A Modified Mass-Conservative Characteristic Finite Element Method for Convection-Diffusion Problems

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***Abstract*—In this paper, a Mass-Conservative rectangular nonconforming finite element method is applied for convection-diffusion problems under anisotropic meshes. In this method, a modified characteristic finite element scheme with a low order Crouzeix-Raviart type nonconforming finite element is used. It is proved that the scheme preserves the mass balance identity and the scheme is unconditionally stable. The error estimate in L^2-norm with respect to the space, is obtained by use of some distinct properties of the interpolation operators. The so-called elliptic projection, which is an indispensable tool in the convergence analysis of the previous literatures, is replaced by the mean value technique. Lastly, numerical examples are provided to confirm the theoretical results.**

***Index Terms*—Mass-conservative, Finite element method, Nonconforming element, Error estimate.**

I. Introduction

We consider the convection-diffusion equation operator ϕ, as shown below. $\Omega \subset R^2$ denotes an open bounded domain with the boundary Γ, $(0,T]$ is the time interval.

$$\mathcal{L}\phi = \frac{\partial \phi(X,t)}{\partial t} + u(X,t)\cdot\nabla\phi + (\nabla\cdot u(X,t))\phi - \nu\triangle\phi \quad (1)$$

The parameters appearing in the equation (1) satisfy the following assumptions.

1) $\phi(X,t)$ denotes, for example, the concentration of a possible substance;

2) $u(X,t)$ represents the velocity of the flow, satisfying

$$|u(X,t)| + |\nabla\cdot u(X,t)| \le C, \quad \forall X \in \Omega,$$

where C is a constant;

3) ν is a diffusion coefficient, and $X=(x,y)$.

4) ∇ and $\nabla\cdot$ denote the gradient and the divergence operators respectively.

The convection-diffusion problem is a mathematical model that describes the combined effects of convection and diffusion in a fluid or a substance. It is commonly encountered in various fields such as fluid dynamics, heat transfer, and mass transport. In many diffusion processes arising in physical problems, convection essentially dominates diffusion. It is natural to seek numerical methods for such problems to reflect their almost hyperbolic nature.

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The finite element method offers several advantages for solving convection-dominant diffusion problems, including its ability of handling complex geometries, adaptivity, and accuracy. This method has been widely used in various applications, such as fluid flow simulations, heat transfer analysis, and pollutant dispersion modeling[1-4]. A lot of schemes of the finite element method have been developed, such as the expanded characteristic-mixed finite element method[5], the staggered discontinuous Galerkin method[6], the least-squares mixed finite element method [7] and the modified method of characteristic-Galerkin finite element (MMOC-Galerkin)[8-10].

The characteristic method is natural from the physical point of view, since it approximates particle movements. And it is attractive from the mathematical point of view, since it symmetrizes the problem[11]. The modified characteristic finite element method was first formulated for scalar parabolic equations by J. Douglas and T. F. Russell in 1982. The main idea is to modify the standard finite element formulation by incorporating the characteristic variables. This modification helps to accurately capture the discontinuities and shocks in the solution. The method combines the advantages of both finite element method and finite volume method, providing the robust and accurate solutions for problems with complex behavior.

For convection-dominated problems, the modified characteristic finite element schemes have much smaller time-truncation errors than the standard methods. Because the solution changes more slowly in the characteristic τ direction than in the t direction. The scheme will permit the use of larger time step [12].

An important property of the convection-diffusion problems possess is the mass balance; the mass should be preserved if there is no source. In the framework of characteristic methods, it it important to maintain this property. Some schemes have been proposed and studied from this point [13-14].

An improved characteristic finite element scheme which preserves the mass balance, is presented[15]. In this method, the time derivative term and the divergence term are approximated directly. Usually the characteristic method is used to approximate the material derivative term, i.e., the time derivative term plus the convection term of non-divergence form. It is proved that the mass balance is satisfied completely.

However, in the studies mentioned above, only conforming finite elements were considered, and the regularity assumption or quasi-uniform assumption [16-17] was required on the meshes in space which is great deficiency in finite element methods. The nonconforming element method is a numerical technique used in finite element analysis to solve partial differential equations. Unlike conforming methods, nonconforming methods allow for discontinuous solutions, and the methods are particularly useful for problems with singularities or sharp gradients. This method is often used in mechanics, electromagnetics, and other fields where the traditional conforming method may not be suitable.

In the present work, a low order Crouzeix-Raviart type nonconforming rectangular element, studied in [18-20], is applied to the convection diffusion problem with the modified Mass-Conservative Characteristic finite element scheme. In this method, we will employ anisotropic meshes with fewer degrees of freedom, which can reflect the anisotropy with a finer mesh size[21-22].

$H^k(\Omega)$ denotes the standard Sobolev space of k-differential functions in $L^2(\Omega)$ with the usual norm $\parallel\cdot\parallel_k$ and semi-norm $|\cdot|_k$ respectively. When $k=0$, $L^2(\Omega)$ denotes the corresponding space defined on Ω with norm $\parallel\cdot\parallel$.

Y is a Sobolev space, and $f(X,t)$ is smooth function defined on $\Omega\times[a,b]$, $[a,b]\subset[0,T]$.

$L^p(a,b;Y)$ and $\parallel f\parallel_{L^p(a,b;Y)}$ are defined as follows,

$$L^p(a,b;Y)=\{f:\int_b^a \parallel f(\cdot,t)\parallel_Y^p\,dt<\infty\},$$

$$\parallel f\parallel_{L^p(a,b;Y)}=(\int_b^a \parallel f(\cdot,t)\parallel_Y^p\,dt)^{\frac{1}{p}},$$

If $p=\infty$, the integral is replaced by the essential supremum.

The remainder of this paper is organized as follows. In Section II, we present the mass-conservative characteristic finite element scheme and show the mass balance identity. In Section III, we analyze the stability and prove the convergence. In Section IV, the corresponding optimal error estimates in L^2 norm are derived. In Section V, some examples are presented to confirm our theoretical analysis.

II. Construction of a Mass-conservation Finite Element Scheme

To describe the important property of the operator $\mathcal{L}$, we consider the following initial boundary value problem. f denotes a source term.

$$\begin{array}{ll}(a)\ \dfrac{\partial\phi}{\partial t}+u(X,t)\cdot\nabla\phi+(\nabla\cdot u(X,t))\phi & \\ \quad -\nu\triangle\phi=f, & in\ \Omega\times[0,T],\\ (b)\ \phi(X,0)=\phi_0(X), & in\quad \Omega,\\ (c)\ \phi(X,t)=0, & on\quad \Gamma\times(0,T).\end{array}\tag{2}$$

The weak form of (2) is as follows: to find $\phi\in H_0^1(\Omega)$, such that

$$\begin{array}{lll}(a) & (\dfrac{\partial\phi}{\partial t},v)+(u(X,t)\cdot\nabla\phi,v)+((\nabla\cdot u(X,t))\phi,v) & \\ & +\nu(\nabla\phi,\nabla v)=(f,v),\quad \forall v\in H_0^1(\Omega), & \\ (b) & \phi(X,0)=\phi_0(X), & \forall X\in\Omega,\\ (c) & \phi(X,t)=0, & on\ \Gamma\ \times(0,T).\end{array}\tag{3}$$

The characteristic direction associated with the operator $\phi_t+u\cdot\nabla\phi$ are usually denoted by $\tau=\tau(X,t)$, where

$$\frac{\partial}{\partial\tau}=\frac{1}{\psi(X,t)}\frac{\partial}{\partial t}+\frac{u}{\psi(X,t)}\cdot\nabla.$$

$$\psi(X,t)=(1+|u|^2)^{\frac{1}{2}}$$

However, there are no characteristic schemes which satisfy the discrete version of the mass balance identity [11]. Our idea is to apply the characteristic approximation to the term

$$\mathcal{L}_0\phi=\frac{\partial\phi}{\partial t}+u(X,t)\cdot\nabla\phi+(\nabla\cdot u(X,t))\phi.\tag{4}$$

In the procedure, we consider a time step $\triangle t>0$ and approximate the solution at times $t^n=n\triangle t,\Delta t=T/N,n=0,1,\ldots,N$, where N is a positive integer.

Let $X:(0,T)$ be a solution of the ordinary differential equation, $\dfrac{dX}{dt}=u(X,t)$.

Subject to an initial condition $X(t^n)=x$, we can get an approximate value of X at t^{n-1} by the Euler method,

$$X_1^n(x)=x-u^n(x)\triangle t.$$

The weak form of (2) is as follows: to find $\phi\in H_0^1(\Omega)$, such that

$$\begin{array}{lll}(a) & (\mathcal{L}_0\phi,v)+\nu(\nabla\phi,\nabla v)=(f,v),\forall v\in H_0^1(\Omega), & \\ (b) & \phi(X,0)=\phi_0(X), & \forall X\in\Omega,\\ (c) & \phi(X,t)=0, & on\ \Gamma\times(0,T).\end{array}\tag{5}$$

Let $\Omega\subset R^2$ be a polygon with boundaries parallelling to the axes, T^h be axis-parallel rectangular meshes of Ω, which doesn't need to satisfy the regularity assumption or quasi-uniform assumption.

Let $\hat{K}=[-1,1]\times[-1,1]$ be the reference element on $\xi-\eta$ plane, the four vertices of $\hat{K}$ are $\hat{d}_1=(-1,-1),\hat{d}_2=(1,-1),\hat{d}_3=(1,1)$ and $\hat{d}_4=(-1,1)$, and the four edges are $\hat{l}_1=\overline{\hat{d}_1\hat{d}_2}$, $\hat{l}_2=\overline{\hat{d}_2\hat{d}_3}$, $\hat{l}_3=\overline{\hat{d}_3\hat{d}_4}$ and $\hat{l}_4=\overline{\hat{d}_4\hat{d}_1}$.

For any $\hat{v}\in H^1(\hat{K})$, the finite element is defined $(\hat{K},\hat{P},\hat{\Sigma})$ on $\hat{K}$ as follows:

$$\hat{\Sigma}=\{\hat{v}^1,\hat{v}^2,\hat{v}^3,\hat{v}^4,\hat{v}^5\},\hat{P}=span\{1,\xi,\eta,\varphi(\xi),\varphi(\eta)\}\ ,$$

where

$$\begin{aligned}\hat{v}^i&=\frac{1}{|\hat{l}_i|}\int_{\hat{l}_i}\hat{v}ds,i=1,2,3,4,\\ \hat{v}^5&=\frac{1}{|\hat{K}|}\int_{\hat{K}}\hat{v}d\xi d\eta,\\ \varphi(t)&=\frac{1}{2}(3t^2-1).\end{aligned}\tag{6}$$

It can be easily checked that interpolation defined above is well-posed and the interpolation function $\hat{I}\hat{v}$ can be expressed as

$$\begin{aligned}\hat{I}\hat{v}=&\hat{v}^5+\frac{1}{2}(\hat{v}^2-\hat{v}^4)\xi+\frac{1}{2}(\hat{v}^3-\hat{v}^1)\eta\\ &+\frac{1}{2}(\hat{v}^2+\hat{v}^4-2\hat{v}^5)\varphi(\xi)+\frac{1}{2}(\hat{v}^3+\hat{v}^1-2\hat{v}^5)\varphi(\eta).\end{aligned}\tag{7}$$

The following important lemma has been proved in [23].

Lemma 2.1 The interpolation operator $\hat{I}$ defined by (7) has the anisotropic interpolation property, i.e. $\forall\hat{v}\in H^2(\hat{K}),\alpha=(\alpha_1,\alpha_2)$ with $|\alpha|=1$, we have

$$\|\hat{D}^\alpha(\hat{v}-\hat{I}\hat{v})\|_{0,\hat{K}}\le\hat{C}|\hat{D}^\alpha\hat{v}|_{1,\hat{K}}.\tag{8}$$

Let $K=[x_K-h_x,x_K+h_x]\times[y_K-h_y,y_K+h_y]$, $h_K=diam(K)$, $\rho_K=\max\limits_{S\subset K}diam(S)$, $h=\max\limits_{K\in T^h}h_K$, $l_k(k=1,2,3,4)$ be the edges of K.

Define the affine mapping $F:\hat{K}\longrightarrow K$ as follows:

$$\left\{\begin{array}{l}x=x_K+h_x\xi,\\ y=y_K+h_y\eta.\end{array}\right.$$

Then the associated finite element space V^h is

$$V^h = \{v|\ \hat{v}|_{\hat{K}} = v|_K \circ F \in \hat{P}, \forall K \in T^h,\ \int_l [v]ds = 0\}, \tag{9}$$

where $[v]$ stands for the jump of v across the edge l if l is an internal edge, and it is equal to v itself if l belongs to $\partial\Omega$.

For any $v \in H^1(\Omega)$, let Π be the associated interpolation operator on V^h, satisfying $\Pi\ |_K = \Pi_K$, $\Pi_K = \hat{I} \circ F_K^{-1}$, then we have

$$\begin{cases} \int_{l_k} (v - \Pi v)ds = 0, \quad k = 1,2,3,4, \\ \int_K (v - \Pi v)dxdy = 0. \end{cases}$$

The mass-conservative characteristic finite element scheme is to find $\phi_h^n \subset V^h$, such that, for $n = 1, \cdots, N_T$,

$$\begin{array}{lll} (a) & (\dfrac{\phi_h^n - \phi_h^{n-1} \circ X_1^n \gamma^n}{\triangle t}, v_h) + (a(X,t^n)\nabla c_h^n, \nabla v_h)_h & \\ & = (f^n, v_h), & \forall v_h \in V^h, \\ (b) & \phi_h^0 = \Pi\phi_0, & \forall X \in \Omega, \\ (c) & \phi_h(X,t) = 0, & on\ \Gamma \times (0,T). \end{array} \tag{10}$$

Where $\phi_h^n = \phi_h(t^n)$,$\phi_h^{n-1} \circ X_1^n$ is a composition defined by$(\phi_h^{n-1} \circ X_1^n)(x) = \phi^{n-1}(X_1^n(x))$ and γ^n is the Jacobian of the transformation X_1^n.

$$\gamma^n = det(\frac{\partial X_1^n}{\partial x}) = det(\delta_{ij} - \triangle t\frac{\partial u_i^n}{\partial x_j}) \tag{11}$$

$\Pi\phi_0$ is the finite element interpolation of ϕ_0, $(u,v)_h = \sum_K \int_K uvdxdy$, and $f^n = f(X,t^n)$.

Theorem 1. Let ϕ_h^n be the solution of (10). Under the argument similar to [8], it holds that, for $m = 1, \cdots, N_T$

$$\int_\Omega \phi_h^m dx = \int_\Omega \phi_h^0 dx + \triangle t\sum_{n=1}^m \int_\Omega f^n dx \tag{12}$$

Proof: Choosing $v_h = 1$ in (10) and multiplying by $\triangle t$, we get

$$(\phi_h^n, 1) - (\phi_h^{n-1} \circ X_1^n\gamma^n, 1) = \triangle t(f^n, 1) \tag{13}$$

By the inverse transformation of X_1^n , we have

$$\int_\Omega \phi_h^{n-1} \circ X_1^n\gamma^n dx = \int_\Omega \phi_h^{n-1} dx \tag{14}$$

$$(\phi_h^n, 1) - (\phi_h^{n-1}, 1) = \triangle t(f^n, 1) \tag{15}$$

Summing up the equations above from $n = 1$ until $n = m$, we get the theorem.

III. The Stability of Discrete Problem

Theorem 1. Let ϕ_h^n be the solution of (10). Under the argument similar to [8], it holds that, for $m = 1, \cdots, N_T$, there exists a positive constant c, which is independent of h and $\triangle t$, such that

$$\begin{aligned} &\| \phi_h \|_{L^\infty(0,T;L^2)} + \triangle t(\sum_{n=1}^m \| \mathcal{L}_0\phi \|)^{\frac{1}{2}} \\ &+\nu^{\frac{1}{2}}\triangle t^{\frac{1}{2}}(\sum_{n=1}^m \| \nabla\phi_h^n \|_h)^{\frac{1}{2}} \\ &\le c \| \phi_h^0 \| + c\triangle t^{\frac{1}{2}} \| f \|_{L^\infty(0,T;L^2)} . \end{aligned} \tag{16}$$

Proof: Choosing $v_h = \phi_h^n$ in (10), we get

$$\begin{aligned} &(\frac{\phi_h^n - \phi_h^{n-1} \circ X_1^n\gamma^n}{\triangle t}, \phi_h^n) + (a(X,t^n)\nabla\phi_h^n, \nabla\phi_h^n)_h \\ &= (f^n, \phi_h^n) \end{aligned} \tag{17}$$

Let $\| \cdot \|_h = (\sum_K | \cdot |_{1,K}^2)^{\frac{1}{2}}$, it is a norm in V^h.

$$\begin{aligned} &\frac{1}{2\triangle t}(\| \phi_h^n \|^2 - \| \phi_h^{n-1} \circ X_1^n\gamma^n \|^2) \\ &+\frac{\triangle t}{2} \| \mathcal{L}_0\phi \|^2 + \nu \| \nabla\phi_h^n \|_h^2 = (f^n, \phi_h^n). \end{aligned} \tag{18}$$

By the inverse transformation of $X_1^n(x)$, it follows that

$$\| \phi_h^{n-1} \circ X_1^n\gamma^n \|^2 \le (1 + c\triangle t) \| \phi_h^{n-1} \|^2 \tag{19}$$

Next we estimate the right hand of (17).

$$(f^n, \phi_h^n) \le \frac{1}{2} \| \phi_h^n \|^2 + \frac{1}{2} \| f^n \|^2 \tag{20}$$

From (17)-(20), we have

$$\begin{aligned} &\frac{1}{2\triangle t}(\| \phi_h^n \|^2 - \| \phi_h^{n-1} \|^2) + \frac{\triangle t}{2} \| \mathcal{L}_0\phi \|^2 \\ &+\frac{\nu}{2} \| \nabla\phi_h^n \|_h^2 \le c \| \phi_h^{n-1} \|^2 + \frac{1}{2} \| \phi_h^n \|^2 + \frac{1}{2} \| f^n \|^2 . \end{aligned} \tag{21}$$

Multiplying (21) by $2\triangle t$, and summing from $n = 1$ to $n = m$, we obtain

$$\begin{aligned} &\| \phi_h^n \|^2 + \triangle t^2\sum_{n=1}^m \| \mathcal{L}_0\phi \|^2 + \nu\triangle t\sum_{n=1}^m \| \nabla\phi_h^n \|_h^2 \\ &\le c\triangle t\sum_{n=1}^m \| \phi_h^i \|^2 + \| \phi_h^0 \|^2 + c\triangle t \| f^n \|^2 \end{aligned} \tag{22}$$

By Gronwall's lemma, it follows that

$$\begin{aligned} &\| \phi_h \|_{L^\infty(0,T;L^2)} + \triangle t(\sum_{n=1}^m \| \mathcal{L}_0\phi \|)^{\frac{1}{2}} \\ &+\nu^{\frac{1}{2}}\triangle t^{\frac{1}{2}}(\sum_{n=1}^m \| \nabla\phi_h^n \|_h)^{\frac{1}{2}} \\ &\le c \| \phi_h^0 \| + c\triangle t^{\frac{1}{2}} \| f \|_{L^\infty(0,T;L^2)} . \end{aligned} \tag{23}$$

The proof is completed.

To get error estimates, we state the following important lemmas.

Lemma 3.1 Under anisotropic meshes, for any $v \in H^2(\Omega)$, we have

$$\begin{aligned} &\| v - \Pi v \| \le Ch^2|v|_2, \\ &\| v - \Pi v \|_h \le Ch|v|_2. \end{aligned}$$

Here and later, the positive C is independent of h_K and $\frac{h_K}{\rho_K}$, which may be different in different places.

Proof: The desired result comes from the interpolation theorem [20].

Lemma 3.2 Under anisotropic meshes, for any $c \in H^1(\Omega)$ and any $v_h \in V^h$, we have

$$(\nabla(c - \Pi c), \nabla v_h)_h = 0, \qquad \forall v_h \in V^h.$$

Proof: By Green's formula and the definition of Π , we get

$$\begin{aligned}(\nabla(c-\Pi c),\nabla v_h)_h &= \sum_{K\in T^h}\int_K \nabla(c-\Pi c)\nabla v_h dxdy\\ &= \sum_{K\in T^h}\int_{\partial K}(c-\Pi c)\frac{\partial v_h}{\partial n}ds\\ &- \sum_{K\in T^h}\int_K (c-\Pi c)\triangle v_h dxdy.\end{aligned}$$

Note that for any $v_h \in V^h$, $\frac{\partial v_h}{\partial n}|_{\partial K}$ and $\triangle v_h|_K$ are constants, thus

$$(\nabla(c-\Pi c),\nabla v_h)_h = 0.$$

Here and later, $n=(n_1,n_2)$ denotes the unit outer norm on ∂K.

The proof is completed.

Lemma 3.3[19] Under anisotropic meshes, for any $u \in (H^3(\Omega)\cap H_0^1(\Omega))^2$, we have

$$\sum_{K\in T^h}\int_{\partial K}\frac{\partial u}{\partial n}v_h ds \le Ch^2|u|_3 \parallel v_h \parallel_h, \quad \forall v_h \in V^h.$$

Lemma 3.4[5] For any $\eta \in L^2(\Omega)$, let $\bar\eta = \eta(X - g(X)\triangle t)$, where function g and its gradient ∇g are bounded, then

$$\parallel \eta - \bar\eta \parallel_{-1} \le C \parallel \eta \parallel \triangle t.$$

IV. Error Estimate

Next we show the error estimate.

Let $e_h = c_h - \Pi c,\ \rho = \Pi c - c$.

Theorem 2. Let ϕ_h, ϕ be the solutions of (2) and (10) respectively, for sufficiently small $\triangle t > 0$, $\frac{\triangle t}{O(h)} \le C$, we have

$$\begin{aligned}&\max_{0\le n\le N} \parallel (\phi_h-\phi)(t^n) \parallel\\ &\le Ch^2|\phi_t|_{L^2(0,T;H^2)} + Ch^2|\phi|_{L^\infty(0,T;H^3)}\\ &+Ch|\phi|_{L^\infty(0,T;H^2)} + Ch|\phi|_{L^\infty(0,T;L^2)}.\end{aligned} \tag{24}$$

Proof: From (3) and (10), we get the error equation as follows:

$$\begin{aligned}&(\frac{e_h^n - e_h^{n-1}\circ X_1^n\gamma^n}{\triangle t}, v_h) + \nu(\nabla e_h^n, \nabla v_h)_h\\ &+\nu(\nabla(\Pi\phi^n - \phi^n),\nabla v_h)_h + \sum_{K\in T^h}\int_{\partial K}\nu\frac{\partial\phi^n}{\partial n}v_h ds\\ &= (\mathcal{L}_0\phi^n - \frac{\phi^n - \phi^{n-1}\circ X_1^n\gamma^n}{\triangle t}, v_h)\\ &-(\frac{\rho^n - \rho^{n-1}\circ X_1^n\gamma^n}{\triangle t}, v_h), \qquad \forall v_h \in V^h.\end{aligned} \tag{25}$$

Choosing $v_h = e_h^n$ in (25) yields

$$\begin{aligned}&(\frac{e_h^n - e_h^{n-1}\circ X_1^n\gamma^n}{\triangle t}, e_h^n) + \nu(\nabla e_h^n, \nabla e_h^n)_h\\ &+\nu(\nabla(\Pi\phi^n - \phi^n),\nabla e_h^n)_h + \sum_{K\in T^h}\int_{\partial K}\nu\frac{\partial\phi^n}{\partial n}e_h^n ds\\ &= (\mathcal{L}_0\phi^n - \frac{\phi^n - \phi^{n-1}\circ X_1^n\gamma^n}{\triangle t}, e_h^n)\\ &-(\frac{\rho^n - \rho^{n-1}\circ X_1^n\gamma^n}{\triangle t}, e_h^n), \qquad \forall v_h \in V^h.\end{aligned} \tag{26}$$

Next we estimate the right hand of (26).

$$\begin{aligned}&\parallel \mathcal{L}_0\phi^n - \frac{\phi^n - \phi^{n-1}\circ X_1^n\gamma^n}{\triangle t} \parallel\\ &= \parallel \frac{\partial\phi^n}{\partial t} + u^n(X,t)\cdot\nabla\phi^n + (\nabla\cdot u^n(X,t))\phi^n\\ &-\frac{\phi^n - \phi^{n-1}\circ X_1^n\gamma^n}{\triangle t} \parallel\\ &\le \parallel \frac{\partial\phi^n}{\partial t} + u^n(X,t)\cdot\nabla\phi^n - \frac{\phi^n - \phi^{n-1}\circ X_1^n}{\triangle t} \parallel\\ &+ \parallel (\nabla\cdot u^n(X,t))\phi^n - \frac{\phi^{n-1}\circ X_1^n(1-\gamma^n)}{\triangle t} \parallel\\ &= E_1 + E_2\end{aligned} \tag{27}$$

E_1 is evaluated with the method in [15], we can get

$$E_1 \le c\triangle t \parallel \phi^n \parallel \tag{28}$$

Due to $\frac{1-\gamma^n}{\triangle t} = \nabla\cdot u^n(X,t) + O(\triangle t)$,

$$\begin{aligned}E_2 = &\parallel \nabla\cdot u^n(X,t)(\phi^n - \phi^{n-1}\circ X_1^n)\\ &+O(\triangle t)\phi^{n-1}\circ X^n \parallel \le c\triangle t \parallel \phi^n \parallel .\end{aligned} \tag{29}$$

The first term on the right hand is estimated as

$$\begin{aligned}&|(\mathcal{L}_0\phi^n - \frac{\phi^n - \phi^{n-1}\circ X_1^n\gamma^n}{\triangle t}, e_h^n)|\\ &\le C \parallel \phi^n \parallel^2_{L^2(t^{n-1},t^n;L^2)} \triangle t^2.\end{aligned} \tag{30}$$

Next, we estimate the second term on the right hand of (26),

$$\begin{aligned}&(\frac{\rho^n - \rho^{n-1}\circ X_1^n\gamma^n}{\triangle t}, e_h^n)\\ &= (\frac{\rho^n - \rho^{n-1}\circ X_1^n}{\triangle t}, e_h^n) + (\frac{\rho^{n-1}\circ X_1^n(1-\gamma^n)}{\triangle t}, e_h^n)\\ &\le (\parallel \frac{\rho^n - \rho^{n-1}\circ X_1^n}{\triangle t} \parallel + c \parallel \rho^{n-1}\circ X_1^n \parallel) \parallel e_h^n \parallel\\ &\le (\parallel \frac{\rho^n - \rho^{n-1}}{\triangle t} \parallel + c \parallel \rho^{n-1} \parallel + c \parallel \rho^{n-1} \parallel_h) \parallel e_h^n \parallel\\ &\le C \parallel e_h^n \parallel^2 + \frac{C}{\triangle t} \parallel \rho_t \parallel^2_{L^2(t^{n-1},t^n;L^2)}\\ &+C \parallel \rho^{n-1} \parallel^2 + C \parallel \rho^{n-1} \parallel_h^2 .\end{aligned} \tag{31}$$

Next we estimate the left hand of (26).

Firstly, the first two terms on the left hand of (26) can be estimated as

$$\begin{aligned}&(\frac{e_h^n - e_h^{n-1}\circ X_1^n\gamma^n}{\triangle t}, e_h^n) + \nu(\nabla e_h^n, \nabla e_h^n)_h\\ &\ge \frac{1}{2\triangle t}[(e_h^n, e_h^n) - (e_h^{n-1}\circ X_1^n\gamma^n, e_h^{n-1}\circ X_1^n\gamma^n)]\\ &+\nu(\nabla e_h^n, \nabla e_h^n)_h \ge \frac{1}{2\triangle t}[(e_h^n, e_h^n) + \nu \parallel e_h^n \parallel_h^2\\ &-(1+C\triangle t)(e_h^{n-1}, e_h^{n-1})],\end{aligned} \tag{32}$$

where the inequality $\parallel e_h^{n-1}\circ X_1^n\gamma^n \parallel^2 \le (1+c\triangle t) \parallel e_h^{n-1} \parallel^2$ is used in the last step.

By Lemma 3.2, the third term of (26) is evaluated as

$$\nu(\nabla(\Pi\phi^n - \phi^n),\nabla e_h^n)_h = 0 \tag{33}$$

The last one can be estimated as

$$|\sum_{K\in T^h}\int_{\partial K}\nu\frac{\partial\phi^n}{\partial n}e_h^n ds| \le Ch^2|\phi^n|_3 \parallel e_h^n \parallel_h . \tag{34}$$

From (27)-(34), we have

$$\frac{1}{2\triangle t}[(e_h^n,e_h^n)-(1+C\triangle t)(e_h^{n-1},e_h^{n-1})]+\nu\parallel e_h^n\parallel_h^2$$
$$\leq Ch^2|\phi^n|_3\parallel e_h^n\parallel_h+C\triangle t^2\parallel\phi\parallel^2_{L^2(t^{n-1},t^n;L^2)}$$
$$+C\parallel e_h^n\parallel^2+\frac{C}{\triangle t}\parallel\rho_t\parallel^2_{L^2(t^{n-1},t^n;L^2)}$$
$$+C\parallel\rho^{n-1}\parallel_h^2+C\parallel\rho^{n-1}\parallel^2 . \quad (35)$$

Multiplying (35) by $2\triangle t$, and summing the above inequality from 1 to N, we obtain

$$\begin{aligned}&\parallel e_h^n\parallel^2\\ &\leq C\triangle t\parallel\rho\parallel_h^2+C\parallel\rho_t\parallel^2_{L^2(0,T;L^2)}\\ &+C\triangle t\parallel\rho\parallel^2_{L^\infty(0,T;L^2)}+Ch^4\triangle t|\phi|^2_{L^\infty(0,T;H^3)}\\ &+C\triangle t^3|\phi|^2_{L^\infty(0,T;L^2)}+C\triangle t\sum_{i=1}^n\parallel e_h^i\parallel^2 .\end{aligned} \quad (36)$$

By Gronwall's lemma, it follows that

$$\begin{aligned}&\parallel e_h^n\parallel\leq\\ &Ch^2\triangle t^{\frac{1}{2}}|\phi|_{L^\infty(0,T;H^3)}+Ch\triangle t^{\frac{1}{2}}|\phi|_{L^\infty(0,T;H^2)}\\ &+C\triangle t^{\frac{1}{2}}\parallel\rho\parallel_{L^\infty(0,T;L^2)}+C\parallel\rho_t\parallel_{L^2(0,T;L^2)}\\ &+C\triangle t^{\frac{3}{2}}|\phi|_{L^\infty(0,T;L^2)}.\end{aligned} \quad (37)$$

Note that $\phi_h^n-\phi^n=e_h^n+\rho^n$, by (37), Lemma 3.1 and the triangle inequality, we complete the proof.

V. Numerical Example

In order to investigate the numerical behavior of the element of this paper, we will give some numerical results to confirm our theoretical analysis.

$$\begin{cases}(a)\quad \phi_t+\phi_x+\phi_y-\nu\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}=f(x,y,t),\\ (b)\quad \phi(x,y,0)=(1-e^{-x(1-x)/\epsilon})(1-e^{-y(1-y)/\epsilon}),\\ (c)\quad \phi(x,y,t)=0,\end{cases} \quad (38)$$

where $\Omega=[0,1]\times[0,1]$, $\nu=0.02$, and the right hand term $f(x,y,t)$ is taken such that $c(x,y,t)=e^{-t}(1-e^{-x(1-x)/\epsilon})(1-e^{-y(1-y)/\epsilon})$ is the exact solution. ϵ denotes the singular perturbation parameter, when $\epsilon=0.05$, the exact solution exhibits four boundary layers.

We subdivide the boundary of Ω parallel to x-axis into n parts by the following two different ways. Mesh 1:square meshes; (illustrated by the Fig.1.) Mesh 2: anisotropic meshes, n+1 points: $(1-cos(i\pi/n))/2$, i=0, 1, ..., n and the same intervals along y-axis. (illustrated by the Fig.2.)

We approximate the integral $\int_K\phi_h^{n-1}\circ X_1^n\psi_h\gamma^n dx$ by the same numerical integration formula in [24].

For different space mesh size h, we give the following numerical results with the rectangular nonconforming finite element,(see Tables I and II), and conforming finite element,(see Tables II and IV).

Numerical results show that our method is of first-order accuracy for ϕ in L^2-norm, which is consistent with theoretical analysis.

Tables I-IV show the results which are in agreement with our investigation in section 4. From the comparison of the errors on the two different meshes, we can see that mesh 2 is in all cases more accurate than mesh 1 in L^2 norm. On square mesh, the results in the boundary region is not very good, because the solution vary significantly near the boundary,

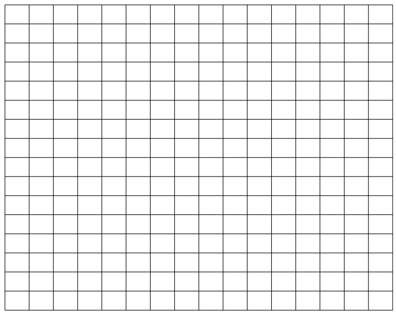

Fig. 1. Square Meshes

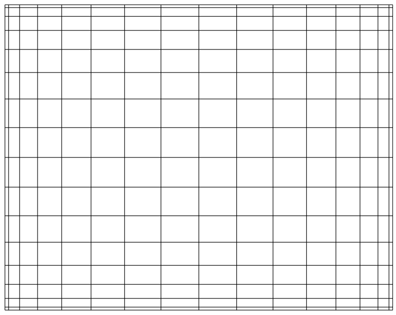

Fig. 2. Anisotropic Meshes

while Mesh 2 use anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution, so the results are more accurate.

We can see that the solution vary significantly near the boundary. By concentrating mesh elements in regions where more detail is required, an anisotropic mesh can reduce numerical errors and improve the solution's accuracy without a significant increase in the total number of elements. For more smaller ϵ, the mesh is more fitted to solving the equation.

VI. Conclusion

We have presented a new mass-conservative characteristic finite element scheme with the anisotropic nonconforming rectangular finite element. The scheme is unconditionally stable. For the problems with significant convection, the approximation of nonconforming finite element is appropriate. The nonconforming finite element has the practical advantage that each degree of freedom belongs to at most two elements, which may result in cheap local communication. The anisotropic interpolation operator combined with the mean value method is used instead of the elliptic projection in the previous literature, which has the practical difficulties in solving simultaneous equations. We have proved the stability and convergence of the scheme in this paper.

The results obtained in this paper are also valid for the rotated Q1 element on square meshes. From the structure of the element, we can see that, for all $v_h\in V^h$, $\frac{\partial v_h}{\partial n}|_{\partial K}$ and $\triangle v_h|_K$ are constants on square meshes, thus v_h satisfies the Lemma 3.2.

TABLE I
APPROXIMATION RESULTS OF ANISOTROPIC MESHES $||\phi - \phi_h||_{0,\Omega}$

$n\backslash t$	0.09566	0.19131	0.28697	0.38262	0.47828
8	0.397081	0.457982	0.562342	0.632980	0.695785
16	0.009454	0.015134	0.016199	0.029779	0.031230
32	0.005452	0.013123	0.017457	0.023817	0.026771
$n\backslash t$	0.57394	0.66960	0.76526	0.86092	0.95658
8	0.745678	0.783452	0.852190	0.897345	0.937651
16	0.033416	0.043598	0.047934	0.053493	0.074387
32	0.032323	0.035378	0.042387	0.046328	0.054523

TABLE II
CONVERGENCE ORDER OF ANISOTROPIC MESHES FOR $||\phi - \phi_h||_{0,\Omega}$

$n\backslash t$	0.09566	0.19131	0.28697	0.38262	0.47828
8	\	\	\	\	\
16	1.005119	1.26745	1.07111	0.68612	0.99684
32	1.010721	1.19911	0.87712	0.89294	1.05107
$n\backslash t$	0.57394	0.66960	0.76526	0.86092	0.95658
8	\	\	\	\	\
16	0.78777	0.95634	0.98865	0.79364	0.963413
32	0.88785	1.04634	1.02012	0.87354	1.055369

TABLE III
APPROXIMATION RESULTS OF SQUARE MESHES $||\phi - \phi_h||_{0,\Omega}$

$n\backslash t$	0.09566	0.19131	0.28697	0.38262	0.47828
8	1.845061	2.111789	2.524345	2.78345	2.790834
16	1.23494	1.245134	1.262349	1.287227	1.293113
32	1.080112	1.083278	1.087345	1.093713	1.096518
$n\backslash t$	0.57394	0.66960	0.76526	0.86092	0.95658
8	2.799958	3.030202	3.12356	3.234489	3.237337
16	1.303416	1.304353	1.304399	1.305265	1.307876
32	1.103789	1.103888	1.10487	0.104632	0.105452

TABLE IV
CONVERGENCE ORDER OF SQUARE MESHES FOR $||\phi - \phi_h||_{0,\Omega}$

$n\backslash t$	0.09566	0.19131	0.28697	0.38262	0.47828
8	\	\	\	\	\
16	0.051192	0.045452	0.01233	0.01524	0.067856
32	0.690231	0.566698	0.508441	0.528554	0.505056
$n\backslash t$	0.57394	0.66960	0.76526	0.86092	0.95658
8	\	\	\	\	\
16	0.076787	0.054356	0.15478	0.153546	0.378887
32	0.794245	0.780486	0.790901	0.786888	0.801223

In the forthcoming paper, we will present a corresponding scheme of nonconforming characteristic mixed finite element for the convection-diffusion problems.

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