(II) If $a_j = \sum_{i=1}^{n} a_i \Big/ n$, then $W_j(\alpha)$ is a constant function with the variable of α.

(III) If $a_j < \sum_{i=1}^{n} a_i \Big/ n$, then $W_j(\alpha)$ is a decreasing function with the variable of α.

From our Theorem 1, all predictions of Lemmas 1-4 are corrected to illustrate the power of analytical approach. Moreover, we point out that Chen and Li [1] implicitly changed the well-known relative weights of Equation (3.2) to a new setting of Equation (2.12) without a proper explanation for their definition.

Based on our Theorem 1, we show that Chen and Li [1] enlarged those relative weights which is greater than the mean of $\{a_j : j = 1,2,...,n\}$ and shrank those relative weights which is smaller than the mean of $\{a_j : j = 1,2,...,n\}$. Hence, our paper provides a good explanation for the relative weights proposed by Chen and Li [1].

IV. A Related Question

In this part, we will study a related question of Ye [7] with respect to two inequalities. For trapezoidal intuitionistic fuzzy sets, we provide a patchwork for Ye [7] with aggregation operators. The previous paper of Ye [7] needed the following two formulas.

Table 1. The intuitionistic fuzzy decision matrix, D.

	x_1	x_2	$\cdots$	x_n
P_1	$(\mu_{11,} v_{11}, \pi_{11})$	$(\mu_{12,} v_{12}, \pi_{12})$	$\cdots$	$(\mu_{1n,} v_{1n}, \pi_{1n})$
P_2	$(\mu_{21,} v_{21}, \pi_{21})$	$(\mu_{22,} v_{22}, \pi_{22})$	$\cdots$	$(\mu_{2n,} v_{2n}, \pi_{2n})$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
P_m	$(\mu_{m1,} v_{m1}, \pi_{m1})$	$(\mu_{m2,} v_{m2}, \pi_{m2})$	$\cdots$	$(\mu_{mn,} v_{mn}, \pi_{mn})$

Table 2. The crisp number decision matrix, D.

	x_1	x_2	$\cdots$	x_n
P_1	E_{11}	E_{12}	$\cdots$	E_{1n}
P_2	E_{21}	E_{22}	$\cdots$	E_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
P_m	E_{m1}	E_{m2}	$\cdots$	E_{mn}

Table 3. The adjusted crisp number decision matrix, D.

	x_1	x_2	$\cdots$	x_n
P_1	h_{11}	h_{12}	$\cdots$	h_{1n}
P_2	h_{21}	h_{22}	$\cdots$	h_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
P_m	h_{m1}	h_{m2}	$\cdots$	h_{mn}

Table 4. Results for $W_k(\alpha)$, for $\alpha = 0, 0.1, 0.2, \ldots, 1$, and $k = 1, 2, \ldots, 6$.

$W_k(\alpha)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$\alpha=0$	0.185	0.199	0.154	0.109	0.199	0.154
$\alpha=0.1$	0.186	0.200	0.153	0.106	0.200	0.153
$\alpha=0.2$	0.187	0.202	0.153	0.104	0.202	0.153
$\alpha=0.3$	0.188	0.203	0.152	0.101	0.203	0.152
$\alpha=0.4$	0.189	0.205	0.152	0.098	0.205	0.152
$\alpha=0.5$	0.190	0.207	0.151	0.095	0.207	0.151
$\alpha=0.6$	0.191	0.209	0.150	0.091	0.209	0.150
$\alpha=0.7$	0.192	0.211	0.149	0.087	0.211	0.149
$\alpha=0.8$	0.194	0.213	0.148	0.083	0.213	0.148
$\alpha=0.9$	0.195	0.216	0.147	0.078	0.216	0.147
$\alpha=1$	0.197	0.219	0.146	0.072	0.219	0.146

If

$$d_1 + p_1 \le 1, \quad (4.1)$$

and

$$d_2 + p_2 \le 1, \quad (4.2)$$

then Ye [7] mentioned that

$$d_1 + d_2 - d_1 d_2 + p_1 p_2 \le 1, \quad (4.3)$$

and

$$1 - (1 - d_1)^{\lambda} + p_1^{\lambda} \le 1. \quad (4.4)$$

According to Equation (4.1), we derive that

$$p_1 \le 1 - d_1. \quad (4.5)$$

On the other hand, referring to Equation (4.2), we obtain that

$$p_2 \le 1 - d_2. \quad (4.6)$$

We combine the results of Equations (4.5) and (4.6), and then we imply that

$$p_1 p_2 \le (1 - d_1)(1 - d_2), \quad (4.7)$$

and then we rewrite the expression of Equation (4.7) to show that

$$p_1 p_2 \le 1 - d_1 - d_2 + d_1 d_2. \quad (4.8)$$

We compare Equations (4.3) and (4.8), then we present a proof for Equation (4.3).

Next, based on logarithm function is an increasing function, owing to Equation (4.5), we show that $p_1 \le 1 - d_1$, then we find that

$$\ln(p_1) \le \ln(1 - d_1). \quad (4.9)$$

Referring to Equation (4.9), with $\lambda > 0$, we derive that

$$\lambda \ln(p_1) \le \lambda \ln(1 - d_1). \quad (4.10)$$

We recall the property of logarithm function that shows

$$a \ln(b) = \ln(b^a). \quad (4.11)$$

Using Equation (4.11), we rewrite Equation (4.10) as follows,

$$\ln(p_1)^{\lambda} \le \ln(1 - d_1)^{\lambda}. \quad (4.12)$$

We apply that logarithm function is an increasing function the second time, to derive that

$$(p_1)^{\lambda} \le (1 - d_1)^{\lambda}. \quad (4.13)$$

We compare Equations (4.4) and (4.13), then we present a proof for Equation (4.4).

V. Further Study for Intuitionistic Fuzzy Sets

Szmidt and Kacprzyk [8] mentioned that if three intuitionistic fuzzy sets A, B and C that satisfy the inclusion relation,

$$A \subseteq B \subseteq C, \quad (5.1)$$

which is defined as

$$t_A(u_i) \le t_B(u_i) \le t_C(u_i), \quad (5.2)$$

and

$$f_A(u_i) \le f_B(u_i) \le f_C(u_i), \quad (5.3)$$

for every element, u_i, in the universe of disclose.

Liu [9] assumed that with respect to two interval-value intuitionistic fuzzy sets, denoted as A, and B, the definition of $A \le B$ if and only if

$$\mu_A(x) \le \mu_B(x), \quad (5.4)$$

and

$$v_A(x) \ge v_B(x), \quad (5.5)$$

for all x in the universe of disclose, X, and the property (P3) $E(A) \le E(B)$ is restricted to two cases:

$$v_A \ge v_B \ge \mu_B \ge \mu_A, \quad (5.6)$$

or

$$\mu_A \ge \mu_B \ge v_B \ge v_A. \quad (5.7)$$

Xu and Xia [10] claimed that for interval-value fuzzy numbers, $A \subseteq B$ if and only if

$$\mu_A^-(x) \le \mu_B^-(x), \quad (5.8)$$

$$\mu_A^+(x) \le \mu_B^+(x), \quad (5.9)$$

$$v_A^-(x) \ge v_B^-(x), \quad (5.10)$$

and

$$v_A^+(x) \ge v_B^+(x), \quad (5.11)$$

for each $x \in X$.

We assume that for three intuitionistic fuzzy sets,

$$A = (b_1 - x, b_2 + z, b_3 + x - z), \quad (5.12)$$

$$B = (b_1, b_2, b_3), \quad (5.13)$$

and

$$C = (b_1 + y, b_2 - w, b_3 - y + w), \quad (5.14)$$

where all components of A, B and C are non-negative crisp numbers with x, y, z and $w \ge 0$. According to the definition, we know that $A \le B \le C$. We will show that under one-norm,

$$d_1(A,B) \le d_1(A,C), \quad (5.15)$$

and

$$d_1(B,C) \le d_1(A,C), \quad (5.16)$$

We derive that

$$d_1(A,B) = \sum_{i=1}^{3} |a_i - b_i|$$
$$= x + z + |x - z| = 2\max\{x, z\}, \quad (5.17)$$

and

$$d_1(A,C) = \sum_{i=1}^{3} |a_i - c_i| = x + y + z + w$$
$$+ |(x + y) - (z + w)| = 2\max\{x + y, z + w\}, \quad (5.18)$$

Similarly, we obtain that

$$d_1(B,C) = \sum_{i=1}^{3} |a_i - b_i|$$
$$= y + w + |y - w| = 2\max\{y, w\}. \quad (5.19)$$

Owing to x, y and z are non-negative crisp numbers, based on our results with respect to Equations (5.17) and (5.18), we show that the inequality of Equation (5.15) is valid.

On the other hand, according to our results of (5.18) and (5.19), and x, y and z are non-negative crisp numbers, we derive that inequality of Equation (5.16) is true.

If we use for, against, and hesitancy degree then triangle property will be preserved. In several papers, researchers use for and hesitancy degree such that they needed to select some special definition to execute the comparison of three intuitionistic fuzzy sets such that they can prove the triangle property for their selected special definition.

Next, we examine the two-norm, whether or not the following inequality,

$$d_2(A,B) \le d_2(A,C), \tag{5.20}$$

is valid?
We obtain that

$$(d_2(A,B))^2 = \sum_{i=1}^{3} |a_i - b_i|^2$$
$$= x^2 + z^2 + (x-z)^2, \tag{5.21}$$

and

$$(d_2(A,C))^2 = \sum_{i=1}^{3} |a_i - c_i|^2$$
$$= (x+y)^2 + (z+w)^2 + [(x+y)-(z+w)]^2, \tag{5.22}$$

Based on our above discussion, we define an auxiliary function, noted as $f(y,w)$, where

$$f(y,w) = (x+y)^2 + (z+w)^2$$
$$+ (z+w)^2 + [(x+y)-(z+w)]^2. \tag{5.23}$$

We begin to consider the minimum problem of Equation (5.23). We find that

$$\frac{\partial}{\partial y} f(y,w) = 0, \tag{5.24}$$

if and only if

$$2(x+y) = z + w. \tag{5.25}$$

We will separate the examination process into two cases, Case 1: $z + w \ge 2x$, and Case 2: $z + w < 2x$.
For Case 1, the minimum point at

$$y = \frac{z+w}{2} - x, \tag{5.26}$$

with the minimum value,

$$f\left(\frac{z+w}{2} - x, w\right) = \left(\frac{3}{2}\right)(z+w)^2. \tag{5.27}$$

Moreover, we consider the minimum problem of Equation (5.27) with respect to the variable, w.
We know that $f\left(\frac{z+w}{2} - x, w\right) = \frac{3(z+w)^2}{2}$ will attain its minimum when $w = 0$ as

$$f\left(\frac{z}{2} - x, 0\right) = \frac{3}{2} z^2. \tag{5.28}$$

Referring to Equation (5.21), we define the second auxiliary function, noted as $g(x)$ where

$$g(x) = x^2 + z^2 + (x-z)^2. \tag{5.29}$$

We solve the minimum problem of Equation (5.29) to imply that $g'(x) = 0$ at $x = \frac{z}{2}$ with

$$g\left(\frac{z}{2}\right) = \frac{3}{2} z^2. \tag{5.30}$$

If we pick $x \ne \frac{z}{2}$ then

$$g(x) > f\left(\frac{z}{2} - x, 0\right), \tag{5.31}$$

such that triangle inequality of Equation (5.20) with respect to two-norm is not hold.

VI. Numerical Example

Hence, in the following, we construct a counter example for two-norm, with

$$A = \left(b_1 - \frac{z}{4}, b_2 + z, b_3 - \frac{3z}{4}\right), \tag{6.1}$$

$$B = (b_1, b_2, b_3), \tag{6.2}$$

and

$$C = \left(b_1 + \frac{z}{4}, b_2, b_3 - \frac{z}{4}\right). \tag{6.3}$$

which satisfy $A \subseteq B \subseteq C$, according to Equations (5.2) and (5.3).
We derive that

$$d_2(A,B) = \sqrt{\frac{1}{2}\left(\frac{z^2}{16} + z^2 + \frac{9}{16} z^2\right)} = \frac{\sqrt{13}}{4} z, \tag{6.4}$$

and

$$d_2(A,C) = \sqrt{\frac{1}{2}\left(\frac{z^2}{4} + z^2 + \frac{1}{4} z^2\right)} = \frac{\sqrt{12}}{4} z. \tag{6.5}$$

From Equations (6.4) and (6.5), we obtain that $d_2(A,B) > d_2(A,C)$, such that we construct a counter example for the two-norm.

At last, not least, we begin to recall a too restricted definition proposed by Park et al. [11].
In Park et al. [11], they defined that $A \le B \le C$ if and only if

$$v_A(x) \ge v_B(x) \ge v_C(x), \tag{6.6}$$

and

$$\pi_A(x) \ge \pi_B(x) \ge \pi_C(x), \tag{6.7}$$

such that

$$\mu_A(x) \le \mu_B(x) \le \mu_C(x), \tag{6.8}$$

appears naturally. Because the definition of Park et al. [11] is too restrict, therefore, Park et al. [11] can provide a proof for the triangle inclusion property. Several related papers of Xu [12], Su et al. [13], Gerogiannis et al. [14], and Wu and Chen [15], that are important for this trend of patter recognition problem with intuitionistic fuzzy numbers.

VII. Discussion of Hesitant Fuzzy Sets

We will present a further discussion with respect to hesitant fuzzy sets that had been examined by Torra [16]. The goal of Torra [16] is to generalize from an expert to several experts or sensors within the sphere of influence to provoke communication. This message arouses the question with respect to data representation and selection of the optimal depiction of a related structure or information through a combined mapping. Each cited data has its relative weight to indicate its trustworthiness. Researchers applied the ordered weighted averaging operator to arrange information based on their ordering. Consequently, by a weighted operator, practitioners measure and aggregate the weight under the consideration of their related weight. Torra [16] proposed a novel operator: weighted ordered weighted averaging operator, to further capture the relative weights among information with respect to their interrelationship. We recall the ordered weighted averaging operator defined as follows,

$$f_{OWA}(a_1,\ldots,a_n)=\sum_{i=1}^{n} w_i a_{\sigma(i)}\ , \qquad (7.1)$$

where $\{\sigma(1),\ldots,\sigma(n)\}$ is a variation of $\{1,\ldots,n\}$ under the constrain,

$$a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(i-1)} \geq a_{\sigma(i)} \geq \ldots \geq a_{\sigma(n)}. \quad (7.2)$$

The background for the weighted averaging is to assume a greater weight for those data occurred in the more recent events under the examination. Researchers provided a weight system among experts to indicate their interrelationship. The ordered weighted averaging operator estimated the data, but not the experts such that all experts has the same contribution to the last decision making. Consequently, the meaning of weights are different in (i) The weighted averaging operator, and (ii) The ordered weighted averaging operator. There two operators have its benefit but also have drawbacks. Hence, Torra [16] developed a novel operator: the weighted ordered weighted averaging operator to absorb the advantage from both point of view under consideration to merge these two operations into a single operator to considering both the data source (as the expert) and data value among them.

Let p and w be weighting vectors of dimension n, where p= $(p_1,\ldots,p_n)$ evaluates the source, and w= $(w_1,\ldots w_n)$ evaluates the data, under the criteria, $\sum_{i=1}^{n} p_i = 1,\ p_i \geq 0$ for $i=1,\ldots,n$, $\sum_{i=1}^{n} w_i = 1,\ w_i \geq 0$ for $i=1,\ldots,n$.

Torra [16] constructed a novel operator which is defined by a function $f_{wowa}: R^n \to R$ is a weighted ordered weighted averaging operator of dimension *n* as follows,

$$f_{wowa}(a_1,\ldots,a_n)=\sum_{i=1}^{n} \lambda_i a_{\sigma(i)}, \qquad (7.3)$$

where $\{\sigma(1),\ldots,\sigma(n)\}$ is a permutation of $\{1,\ldots,n\}$ such that $a_{\sigma(n-1)} \geq a_{\sigma(n)}$ for $i=2,\ldots,n$, and the weight λ_i is defined as

$$\lambda_i = g\left(\sum_{j=1}^{i} p_{\sigma(j)}\right) - g\left(\sum_{j=1}^{i-1} p_{\sigma(j)}\right), \qquad (7.4)$$

where the aggregation function, g, is defined as a monotonic increasing mapping that interposes the points $\left(i/n, \sum_{j=1}^{i} w_j\right)$ along with the initial point $(0,0)$.

We recall that if ***w***= $(1/n,\ldots,1/n)$, then the weighted ordered weighted averaging operator reduced to weighted averaging operator with a weighting vector ***p***. On the other hand, if ***p***= $(1/n,\ldots,1/n)$, then the weighted ordered weighted averaging operator reduced to the ordered weighted averaging operator with a weighting vector ***w***.

We recall a theorem of to claim that for two weighted ordered weighted averaging operators, f_{wowa} and f'_{wowa} with λ and λ' there is a pair j and k such that $1 \leq j < k \leq n$, $\lambda'_j > \lambda_j$, $\lambda'_k < \lambda_k$, and $\lambda'_i = \lambda_i$ for $1 \leq i \leq n$ and $i \neq j$, $i \neq k$. In this conditions:

$$f'_{wowa}(a) \geq f_{wowa}(a), \qquad (7.5)$$

for all $a=(a_1,\ldots,a_n)$.

We can extend the above mentioned theorem such that two weighted ordered weighted averaging operators, f_{wowa} and f'_{wowa} with λ and λ' there is a pair j and k such that $1 \leq j < k \leq n$, $\lambda'_t > \lambda_t$ for $t=1,\ldots,j$, $\lambda'_s < \lambda_s$ for $s=k,\ldots,n$, and $\lambda'_i = \lambda_i$ for $i=j+1,\ldots,k-1$. In this conditions:

$$f'_{wowa}(a) \geq f_{wowa}(a), \qquad (7.6)$$

for all $a=(a_1,\ldots,a_n)$.

We provide a numerical example in the following. with function $g(x)=\sqrt{x/2}$ for $0 \leq x \leq 1/2$ and $g(x)=1-\sqrt{(1-x)/2}$ for $1/2 \leq x \leq 1$. Consequently, we derive that $g(0)=0$, $g(1/2)=1/2$, and then we obtain that

$$g(1/4)=1/2\sqrt{2} \neq 3/8, \qquad (7.7)$$

on the other hand, we evaluate that $g(1/2)=1/2$, $g(1)=1$, to show that

$$g(3/4)=1-1/2\sqrt{2} \neq 5/8. \qquad (7.8)$$

We recall that the quasi-arithmetic mean extends various the averaging operators which is assumed in the following,

$$f_m^{\varphi}(a_1,\ldots,a_n)=\varphi^{-1}\sum_{i=1}^{n}\frac{\varphi(a_i)}{n}. \qquad (7.9)$$

For example, it will simplify the arithmetic mean, under the condition of

$$\varphi(x)=kx+k'. \qquad (7.10)$$

On the other hand, it will extend the geometric mean , with the construction,

$$\varphi(x)=k\ln x+k'. \qquad (7.11)$$

Moreover, it will generate the harmonic mean, with the definition of

$$\varphi(x)=\frac{k}{x}+k'. \qquad (7.12)$$

We have studied some properties of the weighted ordered weighted averaging operator and given an example. We can predict that the weighted ordered weighted averaging operator will be generalized to linguistic environment.

VIII. Direction for Further Study

We offer a brief of some related papers that will help practitioners to locate hot research spots. Based on genetic procedure with parallel similar capacities, Chang and Yao [17] studied the economic lot scheduling problem. With respect to authentication and face recognition, Chen et al. [18] developed a new algorithm for changes of expression, illumination, and pose. Considering freight rate discounts, Lee and Wang [19] constructed a three echelons supply system for lot range decision making problems. Through video coding of artificial stable bit ratio, Fan and Kan [20] examined an analytic procedure with dynamic view. With advertising activity and optimal ordering decision process, Chen [21] studied a supply model under one replenishment policy. Applying graphic searching algorithm, located optimal solution for stack filter. To deal with image smoothing, Sze et al. [22] derived a novel conducted image fluctuation system with practical applications. With an one-time price reduction and defective items, Hsu and Yu [23] investigated economic ordering quantity systems. On the other hand, several recently published papers of applied mathematics, computer sciences, and industrial engineering, are worthy to mention to reveal possible directions for future research. Referring to deep learning, Martadiansyah et al. [24] studied 2D Transcerebellar Images and then converted to 3D by blending hierarchy lattice reform. Based on non-modeled dynamics, Zang et al. [25] examined nonlinear systems feedback presentation control under triggered issue with finite time. Under combined transit conditions, Xu et al. [26] constructed a speedy parking system. Through square norm measure indicator, Zhao et al. [27] derived associated collection for essential activity with vague number ranking tally. With respect to substantial and imitation for elegant lattices, Wankhade and Kottur [28] developed mixture stimulated systems to manage parameters to optimize safety plan. To learn time sequence characteristic, Meng et al. [29] obtained cross concentrated systems with weight monitor without interfering. According to several source consideration and previous training, Yang et al. [30] gained digital health check account for personal identification. For Rose and Hindmarsh form with linear combination, Em [31] acquired diffused and reacted models to identify harmonization under adequate circumstances in the network environment. To realize medium satisfaction, Liu et al. [32] found out relationship between news and knowledge for hostile offense tendency by occurrence mining. Related to vibrant indiscriminate traffic instance, Zhang et al. [33] considered route selection for lasting transit models. Owing to character of boundary points, Fu et al. [34] examined difference Laplacian formulas for the solution existence. Chen and Cheng [35] pointed out questionable results in Caliskan's findings [36] and then presented further revisions. According to our brief reviewing of some published papers, practitioners can find several interesting research topics for their future investigation.

IX. Conclusion

Our paper is developed by rigorous analytic approach. On the other hand, Chen and Li [1] is developed closely related to numerical methods and graphic expressions. We can claim that Chen and Li [1] did not provide a well-designed mathematical structure. Our paper provides a good patch work for Chen and Li [1]. We also presented revisions for Ye [7] and Szmidt and Kacprzyk [8] with respect to two inequalities under the environment of triangular and trapezoidal intuitionistic fuzzy sets.

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