Bipolar Fuzzy (m, n)-Ideals and n-Interior Ideals of Semigroups

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***Abstract*—Lajos studied the concept of (m, n)-ideals of semigroups in 1963. The concepts of bipolar fuzzy semigroups was presented by Kim et al. in 2011. This paper we introduces the notion of bipolar fuzzy (m, n)-ideals in semigroups. We provided basic properties of bipolar fuzzy (m, n)-ideals and the connection between (m, n)-ideals and bipolar fuzzy (m, n)-ideals in semigroups. Moreover, we discuss the properties of bipolar fuzzy n-interior ideals and the connection between n-interior ideals and bipolar fuzzy n-interior ideals in semigroups. We also study weakly n-interior ideals and bipolar fuzzy weakly n-interior ideals.**

***Index Terms*—BF (m, n)-ideals, BF prime (m, n)-ideals, BF semiprime (m, n)-ideals, BF n-interior ideals.**

I. Introduction

The CONCEPTS of fuzzy sets was first considered by L. A. Zadeh in 1965 [1]. The fuzzy set theories developed by Zadeh and others have found many applications in mathematics and elsewhere. In 1981, Kuroki [2] discussed the concept of fuzzy Ssgs and fuzzy generalized bi-ideals in semigroups. The notion of bipolar valued fuzzy set by Zhang [3] in 1994 is an extension of fuzzy sets where the membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 0]$. In 2000, Lee [4] used the term bipolar valued fuzzy sets and applied it to algebraic structures. Kim et al. [5] studied relations of bipolar fuzzy subsemigroups, bipolar fuzzy left (right) ideals, bipolar fuzzy bi-ideals, and bipolar $(1, 2)$ ideals. He provided some necessary and sufficient conditions for a bipolar fuzzy Ssg and a bipolar fuzzy left (right, bi-) ideals of semigroups. Moreover, bipolar fuzzy has many applications in algebraic structures [6], [7], [8], [9], [10]. The theory of (m, n)-ideals in semigroups was studied by Lajos in 1963 [11]. The notion of (m, n)-ideals of semigroups generalized the idea of one-sided ideals of semigroups. In 2019 A. Mahboob [12] studied fuzzzy (m, n)-ideals and proved properties of regular semigroup. Many authors have examined theory in other structures, see, e.g., [13], [14], [15], [16], [17], [18], [19], [21], [20], In 2022, W. Nakkhasen [22] discussed concept picture fuzzy (m, n)-ideals of semigroups and investigated some basic properties of picture fuzzy (m, n)-ideals of semigroups. In the same year, T. Gaketem [23] studied the concept of interval valued fuzzy almost (m, n)-ideals in semigroups. Tiprachot et al. [24] discussed the notion of n-interior ideals as a generalization of interior ideals and characterized many classes of ordered semigroups in terms of (m, n)-ideals and n-interior ideals. In 2023, Tiprachot et al. [25] extend n-interior ideals and (m, n)-ideals to hybrid in ordered semigroups. In 2024 T. Gaketem and P. Khamrot [26] studied concepts interval valued fuzzy (m, n)-ideals in semigroups. Recently P. Khamrot et al. [27] extend concepts fuzzy (m, n)-ideals and n-interior ideals in semigroups to ordered semigroups.

In this paper, we study the concept of bipolar fuzzy (m, n) ideals, minimal bipolar fuzzy (m, n)-ideals, and bipolar fuzzy prime (semiprime) (m, n)-ideals in semigroups. We provide the basic properties and relationship between (m, n)-ideals and bipolar fuzzy (m, n)-ideals in semigroups. Finally, we discuss the properties of bipolar fuzzy n-interior ideals and the relationship between n-interior ideals and bipolar fuzzy n-interior ideals in semigroups. Also, we prove weakly n-interior ideals and bipolar fuzzy weakly n-interior ideals.

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II. Preliminaries

In this section, we introduce certain concepts and findings that will be beneficial in subsequent sections.

Definition 2.1. *Let $\mathcal{E}$ be an semigroup (SG).*

(1) *A* subsemigroup *(Ssg) of $\mathcal{E}$ is a non-empty set $\mathcal{K}$ of $\mathcal{E}$ such that $\mathcal{K}^2 \subseteq \mathcal{K}$.*
(2) *A* left ideal *(Lid) of $\mathcal{E}$ is a non-empty set $\mathcal{K}$ of $\mathcal{E}$ such that $\mathcal{E}\mathcal{K} \subseteq \mathcal{K}$.*
(3) *A* right ideal *(Rid) of $\mathcal{E}$ is a non-empty set $\mathcal{K}$ of $\mathcal{E}$ such that $\mathcal{K}\mathcal{E} \subseteq \mathcal{K}$.*
(4) *By an* ideal *(id) of $\mathcal{K}$, we mean a non-empty set of $\mathcal{E}$, which is both a Lid and a Rid of $\mathcal{E}$.*
(5) *An* interior ideal *(In id) of $\mathcal{E}$ is a non-empty set $\mathcal{K}$ is an Ssg of $\mathcal{E}$ and $\mathcal{E}\mathcal{K}\mathcal{E} \subseteq \mathcal{K}$.*
(6) *A* bi-ideal *(Bid) of $\mathcal{E}$ is a non-empty set $\mathcal{K}$ ois an Ssg of $\mathcal{E}$ and $\mathcal{K}\mathcal{E}\mathcal{K} \subseteq \mathcal{K}$.*

An id $\mathcal{K}$ of an SG $\mathcal{E}$ and m, n are positive integers. We called (m, n)-ideal ((m, n)-id) of an SG $\mathcal{E}$ if $\mathcal{K}^m\mathcal{E}\mathcal{K}^n \subseteq \mathcal{K}$.

A non-empty subset $\mathcal{K}$ of an SG $\mathcal{E}$. We denote the

$$[\mathcal{K}](m, n) = \bigcup_{r=1}^{m+n} \mathcal{K}^r \cap \mathcal{K}^m\mathcal{E}\mathcal{K}^n \text{ is principal } (m, n)\text{-ideal},$$

$$[\mathcal{K}](m, 0) = \bigcup_{r=1}^{m} \mathcal{K}^r \cap \mathcal{K}^m\mathcal{E} \text{ is principal } (m, 0)\text{-ideal},$$

$$[\mathcal{K}](0, n) = \bigcup_{r=1}^{n} \mathcal{K}^r \cap \mathcal{E}\mathcal{K}^n \text{ is the principal } (0, n)\text{-ideal},$$

i.e., the smallest (m, n)-ideal, the smallest $(m, 0)$-ideal and the smallest $(0, n)$-ideal of $\mathcal{E}$ containing $\mathcal{K}$, respectively.

Lemma 2.2. *[16] Let $\mathcal{E}$ be an SG and m, n positive integers, $[\pi]_{(m,n)}$ the principal (m,n)-id generated by the element π. Then*
(1) $([\pi]_{(m,0)})^m\mathcal{E} = \pi^m\mathcal{E}$.
(2) $\mathcal{E}([\pi]_{(0,n)})^n = \mathcal{E}\pi^n$.
(3) $([\pi]_{(m,0)})^m\mathcal{E}([\pi]_{(0,n)})^n = \pi^m\mathcal{E}\pi^n$.

For any $z_i \in [0,1]$, $i \in \mathcal{J}$, define

$$\bigvee_{i\in\mathcal{J}} z_i := \sup_{i\in\mathcal{J}}\{z_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{J}} z_i := \inf_{i\in\mathcal{J}}\{z_i\}.$$

We see that for any $z, r \in [0,1]$, we have

$$z \vee r = \max\{z, r\} \quad \text{and} \quad z \wedge r = \min\{z, r\}.$$

A **fuzzy set** of a non-empty set $\mathcal{T}$ is a function $\vartheta : \mathcal{T} \to [0,1]$.

For any two fuzzy sets ϑ and ξ of a non-empty set $\mathcal{T}$, define the symbol as follows:
(1) $\vartheta \geq \xi \Leftrightarrow \vartheta(z) \geq \xi(z)$ for all $z \in \mathcal{T}$,
(2) $\vartheta = \upsilon \Leftrightarrow \vartheta \geq \xi$ and $\xi \geq \vartheta$,
(3) $(\vartheta \wedge \xi)(z) = \vartheta(z) \wedge \xi(z) = \min\{\rho(z), \xi(z)\}$ for all $z \in \mathcal{T}$,
(4) $(\vartheta \vee \xi)(z) = \vartheta(z) \vee \xi(z) = \max\{\rho(z), \xi(z)\}$ for all $z \in \mathcal{T}$.
For the symbol $\vartheta \leq \xi$, we mean $\xi \geq \vartheta$.

Definition 2.3. *[4] A **bipolar fuzzy set** (BF set) ϑ on a non-empty set $\mathcal{E}$ is an object having the form*

$$\vartheta := \{(h, \vartheta^P(h), \vartheta^N(h)) \mid h \in \mathcal{E}\},$$

where $\vartheta^P : \mathcal{E} \to [0,1]$ and $\vartheta^N : \mathcal{E} \to [-1,0]$.

Remark 2.4. *For the sake of simplicity we shall use the symbol $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ for the BF set*
$\vartheta = \{(h, \vartheta^P(h), \vartheta^N(h)) \mid h \in \mathcal{E}\}$.

The following is an example of a BF set.

Example 2.5. *Let $\mathcal{E} = \{41, 42, 43...\}$. Define $\vartheta^P : \mathcal{E} \to [0,1]$ is a function*

$$\vartheta^P(h) = \begin{cases} 0 & \text{if } h \text{ is old number} \\ 1 & \text{if } h \text{ is even number} \end{cases}$$

and $\vartheta^N : \mathcal{E} \to [-1,0]$ is a function

$$\vartheta^N(h) = \begin{cases} -1 & \text{if } h \text{ is old number} \\ 0 & \text{if } h \text{ is even number.} \end{cases}$$

Then $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is a BF set.

For BF sets $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ and $\xi = (E; \xi^P, \xi^N)$ of $\mathcal{E}$, defined the relation as follows:
(1) $\vartheta \subseteq \xi$ if and only if $\vartheta^P(z) \leq \xi^P(z)$ and $\vartheta^N(z) \geq \xi^N(z)$ for all $z \in \mathcal{E}$,
(2) $\vartheta = \xi$ if and only if $\vartheta \subseteq \xi$ and $\xi \subseteq \vartheta$,
(3) $\vartheta \cap \xi = \vartheta^P(z) \wedge \xi^P(z)$ and $\vartheta^N(z) \vee \xi^N(z)$, for all $z \in \mathcal{E}$,
(4) $\vartheta \cup \xi = \vartheta^P(z) \vee \xi^P(z)$ and $\vartheta^N(z) \wedge \xi^N(z)$, for all $z \in \mathcal{E}$.
For $h \in \mathcal{E}$, define $F_h = \{(h_1, h_2) \in \mathcal{E} \times \mathcal{E} \mid h = h_1h_2\}$.
Define products $\vartheta^P \circ \xi^P$ and $\vartheta^N \circ \xi^N$ as follows: For $h \in \mathcal{E}$

$$(\vartheta^P \circ \xi^P)(h) = \begin{cases} \bigvee_{(h_1,h_2)\in F_h} \{\vartheta^P(h_1) \wedge \xi^P(h_2)\} & \text{if } h = h_1h_2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\vartheta^N \circ \xi^N)(h) = \begin{cases} \bigwedge_{(h_1,h_2)\in F_h} \{\vartheta^N(h_1) \vee \xi^N(h_2)\} & \text{if } h = h_1h_2 \\ 0 & \text{if otherwise.} \end{cases}$$

Definition 2.6. *[5] A BF set $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ on an SG $\mathcal{E}$ is called a **BF subsemigroup** (BF Ssg) on $\mathcal{E}$ if it satisfies the following conditions:*
(1) $\vartheta^P(hr) \geq \vartheta^P(h) \wedge \vartheta^P(r)$
(2) $\vartheta^N(hr) \leq \vartheta^N(h) \vee \vartheta^N(r)$
for all $h, r \in \mathcal{E}$.

The following is an example of a BF Ssg.

Example 2.7. *Let $\mathcal{E}$ be an SG defined by the following table:*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
a	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Define a BF set $\vartheta = (E; \vartheta^P, \vartheta^N)$ on $\mathcal{E}$ as follows :

$\mathcal{E}$	a	b	c	d	e
ϑ^p	0.9	0.8	0.5	0.3	0.3
ϑ^n	-0.8	-0.8	-0.6	-0.5	-0.3

Then $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is a BF Ssg.

Definition 2.8. *[5] A BF set $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ on an SG $\mathcal{E}$ is called a **BF left ideal** (BF Lid) on $\mathcal{E}$ if it satisfies the following conditions:*
(1) $\vartheta^P(hr) \geq \vartheta^P(r)$
(2) $\vartheta^N(hr) \leq \vartheta^N(r)$
for all $h, r \in \mathcal{E}$.

Definition 2.9. *[5] A BF set $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ on an SG $\mathcal{E}$ is called a **BF right ideal** (BF Rid) on $\mathcal{E}$ if it satisfies the following conditions:*
(1) $\vartheta^P(hr) \geq \vartheta^P(h)$
(2) $\vartheta^N(hr) \leq \vartheta^N(h)$
for all $h, r \in \mathcal{E}$.

Definition 2.10. *[5] A BF Ssg $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ on an SG $\mathcal{E}$ is called a **BF bi-ideal** (BF Bid) on $\mathcal{E}$ if it satisfies the following conditions:*
(1) $\vartheta^P(hrk) \geq \vartheta^P(h) \wedge \vartheta^P(k)$
(2) $\vartheta^N(hrk) \leq \vartheta^N(h) \vee \vartheta^N(k)$
for all $h, r, k \in \mathcal{E}$.

Definition 2.11. *[4] Let $\mathcal{K}$ be a non-empty set of an SG $\mathcal{E}$. A **positive characteristic function and a negative characteristic function** are respectively defined by*

$$\lambda^P_{\mathcal{K}} : \mathcal{E} \to [0,1], h \mapsto \lambda^P_{\mathcal{K}}(h) := \begin{cases} 1 & h \in \mathcal{K}, \\ 0 & h \notin \mathcal{K}, \end{cases}$$

and

$$\lambda^N_{\mathcal{K}} : \mathcal{E} \to [-1,0], h \mapsto \lambda^N_{\mathcal{K}}(h) := \begin{cases} -1 & h \in \mathcal{K}, \\ 0 & h \notin \mathcal{K}. \end{cases}$$

Remark 2.12. *For the sake of simplicity we shall use the symbol $\lambda_{\mathcal{K}} = (E; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ for the BF set*
$\lambda_{\mathcal{K}} = \{(h, \lambda^P_{\mathcal{K}}(h), \lambda^N_{\mathcal{K}}(h)) \mid h \in \mathcal{E}\}$.

Lemma 2.13. *[5] Let $\mathcal{K}$ be a non-empty subset of an SG $\mathcal{E}$. Then $\mathcal{K}$ is a Ssg of $\mathcal{E}$ if and only if the characteristic function $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF Ssg of $\mathcal{E}$.*

III. Main Results

In this section, we outline the concept of bipolar fuzzy (m, n)-ideals and explore their properties within semigroups.

Definition 3.1. *A BF Ssg $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ of an SG $\mathcal{E}$ is called a* bipolar fuzzy (m, n)-ideal *(BF (m, n)-id) of $\mathcal{E}$ if*

(1) $\vartheta^P(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \geq \vartheta^P(e_1) \wedge \vartheta^P(e_2) \wedge \cdots \wedge \vartheta^P(e_m) \wedge \vartheta^P(r_1) \wedge \vartheta^P(r_2) \wedge \cdots \wedge \vartheta^P(r_n)$

(2) $\vartheta^N(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \leq \vartheta^N(e_1) \vee \vartheta^N(e_2) \vee \cdots \wedge \vartheta^N(e_m) \vee \vartheta^N(r_1) \vee \vartheta^N(r_2) \vee \cdots \vee \vartheta^N(r_n)$

for all $e_1, e_2, \ldots, e_m, k, r_1, r_2, \ldots r_n$ of $\mathcal{E}$ and $m, n \in \mathbb{N}$.

Theorem 3.2. *Let $\mathcal{E}$ be an SG and m, n be positive integers. Then every BF Bid of $\mathcal{E}$ is a BF (m, n)-ideal of $\mathcal{E}$*

Proof: It is clear. ∎

Theorem 3.3. *Let $\{\vartheta_i \mid i \in \mathcal{J}\}$ be a family of BF (m, n)-ids of an SG $\mathcal{E}$. Then $\bigwedge_{i\in\mathcal{F}} \vartheta_i$ is a BF (m, n)-id of $\mathcal{E}$, where $\vartheta_i = \{(e, \vartheta^P_i, \vartheta^N_i) \mid \mathcal{E} \in \mathcal{E}\}$.*

Proof: Let $e, h \in \mathcal{E}$. Then,

$$\bigwedge_{i\in\mathcal{J}} \vartheta^P_i(eh) \geq \bigwedge_{i\in\mathcal{J}} \{\vartheta^P_i(e) \wedge \vartheta^P_i(h)\} = \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(e) \wedge \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(h)$$

and

$$\bigvee_{i\in\mathcal{J}} \vartheta^N_i(eh) \leq \bigvee_{i\in\mathcal{J}} \{\vartheta^N_i(e) \vee \vartheta^N_i(h)\} = \bigvee_{i\in\mathcal{J}} \vartheta^N_i(e) \vee \bigvee_{i\in\mathcal{J}} \vartheta^N_i(h).$$

Thus, $\bigwedge_{i\in\mathcal{J}} \vartheta_i$ is a BF-Ssg of $\mathcal{E}$.

Let $e_1, e_2, \ldots, \mathcal{E}_m, k, r_1, r_2, \ldots r_n \in \mathcal{E}$. Then,

$$\begin{aligned}
&\bigwedge_{i\in\mathcal{J}} \vartheta^P_i(e_1e_2\cdots\mathcal{E}_mkr_1r_2\cdots r_n)\\
&\geq \bigwedge_{i\in\mathcal{J}} \{\vartheta^P_i(e_1) \wedge \vartheta^P_i(e_2) \cdots \wedge \vartheta^P_i(e_n)\\
&\quad \wedge \vartheta^P_i(r_1) \wedge \vartheta^P_i(r_2) \ldots \vartheta^P_i(r_n)\}\\
&= \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(e_1) \wedge \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(e_2) \cdots \wedge \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(e_n)\\
&\quad \wedge \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(r_1) \wedge \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(r_2) \ldots \bigwedge_{i\in\mathcal{J}} \vartheta^P_i(r_n)
\end{aligned}$$

and

$$\begin{aligned}
&\bigvee_{i\in\mathcal{J}} \vartheta^N_i(e_1e_2\cdots\mathcal{E}_mkr_1r_2\cdots r_n)\\
&\leq \bigvee_{i\in\mathcal{J}} \{\vartheta^N_i(e_1) \vee \vartheta^N_i(e_2) \cdots \vee \vartheta^N_i(e_n)\\
&\quad \wedge \vartheta^N_i(r_1) \vee \vartheta^N_i(r_2) \ldots \vartheta^N_i(r_n)\}\\
&= \bigvee_{i\in\mathcal{J}} \vartheta^N_i(e_1) \vee \bigvee_{i\in\mathcal{J}} \vartheta^N_i(e_2) \cdots \vee \bigvee_{i\in\mathcal{J}} \vartheta^N_i(e_n)\\
&\quad \vee \bigvee_{i\in\mathcal{J}} \vartheta^N_i(r_1) \vee \bigvee_{i\in\mathcal{J}} \vartheta^N_i(r_2) \ldots \bigvee_{i\in\mathcal{J}} \vartheta^N_i(r_n).
\end{aligned}$$

Thus, $\bigwedge_{i\in\mathcal{J}} \vartheta_i$ is a BF (m, n)-id of $\mathcal{E}$. ∎

Theorem 3.4. *Let $\mathcal{K}$ be a non-empty subset of an SG $\mathcal{E}$ and m, n are positive integers. Then $\mathcal{K}$ is an (m, n)-id of $\mathcal{E}$ if and only if the characteristic function $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF (m, n)-id of $\mathcal{E}$.*

Proof: Suppose that $\mathcal{K}$ is an (m, n)-id of $\mathcal{E}$. Then, $\mathcal{K}$ is a Ssg of $\mathcal{E}$. By Lemma 2.13, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF Ssg of $\mathcal{E}$.

Let $e_i, r_j, k, r_1, r_2, \ldots, r_n \in \mathcal{E}$. Then the following cases:

Case 1: If $e_i, r_j \in \mathcal{K}$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, then $e_1e_2\cdots e_mkr_1r_2\cdots r_n \in \mathcal{K}^m\mathcal{E}\mathcal{K}^n$. Thus, $\lambda^P_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) = 1$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) = 0$, $\lambda^P_{\mathcal{K}}(e_i) = 1$ for all $i \in \{1, 2, \ldots, m\}$, $\lambda^P_{\mathcal{K}}(r_j) = 1$ for all $j \in \{1, 2, \ldots, n\}$ and $\lambda^N_{\mathcal{K}}(e_i) = 0$ for all $i \in \{1, 2, \ldots, m\}$, $\lambda^N_{\mathcal{K}}(r_i) = 0$ for all $j \in \{1, 2, \ldots, n\}$. So, we have $\lambda^P_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \geq \lambda^P_{\mathcal{K}}(e_1) \wedge \lambda^P_{\mathcal{K}}(e_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(e_m) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(r_1) \wedge \lambda^P_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(r_n)$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \leq \lambda^N_{\mathcal{K}}((e_1) \vee \lambda^N_{\mathcal{K}}((e_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}((e_m) \vee \cdots \vee \lambda^N_{\mathcal{K}}(r_1) \vee \lambda^N_{\mathcal{K}}(r_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}(r_n)$.

Case 2: If $e_i \notin \mathcal{K}$ or $r_j \notin \mathcal{K}$ for some $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, then $\lambda^P_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \geq \lambda^P_{\mathcal{K}}(e_1) \wedge \lambda^P_{\mathcal{K}}(e_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(e_m) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(r_1) \wedge \lambda^P_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(r_n)$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \leq \lambda^N_{\mathcal{K}}(e_1) \vee \lambda^N_{\mathcal{K}}(e_2) \vee \cdots \wedge \lambda^N_{\mathcal{K}}(e_m) \vee \cdots \vee \lambda^N_{\mathcal{K}}(r_1) \vee \lambda^N_{\mathcal{K}}(r_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}(r_n)$.

Therefore, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF (m, n)-ideal of $\mathcal{E}$.

Conversely, suppose that $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF (m, n)-id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}}$ is a BF Ssg of $\mathcal{E}$. By Lemma 2.13, $\mathcal{K}$ is a Ssg of $\mathcal{E}$.

Let $e_1, e_2, \ldots e_m, k, r_1, r_2, \ldots, r_n \in \mathcal{K}^m\mathcal{E}\mathcal{K}^n$. Then for all $i \in \{1, 2, \ldots m\}$ and $j \in \{1, 2, \ldots, n\}$, $\lambda^P_{\mathcal{K}}(e_i) = 1$, $\lambda^P_{\mathcal{K}}(r_j) = 1$ and $\lambda^N_{\mathcal{K}}(e_i) = -1$, $\lambda^N_{\mathcal{K}}(r_j) = -1$. By assumption, $\lambda^P_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \geq \lambda^P_{\mathcal{K}}(e_1) \wedge \lambda^P_{\mathcal{K}}(e_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(e_m) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(r_1) \wedge \lambda^P_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(r_n)$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \leq \lambda^N_{\mathcal{K}}(e_1) \vee \lambda^N_{\mathcal{K}}(e_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}(e_m) \vee \cdots \vee \lambda^N_{\mathcal{K}}(r_1) \vee \lambda^N_{\mathcal{K}}(r_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}(r_n)$. Thus, $\lambda^P_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) = 1$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots e_mkr_1r_2\cdots r_n) = -1$. It impiles that, $e_1e_2\cdots\mathcal{E}_mkr_1r_2\cdots r_n \in \mathcal{K}$. Hence, $\mathcal{K}^m\mathcal{E}\mathcal{K}^n \subseteq \mathcal{K}$. Therefore, $\mathcal{K}$ is an (m, n)-id of $\mathcal{E}$. ∎

Theorem 3.5. *Let $\mathcal{E}$ be an SG and m, n be positive integers. Then $\mathcal{K}$ is an $(m, 0)$-ideal ($(0, n)$-ideal) of $\mathcal{E}$ if and only if the characteristic function $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF $(m, 0)$-ideal ($(0, n)$-ideal) of $\mathcal{E}$.*

Proof: Suppose that $\mathcal{K}$ is an $(m, 0)$-ideal of $\mathcal{E}$ and let $e_1, e_2, \ldots \mathcal{E}_m, k \in \mathcal{E}$. Then the following cases:

Case 1: If $e_i \notin \mathcal{K}$ for some $i \in \{1, 2, \ldots, m\}$, then $\lambda^P_{\mathcal{K}}(e_i) = 0$ and $\lambda^N_{\mathcal{K}}(e_i) = 0$ for some $i \in \{1, 2, \ldots, m\}$. Thus, $\lambda^P_{\mathcal{K}}(e_1e_2\cdots\mathcal{E}_mk) \geq \lambda^P_{\mathcal{K}}(e_1) \wedge \lambda^P_{\mathcal{K}}(e_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(e_m)$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots\mathcal{E}_mk) \leq \lambda^N_{\mathcal{K}}(e_1) \vee \lambda^N_{\mathcal{K}}(e_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}(e_m)$.

Case 2: If $e_i \in \mathcal{K}$ for each $i \in \{1, 2, \ldots, m\}$, then $\lambda^P_{\mathcal{K}}(e_i) = 1$ and $\lambda^N_{\mathcal{K}}(e_i) = -1$ for each $i \in \{1, 2, \ldots, m\}$. Thus, $\lambda^P_{\mathcal{K}}(e_1e_2\cdots e_mk) \geq \lambda^P_{\mathcal{K}}(e_1) \wedge \lambda^P_{\mathcal{K}}(e_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(e_m)$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots\mathcal{E}_mk) \leq \lambda^N_{\mathcal{K}}(e_1) \vee \lambda^N_{\mathcal{K}}(e_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}(e_m)$

Therefore, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF $(m, 0)$-ideal of $\mathcal{E}$.

Conversely, suppose that $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF $(m, 0)$-ideal of $\mathcal{E}$. Then,

$\lambda_{\mathcal{K}}(e_1e_2\cdots e_mk) \geq \lambda^P_{\mathcal{K}}(e_1) \wedge \lambda^P_{\mathcal{K}}(e_2) \wedge \cdots \wedge \lambda^P_{\mathcal{K}}(e_m)$ and $\lambda^N_{\mathcal{K}}(e_1e_2\cdots\mathcal{E}_mk) \leq \lambda^N_{\mathcal{K}}(e_1) \vee \lambda^N_{\mathcal{K}}(e_2) \vee \cdots \vee \lambda^N_{\mathcal{K}}(e_m)$.

Thus, $\lambda_{\mathcal{K}}^{P}(e_1e_2\cdots\mathcal{E}_mk)=1$ and $\lambda_{\mathcal{K}}^{N}(e_1e_2\cdots\mathcal{E}_mk)=-1$. It implies that, $e_1e_2\cdots e_mk\in\mathcal{K}$. Hence, $\mathcal{K}^m\mathcal{E}\subseteq\mathcal{K}$. Therefore, $\mathcal{K}$ is an $(m,0)$-ideal of $\mathcal{E}$ ∎

Definition 3.6. *Let $\mathcal{E}$ be an SG and m,n be positive integers. Then $\mathcal{E}$ is called* (m,n)-regular *if for each $e\in\mathcal{E}$ there exists $h\in\mathcal{E}$ such that $e=e^mhe^n$ equivalently for each subset $\mathcal{K}$ of $\mathcal{E}$ if $\mathcal{K}\subseteq\mathcal{K}^m\mathcal{E}\mathcal{K}^n$ or for each element $\mathcal{E}$ of $\mathcal{E}$, $e\in\mathcal{E}^m\mathcal{E}e^n$.*

Lemma 3.7. *Let $\mathcal{E}$ be an (m,n)-regular of semigroup and m,n be positive integers. Then every BF (m,n)-id of $\mathcal{E}$ is a BF Bid of $\mathcal{E}$.*

Proof: Suppose that ϑ is a BF (m,n)-id of $\mathcal{E}$ and $i,j,k\in\mathcal{E}$. By assumption, there exists $x,y\in\mathcal{E}$ such that $ijk=i^mxi^njk^myk^n$. Thus,

$$\begin{aligned}\vartheta^P(ijk) &= \vartheta^P(i^mxi^njk^myk^n)\\ &= \vartheta^P(i^m(xi^njk^my)k^n)\\ &\geq \vartheta^P(i^m)\wedge\vartheta^P(k^n)\\ &\geq \vartheta^P(i)\wedge\vartheta^P(k)\end{aligned}$$

and

$$\begin{aligned}\vartheta^N(ijk) &= \vartheta^N(i^mxi^njk^myk^n)\\ &= \vartheta^N(i^m(xi^njk^my)k^n)\\ &\leq \vartheta^N(i^m)\vee\vartheta^N(k^n)\\ &\leq \vartheta^N(i)\vee\vartheta^N(k).\end{aligned}$$

Hence, ϑ is a BF Bid of $\mathcal{E}$. ∎

Definition 3.8. *An (m,n)-id $\mathcal{K}$ of an SG $\mathcal{E}$ is called*

(1) *a* minimal *if for every (m,n)-id of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{J}\subseteq\mathcal{K}$, we have $\mathcal{J}=\mathcal{K}$.*
(2) *a* maximal *if for every (m,n)-id of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{K}\subseteq\mathcal{J}$, we have $\mathcal{J}=\mathcal{K}$.*
(3) *a* 0-minimal *if for every (m,n)-id of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{J}\subseteq\mathcal{K}$, we have $\mathcal{J}=\mathcal{K}$.*

Definition 3.9. *A BF (m,n)-id $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ of an SG $\mathcal{E}$ is*

(1) *a* minimal *if for all BF (m,n)-id $\xi=(\mathcal{E},\xi^P,\xi^N)$ of $\mathcal{E}$ such that $\xi\leq\vartheta$, then $\xi=\vartheta$.*
(2) *a* maximal *if for all BF (m,n)-id $\xi=(\mathcal{E},\xi^P,\xi^N)$ of $\mathcal{E}$ such that $\vartheta\leq\xi$, then $\xi=\vartheta$.*
(3) *a* 0-minimal *if for all BF (m,n)-id $\xi=(\mathcal{E},\xi^P,\xi^N)$ of $\mathcal{E}$ such that $\xi\leq\vartheta$, then $\xi=\vartheta$.*

Lemma 3.10. *For any non-empty subsets $\mathcal{I}$ and $\mathcal{K}$ of an SG $\mathcal{E}$, we have $\mathcal{I}\subseteq\mathcal{K}$ if and only if $\lambda_{\mathcal{I}}^P\leq\lambda_{\mathcal{K}}^P$ and $\lambda_{\mathcal{I}}^N\geq\lambda_{\mathcal{K}}^N$ where $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{I}}^P,\lambda_{\mathcal{I}}^N)$ and $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ are characteristic functions I and $\mathcal{K}$ respective.*

Theorem 3.11. *Let $\emptyset\neq\mathcal{K}\subseteq\mathcal{E}$. Then*

(1) *$\mathcal{K}$ is a minimal (m,n)-id if and only if $\lambda_{\mathcal{K}}=(E;\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is a minimal BF (m,n)-id.*
(2) *$\mathcal{K}$ a maximal (m,n)-id if and only if $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is a maximal BF (m,n)-id.*
(3) *$\mathcal{K}$ a 0-minimal (m,n)-ideal if and only if $\lambda_{\mathcal{K}}=(E;\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is a 0-minimal BF (m,n)-ideal.*

Proof:

(1) Let $\mathcal{K}$ be a minimal (m,n)-id of $\mathcal{E}$. Then $\mathcal{K}$ is an (m,n)-ideal. Thus, by Theorem 3.4, $\lambda_{\mathcal{K}}=(E;\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is a BF (m,n)-id of $\mathcal{E}$. Let $\mathcal{J}$ be an (m,n)-id of $\mathcal{E}$ such that $\mathcal{J}\subseteq\mathcal{K}$. Then by Theorem 3.4, $\lambda_{\mathcal{J}}=(\mathcal{E};\lambda_{\mathcal{J}}^P,\lambda_{\mathcal{J}}^N)$ is a BF (m,n)-id of $\mathcal{E}$ and $\lambda_{\mathcal{J}}\leq\lambda_{\mathcal{K}}$. Since $\mathcal{K}$ is a minimal (m,n)-id of $\mathcal{E}$ we have $\mathcal{J}=\mathcal{K}$. Thus, $\lambda_{\mathcal{J}}=\lambda_{\mathcal{K}}$. Hence, $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is minimal BF (m,n)-id of $\mathcal{E}$.
Conversely, $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is minimal BF (m,n)-id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is a BF (m,n)-id of $\mathcal{E}$. Thus, by Theorem 3.4, $\mathcal{K}$ is an (m,n)-id of $\mathcal{E}$.
Let $\lambda_{\mathcal{J}}$ be a BF (m,n)-id of $\mathcal{E}$ such that $\lambda_{\mathcal{J}}\leq\lambda_{\mathcal{K}}$. Then by Theorem 3.4, $\mathcal{J}$ is an (m,n)-id of $\mathcal{E}$ such that $\mathcal{J}\subseteq\mathcal{K}$. Since $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is minimal BF (m,n)-id of $\mathcal{E}$ we have $\lambda_{\mathcal{J}}=\lambda_{\mathcal{K}}$. Thus, $\mathcal{J}=\mathcal{K}$. Hence, $\mathcal{K}$ is a minimal (m,n)-id of $\mathcal{E}$.
(2) Let $\mathcal{K}$ be a maximal (m,n)-id of $\mathcal{E}$. Then $\mathcal{K}$ is an (m,n)-id. Thus, by Theorem 3.4, $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{K}^P,\lambda_{K}^N)$ is a BF (m,n)-ideal of $\mathcal{E}$. Let $\mathcal{J}$ be an (m,n)-id of $\mathcal{E}$ such that $\mathcal{K}\subseteq\mathcal{J}$. Then by Theorem 3.4, $\lambda_{\mathcal{J}}=(\mathcal{E};\lambda_{\mathcal{J}}^P,\lambda_{\mathcal{J}}^N)$ is a BF (m,n)-id of $\mathcal{E}$ and $\lambda_{\mathcal{K}}\leq\lambda_{\mathcal{J}}$. Since $\mathcal{K}$ is a minimal (m,n)-id of $\mathcal{E}$ we have $\mathcal{J}=\mathcal{K}$. Thus, $\lambda_{\mathcal{J}}=\lambda_{\mathcal{K}}$. Hence, $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is maximal BF (m,n)-id of $\mathcal{E}$.
Conversely, $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is maximal BF (m,n)-id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is a BF (m,n)-id of $\mathcal{E}$. Thus, by Theorem 3.4, $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{E}$. Let $\lambda_{\mathcal{J}}$ be a BF (m,n)-id of $\mathcal{E}$ such that $\lambda_{\mathcal{K}}\leq\lambda_{\mathcal{J}}$. Then by Theorem 3.4, $\mathcal{J}$ is an (m,n)-ideal of $\mathcal{E}$ such that $\mathcal{K}\subseteq\mathcal{J}$. Since $\lambda_{\mathcal{K}}=(\mathcal{E};\lambda_{\mathcal{K}}^P,\lambda_{\mathcal{K}}^N)$ is maximal BF (m,n)-id of $\mathcal{E}$ we have $\lambda_{\mathcal{J}}=\lambda_{\mathcal{K}}$. Thus, $\mathcal{J}=\mathcal{K}$. Hence, $\mathcal{K}$ is a maximal (m,n)-id of $\mathcal{E}$.
(3) It follows from (1).

∎

Let $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ be a BF set and $(s,t)\in[0,1]\times[-1,0]$. Define the set $U_{\vartheta}^{(s,t)}:=\{e\in\mathcal{E}\mid\vartheta^P(e)\geq s,\vartheta^N\leq t\}$ is called an (s,t)-level subset of BF set of $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$.

Lemma 3.12. *[5] A BF set $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ is a BF Ssg of an SG $\mathcal{E}$ if and only if the level set $U_{\vartheta}^{(s,t)}$ is a Ssg of $\mathcal{E}$ for all $(s,t)\in[0,1]\times[-1,0]$.*

Theorem 3.13. *A BF set $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ is a BF (m,n)-ideal of an SG $\mathcal{E}$ if and only if the level set $U_{\vartheta}^{(s,t)}$ is an (m,n)-id of $\mathcal{E}$ for all $(s,t)\in[0,1]\times[-1,0]$.*

Proof: Let $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ be a BF (m,n)-id of $\mathcal{E}$. Then $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ is a BF Ssg of $\mathcal{E}$. By Lemma 3.12, $U_{\vartheta}^{(s,t)}$ is a Ssg of $\mathcal{E}$. Let $e_1,e_2,\dots e_m,k,r_1,r_2,\dots,r_n\in U_{\vartheta}^{(s,t)}$. Then $\vartheta^P(e_i)\geq s$, $\vartheta^P(r_j)\geq s$ and $\vartheta^N(e_i)\leq t$, $\vartheta^N(r_j)\leq t$ for some $i\in\{1,2,\dots,m\}$ and $j\in\{1,2,\dots,m\}$. By assumption, $\vartheta^P(e_1e_2\cdots\mathcal{E}_mkr_1r_2\cdots r_n)\geq\vartheta^P(e_1)\wedge\vartheta^P(e_2)\wedge\cdots\wedge\vartheta^P(e_m)\wedge\cdots\wedge\vartheta^P(r_1)\wedge\vartheta^P(r_2)\wedge\cdots\wedge\vartheta^P(r_n)$ and $\vartheta^N(e_1e_2\cdots e_mkr_1r_2\cdots r_n)\leq\vartheta^N(e_1)\vee\vartheta^N(e_2)\vee\cdots\vee\vartheta^N(e_m)\vee\cdots\vee\vartheta^N(r_1)\vee\vartheta^N(r_2)\vee\cdots\vee\vartheta^N(r_n)$. Thus, $\vartheta^P(e_1e_2\cdots e_mkr_1r_2\cdots r_n)\geq s$ and $\vartheta^N(e_1e_2\cdots e_mkr_1r_2\cdots r_n)\leq t$. It implies that, $e_1e_2\cdots\mathcal{E}_mkr_1r_2\cdots r_n\in U_{\vartheta}^{(s,t)}$. Hence, $U_{\vartheta}^{(s,t)}$ is an (m,n)-id of $\mathcal{E}$.

Conversely, suppose that $U_{\vartheta}^{(s,t)}$ is an (m,n)-id of $\mathcal{E}$. Then $U_{\vartheta}^{(s,t)}$ is a Ssg of $\mathcal{E}$. By Lemma 3.12, $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ is a BF Ssg of an SG $\mathcal{E}$. If $\vartheta=(\mathcal{E};\vartheta^P,\vartheta^N)$ is not a BF (m,n)-id of $\mathcal{E}$, then there exists $e_i,k,r_j\in\mathcal{E}$ such that $\vartheta^P(e_1e_2\cdots e_mkr_1r_2\cdots r_n)<\vartheta^P(e_1)\wedge\vartheta^P(e_2)\wedge\cdots\wedge\vartheta^P(e_m)\wedge\cdots\wedge\vartheta^P(r_1)\wedge\vartheta^P(r_2)\wedge\cdots\wedge\vartheta^P(r_n)$ or $\vartheta^N(e_1e_2\cdots e_mkr_1r_2\cdots r_n)>\vartheta^N(e_1)\vee\vartheta^N(e_2)\vee\cdots\vee\vartheta^N(e_m)\vee\cdots\vee\vartheta^N(r_1)\vee\vartheta^N(r_2)\vee\cdots\vee\vartheta^N(r_n)$.

By assumption, we have $e_1e_2\cdots e_mkr_1r_2\cdots r_n \in U_{\vartheta}^{(s,t)}$. Thus $\vartheta^P(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \geq \vartheta^P(e_1) \wedge \vartheta^P(e_2) \wedge \cdots \wedge \vartheta^P(e_m) \wedge \cdots \wedge \vartheta^P(r_1) \wedge \vartheta^P(r_2) \wedge \cdots \wedge \vartheta^P(r_n)$ or $\vartheta^N(e_1e_2\cdots e_mkr_1r_2\cdots r_n) \leq \vartheta^N(e_1) \vee \vartheta^N(e_2) \vee \cdots \vee \vartheta^N(e_m) \vee \cdots \vee \vartheta^N(r_1) \vee \vartheta^N \vee \cdots \vee \vartheta^N(r_n)$. It is a contradiction. Hence, $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ be a BF (m,n)-id of $\mathcal{E}$. ■

Next, we give the relationship between prime, semiprime (m,n)-ideals and prime, semiprime BF (m,n)-ideals.

Definition 3.14. *Let $\mathcal{K}$ be an (m,n)-id of an SG $\mathcal{E}$ is called*

(1) prime *if $eh \in \mathcal{K}$ implies $e \in \mathcal{K}$ or $h \in \mathcal{K}$ for all $e,h \in \mathcal{E}$,*
(2) semiprime *if $e^2 \in \mathcal{K}$ implies $e \in \mathcal{K}$ for all $e \in \mathcal{E}$.*

Definition 3.15. *Let $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ be a BF (m,n)-id of an SG is called*

(1) prime *if $\vartheta^P(eh) \leq \vartheta^P(e) \vee \vartheta^P(h)$ and $\vartheta^N(eh) \geq \vartheta^N(e) \wedge \vartheta^N(h)$ for all $e,h \in \mathcal{E}$,*
(2) semiprime *if $\vartheta^P(e^2) \leq \vartheta^P(e)$ and $\vartheta^N(e^2) \geq \vartheta^N(e)$ for all $e \in \mathcal{E}$.*

Remark 3.16. *Every prime (m,n)-id is semiprime (m,n)-id in an SG.*

Theorem 3.17. *Let $\mathcal{K}$ be a non-empty subset of an SG $\mathcal{E}$. Then*

(1) *$\mathcal{K}$ is a prime (m,n)-id of $\mathcal{E}$ if and only if $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda_{\mathcal{K}}^P, \lambda_{\mathcal{K}}^N)$ is a prime BF (m,n)-id of $\mathcal{E}$.*
(2) *$\mathcal{K}$ is a semiprime (m,n)-id of $\mathcal{E}$ if and only if $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda_{\mathcal{K}}^P, \lambda_{\mathcal{K}}^N)$ is a semiprime BF (m,n)-id of $\mathcal{E}$.*

Proof:

(1) Suppose that $\mathcal{K}$ is a prime (m,n)-id of $\mathcal{E}$. Then $\mathcal{K}$ is an (m,n)-id of $\mathcal{E}$. Thus, by Theorem 3.4 $\lambda_{\mathcal{K}}$ is a BF (m,n)-id of $\mathcal{E}$. Let $e,h \in \mathcal{E}$.
Case 1: If $eh \in \mathcal{K}$, then $e \in \mathcal{K}$ or $h \in \mathcal{K}$. Thus, $\lambda_{\mathcal{K}}^P(eh) = 1 = \lambda_{\mathcal{K}}^P(e)$ and $\lambda_{\mathcal{K}}^N(eh) = -1 = \lambda_{\mathcal{K}}^N(e)$ or $\lambda_{\mathcal{K}}^P(h) = 1 = \lambda_{\mathcal{K}}^P(eh)$ and $\lambda_{\mathcal{K}}^N(h) = -1 = \lambda_{\mathcal{K}}^N(eh)$. Hence, $\lambda_{\mathcal{K}}^P(eh) \leq \lambda_{\mathcal{K}}^P(e) \vee \lambda_{\mathcal{K}}^P(h)$ and $\lambda_{\mathcal{K}}^N(eh) \geq \lambda_{\mathcal{K}}^N(e) \wedge \lambda_{\mathcal{K}}^N(h)$.
Case 2: If $eh \notin \mathcal{K}$, then $\lambda_{\mathcal{K}}^P(eh) = 0$ and $\lambda_{\mathcal{K}}^N(eh) = 0$. Thus, $\lambda_{\mathcal{K}}^P(eh) \leq \lambda_{\mathcal{K}}^P(e) \vee \lambda_{\mathcal{K}}^P(h)$ and $\lambda_{\mathcal{K}}^N(eh) \geq \lambda_{\mathcal{K}}^N(e) \wedge \lambda_{\mathcal{K}}^N(h)$.
Therefore, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda_{\mathcal{K}}^P, \lambda_{\mathcal{K}}^N)$ is a prime BF (m,n)-id of $\mathcal{E}$.
Conversely, suppose that $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda_{\mathcal{K}}^P, \lambda_{\mathcal{K}}^N)$ is a prime BF (m,n)-id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}}$ is a BF (m,n)-ideal of $\mathcal{E}$. Thus, by Theorem 3.4, $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{E}$. Let $e,h \in \mathcal{E}$ with $eh \in \mathcal{K}$. Then, $\lambda_{\mathcal{K}}^P(eh) = 1$ and $\lambda_{\mathcal{K}}^N(eh) = -1$. If $e \notin \mathcal{K}$ and $h \notin \mathcal{K}$, then $\lambda_{\mathcal{K}}^P(e) = 0 = \lambda_{\mathcal{K}}^P(h)$ and $\lambda_{\mathcal{K}}^N(e) = 0 = \lambda_{\mathcal{K}}^N(h)$. By assumption, $\lambda_{\mathcal{K}}^P(eh) \leq \lambda_{\mathcal{K}}^P(e) \vee \lambda_{\mathcal{K}}^P(h)$ and $\lambda_{\mathcal{K}}^N(eh) \geq \lambda_{\mathcal{K}}^N(e) \wedge \lambda_{\mathcal{K}}^N(h)$. Thus, $\lambda_{\mathcal{K}}^P(eh) = 0$ and $\lambda_{\mathcal{K}}^N(eh) = 0$. It is a contradiction, so $e \in \mathcal{K}$ or $h \in \mathcal{K}$. Hence, $\mathcal{K}$ is a prime (m,n)-id of $\mathcal{E}$.
(2) Suppose that $\mathcal{K}$ is a semiprime (m,n)-id of $\mathcal{E}$. Then $\mathcal{K}$ is an (m,n)-id of $\mathcal{E}$. Thus, by Theorem 3.4 $\lambda_{\mathcal{K}}$ is a BF (m,n)-id of $\mathcal{E}$. Let $e \in \mathcal{E}$.
Case 1: If $e^2 \in \mathcal{K}$, then $e \in \mathcal{K}$. Thus, $\lambda_{\mathcal{K}}^P(e^2) = 1 = \lambda_{\mathcal{K}}^P(e)$ and $\lambda_{\mathcal{K}}^N(e^2) = -1 = \lambda_{\mathcal{K}}^N(e)$. Hence, $\lambda_{\mathcal{K}}^P(e^2) \leq \lambda_{\mathcal{K}}^P(e)$ and $\lambda_{\mathcal{K}}^N(e^2) \geq \lambda_{\mathcal{K}}^N(e)$.
Case 2: If $e^2 \notin \mathcal{K}$, then $\lambda_{\mathcal{K}}^P(e^2) = 0$ and $\lambda_{\mathcal{K}}^N(e^2) = 0$. Thus, $\lambda_{\mathcal{K}}^P(e^2) \leq \lambda_{\mathcal{K}}^P(e)$ and $\lambda_{\mathcal{K}}^N(e^2) \geq \lambda_{\mathcal{K}}^N(e)$.
Therefore, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda_{\mathcal{K}}^P, \lambda_{\mathcal{K}}^N)$ is a prime BF (m,n)-id of $\mathcal{E}$.
Conversely, suppose that $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda_{\mathcal{K}}^P, \lambda_{\mathcal{K}}^N)$ is a prime BF (m,n)-id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}}$ is a BF (m,n)-id of $\mathcal{E}$. Thus, by Theorem 3.4, $\mathcal{K}$ is an (m,n)-idl of $\mathcal{E}$. Let $e \in \mathcal{E}$ with $e^2 \in \mathcal{K}$. Then, $\lambda_{\mathcal{K}}^P(e^2) = 1$ and $\lambda_{\mathcal{K}}^N(e^2) = -1$. If $e \notin \mathcal{K}$, then $\lambda_{\mathcal{K}}^P(e) = 0$ and $\lambda_{\mathcal{K}}^N(e) = 0$. By assumption, $\lambda_{\mathcal{K}}^P(e^2) \leq \lambda_{\mathcal{K}}^P(e)$ and $\lambda_{\mathcal{K}}^N(e^2) \geq \lambda_{\mathcal{K}}^N(e)$. Thus, $\lambda_{\mathcal{K}}^P(eh) = 0$ and $\lambda_{\mathcal{K}}^N(e^2) = 0$. It is a contradiction, so $e \in \mathcal{K}$. Hence, $\mathcal{K}$ is a semiprime (m,n)-id of $\mathcal{E}$.

■

IV. Bipolar fuzzy n-interior ideals

Before, we will review the definition of n-interior ideals and weakly n-interior ideals in Sgs.

Definition 4.1. *[24] A Ssg $\mathcal{K}$ of an SG $\mathcal{E}$ is said to be an* n-interior ideal *(n-In id) of $\mathcal{E}$ if $\mathcal{E}\mathcal{K}^n\mathcal{E} \subseteq \mathcal{K}$, where n is an integer.*

Definition 4.2. *A non-empty subset $\mathcal{K}$ of an SG $\mathcal{E}$ is called a* weakly n-interior ideal *(W n-In id) of $\mathcal{E}$ if $\mathcal{E}\mathcal{K}^n\mathcal{E} \subseteq \mathcal{K}$, where n is an integer.*

Next, we defined bipolar fuzzy n-interior ideals and bipolar fuzzy weakly n-interior ideals in SGs.

Definition 4.3. *A BF Ssg $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ in an SG $\mathcal{E}$ is called* BF n-interior ideal *(BF n-In id) of $\mathcal{E}$ if*

(1) $\vartheta^P(hr_i^nk) \geq \vartheta^P(r_i) \wedge \vartheta^P(r_2) \wedge \cdots \wedge \vartheta^P(r_n)$
(2) $\vartheta^N(hr_i^nk) \leq \vartheta^N(r_i) \vee \vartheta^N(r_2) \vee \cdots \vee \vartheta^N(r_n)$

for all $h, r_i, k \in \mathcal{E}$ and where $i \in \{1,2,\ldots,n\}$.

Definition 4.4. *A BF set $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ in an SG $\mathcal{E}$ is called* BF weakly n-interior ideal *(BF W n-In id) of $\mathcal{E}$ if*

(1) $\vartheta^P(hr_i^nk) \geq \vartheta^P(r_i) \wedge \vartheta^P(r_2) \wedge \cdots \wedge \vartheta^P(r_n)$
(2) $\vartheta^N(hr_i^nk) \leq \vartheta^N(r_i) \vee \vartheta^N(r_2) \vee \cdots \vee \vartheta^N(r_n)$

for all $h, r_i, k \in \mathcal{E}$ and where $i \in \{1,2,\ldots,n\}$.

Theorem 4.5. *Let $\{\vartheta_i \mid i \in \mathcal{J}\}$ be a family of BF n-interior ideals (BF n-In id) of an SG $\mathcal{E}$. Then $\bigwedge_{i\in\mathcal{F}} \vartheta_i$ is a BF n-In id of $\mathcal{E}$, where $\vartheta_i = \{(e, \vartheta_i^P, \vartheta_i^N) \mid \mathcal{E} \in \mathcal{E}\}$.*

Proof: Let $e,h \in \mathcal{E}$. Then,

$$\begin{aligned}\bigwedge_{i\in\mathcal{J}} \vartheta_i^P(eh) &\geq \bigwedge_{i\in\mathcal{J}} \{\vartheta_i^P(e) \wedge \vartheta_i^P(h)\} \\ &= \bigwedge_{i\in\mathcal{J}} \vartheta_i^P(e) \wedge \bigwedge_{i\in\mathcal{J}} \vartheta_i^P(h)\end{aligned}$$

and

$$\begin{aligned}\bigvee_{i\in\mathcal{J}} \vartheta_i^N(eh) &\leq \bigvee_{i\in\mathcal{J}} \{\vartheta_i^N(e) \vee \vartheta_i^N(h)\} \\ &= \bigvee_{i\in\mathcal{J}} \vartheta_i^N(e) \vee \bigvee_{i\in\mathcal{J}} \vartheta_i^N(h).\end{aligned}$$

Thus, $\bigwedge_{i\in\mathcal{J}} \vartheta_i$ is a BF-Ssg of $\mathcal{E}$.

Let $h, r_i^n, k \in \mathcal{E}$ for all $i \in \{1, 2, \ldots, n\}$. Then,

$$\begin{aligned}&\bigwedge_{i\in\mathcal{J}} \vartheta_i^P(hr_i^n k)\\ &\quad\geq \bigwedge_{i\in\mathcal{J}} \{\vartheta_i^P(r_1) \wedge \vartheta_i^P(r_2) \cdots \wedge \vartheta_i^P(r_n)\}\\ &\quad= \bigwedge_{i\in\mathcal{J}} \vartheta_i^P(r_1) \wedge \bigwedge_{i\in\mathcal{J}} \vartheta_i^P(r_2) \cdots \wedge \bigwedge_{i\in\mathcal{J}} \vartheta_i^P(r_n)\end{aligned}$$

and

$$\begin{aligned}&\bigvee_{i\in\mathcal{J}} \vartheta_i^N(hr_i^n k)\\ &\quad\geq \bigvee_{i\in\mathcal{J}} \{\vartheta_i^N(r_1) \vee \vartheta_i^N(r_2) \cdots \vee \vartheta_i^N(r_n)\}\\ &\quad= \bigvee_{i\in\mathcal{J}} \vartheta_i^N(r_1) \wedge \bigvee_{i\in\mathcal{J}} \vartheta_i^N(r_2) \cdots \vee \bigvee_{i\in\mathcal{J}} \vartheta_i^N(r_n).\end{aligned}$$

Thus, $\bigwedge_{i\in\mathcal{J}} \vartheta_i$ is a BF n-In id of $\mathcal{E}$. ■

Theorem 4.6. *Let $\{\vartheta_i \mid i \in \mathcal{J}\}$ be a family of BF W n-In id s of an SG $\mathcal{E}$. Then $\bigwedge_{i\in\mathcal{F}} \vartheta_i$ is a BF W n-In id of $\mathcal{E}$, where $\vartheta_i = \{(e, \vartheta_i^P, \vartheta_i^N) \mid \mathcal{E} \in \mathcal{E}\}$.*

Proof: It follows from Theorem 4.5. ■

Theorem 4.7. *Let $\mathcal{K}$ be an non-empty subset of an SG $\mathcal{E}$ and m, n are positive integers. Then the following statements hold*

(1) *$\mathcal{K}$ is an n-In id of $\mathcal{E}$ if and only if the characteristic function $\lambda_\mathcal{K} = (\mathcal{E}; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a BF n-In id of $\mathcal{E}$.*
(2) *$\mathcal{K}$ is a W n-In id of $\mathcal{E}$ if and only if the characteristic function $\lambda_\mathcal{K} = (\mathcal{E}; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a BF W n-In id of $\mathcal{E}$.*

Proof:

(1) Suppose that $\mathcal{K}$ is an n-interior ideal of $\mathcal{E}$. Then $\mathcal{K}$ is a Ssg of $\mathcal{E}$. Thus, by Theorem 2.13, $\lambda_\mathcal{K} = (\mathcal{E}; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a BF Ssg of $\mathcal{E}$. Let $h, r_i, k \in \mathcal{E}$ where $i \in \{1, 2, \ldots, n\}$. If $r_i \in \mathcal{K}$ for all $i \in \{1, 2, \ldots, n\}$, then $hr_i^n k \in \mathcal{K}$. Thus, $\lambda_\mathcal{K}^P(r_i) = \lambda_\mathcal{K}^P(hr_i^n k) = 1$ and $\lambda_\mathcal{K}^N(r_i) = \lambda_\mathcal{K}^N(hr_i^n k) = -1$ for all $i \in \{1, 2, \ldots, n\}$. Hence, $\lambda_\mathcal{K}^P(hr_i^n k) \geq \lambda_\mathcal{K}^P(r_i) \wedge \lambda_\mathcal{K}^P(r_2) \wedge \cdots \wedge \lambda_\mathcal{K}^P(r_n)$ and $\lambda_\mathcal{K}^N(hr_i^n k) \leq \lambda_\mathcal{K}^N(r_i) \vee \lambda_\mathcal{K}^N(r_2) \vee \cdots \vee \lambda_\mathcal{K}^N(r_n)$.
If $r_i \notin \mathcal{K}$ for some $i \in \{1, 2, \ldots, n\}$, then $\lambda_\mathcal{K}^P(r_i) = 0$ and $\lambda_\mathcal{K}^N(r_i) = 0$ for some $i \in \{1, 2, \ldots, n\}$. Thus, $\lambda_\mathcal{K}^P(hr_i^n k) \geq \lambda_\mathcal{K}^P(r_i) \wedge \lambda_\mathcal{K}^P(r_2) \wedge \cdots \wedge \lambda_\mathcal{K}^P(r_n)$ and $\lambda_\mathcal{K}^N(hr_i^n k) \leq \lambda_\mathcal{K}^N(r_i) \vee \lambda_\mathcal{K}^N(r_2) \vee \cdots \vee \lambda_\mathcal{K}^N(r_n)$.
Therefore, $\lambda_\mathcal{K} = (\mathcal{E}; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a BF n-In id of $\mathcal{E}$.
Conversely, suppose that $\lambda_\mathcal{K} = (E; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a BF n-In id of $\mathcal{E}$. Then $\lambda_\mathcal{K} = (\mathcal{E}; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a BF Ssg of $\mathcal{E}$. Thus, by Theorem 2.13, $\mathcal{K}$ is a Ssg of $\mathcal{E}$. Let $r_i^n \in \mathcal{E}\mathcal{K}^n\mathcal{E}$ where n is an integer and for all $i \in \{1, 2, \ldots, n\}$. Then $\lambda_\mathcal{K}^P(r_i^n) = 1$ and $\lambda_\mathcal{K}^n(r_i^n) = 0$ for all $i \in \{1, 2, \ldots, n\}$. By assumption, $\lambda_\mathcal{K}^P(hr_i^n k) \geq \lambda_\mathcal{K}^P(r_i) \wedge \lambda_\mathcal{K}^P(r_2) \wedge \cdots \wedge \lambda_\mathcal{K}^P(r_n)$ and $\lambda_\mathcal{K}^N(hr_i^n k) \leq \vartheta_\mathcal{K}^N(r_i) \vee \lambda_\mathcal{K}^N(r_2) \vee \cdots \vee \lambda_\mathcal{K}^N(r_n)$. Thus, $\lambda_\mathcal{K}^P(hr_i^n k) = 1$ for all $i \in \{1, 2, \ldots, n\}$ and $\lambda_\mathcal{K}^N(hr_i^n k) = 0$. Hence, $r_i^n \in \mathcal{K}$ for all $i \in \{1, 2, \ldots, n\}$. Therefore, $\mathcal{K}$ is an n-In id of $\mathcal{E}$.
(2) It follows from (1).

■

Theorem 4.8. *A BF set $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is a BF n-In id of an SG $\mathcal{E}$ if and only if the level set $U_\vartheta^{(s,t)}$ is an n-In id of $\mathcal{E}$ for all $(s, t) \in [0, 1] \times [-1, 0]$.*

Proof: Let $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ be a BF n-In id of $\mathcal{E}$. Then $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is a BF Ssg of $\mathcal{E}$. By Lemma 3.12, $U_\vartheta^{(s,t)}$ is a Ssg of $\mathcal{E}$. Let $r_1, r_2, \ldots r_m, k, h \in U_\vartheta^{(s,t)}$. Then $\vartheta^P(r_i) \geq s$ and $\vartheta^N(r_i) \leq t$ for some $i \in \{1, 2, \ldots, n\}$. By assumption, $\vartheta^P(hr_i^n k) \geq \vartheta^P(r_1) \wedge \vartheta^P(r_2) \wedge \cdots \wedge \vartheta^P(r_n)$ and $\vartheta^N(hr_i^n k) \leq \vartheta^N(r_1) \vee \vartheta^N(r_2) \vee \cdots \vee \vartheta^N(r_m)$. Thus, $\vartheta^P(hr_i^n k) \geq s$ and $\vartheta^N(hr_i^n k) \leq t$. It implies that, $r_i^n \in U_\vartheta^{(s,t)}$. Hence, $U_\vartheta^{(s,t)}$ is an n-In id of $\mathcal{E}$.

Conversely, suppose that $U_\vartheta^{(s,t)}$ is an n-interior ideal of $\mathcal{E}$. Then $U_\vartheta^{(s,t)}$ is a Ssg of $\mathcal{E}$. By Lemma 3.12, $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is a BF Ssg of an SG $\mathcal{E}$. If $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is not a BF n-interior ideal of $\mathcal{E}$, then there exists $r_i, k, h \in \mathcal{E}$ such that $\vartheta^P(hr_i^n k) < \vartheta^P(r_1) \wedge \vartheta^P(r_2) \wedge \cdots \wedge \vartheta^P(r_n)$ or $\vartheta^N(hr_i^n k) > \vartheta^N(r_1) \vee \vartheta^N(r_2) \vee \cdots \vee \vartheta^N(r_n)$. By assumption, we have $hr_i^n k \in U_\vartheta^{(s,t)}$. Thus, $\vartheta^P(hr_i^n k) \geq \vartheta^P(r_1) \wedge \vartheta^P(r_2) \wedge \cdots \wedge \vartheta^P(r_n)$ or $\vartheta^N(hr_i^n k) \leq \vartheta^N(r_1) \vee \vartheta^N(r_2) \vee \cdots \vee \vartheta^N(r_n)$. It is a contradiction. Hence, $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is a BF n-In id of $\mathcal{E}$. ■

Corollary 4.9. *A BF set $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ is a BF W n-In id of an SG $\mathcal{E}$ if and only if the level set $U_\vartheta^{(s,t)}$ is a W n-In id of $\mathcal{E}$ for all $(s, t) \in [0, 1] \times [-1, 0]$.*

Definition 4.10. *An n-interior ideal $\mathcal{K}$ of an SG $\mathcal{E}$ is called*

(1) *a* minimal *if for every n-In id of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{J} \subseteq \mathcal{K}$, we have $\mathcal{J} = \mathcal{K}$,*
(2) *a* maximal *if for every n-In id of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{K} \subseteq \mathcal{J}$, we have $\mathcal{J} = \mathcal{K}$.*
(3) *a* 0-minimal *if for every n-interior ideal of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{J} \subseteq \mathcal{K}$, we have $\mathcal{J} = \mathcal{K}$.*

Definition 4.11. *A BF n-interior ideal $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ of an SG $\mathcal{E}$ is*

(1) *a* minimal *if for all BF n-In id $\xi = (\mathcal{E}; \xi^P, \xi^N)$ of $\mathcal{E}$ such that $\xi \leq \vartheta$, then $\xi = \vartheta$,*
(2) *a* maximal *if for all BF n-In id $\xi = (\mathcal{E}; \xi^P, \xi^N)$ of $\mathcal{E}$ such that $\vartheta \leq \xi$, then $\xi = \vartheta$.*
(3) *a* 0-minimal *if for all BF n-interior ideal$\xi = (\mathcal{E}, \xi^P, \xi^N)$ of $\mathcal{E}$ such that $\xi \leq \vartheta$, then $\xi = \vartheta$.*

Theorem 4.12. *A non-empty subset $\mathcal{K}$ of an SG $\mathcal{E}$. Then the following statements hold*

(1) *$\mathcal{K}$ is a minimal n-In id if and only if $\lambda_\mathcal{K} = (E; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a minimal BF n-In id of $\mathcal{E}$.*
(2) *$\mathcal{K}$ is a maximal n-In id if and only if $\lambda_\mathcal{K} = (E; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a maximal BF n-In id of $\mathcal{E}$.*
(3) *$\mathcal{K}$ is a 0-minimal n-In id if and only if $\lambda_\mathcal{K} = (E; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a 0-minimal BF n-In id of $\mathcal{E}$.*

Proof:

(1) Let $\mathcal{K}$ be a minimal n-In id of $\mathcal{E}$. Then $\mathcal{K}$ is an n-In id of $\mathcal{E}$. Thus, by Theorem 4.7, $\lambda_\mathcal{K} = (\mathcal{E}; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is a BF n-In id of $\mathcal{E}$. Let $\mathcal{J}$ be an n-interior ideal of $\mathcal{E}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Then by Theorem 4.7, $\lambda_\mathcal{J} = (\mathcal{E}; \lambda_\mathcal{J}^P, \lambda_\mathcal{J}^N)$ is a BF n-interior ideal of $\mathcal{E}$ and $\lambda_\mathcal{J} \leq \lambda_\mathcal{K}$. Since $\mathcal{K}$ is a minimal n-interior ideal of $\mathcal{E}$ we have $\mathcal{J} = \mathcal{K}$. Thus, $\lambda_\mathcal{J} = \lambda_\mathcal{K}$. Hence, $\lambda_\mathcal{K} = (\mathcal{E}; \lambda_\mathcal{K}^P, \lambda_\mathcal{K}^N)$ is minimal BF n-In id of $\mathcal{E}$.

Conversely, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is minimal BF n-In id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF n-interior ideal of $\mathcal{E}$. Thus, by Theorem 4.7, $\mathcal{K}$ is an n-In id of $\mathcal{E}$. Let $\lambda_{\mathcal{J}} = (\mathcal{E}; \lambda^P_{\mathcal{J}}, \lambda^N_{\mathcal{J}})$ be a BF n-In id of $\mathcal{E}$ such that $\lambda_{\mathcal{J}} \leq \lambda_{\mathcal{K}}$. Then by Theorem 4.7, $\mathcal{J}$ is an n-In id of $\mathcal{E}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Since $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is minimal BF n-In id of $\mathcal{E}$ we have $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathcal{K}$. Hence, $\mathcal{K}$ is a minimal n-In id of $\mathcal{E}$.

(2) Let $\mathcal{K}$ be a maximal n-In id of $\mathcal{E}$. Then $\mathcal{K}$ is an n-In id. Thus, by Theorem 4.7, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF n-In id of $\mathcal{E}$. Let $\mathcal{J}$ be an n-In id of $\mathcal{E}$ such that $K \subseteq J$. Then by Theorem 4.7, $\lambda_{\mathcal{J}} = (\mathcal{E}; \lambda^P_{\mathcal{J}}, \lambda^N_{\mathcal{J}})$ is a BF n-In id of $\mathcal{E}$ and $\lambda_{\mathcal{K}} \leq \lambda_{\mathcal{J}}$. Since $\mathcal{K}$ is a maximal n-In id of $\mathcal{E}$ we have $\mathcal{J} = \mathcal{K}$. Thus, $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Hence, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is maximal BF n-In id of $\mathcal{E}$.
Conversely, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is maximal BF n-In id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF n-In id of $\mathcal{E}$. Thus, by Theorem 4.7, $\mathcal{K}$ is an n-In id of $\mathcal{E}$. Let $\lambda_{\mathcal{J}} = (\mathcal{E}; \lambda^P_{\mathcal{J}}, \lambda^N_{\mathcal{J}})$ be a BF n-In id of $\mathcal{E}$ such that $\lambda_{\mathcal{K}} \leq \lambda_{\mathcal{J}}$. Then by Theorem 4.7, $\mathcal{J}$ is an n-In id of $\mathcal{E}$ such that $\mathcal{K} \subseteq \mathcal{J}$. Since $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is maximal BF n-In id of $\mathcal{E}$ we have $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathcal{K}$. Hence, $\mathcal{K}$ is a maximal n-In id of $\mathcal{E}$.

(3) It follows from (1).

∎

Definition 4.13. *A W n-In id $\mathcal{K}$ of an SG $\mathcal{E}$ is called*

(1) *a* minimal *if for every W n-In idof $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{J} \subseteq \mathcal{K}$, we have $\mathcal{J} = \mathcal{K}$,*
(2) *a* maximal *if for every W n-In id of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{K} \subseteq \mathcal{J}$, we have $\mathcal{J} = \mathcal{K}$.*
(3) *a* 0-minimal *if for every W n-In id of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{J} \subseteq \mathcal{K}$, we have $\mathcal{J} = \mathcal{K}$.*

Definition 4.14. *A BF W n-In id $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ of an SG $\mathcal{E}$ is*

(1) *a* minimal *if for all BF W n-In id $\xi = (\mathcal{E}; \xi^P, \xi^N)$ of $\mathcal{E}$ such that $\xi \leq \vartheta$, then $\xi = \vartheta$,*
(2) *a* maximal *if for all BF W n-In id $\xi = (\mathcal{E}; \xi^P, \xi^N)$ of $\mathcal{E}$ such that $\vartheta \leq \xi$, then $\xi = \vartheta$.*
(3) *a* 0-minimal *if for all BF W n-In id $\xi = (\mathcal{E}, \xi^P, \xi^N)$ of $\mathcal{E}$ such that $\xi \leq \vartheta$, then $\xi = \vartheta$.*

Theorem 4.15. *A non-empty subset $\mathcal{K}$ of an SG $\mathcal{E}$. Then the following statements hold*

(1) *$\mathcal{K}$ is a minimal W n-In id if and only if $\lambda_{\mathcal{K}} = (E; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a minimal BF W n-In id of $\mathcal{E}$.*
(2) *$\mathcal{K}$ is a maximal W n-In id if and only if $\lambda_{\mathcal{K}} = (E; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a maximal BF W n-In id of $\mathcal{E}$.*

The following theorem we can prove according to the theorem 4.12.

Theorem 4.16. *A non-empty subset $\mathcal{K}$ of an SG $\mathcal{E}$ is a 0-minimal weakly n-interior ideal if and only if $\lambda_{\mathcal{K}} = (E; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a 0-minimal BF weakly n-interior ideal.*

Proof: It follows from Theorem 4.12. ∎

Next, we give the relationship between prime, semiprime n-In ids and prime, semiprime BF n-In ids.

Definition 4.17. *Let $\mathcal{K}$ be an n-In id of an SG $\mathcal{E}$ is called*

(1) prime *if $eh \in \mathcal{K}$ implies $e \in \mathcal{K}$ or $h \in \mathcal{K}$ for all $e, h \in \mathcal{E}$,*
(2) semiprime *if $e^2 \in \mathcal{K}$ implies $e \in \mathcal{K}$ for all $e \in \mathcal{E}$.*

Definition 4.18. *Let $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ be a BF n-In id of an SG $\mathcal{E}$ is called*

(1) prime *if $\vartheta^P(eh) \leq \vartheta^P(e) \vee \vartheta^P(h)$ and $\vartheta^N(eh) \geq \vartheta^N(e) \wedge \vartheta^N(h)$ for all $e, h \in \mathcal{E}$,*
(2) semiprime *if $\vartheta^P(e^2) \leq \vartheta^P(e)$ and $\vartheta^N(e^2) \geq \vartheta^N(e)$ for all $e \in \mathcal{E}$.*

Remark 4.19. *Every prime n-In id is semiprime n-In id in an SG.*

Theorem 4.20. *Let $\mathcal{K}$ be a non-empty subset of an SG $\mathcal{E}$. Then the following statements hold*

(1) *$\mathcal{K}$ is a prime n-In id of $\mathcal{E}$ if and only if $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a prime BF n-In id of $\mathcal{E}$.*
(2) *$\mathcal{K}$ is a semiprime n-In id of $\mathcal{E}$ if and only if $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a semiprime BF n-In id of $\mathcal{E}$.*

Proof:

(1) Suppose that $\mathcal{K}$ is a prime n-In id of $\mathcal{E}$. Then $\mathcal{K}$ is an n-In id of $\mathcal{E}$. Thus, by Theorem 4.7 $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF n-In id of $\mathcal{E}$. Let $e, h \in \mathcal{E}$.
Case 1: If $eh \in \mathcal{K}$, then $e \in \mathcal{K}$ or $h \in \mathcal{K}$. Thus $\lambda^P_{\mathcal{K}}(eh) = 1 = \lambda^P_{\mathcal{K}}(e)$ and $\lambda^N_{\mathcal{K}}(eh) = -1 = \lambda^N_{\mathcal{K}}(e)$ or $\lambda^P_{\mathcal{K}}(h) = -1 = \lambda^P_{\mathcal{K}}(eh)$ or $\lambda^N_{\mathcal{K}}(eh) = -1 = \lambda^N_{\mathcal{K}}(h)$. Hence, $\lambda^P_{\mathcal{K}}(eh) \leq \lambda^P_{\mathcal{K}}(e) \vee \lambda^P_{\mathcal{K}}(h)$ and $\lambda^N_{\mathcal{K}}(eh) \geq \lambda^N_{\mathcal{K}}(e) \wedge \lambda^N_{\mathcal{K}}(h)$.
Case 2: If $eh \notin \mathcal{K}$, then $\lambda^P_{\mathcal{K}}(eh) = 0$ and $\lambda^N_{\mathcal{K}}(eh) = 0$. Thus, $\lambda^P_{\mathcal{K}}(eh) \leq \lambda^P_{\mathcal{K}}(e) \vee \lambda^P_{\mathcal{K}}(h)$ and $\lambda^N_{\mathcal{K}}(eh) \geq \lambda^N_{\mathcal{K}}(e) \wedge \lambda^N_{\mathcal{K}}(h)$.
Therefore, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a prime BF n-In id of $\mathcal{E}$.
Conversely, suppose that $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a prime BF n-In id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}}$ is a BF n-In id of $\mathcal{E}$. Thus, by Theorem 4.7, $\mathcal{K}$ is an n-In id of $\mathcal{E}$. Let $e, h \in \mathcal{E}$ with $eh \in \mathcal{K}$. Then, $\lambda^P_{\mathcal{K}}(eh) = 1$ and
$\lambda^N_{\mathcal{K}}(eh) = -1$. If $e \notin \mathcal{K}$ and $h \notin \mathcal{K}$, then $\lambda^P_{\mathcal{K}}(e) = 0 = \lambda^P_{\mathcal{K}}(h)$ and $\lambda^N_{\mathcal{K}}(e) = 0 = \lambda^N_{\mathcal{K}}(h)$. By assumption, $\lambda^P_{\mathcal{K}}(eh) \leq \lambda^P_{\mathcal{K}}(e) \vee \lambda^P_{\mathcal{K}}(h)$ and $\lambda^N_{\mathcal{K}}(eh) \geq \lambda^N_{\mathcal{K}}(e) \wedge \lambda^N_{\mathcal{K}}(h)$. Thus,$\lambda^P_{\mathcal{K}}(eh) = 0$ and $\lambda^N_{\mathcal{K}}(eh) = 0$. It is a contradiction, so $e \in \mathcal{K}$ or $h \in \mathcal{K}$. Hence, $\mathcal{K}$ is a prime n-In id of $\mathcal{E}$.

(2) Suppose that $\mathcal{K}$ is a semiprime n-In id of $\mathcal{E}$. Then $\mathcal{K}$ is an n-In id of $\mathcal{E}$. Thus, by Theorem 4.7 $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a BF n-In id of $\mathcal{E}$. Let $e, h \in \mathcal{E}$.
Case 1: If $e^2 \in \mathcal{K}$, then $e \in \mathcal{K}$. Thus $\lambda^P_{\mathcal{K}}(e^2) = 1 = \lambda^P_{\mathcal{K}}(e)$ and $\lambda^N_{\mathcal{K}}(e^2) = -1 = \lambda^N_{\mathcal{K}}(e)$ Hence, $\lambda^P_{\mathcal{K}}(e^2) \leq \lambda^P_{\mathcal{K}}(e)$ and $\lambda^N_{\mathcal{K}}(e^2) \geq \lambda^N_{\mathcal{K}}(e)$.
Case 2: If $e^2 \notin \mathcal{K}$, then $\lambda^P_{\mathcal{K}}(e^2) = 0$ and $\lambda^N_{\mathcal{K}}(eh) = 0$. Thus, $\lambda^P_{\mathcal{K}}(e^2) \leq \lambda^P_{\mathcal{K}}(e)$ and $\lambda^N_{\mathcal{K}}(e^2) \geq \lambda^N_{\mathcal{K}}(e)$.
Therefore, $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a prime BF n-In id of $\mathcal{E}$.
Conversely, suppose that $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a prime BF n-In id of $\mathcal{E}$. Then $\lambda_{\mathcal{K}}$ is a BF n- in id of $\mathcal{E}$. Thus, by Theorem 4.7, $\mathcal{K}$ is an n-In id of $\mathcal{E}$. Let $e \in \mathcal{E}$ with $e^2 \in \mathcal{K}$. Then, $\lambda^P_{\mathcal{K}}(e^2) = 1$ and
$\lambda^N_{\mathcal{K}}(e^2) = -1$. If $e \notin \mathcal{K}$, then $\lambda^P_{\mathcal{K}}(e) = 0$ and $\lambda^N_{\mathcal{K}}(e) = 0$. By assumption, $\lambda^P_{\mathcal{K}}(e^2) \leq \lambda^P_{\mathcal{K}}(e)$ and $\lambda^N_{\mathcal{K}}(e^2) \geq \lambda^N_{\mathcal{K}}(e)$. Thus,$\lambda^P_{\mathcal{K}}(e^2) = 0$ and $\lambda^N_{\mathcal{K}}(e^2) = 0$. It is a contradiction, so $e \in \mathcal{K}$. Hence, $\mathcal{K}$ is a prime n-In id of $\mathcal{E}$.

■

Definition 4.21. *Let $\mathcal{K}$ be a W n-In id of an SG $\mathcal{E}$ is called*

(1) prime *if $eh \in \mathcal{K}$ implies $e \in \mathcal{K}$ or $h \in \mathcal{K}$ for all $e, h \in \mathcal{E}$,*

(2) semiprime *if $e^2 \in \mathcal{K}$ implies $e \in \mathcal{K}$ for all $e \in \mathcal{E}$.*

Definition 4.22. *Let $\vartheta = (\mathcal{E}; \vartheta^P, \vartheta^N)$ be a BF W n-In id of an SG $\mathcal{E}$ is called*

(1) prime *if $\vartheta^P(eh) \leq \vartheta^P(e) \vee \vartheta^P(h)$ and $\vartheta^N(eh) \geq \vartheta^N(e) \wedge \vartheta^N(h)$ for all $e, h \in \mathcal{E}$,*

(2) semiprime *if $\vartheta^P(e^2) \leq \vartheta^P(e)$ and $\vartheta^N(e^2) \geq \vartheta^N(e)$ for all $e \in \mathcal{E}$.*

Theorem 4.23. *Let $\mathcal{K}$ be a non-empty subset of an SG $\mathcal{E}$. Then the following statements hold*

(1) *$\mathcal{K}$ is a prime W n-In id of $\mathcal{E}$ if and only if $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a prime BF W n-In id of $\mathcal{E}$.*

(2) *$\mathcal{K}$ is a semiprime W n-In id of $\mathcal{E}$ if and only if $\lambda_{\mathcal{K}} = (\mathcal{E}; \lambda^P_{\mathcal{K}}, \lambda^N_{\mathcal{K}})$ is a semiprime BF W n-In id of $\mathcal{E}$.*

Proof: It follows from Theorem 4.20. ■

V. Conclusion

In this paper, we introduce the concept of bipolar fuzzy (m, n)-ideals in semigroups and investigate their properties. Additionally, we establish the relationship between (m, n)-ideals and bipolar fuzzy (m, n)-ideals. Furthermore, we define bipolar fuzzy n-interior ideals in semigroup and prove the relationship between n-interior ideals and bipolar fuzzy n-interior ideals. Also, we prove weakly n-interior ideals and bipolar fuzzy weakly n-interior ideals. In the future, we plan to explore hybrid almost (m, n)-ideals and n-interior ideals in semigroups or within the algebraic context.

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