Approximation of Internal Rate of Return by Comparing Centroid-Based and Midpoint-Based Newton Raphson Methods

Sri Purwani, Nurnisaa, Moh. Alfi Amal, and Dwi Susanti

Abstract—**Internal Rate of Return (IRR) is a method that is used to assess how profitable a project or investment would be and it plays a very crucial role in determining whether an investment can provide adequate returns. A repeated root-finding algorithm in IRR calculations can be employed to find the root of the equation that models the cash flow of the investment. Meanwhile, Newton-Raphson algorithm is generally used because it is more easy to use and it is very efficient in carrying out all the repeated calculations that are involved. However, there are associated difficulties most especially when the initial prediction that was made for IRR is far from the actual value. This makes it to be very difficult for the algorithm to converge to an accurate solution and creating uncertainty in the investment assessment. In order to find solution to these problems, midpoint-based methods are now being used to help increase convergence accuracy, but the methods do not really adjust well to varying cash flows. In this regard, it becomes very crucial to improve the performance by carrying out a comparison with innovative models using centroid-based Newton-Raphson algorithm, which uses the center of mass of the cash flow distribution as the starting point. The results from using data from Apple Inc. showed the accuracy was improved by 33.97% with centroid-based compared to midpoint-based. This shows an effectiveness in improving the accuracy and reliability of investment assessments.**

Index Terms—**IRR, centroid-based Newton-Raphson, midpoint-based Newton-Raphson, Newton-Raphson algorithm**

I. Introduction

INTERNAL Rate of Return (IRR) is considered as a very good parameter that is used for evaluating investment and financial projects, and can offer an idea into expected rate of return, which is necessary when making decisions for a particular business [1]. Nevertheless, because the parameters play a critical role in having an influence on resource allocation and strategies to invest, it is important to have an algorithm that is used to calculate IRR [2]. As the field evolves, previous studies have primarily paid attention to focus on improving the accuracy and efficiency of RRR calculation methods. This has caused several questions to be asked regarding the proper method to be used to achieve the objective. Therefore, this study looked at two ways through which IRR can be calculated using Newton-Raphson method, namely centroid-based and midpoint-based.

The commonly used repeated method for calculating IRR is perceived to be a trial-and-error process even though it is quite popular [3]. In order to increase the ability of the IRR calculations to produce accurate outcomes, Newton-Raphson method has been used recently [4]. This method helps to make the process very simple by removing the use of two initial guesses and achieve quadratic convergence, which plays a crucial role to enhance the accurate result [3]. Also, the method is very popular worldwide to be a tool to find the roots of mathematical equations [5], [6], [7], [8], [9]. Some of the many usages include using it to determine IRR in financial evaluations [10]. Nevertheless, this study revealed the major differences between the two Newton-Raphson approaches in estimating IRR.

Centroid-based method helps in offering initial values that are very close to the actual solution by employing the center of mass of the cashflow distribution, which serves as the starting point. This is expected to enhance the algorithm convergence and the accuracy of IRR estimates. Meanwhile, midpoint-based helps to provide a comparison and it is used worldwide in literature on finance. It uses the median value of the cash flow period as the starting point. Making a comparison of the performance with centroid-based method gives a good overview regarding the advantages of the two methods.

Patrick et al., reported that midpoint-based can be used as a substitute to Newton-Raphson method when looking for a solution to solve nonlinear equations [11]. In this context, Newton midpoint method is very unique due to the cubic convergence order, which helps to speed up the convergence process [11]. The process is started by using Newton's method to provide the initial solution to a system of nonlinear equations. It is then subsequently integrated into midpoint-based to get a better convergence. Also, the

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repeated process is stopped when the iteration error reaches a specified tolerance level.

Centroid-based Newton-Raphson algorithm is regarded as a new and improved solution which makes use of a dynamic initial guess based on the actual cash flow distribution [5]. The major purpose of this method is to tackle the challenges of the traditional approaches, as well as provide better convergence in the process of IRR estimation.

An accurate initial IRR value can lead to smaller errors in several instances, whereas inaccurate initial IRR value produces a larger error. Therefore, using the method that offers an initial value that is close to the actual value is very important to reduce or minimize error. When the initial IRR is close to the true value, the outcome would be more optimal. In addition, the number of iterations that is required for each method can be used to assess the efficiency. Lesser iterations to achieve convergence indicate a good method in estimating the initial IRR value. This study aimed to increase the understanding of the accuracy of both methods. It also analyzed simulation results to assess the performance differences between centroid-based and midpoint-based methods, which help to contribute to the current understanding of IRR calculations in the financial domain.

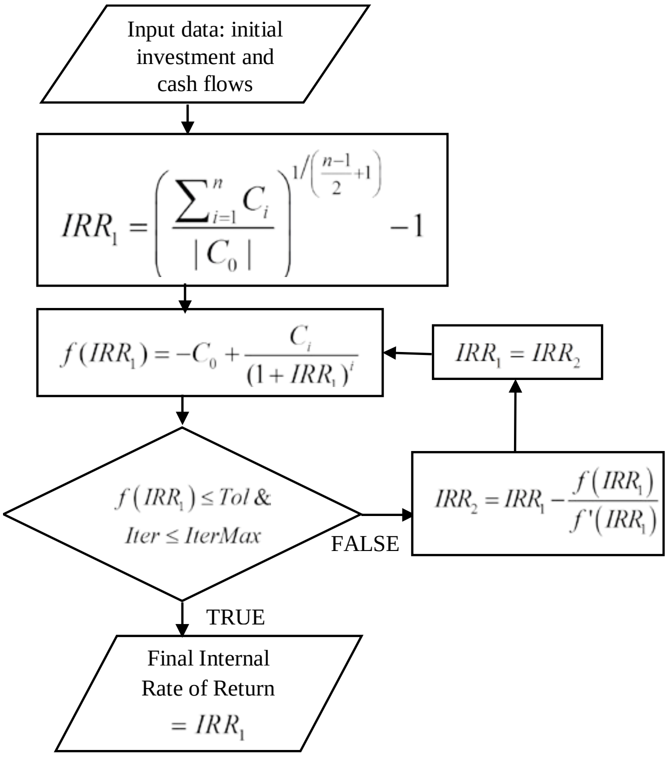

Fig. 1. Midpoint-based Newton-Raphson algorithm process.

II. Materials and Methods

A. Internal Rate of Return (IRR)

IRR is regarded as the discounted rate that makes Net Present Value (NPV) of a project's or investment's net cash equal to zero, NPV = 0 [12]. This means that IRR equalizes the present value of a project's cash inflows or outflows [3]. The general formula for IRR can be expressed through NPV equation as follows.

$$NPV = \sum_{t=1}^{n} \frac{CF_t}{(1+IRR)^t} - C_0 = 0 \tag{1}$$

IRR : Internal rate or return

C_0 : Initial investment

CF_t : Net cash flow = cash inflow-cash outflow

t : Period

NPV is a financial method that is employed in evaluating the profitability of a project or investment. It is based on the principle that the value of money changes over time, hence, all cash flows are measured in present value [13].

The decision-making rules based on NPV are as follows [13]:

a. When NPV > 0: The project or investment is considered feasible because it produces net profits.
b. When NPV = 0: The project or investment is at break-even, producing sufficient profit to cover capital costs without additional gains.
c. When NPV < 0: The project or investment is considered unfeasible, as it results in a net loss.

The decision rules based on IRR are as follows [13]:

a. When IRR > the discount or specified interest rates, the project or investment is considered feasible.
b. When IRR < the discount rate, the project or investment is considered unfeasible.

IRR and NPV should provide consistent decisions, hence, when IRR shows the investment is feasible, NPV should be positive, and vice versa. IRR value can be calculated by solving NPV equation (1) in the form of the nonlinear equation $f(IRR)=0$. This can be solved using a root finding Newton-Raphson method, an iterative algorithm that produces a sequence of IRR values until the iteration error reaches a specified tolerance level meaning NPV reaches zero.

B. Midpoint-based Newton Raphson

Pascual et al. proposed an adjustment to Newton-Raphson algorithm for IRR estimation [14]. This adjustment automatically produces an initial value using midpoint of the cash flows over the period $\frac{n-1}{2}+1$

$$IRR = \left(\frac{\sum_{i=1}^{n} C_i}{|C_0|} \right)^{1/\left(\frac{n-1}{2}+1\right)} - 1 \tag{2}$$

C_0 : Initial investment

C_i : Subsequent cash flow

i : Period

n : Total period

The method aims to improve the convergence rate and precision of the original Newton-Raphson algorithm, producing results that closely approximate the true solution without requiring the user to provide an initial guess value for IRR. However, a significant limitation of midpoint-based is the reduced accuracy during the early stages of IRR calculations when cash flows are uneven. Although this condition can increase the number of iterations required to achieve convergence, the method remains valuable as a reference point, offering suggestions for improving algorithm development. The process of Newton-Raphson algorithm with midpoint method is presented in Figure 1

$$f(IRR) = -C_0 + \sum_{i=1}^{n} \frac{C_i}{(1+IRR)^i}$$
$$f'(IRR) = -\sum_{i=1}^{n} \frac{i \times C_i}{(1+IRR)^{i+1}} \tag{3}$$

C. *Centroid-based Newton Raphson*

The proposed centroid method, shown in Figure 2, is an improvement of midpoint-based Newton-Raphson technique [15]. This improvement involves replacing midpoint of the cash flow period midpoint-based IRR with a cash flow period center of $\frac{\sum_{i=1}^{n} C_i x_i}{\sum_{i=1}^{n} C_i}$

$$IRR = \left(\frac{\sum_{i=1}^{n} C_i}{|C_0|} \right)^{1 / \frac{\sum_{i=1}^{n} C_i x_i}{\sum_{i=1}^{n} C_i}} - 1 \tag{4}$$

x_i : Distance C_i from the first investment period
C_i : Subsequent cash flow
C_0 : Initial investment
i : Period
n : Total period

In centroid-based method, the center of mass is used as the initial guess, which tends to provide a more accurate and closer estimate of the actual value of IRR. With an initial guess closer to the true solution, Newton-Raphson algorithm

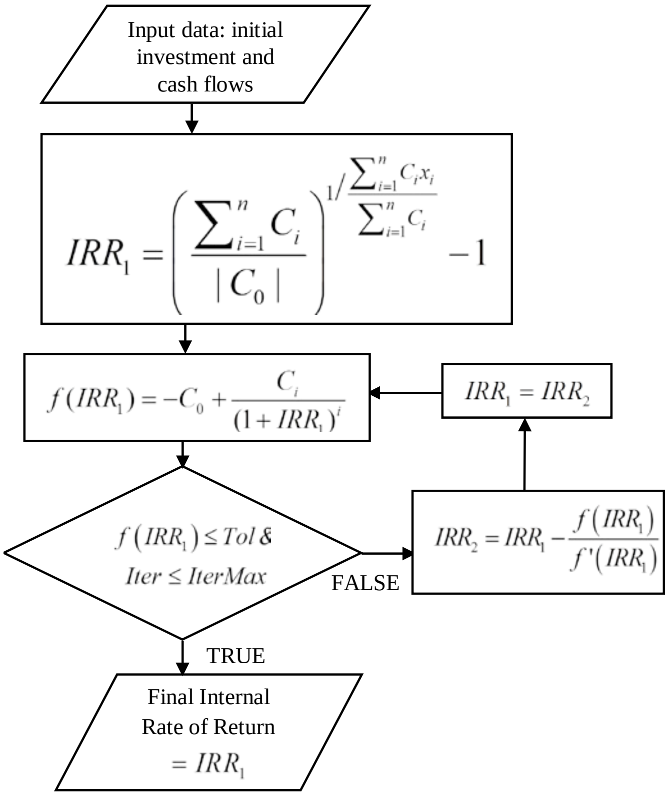

Fig. 2. Centroid-based Newton-Raphson algorithm process.

is more likely to converge quickly, reducing the number of iterations needed to reach an accurate IRR value. This results in a more computationally efficient algorithm. The process of Newton-Raphson algorithm with centroid-based is presented in Figure 2, where $f(IRR_1)$ and $f'(IRR_1)$ refer to (3).

D. *Newton Raphson Algorithm Process*

This study applied both midpoint-based and centroid-based methods in Newton-Raphson algorithm. The objective is to compare the accuracy of both methods. Although the processes are nearly identical, the main difference lies in the use of midpoint and centroid formulas. Figure 1 presents the

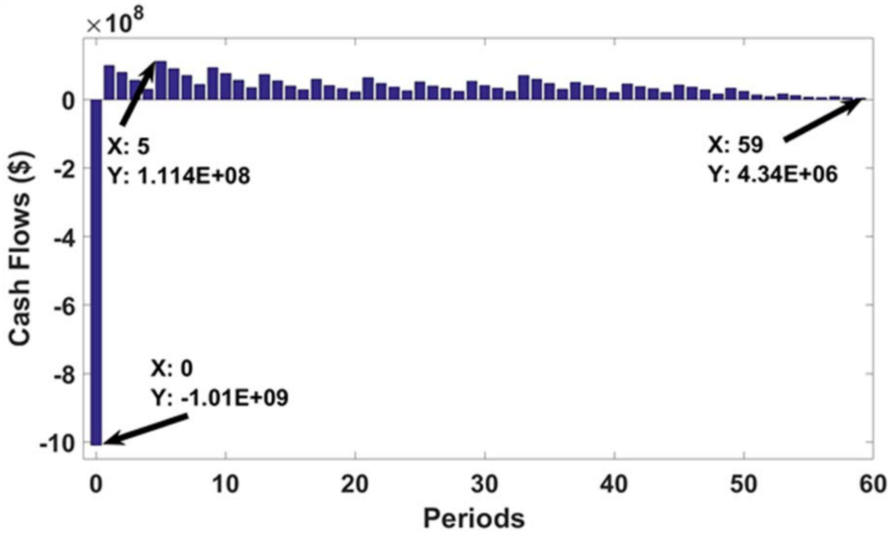

Fig. 3. Apple Annually Cash Flow 2009-2023.

TABLE I
APPLE QUARTERLY CASH FLOW 2009-2023

t	Cashflow ($)	t	Cashflow ($)	t	Cashflow ($)
0	-1009870000.00	20	23335000.00	40	20685000.00
1	99584000.00	21	64121000.00	41	45501000.00
2	80149000.00	22	47639000.00	42	37548000.00
3	55862000.00	23	36418000.00	43	31605000.00
4	30218000.00	24	25483000.00	44	21109000.00
5	111443000.00	25	51774000.00	45	42561000.00
6	90605000.00	26	39890000.00	46	36886000.00
7	69815000.00	27	33495000.00	47	28753000.00
8	44163000.00	28	23900000.00	48	16233000.00
9	92953000.00	29	53497000.00	49	33269000.00
10	75976000.00	30	40941000.00	50	24485000.00
11	56975000.00	31	33116000.00	51	14154000.00
12	35263000.00	32	23851000.00	52	8559000.00
13	73365000.00	33	70019000.00	53	16590000.00
14	54573000.00	34	60162000.00	54	11667000.00
15	39867000.00	35	47217000.00	55	7461000.00
16	28409000.00	36	30505000.00	56	5405000.00
17	58896000.00	37	50142000.00	57	9015000.00
18	41763000.00	38	40718000.00	58	6364000.00
19	32127000.00	39	32841000.00	59	4340000.00

specific steps for midpoint-based, while centroid-based replaces formula (2) with formula (4), and the rest of the process remains the same, as shown in Figure 2.

III. RESULT AND DISCUSSION

This section presents the simulation results of centroid-based Newton-Raphson algorithm compared to midpoint-based Newton-Raphson algorithm, focusing on precision and accuracy. The data analysis was based on the comprehensive dataset, presented in Figure 3, obtained directly from Apple Inc. The processed data are presented in Table 1. This dataset covered the period from 2009 to 2023, was recorded quarterly, and was carefully processed using officially recognized data sources, namely Yahoo Finance and Macrotrends. The negative value in Figure 3 showed the initial investment C_0, amounting to $1,009,870,000.00, while the positive values reflected subsequent cash flows, C_1, C_2, ... , C_n.

Midpoint-based method automatically calculated the initial value using midpoint of the cash flow period of $\frac{n-1}{2}+1$, which in this case was at period 30 (Figure 4). Meanwhile, centroid-based replaced this with the cash flow period center of $\frac{\sum_{i=1}^{n} C_i x_i}{\sum_{i=1}^{n} C_i}$, calculated at 22.5025, as shown in Figure 4. Centroid-based's ability to adjust for fluctuations in cash allows it to perform better than midpoint-based, presented in Table II.

Based on Table II, centroid-based method, which improved upon midpoint-based IRR method, automatically produced an initial value closer to the actual IRR root compared to the previous algorithm. This resulted in a reduction in the number of iterations required to find the final IRR, as shown in Table III. With an error tolerance of 10^{-16}, centroid-based method achieved a 20% reduction in the number of iterations compared to midpoint-based.

Centroid-based required only basic data, such as the principal amount (initial investment C_0) and subsequent cash flows, C_1, C_2, ... , C_n. Using data from Apple Inc., with an initial investment of $1,009,870,000.00 over a 15-year period, IRR was estimated for economic engineering analysis.

A. *Results of Centroid and Midpoint-based Methods in Automating Initial IRR Calculations*

In this simulation, the relative error in calculating initial IRR was analyzed. The relative error was determined by measuring the difference between the initial and the final converged IRR values. More precisely, the relative error was calculated as the difference between the initial and final IRR values, divided by the final IRR value, which served as the reference. The presentation of the simulation results aimed to provide insight into how accurate the initial IRR estimate is compared to the final IRR value.

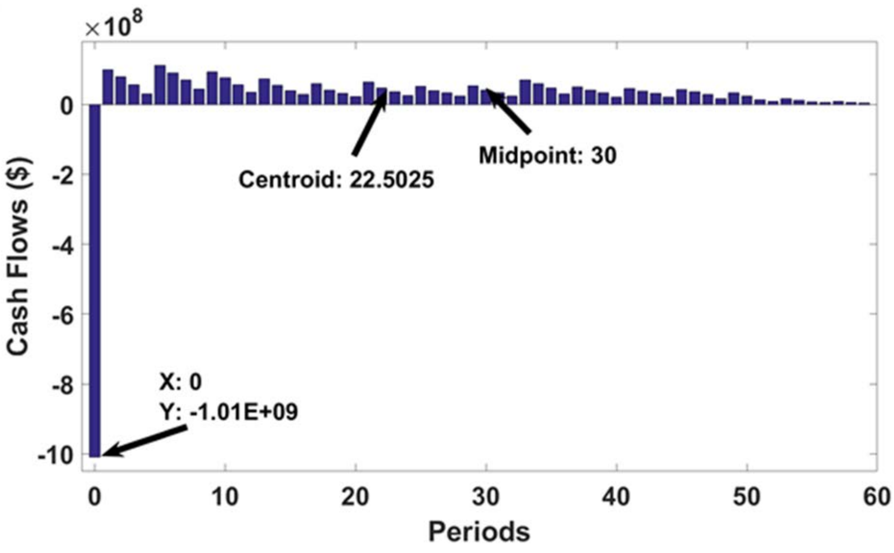

Fig. 4. A cash flow period midpoint (Midpoint) and a cash flow period center (Centroid).

TABLE II
ACCURACY OF APPROXIMATING INITIAL IRR

Method	Final IRR	Initial IRR	Initial IRR Relative Error	Accuracy
Midpoint	0.052109047788149	0.029606205499595	0.431841364287453	56.82%
Centroid	0.052109047788149	0.039663947510356	0.238828011756988	76.12%

$$\text{Initial IRR Relative Error} = \frac{| IRR_1 - IRR_f |}{IRR_f} \tag{5}$$

Based on the experimental analysis shown in Table II, centroid-based method produced a smaller relative error compared to midpoint-based in the initial IRR calculation. Specifically, with an error tolerance of 10^{-16}, midpoint method showed a relative error of 0.431841364287453 (43.18%), while centroid had a relative error of 0.238828011756988 (23.88%). This corresponds to accuracies of 56.82% and 76.12% for midpoint and centroid-based, respectively. Therefore, centroid-based achieved a significant improvement in accuracy, with an increase of 33.97%.

The initial IRR error refers to the difference between the estimated initial IRR and the actual IRR value. This value was influenced by how an algorithm or IRR calculation method approximated the initial IRR. In some cases, a more accurate initial IRR could result in a smaller error, while a less accurate initial IRR could lead to a larger error.

In general, the closer the initial IRR value is to accuracy, the smaller the resulting error. Therefore, selecting a method that can provide an initial IRR close to the actual value minimizes the initial IRR error. In other words, the accuracy of the initial IRR calculation significantly influenced the size of the resulting error. A closer initial IRR value results in a more optimal estimate. Centroid-based IRR method outperformed midpoint-based in providing a more accurate initial IRR estimate.

The results reinforced the current understanding of IRR calculations in the financial environment of Apple Inc. The application of midpoint-based Newton-Raphson method is more optimized to suit the company's specific financial conditions. Similarly, centroid-based Newton-Raphson method could be adapted to better address Apple's financial characteristics, including market volatility, currency risk, or other factors affecting the technology companies.

In summary, centroid-based method is more better compared to midpoint-based from two major perspectives, namely higher accuracy and convergence stability. Centroid gave more accurate estimates by accounting for the probability distribution and showed stable convergence properties, particularly beneficial in a dynamic financial context. However, these contributions rely on the specific conditions of Apple Inc., which necessitates the careful consideration of the implementation of numerical methods to ensure relevance and reliability.

TABLE III
THE NUMBER OF ITERATIONS IN APPROXIMATING IRR

Method	Iteration Number
Midpoint	6
Centroid	5
Reduction in Iteration Number	20%

IV. CONCLUSION

In conclusion, the algorithm of centroid-based Newton-Raphson replaced midpoint formula with centroid. This showed a better performance compared to midpoint-based Newton-Raphson algorithm. In fact, it can easily provide an initial IRR value that is close to the final root, while dynamically considering cash flow variations with low error. Unlike static midpoint, which was less responsive to cash flow fluctuations, centroid-based algorithm was more adaptable. This was reflected in the lower initial IRR relative error of 23.88% compared to 43.18% for midpoint-based, showing an accuracy improvement of 33.97%. Therefore, centroid-based algorithm offered a more accurate and stable initial IRR estimate. In the context of investment decision-making, for both individuals and companies, this approach was highly recommended. It could help determine more realistic interest rates or rates of return on investment (RROI), supporting decision-making with a stronger basis. The results provided an opportunity for deeper understanding and increased effectiveness in IRR calculations.

An in-depth understanding of Apple Inc.'s specific financial context was also essential to assess the contributions of numerical methods like centroid-based Newton-Raphson algorithm. Its successful application depended on accurately accounting for the unique financial dynamics that Apple or any company might face in an ever-changing environment. In addition, appropriate implementation should consider data accuracy, an intensive understanding of the company's financial characteristics, and the involvement of financial experts, well-versed in the adopted methodology.

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