Ordering Fuzzy Numbers by a Novel Defuzzification Technique Using Volumes-An Application to Type 2 Diabetes Risk Assessment

P.N.V.L. Sasikala, Peddi Phani Bushan Rao*, Lazim Abdullah, Akiri Sridhar

Abstract—**Fuzzy numbers (FNs) are essential for addressing uncertainty and ambiguity in decision-making problems. Ranking FNs facilitates evaluating and comparing alternatives, enabling informed decision-making under uncertainty. This paper presents a novel approach for ranking Generalized Trapezoidal Fuzzy Numbers (GTrFNs) with different left-right heights. The proposed method utilizes a defuzzification technique that calculates the volume of the solid generated by revolving the GTrFN's membership function (MF) around an axis. A scoring function is developed to determine the defuzzified value of GTrFNs with different left-right heights, considering both positive and negative side volumes relative to a benchmark FN and the GTrFN's centroid. This scoring function serves as a criterion for evaluating and comparing alternatives. The proposed approach overcomes the limitations of existing ranking methods for GTrFNs with different left-right heights. Furthermore, the method's applicability is demonstrated through a fuzzy risk analysis case study, specifically addressing the likelihood of developing Type 2 diabetes in individuals with diverse risk profiles.**

Index Terms— **Centroid, Fuzzy Risk Analysis, Generalized Fuzzy Numbers, Score Function, Type 2 Diabetes, Volume of Solid.**

I. INTRODUCTION

Fuzzy set theory was introduced by [1] to address the challenges of handling uncertain, vague, or imprecise data. It provides a powerful framework for modeling and analyzing uncertainty and vagueness in real-world problems. The applications of fuzzy set theory are diverse, ranging from image processing to decision support systems and optimization problems. The ranking of FNs using defuzzification techniques allows for comparing and selecting alternatives by capturing the information contained in a fuzzy set in terms of a crisp number. [2] proposed ranking FNs using the X-coordinate of the centroid point, [3] suggested ranking GTrFNs with different left-right heights for fuzzy risk analysis, [4] suggested an improved method based on the area between the centroid and the FNs' original point, [5] suggested classifying FNs by spreads and heights, and [6] proposed a novel ranking system based on FNs' left and right sides. [7] proposed a ranking methodology for FNs that was applied to risk analysis concerns based on varying left-right height scores, whereas [8] identified the limitations of [3] and introduced an alternative ranking procedure for FNs with different left-right heights. A novel ranking method on centroid points, fuzziness levels, and spreads of FNs was introduced by [9], and [10] provided a counter-example to identify the limitations of the technique in [7] and proposed an alternative ranking system based on scores employing standard deviation. [11] presented ranking p-norm GTrFNs with different left-right heights, which computes the total integral value based on left and right integral and middle integral values. Further, based on the idea of parametric form FNs with defuzzifiers at different heights, [12] developed a novel way of ranking FNs utilizing value and ambiguity at distinct decision levels. Later, [13] proposed a unique technique for ranking generalized fuzzy numbers (GFNs) using an ordered weighted averaging operator that considers the relative relevance of three scoring factors: defuzzified value, height, and spread. Additionally, [14] addressed a risk analysis issue in poultry farming by proposing a novel similarity measure based on geometric distance, height, and radius of gyration point of FNs with varying left and right heights. A novel method based on the ideas of centroid point, rank index value, and height of FNs was presented by [15]. [16] suggested a new ranking system based on the center of gravity of FNs of varied heights, [17] propose a novel ranking concept based on a geometric approach for solving decision-making issues by measuring the distance between the centroids and midpoints of GTrFN diagonals. [18] introduced a unique approach for ranking GFNs based on the normalized height coefficient as well as benefit and cost areas, and a comprehensive ranking strategy for GTrFNs was presented by [19], utilizing five scoring functions to determine a total ordering among them. [20] used distance and area index to rank virtual company partners. [21] developed a fuzzy ranking score variance model to evaluate portfolio attributes. [22] developed a score function to determine the best and worst criteria in decision-making. [23] devised a fuzzy model with vector quantization. [24], [25] used GTrFNs with different left-right heights to assess the production system's efficiency and profit analysis of skimmed milk powder for a milk factory.

Every fuzzy ranking method has its own advantages and disadvantages. However, the prime concern is that the ranking methods should fairly rank FNs and their images, crisp numbers, and symmetric FNs. Many of the existing fuzzy ranking methods failed to do so. Therefore, this study is developed to address the above issues. This study uses defuzzification to introduce a novel ranking method for GTrFNs with different left-right heights. To find the defuzzified value of GTrFNs with different left-right heights a score function is defined using volumes, calculated from a benchmark FN and centroid. The volumes are an amalgam of positive and negative side volumes of GTrFN with different

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left-right heights obtained by revolving the MF of the GTrFN with different left-right heights about the x-axis. Subsequently, the FNs score is defined by combining the volumes of the left and right positive, left and right negative sides of the GTrFN with different left-right heights along with its centroid. This score function evaluates the defuzzified value of GTrFN, a valued used to rank FNs. The comparative study presented in the paper, notifies that the suggested approach overcomes the limitations of existing techniques on ranking GTrFNs with different left-right heights. Furthermore, the proposed method is utilized to address a fuzzy risk analysis problem focused on determining the likelihood of developing Type 2 diabetes in individuals with diverse risk profiles.

The structure of the paper is as follows: Section II presents the preliminaries required for the study, Section III presents the new defuzzification technique for ranking GTrFNs with different left-right heights, Section IV discusses the properties of the score function, Section V addresses the reasonable properties for ranking FNs, while Section VI presents numerical examples to illustrate the proposed research. A comparative study is presented in Section VII to compare the suggested technique with some existing methods in the literature, Section VIII introduces a new risk analysis method, to identify the risk of getting affected by Type 2 diabetes in persons with different risk-prone parameters, the conclusions are presented in Section IX and finally, the limitations and future scope of this study are presented in Section X.

II. PRELIMINARIES

The definitions required for this study are derived from the work presented in [26].

Definition 1 If H is a universe of discourse and h be any element of H. The fuzzy set $\bar{X}$ defined on H is a collection of ordered pairs,

$$\bar{X} = \{(h, \mu_{\bar{X}}(h)) | h \in H\}$$

where $\mu_{\bar{X}}: H \to [0,1]$ is the membership function of H in $\bar{X}$.

Definition 2 A FN $\bar{G}$ shown in Fig 1. is a fuzzy subset of real line R with MF $f_{\bar{G}}$ satisfying the below properties:

1. $f_{\bar{G}}$ is continuous from R to $[0, w]$,
2. $f_{\bar{G}}$ is strictly increasing on $[p_1, p_2]$,
3. $f_{\bar{G}}(x) = w$, for all $x \in [p_2, p_3]$,
4. $f_{\bar{G}}$ is strictly decreasing on $[p_3, p_4]$,
5. $f_{\bar{G}}(x) = 0$, otherwise

The MF of $f_{\bar{G}}$ can be expressed as:

$$f_{\bar{G}} = \begin{cases} f_{\bar{G}}^L(x); & p_1 \le x \le p_2, \\ w; & p_2 \le x \le p_3, \\ f_{\bar{G}}^R(x); & p_3 \le x \le p_4, \\ 0; & otherwise. \end{cases} \quad (1)$$

where $f_{\bar{G}}^L : [p_1, p_2] \to [0, w]$, and $f_{\bar{G}}^R : [p_3, p_4] \to [0, w]$.

Definition 3 A GTrFN $\bar{G} = (p_1, p_2, p_3, p_4; w)$, shown in Fig 2., is a fuzzy subset of the real line R with MF defined as

$$f_{\bar{G}}(x) = \begin{cases} w\left(\frac{x-p_1}{p_2-p_1}\right), if\ p_1 \le x \le p_2, \\ w\ \ if\ p_2 \le x \le p_3, \\ w\left(\frac{p_4-x}{p_4-p_3}\right), if\ p_3 \le x \le p_4, \\ 0,\ \ otherwise. \end{cases} \quad (2)$$

Here p_1, p_2, p_3, p_4 are real numbers, and $0 \le w \le 1$.

If $w = 1$, then $\bar{G}$ is called a normal trapezoidal fuzzy number (TrFN), and if $p_2 = p_3$, then $\bar{G} = (p_1, p_2, p_3; w)$ is a generalized triangular fuzzy number (GTFN).

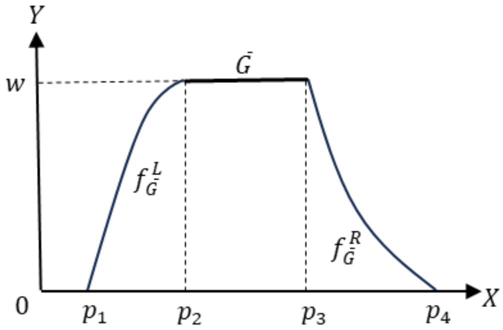


Fig.1: Fuzzy number

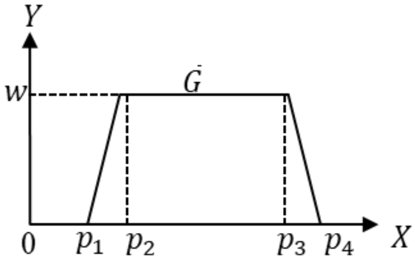


Fig.2: Generalized trapezoidal fuzzy number $\bar{G}$

Definition 4 A GTrFN $\bar{G} = (p_1, p_2, p_3, p_4; w_1, w_2)$, with different left-right heights shown in Fig 3., has membership function defined as:

$$\mu_{\bar{G}}(x) = \begin{cases} s_1 = \frac{w_1(x-p_1)}{p_2-p_1},\ if\ p_1 \le x \le p_2, \\ s_2 = \frac{w_1(p_3-p_2)+(w_2-w_1)(x-p_2)}{p_3-p_2}, if\ p_2 \le x \le p_3, \\ s_3 = \frac{w_2(p_4-x)}{p_4-p_3},\ if\ p_3 \le x \le p_4, \\ 0,\quad otherwise. \end{cases} \quad (3)$$

here w_1 is the left height, and w_2 is the right height of $\bar{G}$, $w_1 \in [0,1]$ & $w_2 \in [0,1]$, $s_1: [p_1, p_2] \to [0, 1]$, $s_2: [p_2, p_3] \to [0, 1]$, $s_3: [p_3, p_4] \to [0, 1]$.

If $w_1 = w_2$, then $\bar{G} = (p_1, p_2, p_3, p_4)$ is a trapezoidal fuzzy number.

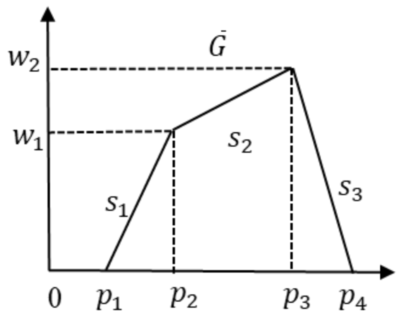


Fig.3: Generalized trapezoidal fuzzy number $\bar{G}$ with different left-right heights

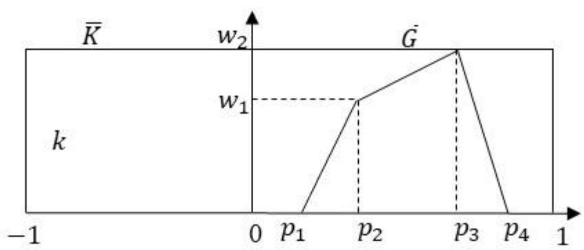


Fig.4: Fuzzy number $\bar{K}$ and generalized trapezoidal fuzzy number with different left-right heights

Fig 4. shows a FN $\bar{K}$ with MF $\mu_{\bar{K}}$ along with GTrFN with different left-right heights defined as:

$$\mu_{\bar{K}}(x) = \begin{cases} k(x) = \max(w_1, w_2)\,; & -1 \le x \le 1, \\ 0\ ;\ otherwise. \end{cases} \quad (4)$$

Definition 5 [27]: The volume of the solid generated by revolving an area bounded by the curve $y = f(x)$, the x-axis and the ordinates, $x = a, x = b$, about the x-axis is:

$$V = \pi \int_a^b y^2\, dx \quad (5)$$

Definition 6: The volume of a solid obtained by revolving the MF s_1 of the GTrFN $\bar{G}$ with different left-right heights given by (3), about x-axis is:

$$v_1 = \frac{\pi}{3} w_1^2 (p_2 - p_1) \quad (6)$$

Definition 7: The volume of a solid obtained by revolving the MF s_2 of the GTrFN $\bar{G}$ with different left-right heights given by (3), about x-axis is:

$$v_2 = \frac{\pi}{3}(p_3 - p_2)(w_1^2 + w_1 w_2 + w_2^2) \quad (7)$$

Definition 8: The volume of a solid obtained by revolving the MF s_3 of the GTrFN $\bar{G}$ with different left-right heights given by (3), about x-axis is:

$$v_3 = \frac{\pi}{3} w_2^2 (p_4 - p_3) \quad (8)$$

Definition 9: Arithmetic Operations

Arithmetic Operations between two GTrFNs with different left-right heights.

Let $\bar{I} = \{i_1, i_2, i_3, i_4; h_{i1}, h_{i2}\}$, & $\bar{J} = \{j_1, j_2, j_3, j_4; h_{j1}, h_{j2}\}$ be two GTrFNs with different left-right heights where $0 \le h_{i1}, h_{i2} \le 1$ and $0 \le h_{j1}, h_{j2} \le 1$, then addition, subtraction, multiplication, and division are defined as follows:

Addition of GTrFNs $\oplus$:

$$\bar{I} \oplus \bar{J} = (i_1, i_2, i_3, i_4; h_{i1}, h_{i2}) \oplus (j_1, j_2, j_3, j_4; h_{j1}, h_{j2}) = \begin{pmatrix} i_1 + j_1, i_2 + j_2, i_3 + j_3, i_4 + j_4; \\ \min(h_{i1}, h_{j1}), \min(h_{i2}, h_{j2}) \end{pmatrix} \quad (9)$$

Subtraction of GTrFNs $\ominus$:

$$\bar{I} \ominus \bar{J} = (i_1, i_2, i_3, i_4; h_{i1}, h_{i2}) \ominus (j_1, j_2, j_3, j_4; h_{j1}, h_{j2}) = \begin{pmatrix} i_1 - j_4, i_2 - j_3, i_3 - j_2, i_4 - j_1; \\ \min(h_{i1}, h_{j1}), \min(h_{i2}, h_{j2}) \end{pmatrix} \quad (10)$$

Multiplication of GTrFNs $\otimes$:

$$\bar{I} \otimes \bar{J} = (i_1, i_2, i_3, i_4; h_{i1}, h_{i2}) \otimes (j_1, j_2, j_3, j_4; h_{j1}, h_{j2}) = \begin{pmatrix} i_1 \times j_1, i_2 \times j_2, i_3 \times j_3, i_4 \times j_4; \\ \min(h_{i1}, h_{j1}), \min(h_{i2}, h_{j2}) \end{pmatrix} \quad (11)$$

Division of GTrFNs $\oslash$:

$$\bar{I} \oslash \bar{J} = (i_1, i_2, i_3, i_4; h_{i1}, h_{i2}) \oslash (j_1, j_2, j_3, j_4; h_{j1}, h_{j2}) = \left(\frac{i_1}{j_4}, \frac{i_2}{j_3}, \frac{i_3}{j_2}, \frac{i_4}{j_1}; \min(h_{i1}, h_{j1}), \min(h_{i2}, h_{j2})\right) \quad (12)$$

III. Proposed Method

This section outlines a new method for ranking GTrFNs with different left-right heights. The approach calculates the volumes of the positive and negative regions of each GTrFN from a benchmark FN, by rotating its left and right membership functions around the x-axis. A scoring function combining these volumes and the centroid of the GTrFN is defined to find the defuzzified value of each GTrFN. This defuzzified value is a crisp value and is the basis for ranking the GTrFNs.

Consider a GTrFN, $\bar{G}_i = (p_1, p_2, p_3, p_4; w_1, w_2)$, $-\infty < p_1 \le p_2 \le p_3 \le p_4 < \infty; w_1 \in [0,1], w_2 \in [0,1]$, with different left-right heights, w_1 indicates the left height and w_2 indicates the right height. The suggested method is presented through the following steps.

Step 1: Calculation of left negative and right positive volumes:

(i) Left Negative Volume $LNV(\bar{G}_i)$: This is defined as the volume from $(-1, -1, -1, -1; \max(w_1, w_2), \max(w_1, w_2))$ benchmark FN to the MF curves s_1 *or* s_2 of the GTrFN with different left-right heights. This is shown in Fig. 5(a), 6(a), and 7(a).

Case 1: If $w_1 > w_2$ or $w_1 = w_2$, then

$$LNV(\bar{G}_i) = \int_{-1}^{p_2} [k(x)]^2 dx - \int_{p_1}^{p_2} [s_1(x)]^2 dx \quad (13)$$

$$= \int_{-1}^{p_2} \pi (w_1)^2 dx - \int_{p_1}^{p_2} \pi \left(\frac{w_1(x - p_1)}{p_2 - p_1}\right)^2 dx$$

$$= \frac{\pi}{3} w_1^2 (2p_2 + p_1 + 3) \quad (14)$$

Case 2: If $w_1 < w_2$, then

$$LNV(\bar{G}_i) = \int_{-1}^{p_3} [k(x)]^2 dx - \int_{p_1}^{p_2} [s_1(x)]^2 dx + \int_{p_2}^{p_3} [s_2(x)]^2 dx \quad (15)$$

$$= \int_{-1}^{p_3} \pi (w_2)^2 dx - \left[\int_{p_1}^{p_2} \pi \left(\frac{w_1(x - p_1)}{p_2 - p_1}\right)^2 dx\right] + \int_{p_2}^{p_3} \pi \left(\frac{w_1(p_3 - p_2) + (w_2 - w_1)(x - p_2)}{p_3 - p_2}\right)^2 dx$$

$$= \frac{\pi}{3}[w_2^2(2p_3 + p_2 + 3) - w_1^2(p_3 - p_1) - w_1 w_2 (p_3 - p_2)] \quad (16)$$

(ii) Right Negative Volume $RNV(\bar{G}_i)$: This is defined as the volume from $(-1, -1, -1, -1; \max(w_1, w_2), \max(w_1, w_2))$ benchmark FN to the MF curve s_3, of the GTrFN with different left-right heights. This is shown in Fig. 5(b), 6(b), and 7(b).

Case 1: If $w_1 < w_2$ or $w_1 = w_2$ then

$$RNV(\bar{G}_i) = \int_{-1}^{p_3} [k(x)]^2 dx + \int_{p_3}^{p_4} [s_3(x)]^2 dx \quad (17)$$

$$= \int_{-1}^{p_3} \pi (w_2)^2 dx + \int_{p_3}^{p_4} \pi \left(\frac{w_2(p_4 - x)}{p_4 - p_3}\right)^2 dx$$

$$= \frac{\pi}{3} w_2^2 (2p_3 + p_4 + 3) \quad (18)$$

Case 2: If $w_1 > w_2$, then

$$RNV(\bar{G}_i) = \int_{-1}^{p_2} [k(x)]^2 dx + \int_{p_2}^{p_3} [s_2(x)]^2 dx + \int_{p_3}^{p_4} [s_3(x)]^2 dx \quad (19)$$

$$= \int_{-1}^{p_2} \pi (w_1)^2 dx + \int_{p_2}^{p_3} \pi \left(\frac{w_1(p_3 - p_2) + (w_2 - w_1)(x - p_2)}{p_3 - p_2}\right)^2 dx + \int_{p_3}^{p_4} \pi \left(\frac{w_2(p_4 - x)}{p_4 - p_3}\right)^2$$

$$= \frac{\pi}{3}[w_1^2(2p_2 + 3 + p_3) + w_2^2(p_4 - p_2) + w_1 w_2 (p_3 - p_2)] \quad (20)$$

(iii) Left Positive Volume $LPV(\bar{G}_i)$: This is defined as the volume from the MF curve of s_1 of the GTrFN with different left-right heights to $(1,1,1,1; \max(w_1, w_2), \max(w_1, w_2))$ benchmark FN. This is shown in Fig. 5(c), 6(c), and 7(c).

Case 1: If $w_1 > w_2$ or $w_1 = w_2$, then

$$LPV(\bar{G}_i) = \int_{p_2}^{1} [k(x)]^2 dx + \int_{p_1}^{p_2} [s_1(x)]^2 dx \quad (21)$$

$$= \int_{p_2}^{1} \pi (w_1)^2 dx + \int_{p_1}^{p_2} \pi \left(\frac{w_1(x - p_1)}{p_2 - p_1}\right)^2 dx$$

$$= \frac{\pi}{3} w_1^2 (3 - 2p_2 - p_1) \quad (22)$$

Case 2: If $w_1 < w_2$, then

$$LPV(\bar{G}_i) = \int_{p_3}^{1} [g(x)]^2 dx + \int_{p_1}^{p_2} [s_1(x)]^2 dx + \int_{p_2}^{p_3} [s_2(x)]^2 dx \quad (23)$$

$$= \int_{p_3}^{1} \pi (w_2)^2 dx + \int_{p_1}^{p_2} \pi \left(\frac{w_1(x - p_1)}{p_2 - p_1}\right)^2 dx + \int_{p_2}^{p_3} \pi \left(\frac{w_1(p_3 - p_2) + (w_2 - w_1)(x - p_2)}{p_3 - p_2}\right)^2 dx$$

$$= \frac{\pi}{3}[w_1^2(p_3 - p_1) + w_2^2(3 - 2p_3 - p_2) + w_1 w_2 (p_3 - p_2)] \quad (24)$$

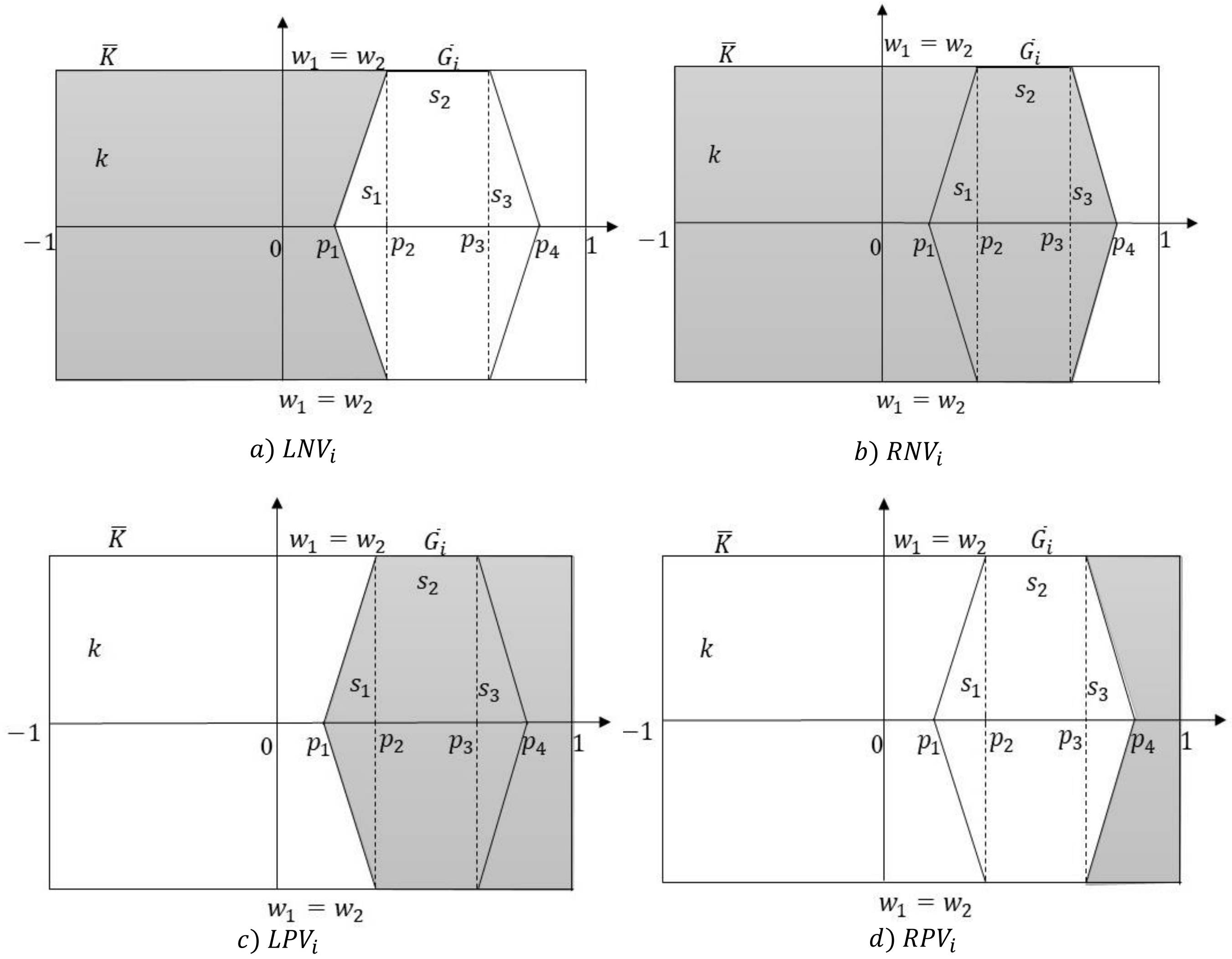


Fig. 5: The volumes of positive and negative sides of generalized trapezoidal fuzzy number $\bar{G}_i$, from benchmark fuzzy number $\bar{K}$ for $w_1 = w_2$

(iv) Right Positive Volume $RPV(\bar{G}_i)$: This is defined as the volume from the MF curves of s_2 or s_3 of the GTrFN with different left-right heights to the benchmark FN $(1,1,1,1; \max(w_1, w_2), \max(w_1, w_2))$. This is shown in Fig. 5(d), 6(d), and7(d).

Case 1: If $w_1 < w_2$ or $w_2 = w_2$ then

$$RPV(\bar{G}_i) = \int_{p_3}^{1} k(x)dx - \int_{p_3}^{p_4} s_3(x)dx \tag{25}$$

$$= \int_{p_3}^{1} \pi(w_2)^2 dx - \int_{p_3}^{p_4} \pi\left(\frac{w_2(p_4-x)}{p_4-p_3}\right)^2 dx$$

$$= \frac{\pi}{3} w_2^2(3 - 2p_3 - p_4) \tag{26}$$

Case 2: If $w_1 > w_2$, then

$$RPV(\bar{G}_i) = \int_{p_2}^{1} k(x)dx - \left(\int_{p_2}^{p_3} s_2(x)dx + \int_{p_3}^{p_4} s_3(x)dx\right) \tag{27}$$

$$= \int_{p_2}^{1} \pi(w_1)^2 dx - \int_{p_2}^{p_3} \pi\left(\frac{w_1(p_3-p_2)+(w_2-w_1)(x-p_2)}{p_3-p_2}\right)^2 dx$$

$$- \int_{p_3}^{p_4} \pi\left(\frac{w_2(p_4-x)}{p_4-p_3}\right)^2 dx$$

$$= \frac{\pi}{3}[w_1^2(3 - 2p_2 - p_3) - w_2^2(p_4 - p_2) - w_1 w_2(p_3 - p_2)] \tag{28}$$

Step 2: Find the sum of negative volumes $M(\bar{G}_i)$ and the sum of positive volumes $N(\bar{G}_i)$ of the GTrFN $\bar{G}_i$ with different left-right heights.

$$M(\bar{G}_i) = LNV(\bar{G}_i) + RNV(\bar{G}_i) \tag{29}$$

$$N(\bar{G}_i) = LPV(\bar{G}_i) + RPV(\bar{G}_i) \tag{30}$$

Step 3: Calculate the centroid $C(\bar{G}_i)$ of each GTrFN $\bar{G}_i$ as given below:

For $w_1 = w_2$

$$C(\bar{G}_i) = \frac{1}{3}\left[p_1 + p_2 + p_3 + p_4 - \frac{p_4 p_3 - p_1 p_2}{(p_4+p_3)-(p_1+p_2)}\right] \tag{31}$$

For $w_1 < w_2$ or $w_1 > w_2$, use the following:

$$C(\bar{G}_i) = \frac{\frac{1}{3}[w_1(-p_1^2+p_3^2-p_1p_2+p_2p_3)+w_2(-p_2^2+p_4^2-p_2p_3+p_3p_4)]}{w_1(-p_1+p_3)+w_2(p_4-p_2)} \tag{32}$$

For a crisp FN $\bar{G}_i = (p_i, p_i, p_i, p_i; 1)$, the centroid is defined as $C(\bar{G}_i) = \frac{\sum_i \mu_i p_i}{\sum_i \mu_i}$ (33)

Step 4: The ranking $score(\bar{G}_i)$ of the GTrFN $\bar{G}_i$ with different left-right heights is defined as follows:

$$score(\bar{G}_i) = \frac{M_i - N_i}{M_i + N_i + (1 - |C(\bar{G}_i)|)} \tag{34}$$

Therefore, if $\bar{G}_i = (p_1, p_2, p_3, p_4; w_1, w_2)$ is a GTrFN with different left-right heights, then:

For $w_1 < w_2$,

$$score(\bar{G}_i) = \frac{\frac{2\pi}{3}[w_1^2(p_1-p_3)+w_2^2(p_2+4p_3+p_4)+w_1w_2(p_2-p_3)]}{4\pi w_2^2+1-\left|\frac{\frac{1}{3}[w_1(-p_1^2+p_3^2-p_1p_2+p_2p_3)+w_2(-p_2^2+p_4^2-p_2p_3+p_3p_4)]}{w_1(-p_1+p_3)+w_2(p_4-p_2)}\right|} \tag{35}$$

For $w_1 > w_2$,

$$score(\bar{G}_i) = \frac{\frac{2\pi}{3}[w_1^2(p_1+4p_2+p_3)+w_2^2(p_4-p_2)+w_1w_2(p_3-p_2)]}{4\pi w_1^2+1-\left|\frac{\frac{1}{3}[w_1(-p_1^2+p_3^2-p_1p_2+p_2p_3)+w_2(-p_2^2+p_4^2-p_2p_3+p_3p_4)]}{w_1(-p_1+p_3)+w_2(p_4-p_2)}\right|} \tag{36}$$

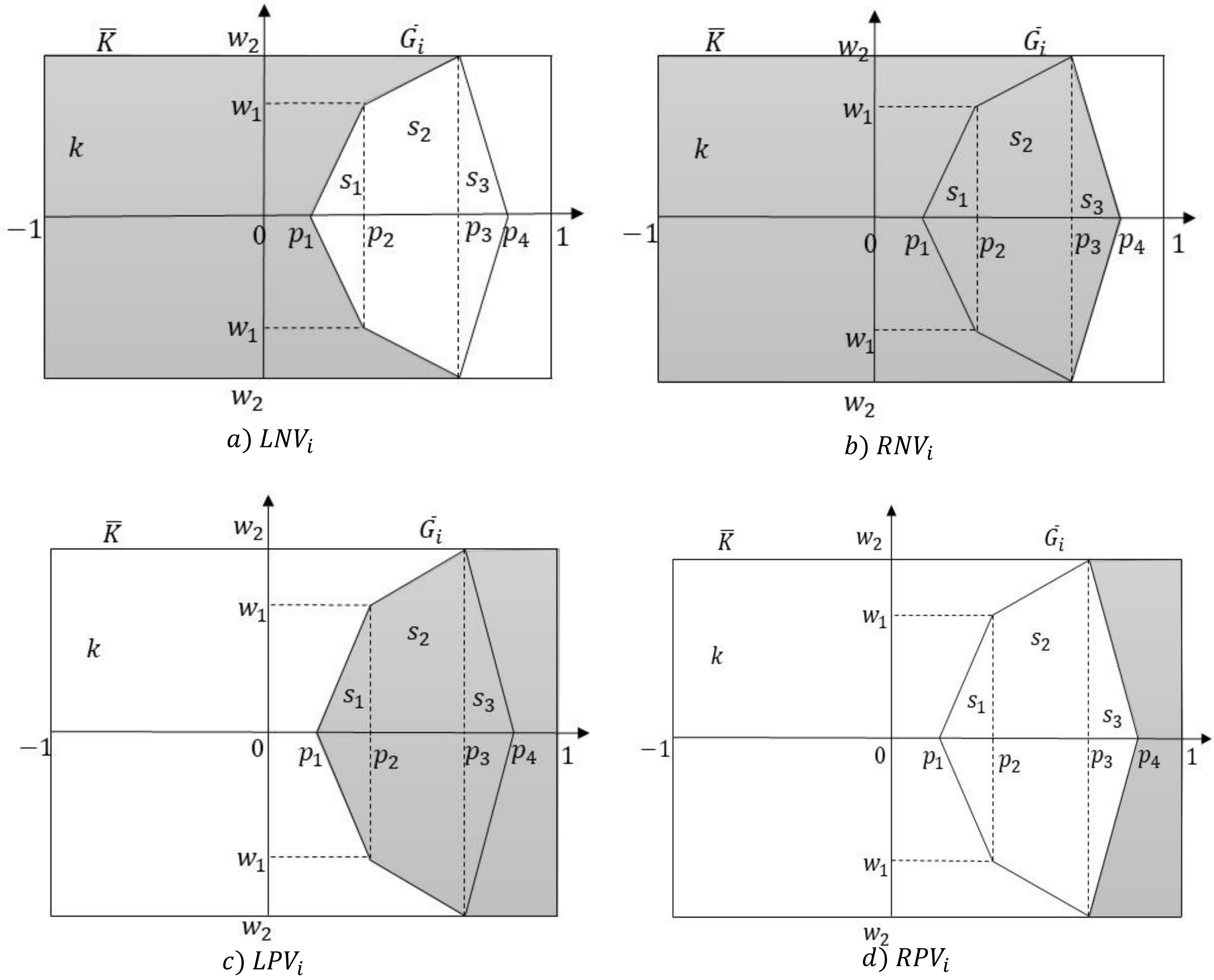


Fig. 6: The volumes of positive and negative sides of generalized trapezoidal fuzzy number $\bar{G}_i$ from benchmark fuzzy number $\bar{K}$ for $w_1 < w_2$

For $w_1 = w_2 = w$, then

$$score(\bar{G}_i) = \frac{\frac{2\pi}{3}w^2(p_1+2p_2+2p_3+p_4)}{4\pi w^2+1-\left|\frac{1}{3}\left[p_1+p_2+p_3+p_4-\frac{p_4p_3-p_1p_2}{(p_4+p_3)-(p_1+p_2)}\right]\right|} \quad (37)$$

Ranking procedure:

If $\bar{G}_1$ and $\bar{G}_2$ are two GTrFNs with different left-right heights; the following ranking order is defined by calculating the respective scores by Equations (35), (36), and (37).

i) $\bar{G}_1$ is less preferred to $\bar{G}_2$, expressed as $\bar{G}_1 \prec \bar{G}_2$, if $score(\bar{G}_1) < score(\bar{G}_2)$

ii) $\bar{G}_1$ is more preferred to $\bar{G}_2$, expressed as $\bar{G}_1 \succ \bar{G}_2$, if $score(\bar{G}_1) > score(\bar{G}_2)$

iii) $\bar{G}_1$ is equal to $\bar{G}_2$, expressed as $\bar{G}_1 \approx \bar{G}_2$, if $score(\bar{G}_1) = score(\bar{G}_2)$.

IV. Properties Of Score Function

This section analyzes the theoretical properties of the proposed fuzzy ranking approach.

Property 1

If $\bar{G}_1 = (p_1, p_2, p_3, p_4; w)$ is a GTrFN and $p_1 + p_2 + p_3 + p_4 = 0$ and $-1 \le p_1 \le p_2 \le p_3 \le p_4 \le 1$, then $score(\bar{G}_1) = 0$

Proof: To satisfy the equation $p_1 + p_2 + p_3 + p_4 = 0$, put $p_3 = -p_2$ & $p_4 = -p_1$ in (37), we get

$$score(\bar{G}_1) = \frac{0}{4\pi h^2 + 1} = 0$$

Property 2

If $\bar{G}_1 = (1,1,1,1; w_1, w_2)$, is a GTrFN with different left-right heights, then $score(\bar{G}_1) = 1$.

Proof: Given $\bar{G}_1 = (1,1,1,1; w_1, w_2)$ and from (33), we have $C(\bar{G}_1) = 1$.

Substituting $C(\bar{G}_1) = 1$ in (35), (36), and (37) for $w_1 < w_2$, $w_1 > w_2$, and $w_1 = w_2$ cases, we get $score(\bar{G}_1) = 1$.

Property 3

If $\bar{G}_1 = (0,0,0,0; w_1, w_2)$, is a GTrFN with different left-right heights, then $score(\bar{G}_1) = 0$.

Proof: Given $\bar{G}_1 = (0,0,0,0; w_1, w_2)$ and from (33), we have $C(\bar{G}_1) = 0$.

Substituting $C(\bar{G}_1) = 0$ in (35), (36), and (37) for $w_1 < w_2$, $w_1 > w_2$, and $w_1 = w_2$ cases, we get $score(\bar{G}_1) = 0$.

Property 4

If $\bar{G}_1 = (-1,-1,-1,-1; w_1, w_2)$, is a GTrFN with different left-right heights, then $score(\bar{G}_1) = -1$.

Proof: Given $\bar{G}_1 = (-1,-1,-1,-1; w_1, w_2)$ and from (33) we have $C(\bar{G}_1) = -1$.

Substituting $C(\bar{G}_1) = -1$ in (35), (36), and (37) for $w_1 < w_2$, $w_1 > w_2$, and $w_1 = w_2$ cases, we get $score(\bar{G}_1) = -1$.

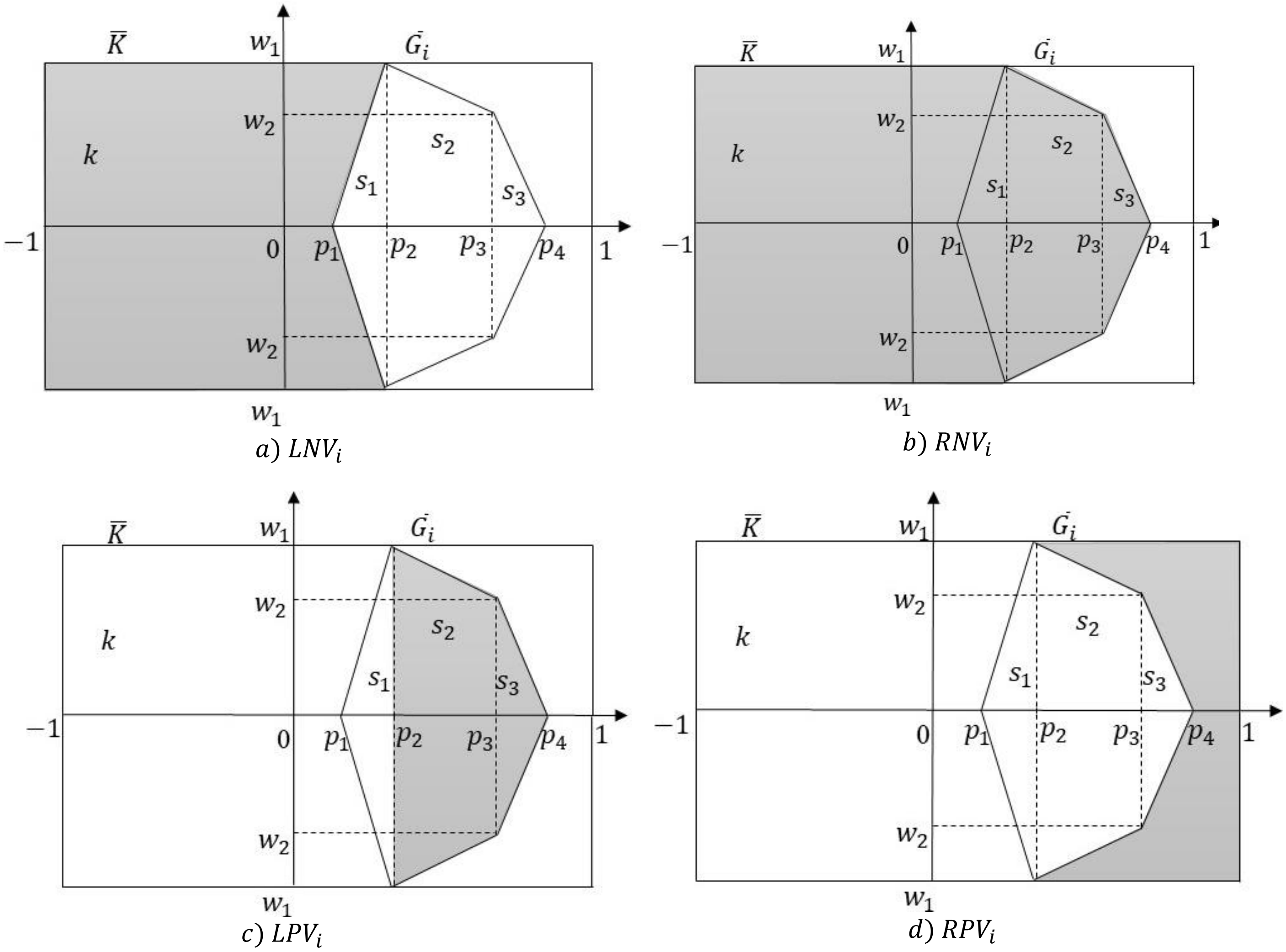


Fig. 7: The volumes of positive and negative sides of generalized trapezoidal fuzzy number $\bar{G}_i$, from benchmark fuzzy number for $w_1 > w_2$

Property 5

Let $\bar{G}_1 = (c, c, c, c; 1,1)$ *and* $\bar{G}_2 = (d, d, d, d; 1,1)$ are two GTrFNs, and if $-1 \le c, d \le 1$ & $c + d = 1$, then $score(\bar{G}_1) + score(\bar{G}_2)$=1.

Proof: Given $\bar{G}_1 = (c, c, c, c; 1,1)$ *and* $\bar{G}_2 = (d, d, d, d; 1,1)$ and $-1 \le c, d \le 1$ & $c + d = 1$,

From (33) we have $C(\bar{G}_1) = 1$, and $C(\bar{G}_2) = 1$

Substituting $C(\bar{G}_1) = 1$ and $C(\bar{G}_2) = 1$ in (35), (36), and (37) for $w_1 < w_2$, $w_1 > w_2$, and $w_1 = w_2$ cases, we get $score(\bar{G}_1) = c$, and $score(\bar{G}_2) = d$.

Therefore, $score(\bar{G}_1) + score(\bar{G}_2) = c + d = 1$.

Property 6

If $\bar{G}_1 = (p_1, p_2, p_3, p_4; w)$ is a GTrFN and $-\bar{G}_1 = (-p_4, -p_3, -p_2, -p_1; w)$, is the image of $\bar{G}_1$ where $-1 \le p_1 \le p_2 \le p_3 \le p_4 \le 1$, then $score(-\bar{G}_1) = -score(\bar{G}_1)$.

Proof: Given $\bar{G}_1 = (p_1, p_2, p_3, p_4, w)$, and from (37)

$$score(\bar{G}_1) = \frac{\frac{2\pi}{3}w^2(p_1+2p_2+2p_3+p_4)}{4\pi w^2+1-\left|\frac{1}{3}[p_1+p_2+p_3+p_4-\frac{p_4p_3-p_1p_2}{(p_4+p_3)-(p_1+p_2)}]\right|}$$

For $-\bar{G}_1 = (-p_4, -p_3, -p_2, -p_1, w)$, and from (37),

$$score(-\bar{G}_1) = \frac{-(\frac{2\pi}{3}w^2(p_1+2p_2+2p_3+p_4))}{4\pi w^2+1-\left|\frac{1}{3}[p_1+p_2+p_3+p_4-\frac{p_4p_3-p_1p_2}{(p_4+p_3)-(p_1+p_2)}]\right|}$$

Therefore, $score(-\bar{G}_1) = -score(\bar{G}_1)$.

Property 7

If $\bar{G}_1$and $\bar{G}_2$ are two GTrFNs such that $\bar{G}_1 \succ \bar{G}_2$ then $-\bar{G}_1 \prec -\bar{G}_2$.

Proof: Given that $\bar{G}_1 \succ \bar{G}_2$, by the ranking procedure given above, we have $score(\bar{G}_1) > score(\bar{G}_2)$.

$\Longrightarrow -score(\bar{G}_1) < -score(\bar{G}_2)$

$\Longrightarrow score(-\bar{G}_1) < score(-\bar{G}_2)$ (by property 6)

Hence, $-\bar{G}_1 \prec -\bar{G}_2$.

Property 8

If $\bar{G}_1$,$\bar{G}_2$and $\bar{G}_3$ are three GTrFNs such that $\bar{G}_1 \prec \bar{G}_2$ & $\bar{G}_2 \prec \bar{G}_3$ then $\bar{G}_1 \prec \bar{G}_3$.

Proof: Given that $\bar{G}_1 \prec \bar{G}_2$ & $\bar{G}_2 \prec \bar{G}_3$, by the ranking procedure given above, we have $score(\bar{G}_1) < score(\bar{G}_2)$ and $score(\bar{G}_2) < score(\bar{G}_3)$

$\Rightarrow score(\bar{G}_1) < score(\bar{G}_3)$,

Hence $\bar{G}_1 \prec \bar{G}_3$.

V. Reasonable Properties

This section reviews the reasonable properties for ordering FNs, as outlined by [28], which serve as a benchmark for evaluating the performance of fuzzy ranking methods.

Let N denote the ordering approach and V denote the set of fuzzy quantities that may be ordered using N. $\boldsymbol{T}$ is a finite subset of V, with $\bar{X}$ and $\bar{Y}$ being two elements in $\boldsymbol{T}$.

Theorem 1

Let $\boldsymbol{T}$ be a finite subset of V and $\bar{X} \in \boldsymbol{T}$, $\bar{X} \succ \bar{X}$ by N on $\boldsymbol{T}$.
Proof: For any arbitrary FN $\bar{X} \in \boldsymbol{T}$, $score(\bar{X})$ is a real value, say p
As $p \geq p$, we have $\bar{X} \succcurlyeq \bar{X}$.

Theorem 2

Let $\boldsymbol{T}$ be a finite subset of V and $(\bar{X}, \bar{Y}) \in \boldsymbol{T}^{\boldsymbol{2}}$, $\bar{X} \succcurlyeq \bar{Y}$ and $\bar{Y} \succcurlyeq \bar{X}$, by N on $\boldsymbol{T}$, then $\bar{X} \sim \bar{Y}$ by N on $\boldsymbol{T}$.
Proof: Consider $(\bar{X}, \bar{Y}) \in \boldsymbol{T}^{\boldsymbol{2}}$ with $\bar{X} \succcurlyeq \bar{Y}$, $\bar{Y} \succcurlyeq \bar{X}$.
Let $score(\bar{X}) = a$ and $score(\bar{Y}) = b$;
Now, $\bar{X} \succcurlyeq \bar{Y}$
$\Rightarrow score(\bar{X}) \geq score(\bar{Y}) \Rightarrow a \geq b$.
Now, $\bar{Y} \succcurlyeq \bar{X}$
$\Rightarrow score(\bar{Y}) \geq score(\bar{X}) \Rightarrow b \geq a$.
The above two inequalities satisfy only if $a = b$.
Hence, $\bar{X} \sim \bar{Y}$.

Theorem 3

Let $\boldsymbol{T}$ be a finite subset of V and $(\bar{X}, \bar{Y}, \bar{Z}) \in \boldsymbol{T}^{\boldsymbol{3}}$, $\bar{X} \succcurlyeq \bar{Y}$ and $\bar{Y} \succcurlyeq \bar{Z}$ by N on $\boldsymbol{T}$, then $\bar{X} \succcurlyeq \bar{Z}$ by N on $\boldsymbol{T}$.
Proof: Consider three FNs $(\bar{X}, \bar{Y}, \bar{Z}) \in \boldsymbol{T}^{\boldsymbol{3}}$ with $\bar{X} \succcurlyeq \bar{Y}$ *and* $\bar{Y} \succcurlyeq \bar{Z}$ by N on $\boldsymbol{T}$.
Let $score(\bar{X}) = p, score(\bar{Y}) = q, score(\bar{Z}) = r$
We know that $\bar{X} \succcurlyeq \bar{Y}$ *and* $\bar{Y} \succcurlyeq \bar{Z}$ then
$\Rightarrow score(\bar{X}) \geq score(\bar{Y})$; $score(\bar{Y}) \geq score(\bar{Z})$
$\Rightarrow p \geq q$; $q \geq r$
$\Rightarrow p \geq r \Rightarrow \bar{X} \succcurlyeq \bar{Z}$.

Theorem 4

Let $\boldsymbol{T}$ be a finite subset of V and $(\bar{X}, \bar{Y}) \in \boldsymbol{T}^{\boldsymbol{2}}$,
(i) if $\inf Supp(\bar{X}) > \sup Supp(\bar{Y})$, then $\bar{X} \succcurlyeq \bar{Y}$ by N on $\boldsymbol{T}$.
(ii) if $\inf Supp(\bar{X}) > \sup Supp(\bar{Y})$, then $\bar{X} \succ \bar{Y}$ by N on $\boldsymbol{T}$ (a stronger version of (i)).
Proof: Since (ii) is stronger than (i), (ii) is proved.
Let $\boldsymbol{T}$ be a finite subset of V and $(\bar{X}, \bar{Y}) \in \boldsymbol{T}^{\boldsymbol{2}}$ with $\inf Supp(\bar{X}) > \sup Supp(\bar{Y})$.
Clearly, $score(\bar{X}) \geq \inf Supp(\bar{X})$
And $score(\bar{Y}) \leq \sup Supp(\bar{Y})$
Therefore
$score(\bar{X}) \geq \inf Supp(\bar{X}) > \sup Supp(\bar{Y}) \geq score(\bar{Y})$.
Hence $\bar{X} \succ \bar{Y}$.

VI. Numerical Examples

This section provides numerical examples to demonstrate the efficacy and applicability of the proposed ranking procedure.

Example 1

Let $\bar{P} = (0.1, 0.2, 0.4, 0.5; 1.0, 1.0)$,
$\bar{Q} = (0.1, 0.2, 0.4, 0.5; 0.8, 1.0)$,
$\bar{R} = (0.1, 0.2, 0.4, 0.5; 1.0, 0.8)$ be GTrFNs with different left-right heights taken from [10] shown in Fig. 8.
Using (33), $score(\bar{P}) = 0.2841$, and using (31) & (32), $score(\bar{Q}) = 0.3077$, $score(\bar{R}) = 0.2606$ and the ranking order is $\bar{R} \prec \bar{P} \prec \bar{Q}$. Our ranking results coincide with [10].

Example 2

Let $\bar{P} = (-0.5, -0.4, -0.2, -0.1; 1.0, 1.0)$,
$\bar{Q} = (-0.5, -0.4, -0.2, -0.1; 1.0, 0.8)$,
$\bar{R} = (-0.5, -0.4, -0.2, -0.1; 0.8, 1.0)$ be GTrFNs with different left-right heights taken from [10] shown in Fig. 9.

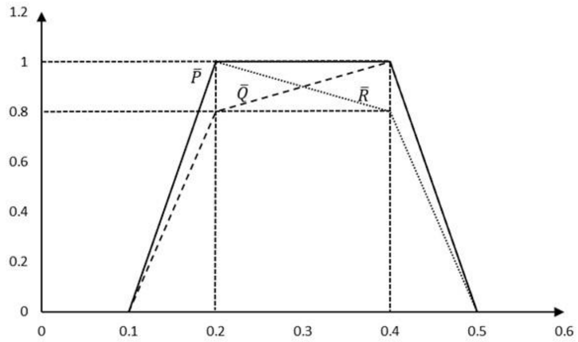


Fig. 8: $\bar{P} = (0.1, 0.2, 0.4, 0.5; 1.0, 1.0)$, $\bar{Q} = (0.1, 0.2, 0.4, 0.5; 0.8, 1.0)$
$\bar{R} = (0.1, 0.2, 0.4, 0.5; 1.0, 0.8)$

Using (33), $score(\bar{P}) = -0.2718$, and using (31) & (32), $score(\bar{Q}) = -0.2940, score(\bar{R}) = -0.2496$ and the ranking order is $\bar{Q} \prec \bar{P} \prec \bar{R}$. Our ranking results coincide with [10].

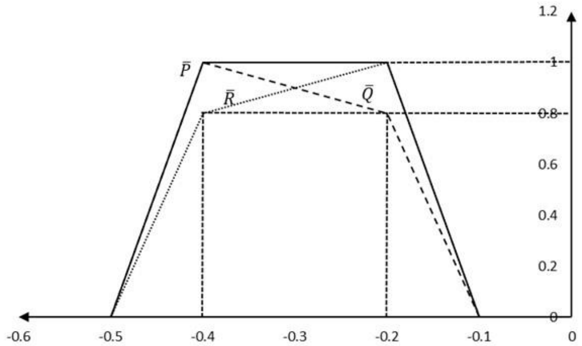


Fig. 9: $\bar{P} = (-0.5, -0.4, -0.2, -0.1; 1.0, 1.0)$,
$\bar{Q} = (-0.5, -0.4, -0.2, -0.1; 1.0, 0.8)$,
$\bar{R} = (-0.5, -0.4, -0.2, -0.1; 0.8, 1.0)$

Example 3

Let $\bar{P} = (0.1659, 0.2803, 0.7463, 1.154; 0.5, 0.6)$,
$\bar{Q} = (0.1611, 0.2475, 0.5696, 0.8187; 0.4, 0.5)$,
$\bar{R} = (0.1645, 0.2445, 0.5869, 0.8894; 0.5, 0.6)$ be GTrFNs with different left-right heights taken from [11], shown in Fig. 10.
Using (31), $score(\bar{P}) = 0.5566$, $score(\bar{Q}) = 0.4024$, $score(\bar{R}) = 0.4346$ and the ranking order is $\bar{Q} \prec \bar{R} \prec \bar{P}$. Our ranking results coincide with [11].

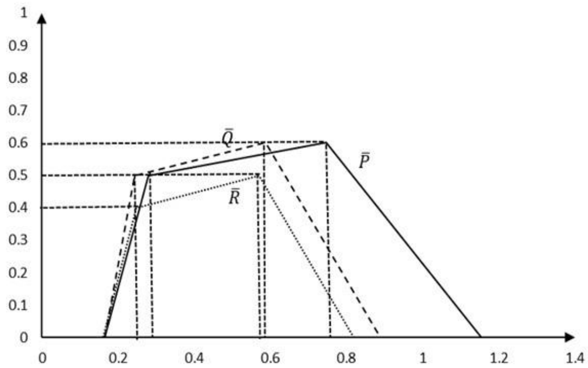


Fig. 10: $\bar{P} = (0.1659, 0.2803, 0.7463, 1.154; 0.5, 0.6)$,
$\bar{Q} = (0.1611, 0.2475, 0.5696, 0.8187; 0.4, 0.5)$,
$\bar{R} = (0.1645, 0.2445, 0.5869, 0.8894; 0.5, 0.6)$

Example 4

Let $\bar{P} = (2, 5, 6, 7; 0.6, 0.4)$, $\bar{Q} = (3, 4, 5, 6; 0.8, 0.6)$ be two GTrFNs with different left-right heights taken from [8], shown in Fig. 11.
Using (31), $score(\bar{P}) = 28.7391$, $score(\bar{Q}) = 7.514$, and the ranking order is $\bar{Q} \prec \bar{P}$. But [8] method's ranking order,

i.e. $\bar{P} \prec \bar{Q}$ does not coincide with intuition, though the x-coordinate centroid value of FN $\bar{Q}$ is less than the x-coordinate centroid value of FN $\bar{P}$.

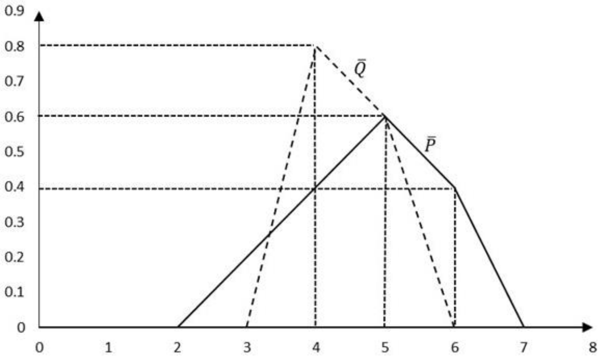


Fig. 11: $\bar{P} = (2,5,6,7; 0.6,0.4)$, $\bar{Q} = (3,4,5,6; 0.8,0.6)$

Example 5

Let $\bar{P} = (1,2,3,4; 0.6,0.4)$ $\bar{Q} = (0,3,4,5; 0.4,0.2)$ be two GTrFNs with different left-right heights taken from [8], shown in Fig. 12.
Using (31), $score(\bar{P}) = 3.271$ and $score(\bar{Q}) = 16.5039$ and the ranking order is $\bar{P} \prec \bar{Q}$.
Our ranking results coincide with [8].

Example 6

Let $\bar{P} = (0,0.225,0.225,0.45; 0.225,0.225)$ &
$\bar{Q} = (0.45,0.675,0.675,0.9; 0.225,0.225)$ be two GTrFNs taken from [14] illustrated in Fig. 13.
Using (33), we get $score(\bar{P}) = 0.1014$, and $score(\bar{Q}) = 0.4468$ and the ranking order is $\bar{P} \prec \bar{Q}$. But [14] method ranking order, i.e. $\bar{P} = \bar{Q}$ does not coincide with intuition, though the x-coordinate centroid value of FN $\bar{P}$ is less than the x-coordinate centroid value of FN $\bar{Q}$.

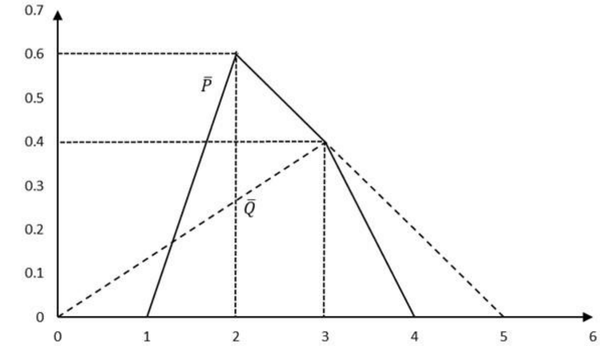


Fig. 12: $\bar{P} = (1,2,3,4; 0.6,0.4)$, $\bar{Q} = (0,3,4,5; 0.4,0.2)$

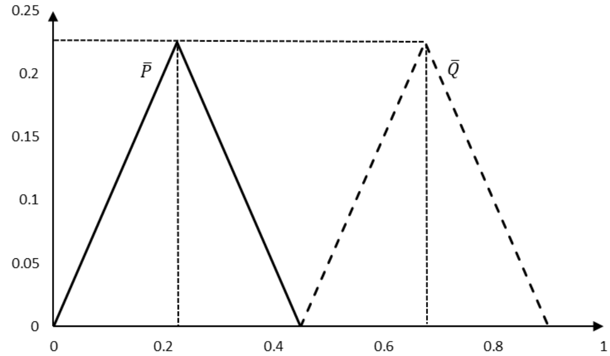


Fig. 13: $\bar{P} = (0,0.225,0.225,0.45; 0.225,0.225)$,
$\bar{Q} = (0.45,0.675,0.675,0.9; 0.225,0.225)$

Example 7

Let $\bar{P} = (0.1,0.2,0.3,0.4; 0.6,0.8)$ &
$\bar{Q} = (0.1,0.2,0.3,0.4; 0.8,0.6)$ be two GTrFNs with different left-right heights taken from [14] illustrated in Fig. 14.
Using (31) & (32), $score(\bar{P}) = 0.2460$ & $score(\bar{Q}) = 0.2113$ and the ranking order is $\bar{Q} \prec \bar{P}$. But [14] method ranking order, i.e. $\bar{P} = \bar{Q}$ does not coincide with intuition, though the x-coordinate centroid value of FN $\bar{Q}$ is less than the x-coordinate centroid value of FN $\bar{P}$.

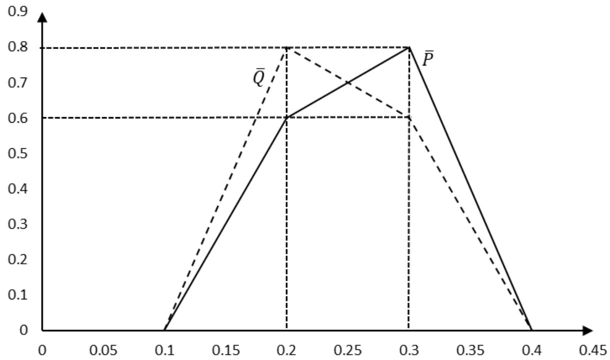


Fig. 14: $\bar{P} = (0.1,0.2,0.3,0.4; 0.6,0.8)$, $\bar{Q} = (0.1,0.2,0.3,0.4; 0.8,0.6)$

VII. Comparative Study of the Proposed Method with Some Existing Methods

A. Comparative study I

A comparative analysis is undertaken involving five distinct sets of FNs, as presented in [11] and illustrated in Fig. 15. The proposed method is contrasted with existing methodologies ([11], [29], [30]), and the comparative results are tabulated in Table I.
Set I: $\bar{M} = (0.4,0.5,0.5,0.6; 1,1)$, $\bar{N} = (0.2,0.4,0.6,0.8; 1,1)$.
Set II: $\bar{M} = (0.2,0.3,0.3,0.6; 1,1)$, $\bar{N} = (0.2,0.4,0.4,0.6; 1,1)$,
$\bar{O} = (0.2,0.5,0.5,0.6; 1,1)$.
Set III: $\bar{M} = (0.2,0.4,0.4,0.6; 0.9,0.9)$,
$\bar{N} = (0.2,0.4,0.4,0.6; 1,1)$,
$\bar{O} = (0.2,0.4,0.4,0.6; 0.5,0.5)$.
Set IV: $\bar{M} = (0.4,0.5,0.5,0.6; 0.5,0.5)$,
$\bar{N} = (0.2,0.4,0.6,0.8; 0.6,0.6)$.
Set V: $\bar{M} = (0.4,0.5,0.5,0.6; 0.6,0.7)$,
$\bar{N} = (0.2,0.4,0.6,0.8; 0.5,0.6)$.
From Table I, the following are the conclusions:
Set I: The ranking order of the proposed method is consistent with all other fuzzy ranking methods.
Set II: The ranking order of the proposed method is consistent with all other fuzzy ranking methods.
Set III: The ranking order of the proposed method is consistent with [30], [11], but [29] failed to rank the FNs, as the FNs are non-normal.
Set IV: The ranking order of the proposed method is consistent with [30], [11], but [29] failed to rank the FNs, as the FNs are non-normal.
Set V: The ranking order of the proposed method is consistent with [30], [11], but [29] failed to rank GTrFNs having different left-right heights.

B. Comparative Study II

Five distinct sets of FNs, sourced from [31] and visualized in Fig. 16, are considered for a comparative study. The proposed method is compared with [31], and the results are tabulated in Table II.

Set 1: $A = (0.09,0.14,0.36,0.52; 0.5,0.6)$,
$B = (0.16,0.24,0.56,0.81; 0.4,0.5)$
$C = (0.16,0.24,0.58,0.88; 0.5,0.6)$
Set 2: $A = (0.2,0.4,0.6,0.8; 0.35)$,
$B = (0.1,0.2,0.3,0.4; 0.7)$
Set 3: $A = (0.1,0.3,0.3,0.8; 0.5)$,
$B = (0.4,0.5,0.5,0.6; 0.5)$
Set 4: $A = (0.2,0.4,0.6,0.8; 0.35)$,
$B = (0.1,0.2,0.3,0.4; 0.7)$
Set 5: $A = (0.23,0.37,0.48,0.59; 0.55)$,
$B = (0.37,0.51,0.64,0.78; 0.55)$,
$C = (0.29,0.41,0.54,0.64; 0.68)$

From Table II, we can observe that for Set 1, and Set 3, the ranking order of the proposed method is consistent with [31] ranking method.

For Set 2, and Set 4, the ranking order produced by [31] is inconsistent with intuition, as the x-coordinate centroid value of FN exceeds that of other FN. In contrast, the proposed method yields a more accurate ranking.

Set 5: The ranking order produced by [31] is inconsistent with intuition, as the x-coordinate centroid value of FN exceeds that of other FNs. In contrast, the proposed method yields a more accurate ranking.

VIII. Risk analysis

This section introduces a new risk analysis method to identify the risk of being affected by Type 2 diabetes in persons with different risk-prone parameters based on the proposed ranking method.

Diabetes is a chronic illness that develops when the pancreas cannot create enough insulin or when the body cannot properly use the insulin that is produced. Type 2 diabetes particularly impairs the body's capacity to utilize glucose for energy, resulting in inappropriate insulin usage. More than 95% of people with diabetes have Type 2 diabetes (WHO [32]). Type 2 diabetes is associated with a number of risk factors, including blood lipid levels, age, weight, and a family history of the disease.

Five persons $(D_1, D_2, D_3, D_4, D_5)$ data have been collected based on the factors affecting Type 2 diabetes from Sri Sadguru Medical Labs, Visakhapatnam, Andhra Pradesh, India, and the data is shown in Table III. The data corresponds to different ages, heights & weights, family history (FamH), Fasting Blood Sugar (FBS), HbA1C test levels, Blood triglyceride level (BTLev), and Blood Pressure (BP). To measure the above factors, the nine-member linguistic terms set represented by GTrFNs is taken from [33], and is presented in Table IV. The factors that contribute to Type 2 diabetes are given below: (CDC [34,35], NHLBI [36])

1. Age: Age acts as a major factor in diabetes. The probability of being affected by diabetes and its severity of physical weakness for different ages are:

a) Under 30 years, the probability of being affected is very low, and the severity of physical weakness is also very low.

b) Between 30-50 years, the probability of being affected is high, and the severity of physical weakness is fairly high.

c) Above 50 years, the probability of being affected is medium, and the severity of physical weakness is fairly low.

2. Body Mass Index (BMI): BMI is one of the most effective techniques to determine whether a person is overweight. The BMI calculation takes into account the height and body weight of the person. One of the leading causes of Type 2 diabetes is uneven fat distribution in the body. Furthermore, fat buildup, particularly in the abdomen rather than the hips and thighs, implies an increased risk. A person with an 18.5-24.9 BMI is considered normal, the probability of being affected is low, and the severity of physical weakness is very low. A person with a 25.0-39.9 BMI is considered overweight, the probability of being affected is medium, and the severity of a loss is fairly low. A person with 40 and A person with 40 and above BMI is considered obese, the probability of being affected is fairly high, and the severity of physical weakness is medium.

3. Family History (FamH): Family history plays a significant role in developing Type 2 diabetes. If the person has a family history, then the probability of being affected by diabetes is fairly high, and the severity of physical weakness is fairly high. If there is no family history, then the probability of being affected is low, and the severity of physical weakness is also low.

4. Fasting Blood Sugar (FBS): This test checks blood sugar following an overnight fast. According to Centres for Disease Control and Prevention (CDS), if the sugar level of a person is less than or equal to 99 milligrams per deciliter (mg/dL), then the person is considered nondiabetic. So, the probability of being affected is low, and the severity of physical weakness is very low. If the sugar levels are between 100 mg/dL and 125 mg/dL, then the person is considered prediabetic. Therefore, the probability of being affected is high, and the severity of physical weakness is fairly high. If the sugar levels are greater than 126mg/dL, then it is considered diabetic, so the probability of being affected is very high, and the severity of physical weakness is high.

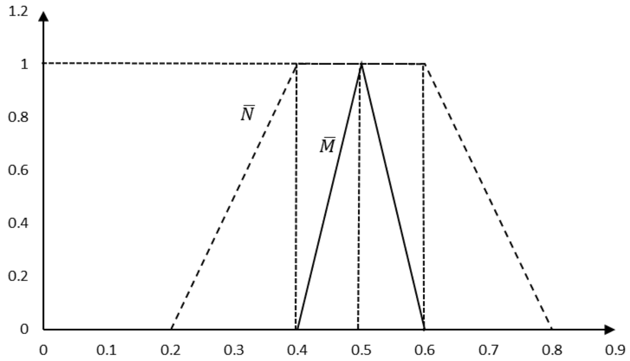
5. HbA1C test: The A1C test evaluates the average blood sugar level over the previous two or three months. As per the CDC [35], an A1C of less than 5.7% is considered normal, so the probability of being affected is low, and the severity of physical weakness is very low. If A1C is between 5.7% and 6.4%, it implies prediabetes, so the probability of being affected and the severity of physical weakness is fairly high. If A1C is 6.5% or more, it indicates diabetes, so the probability of being affected is very high, and the severity of physical weakness is high.

6. Blood Triglycerides level (BTLev): Triglycerides are lipids, or fats, present in the blood. As per NHLBI [36], if BTLev is less than or equal to 150mg/dL, it is considered healthy, so the probability of being affected is low, and its severity of physical weakness is very low. If BTLev is between 150 and 199mg/dL, it is considered borderline high, so the probability of being affected is fairly high, and its severity of physical weakness is medium. If BTlev is between 200 and 499mg/dL, then it is considered as high, so the probability of being affected is high, and the severity of physical weakness is fairly high. If BTLev is more than 500mg/dL, it is considered very high, so the probability of being affected is very high, and the severity of physical weakness is high.

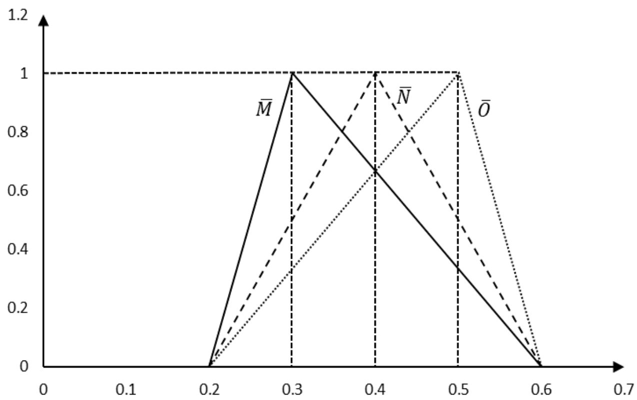
7. Blood Pressure (BP): Blood pressure rises due to hypertension. Diabetes is caused by it as well. According to the CDC [34], a systolic pressure (SP) is less than or equal to 120 millimeters of mercury (mmHg) and diastolic pressure (DP) less than or equal to 80mmHg respectively, is considered normal, and the probability of being affected is low, the severity of physical weakness is very low. If SP is between 120 and 129mmHg and DP is less than or equal to 80mmHg, it is considered prehypertension, so the probability of being affected is medium, and the severity of physical

weakness is fairly low. If SP is more than 130mmHg and DP is more than 80mmHg, then it is considered hypertension, so the probability of being affected is very high, and the severity of physical weakness is high.

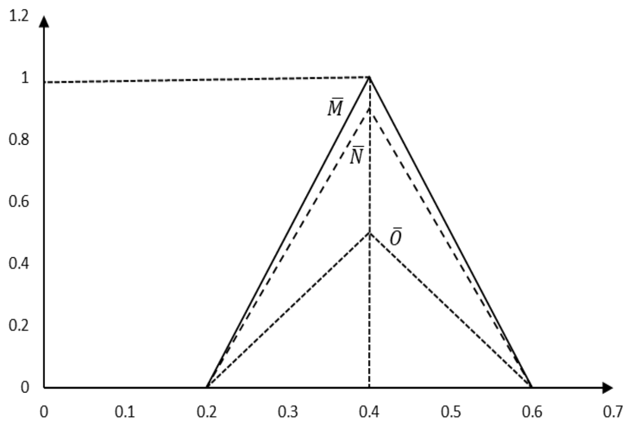
Based on the proposed ranking approach, we present a novel ranking approach for identifying the risk [37] of being affected by Type 2 diabetes. Consider the data of five persons $(D_1, D_2, D_3, D_4, D_5)$ with different ages, different heights & weights, family history (FamH) of diabetics, fasting blood sugar (FBS) levels, HbA1C test levels, blood triglyceride levels (BTLev), and blood pressure (BP) readings. These are called as sub-components, denoted by $\bar{L}_n, n = 1,2,3,\dots,7$. For each sub-component, there is an associated probability of being affected $\bar{R}_n$ and corresponding severity of physical weakness $\bar{S}_n$.



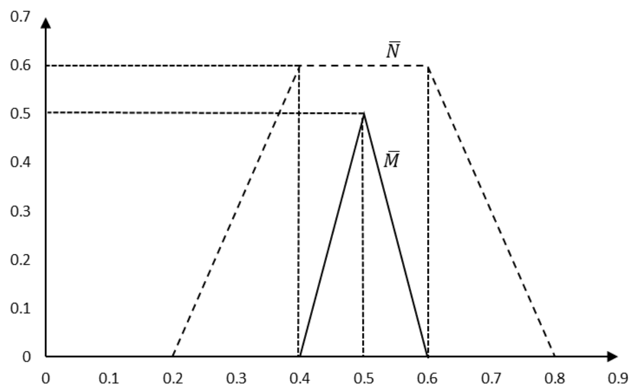
Set I $\bar{M} = (0.4,0.5,0.5,0.6; 1,1)$,
$\bar{N} = (0.2,0.4,0.6,0.8; 1,1)$



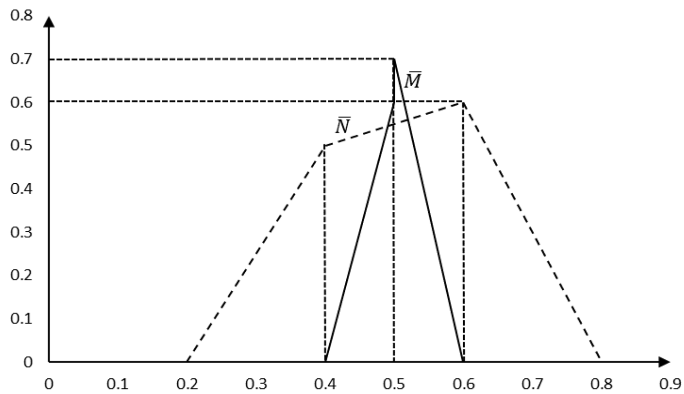
Set II $\bar{M} = (0.2,0.3,0.3,0.6; 1,1)$,
$\bar{N} = (0.2,0.4,0.4,0.6; 1,1)$,
$\bar{O} = (0.2,0.5,0.5,0.6; 1,1)$



Set III $\bar{M} = (0.2,0.4,0.4,0.6; 0.9,0.9)$,
$\bar{N} = (0.2,0.4,0.4,0.6; 1,1)$,
$\bar{O} = (0.2,0.4,0.4,0.6; 0.5,0.5)$



Set IV $\bar{M} = (0.4,0.5,0.5,0.6; 0.5,0.5)$,
$\bar{N} = (0.2,0.4,0.6,0.8; 0.6,0.6)$



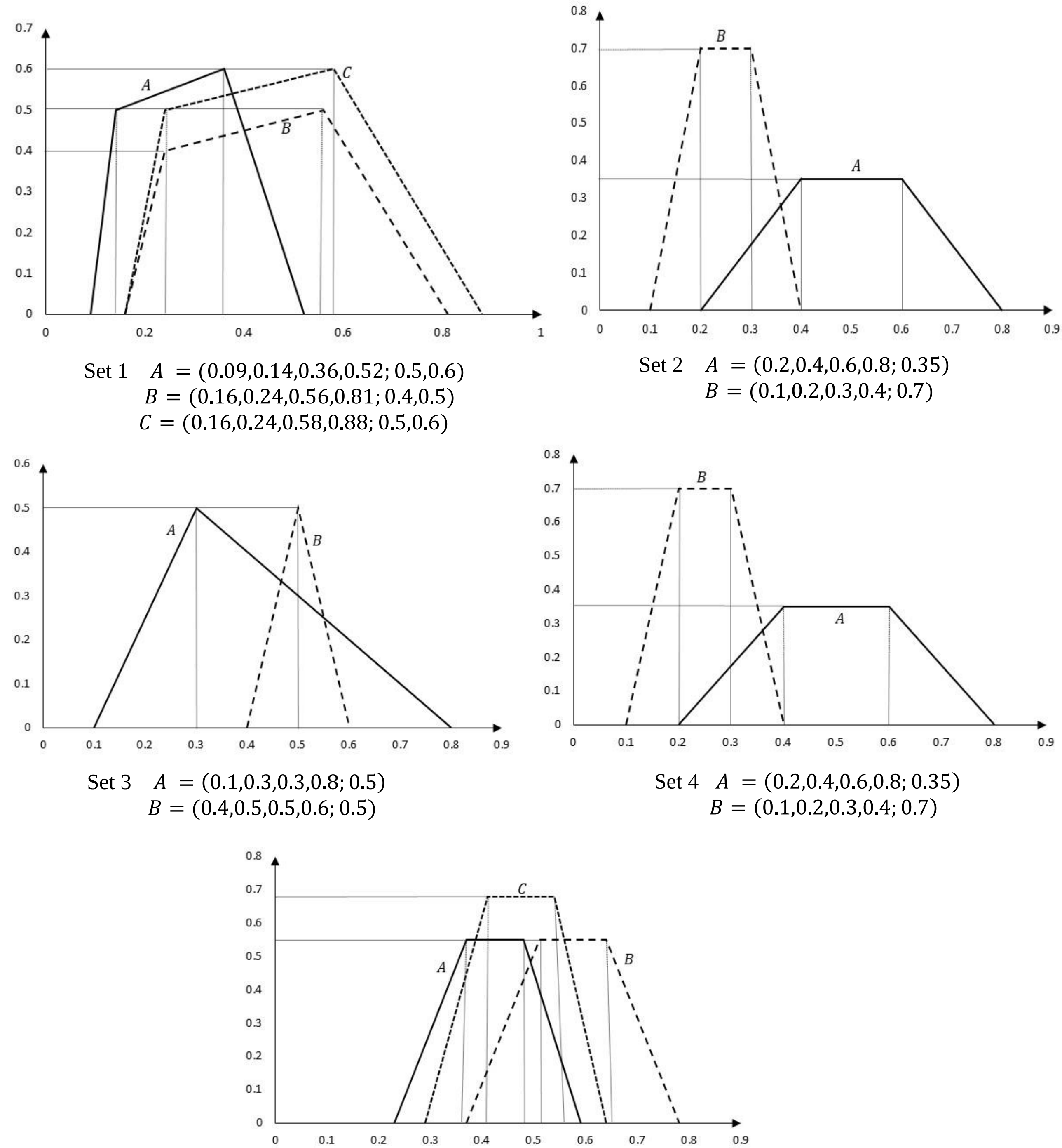
Set V $\bar{M} = (0.4,0.5,0.5,0.6; 0.6,0.7)$,
$\bar{N} = (0.2,0.4,0.6,0.8; 0.5,0.6)$

Fig. 15: Five sets of fuzzy numbers taken from (Chutia & Gogoi [11])

TABLE I
COMPARATIVE STUDY I OF THE PROPOSED METHOD WITH EXISTING TECHNIQUES

Sets	Kim & Park [29]			Kumar [30]			Chutia & Gogoi [11]			Proposed approach		
	$\bar{M}$	$\bar{N}$	$\bar{O}$	$\bar{M}$	$\bar{N}$	$\bar{O}$	$\bar{M}$	$\bar{N}$	$\bar{O}$	$\bar{M}$	$\bar{N}$	$\bar{O}$
Set I	0.571	0.750	#	0.550	0.700	#	0.550	0.700	#	0.480	0.490	#
Set II	0.571	0.600	0.800	0.450	0.500	0.550	0.450	0.500	0.550	0.317	0.381	0.446
Set III	#	#	#	0.450	0.500	0.250	0.450	0.500	0.250	0.377	0.381	0.335
Set IV	#	#	#	0.275	0.350	#	0.275	0.350	#	0.431	0.450	#
Set V	#	#	#	#	#	#	0.380	0.470	#	0.46	0.474	#

means that the method cannot rank the FNs



Set 1 $A = (0.09,0.14,0.36,0.52; 0.5,0.6)$
$B = (0.16,0.24,0.56,0.81; 0.4,0.5)$
$C = (0.16,0.24,0.58,0.88; 0.5,0.6)$

Set 2 $A = (0.2,0.4,0.6,0.8; 0.35)$
$B = (0.1,0.2,0.3,0.4; 0.7)$

Set 3 $A = (0.1,0.3,0.3,0.8; 0.5)$
$B = (0.4,0.5,0.5,0.6; 0.5)$

Set 4 $A = (0.2,0.4,0.6,0.8; 0.35)$
$B = (0.1,0.2,0.3,0.4; 0.7)$

Set 5 $A = (0.23,0.37,0.48,0.59; 0.55)$
$B = (0.37,0.51,0.64,0.78; 0.55)$
$C = (0.29,0.41,0.54,0.64; 0.68)$

Fig 16: Five sets of fuzzy numbers taken from (Marimuthu & Mahapatra [31])

TABLE II
COMPARATIVE STUDY II OF THE PROPOSED METHOD WITH [31]

Fuzzy Sets	Ranking result of [31]	Proposed method
Set 1	$A \prec B \prec C$	$A \prec B \prec C$
Set 2	$A \sim B$	$A \succ B$
Set 3	$A \prec B$	$A \prec B$
Set 4	$A \sim B$	$A \succ B$
Set 5	$A \prec B \prec C$	$A \prec C \prec B$

These $\bar{R}_n, \bar{S}_n$ are considered in a linguistic form, given by GTrFNs. The linguistic representation of the collected data of five persons for the seven sub-components is presented in Tables V, VI, VII, VIII, and IX.

The proposed method for risk analysis is given in the following steps:

Step 1: Calculate the probability of being affected $\bar{R}_n$ and severity of physical weakness $\bar{S}_n$ for each sub-component $\bar{L}_n, n = 1,2,3,\ldots,7$ using the fuzzy weighted mean method and the GTrFNs arithmetic operations (Definition 9) to get the probability of being affected by Type 2 diabetes $\bar{R}_a$ of each person.

$$\bar{R}_a = \frac{\sum_{n=1}^{7} \bar{R}_n \otimes \bar{S}_n}{\sum_{n=1}^{7} \bar{S}_n} = (r_{a1}, r_{a2}, r_{a3}, r_{a4}, w_{a1}, w_{a2}) \qquad (38)$$

As $\bar{R}_n$ & $\bar{S}_n$ are GTrFNs, so is $\bar{R}_a, a = 1,2,\ldots,5$.

$\bar{R}_1 = [(High \otimes Fairly\ high) \oplus (Low \otimes Very\ low) \oplus (Fairly\ high \otimes Medium) \oplus (Very\ high \otimes High) \oplus (Very\ high \otimes High) \oplus (High \otimes Fairly\ high) \oplus (Very\ high \otimes High) \oslash [Fairly\ high \oplus Very\ low \oplus Medium \oplus High \oplus High \oplus Fairly\ high \oplus High]$

$\bar{R}_1 = (0.5662, 0.7180, 1.1844, 1.4157; 0.7, 0.7)$.

$\bar{R}_2 = [(High \otimes Fairly\ high) \oplus (Low \otimes Very\ low) \oplus (Fairly\ high \otimes Medium) \oplus (Very\ high \otimes High) \oplus (High \otimes Fairly\ high) \oplus (High \otimes Fairly\ high) \oplus (Very\ high \otimes High)] \oslash [Fairly\ high \oplus Very\ low \oplus Medium \oplus High \oplus Fairly\ high \oplus Fairly\ high \oplus High]$

$\bar{R}_2 = (0.5300, 0.6794, 1.1827, 1.4336; 0.7, 0.7)$.

$\bar{R}_3 = [(High \otimes Fairly\ high) \oplus (Medium \otimes Fairly\ low) \oplus (Low \otimes Very\ low) \oplus (Very\ high \otimes High) \oplus (Very\ high \otimes High) \oplus (Medium \otimes Fairly\ low) \oplus (Low \otimes Very\ low) \oslash [Fairly\ high \oplus Fairly\ low \oplus Very\ low \oplus High \oplus High \oplus Fairly\ low \oplus Very\ low]$

$\bar{R}_3 = (0.4935, 0.6472, 1.1790, 1.4205; 0.7, 0.7)$.

$\bar{R}_4 = [(Medium \otimes Fairly\ low) \oplus (Low \otimes Very\ low) \oplus (Low \otimes Very\ low) \oplus (Low \otimes Very\ low) \oplus (Low \otimes Very\ low) \oplus (Fairly\ high \otimes Medium) \oplus (Very\ high \otimes High)] \oslash [Fairly\ low \oplus Very\ low \oplus Very\ low \oplus Very\ low \oplus Very\ low \oplus Medium \oplus High]$

$\bar{R}_4 = (0.3920, 0.5736, 1.1753, 1.5424; 0.7, 0.7)$.

$\bar{R}_5 = [(Medium \otimes Fairly\ low) \oplus (Medium \otimes Fairly\ low) \oplus (Low \otimes Very\ low) \oplus (Low \otimes Very\ low) \oplus (Low \otimes Very\ low) \oplus (Fairly\ high \otimes Medium) \oplus (Low \otimes Very\ low)] \oslash [Fairly\ low \oplus Fairly\ low \oplus Very\ low \oplus Very\ low \oplus Very\ low \oplus Medium \oplus Very\ low]$

$\bar{R}_5 = (0.1663, 0.3178, 1.0541, 1.7718; 0.7, 0.7)$.

Step 2: Using the proposed score function (34), the score value of each $\bar{R}_a$ obtained in Step 1 is calculated to find the highest risk of getting affected by Type 2 diabetes.

$score(\bar{R}_1) = 0.9602$, $score(\bar{R}_2) = 0.9416$, $score(\bar{R}_3) = 0.9183$, $score(\bar{R}_4) = 0.8945$, $score(\bar{R}_5) = 0.7611$.

Step 3: The larger is the score value of $\bar{R}_a$, the higher is the probability of being affected by Type 2 diabetes.

From the above score values in Step 2, we can see that, $score(\bar{R}_1) > score(\bar{R}_2) > score(\bar{R}_3) > score(\bar{R}_4) > score(\bar{R}_5)$.

So, $D_1 > D_2 > D_3 > D_4 > D_5$ is the ranking order for identifying the person with the highest risk of getting affected among the five persons. Therefore person D_1 has the highest risk of getting affected by Type 2 diabetes.

TABLE III
COLLECTED LAB DATA OF SUB-COMPONENTS OF FIVE PERSONS

Components	D_1	D_2	D_3	D_4	D_5
Age	38	47	49	53	60
Height	5.9	5.4	5.6	5.7	5.4
Weight	74	54	72	68	68
FamH	Yes	Yes	No	No	No
FBS	238	156	161	91	96
HbA1C	7.8	6.1	6.6	4.8	4
BTLev	368	244	282	173	156
BP	140-90	150-100	110-80	130-90	120-80

TABLE IV
LINGUISTIC TERM SET (CHEN [26])

Linguistic terms	GFNs
Absolutely low	(0.0,0.0,0.0,0.0; 1.0,1.0)
Very low	0.0,0.0,0.02,0.07; 1.0,1.0
Low	(0.04,0.10,0.18,0.23; 1.0,1.0)
Fairly low	(0.17,0.22,0.36,0.42; 1.0,1.0)
Medium	(0.32,0.41,0.58,0.65; 1.0,1.0)
Fairly high	(0.58,0.63,0.80,0.86; 1.0,1.0)
High	(0.72,0.78,0.92,0.97; 1.0,1.0)
Very high	(0.93,0.97,1.0,1.0; 1.0,1.0)
Absolutely high	(1.0,1.0,1.0,1.0; 1.0,1.0)

TABLE V
LINGUISTIC REPRESENTATION OF RISK FACTORS FOR D_1

Components	Probability of being affected	Severity of physical weakness
Age	High	Fairly high ($w_1 = w_2 = 0.9$)
BMI	Low	Very low ($w_1 = w_2 = 0.7$)
FamH	Fairly high	Medium ($w_1 = w_2 = 0.85$)
FBS	Very high	High ($w_1 = w_2 = 0.95$)
A1C	Very high	High ($w_1 = w_2 = 0.95$)
BTLev	High	Fairly high ($w_1 = w_2 = 0.9$)
BP	Very high	High ($w_1 = w_2 = 0.95$)

TABLE VI
LINGUISTIC REPRESENTATION OF RISK FACTORS FOR D_2

Components	Probability of being affected	Severity of physical weakness
Age	High	Fairly high ($w_1 = w_2 = 0.9$)
BMI	Low	Very low ($w_1 = w_2 = 0.7$)
FamH	Fairly high	Medium ($w_1 = w_2 = 0.85$)
FBS	Very high	High ($w_1 = w_2 = 0.95$)
HbA1C	High	Fairly high ($w_1 = w_2 = 0.9$)
BTLev	High	Fairly high ($w_1 = w_2 = 0.9$)
BP	Very high	High ($w_1 = w_2 = 0.95$)

TABLE VII
LINGUISTIC REPRESENTATION OF RISK FACTORS FOR D_3

Components	Probability of being affected	Severity of physical weakness
Age	High	Fairly high ($w_1 = w_2 = 0.9$)
BMI	Medium	Fairly low ($w_1 = w_2 = 0.8$)
FamH	Low	Very low ($w_1 = w_2 = 0.7$)
FBS	Very high	High ($w_1 = w_2 = 0.95$)
HbA1C	Very high	High ($w_1 = w_2 = 0.95$)
BTLev	Medium	Fairly low ($w_1 = w_2 = 0.8$)
BP	Low	Very low ($w_1 = w_2 = 0.7$)

TABLE VIII
LINGUISTIC REPRESENTATION OF RISK FACTORS FOR D_4

Components	Probability of being affected	Severity of physical weakness
Age	Medium	Fairly low ($w_1 = w_2 = 0.8$)
BMI	Low	Very low ($w_1 = w_2 = 0.7$)
FamH	Low	Very low ($w_1 = w_2 = 0.7$)
FBS	Low	Very low ($w_1 = w_2 = 0.7$)
HbA1C	Low	Very low ($w_1 = w_2 = 0.7$)
BTLev	Fairly high	Medium ($w_1 = w_2 = 0.85$)
BP	Very high	High ($w_1 = w_2 = 0.95$)

TABLE IX
LINGUISTIC REPRESENTATION OF RISK FACTORS FOR D_5

Components	Probability of being affected	Severity of physical weakness
Age	Medium	Fairly low ($w_1 = w_2 = 0.8$)
BMI	Medium	Fairly low ($w_1 = w_2 = 0.8$)
FamH	Low	Very low ($w_1 = w_2 = 0.7$)
FBS	Low	Very low ($w_1 = w_2 = 0.7$)
HbA1C	Low	Very low ($w_1 = w_2 = 0.7$)
BTLev	Fairly high	Medium ($w_1 = w_2 = 0.85$)
BP	Low	Very low ($w_1 = w_2 = 0.7$)

IX. CONCLUSIONS

This research introduces a novel defuzzification method for ranking Generalized Trapezoidal Fuzzy Numbers (GTrFNs) with varying left-right heights. The proposed approach leverages a unique defuzzification technique that integrates the volumes generated by rotating the left and right membership functions of GTrFNs around the x-axis, and finding positive and negative side volumes considering a benchmark fuzzy number. A scoring function is derived from these volumes and the GTrFN's centroid, accommodating varying left-right heights for effective fuzzy number ranking. This method overcomes existing approaches' limitations and performs better in ranking diverse fuzzy numbers, including crisp values. Its applicability is demonstrated through a fuzzy risk analysis case study, effectively identifying the risk of developing Type 2 diabetes in individuals with varying risk profiles. The results highlight the efficacy of this novel technique, suggesting its potential for diverse applications in decision-making, optimization, and fuzzy-based problem-solving.

X. LIMITATIONS AND FUTURE SCOPE

This study focuses exclusively on triangular and trapezoidal fuzzy numbers, considering only the degree of membership for each element. Consequently, this proposed technique can be further extended to accommodate fuzzy numbers that incorporate the degree of non-membership, such as intuitionistic fuzzy numbers and Pythagorean fuzzy numbers. The suggested approach has the potential for broad application in various decision-making problems within fuzzy environments.

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