Type-2 Picture Fuzzy Geometric Heronian Mean Operators with Applications in Decision Making

Yongwei Yang, Hongwei Wang, and Ting Qian

Abstract—**The type-2 picture fuzzy set represents an extension of Cuong's picture fuzzy set, where the aggregation operator is pivotal in aggregating the type-2 picture fuzzy information provided by decision makers. The Heronian mean not only takes into account the interrelationships among attribute values, but also considers the correlation between input arguments and themselves. Therefore, in order to aggregate type-2 picture fuzzy information more effectively, we put forward the Heronian mean under the type-2 picture fuzzy environment with the help of operations on type-2 picture fuzzy numbers. In this paper, the geometric Heronian mean operator in the classical category is extended to the type-2 picture fuzzy geometric Heronian mean aggregation operator and the type-2 picture fuzzy weighted geometric Heronian mean operator. Various fundamental properties of these operators, such as idempotency, monotonicity, and boundedness, are investigated. Additionally, a novel multiple attribute decision-making approach based on the type-2 picture fuzzy weighted geometric Heronian mean operator is introduced. Finally, an illustrative example of evaluating and selecting the best financial products to reduce risks for risk reduction in the type-2 picture fuzzy environment is given to demonstrate the usefulness and effectiveness of the developed method. After that, the comparative analysis with other techniques is utilized to demonstrate the consistency and superiority of the recommended approach.**

Index Terms—**Tyep-2 picture fuzzy set, Aggregation operator, Heronian mean operator, MADM.**

I. Introduction

THE multiple attribute decision making (MADM) technology is a complex decision-making process that evaluates and selects an optimal and reliable scheme based on specific criteria or attributes. During the MADM process, decision makers need to analyze and evaluate different schemes according to their individual preferences, so as to choose the most appropriate option. However, the MADM process often encounters many complexities and challenges, with a major problem being the presence of incomplete and redundant information. Particularly when dealing with human opinions and subjective judgments, this uncertainty becomes more pronounced. Decision makers may have to navigate through vague, uncertain, and even conflicting information to arrive at the most favorable decision. To tackle the ambiguity and uncertainty inherent in human opinions, Zadeh introduced the concept of fuzzy sets [1]. In the fields of artificial intelligence and data processing, fuzzy sets serve as a crucial mathematical tool for handling cases with unclear or ambiguous boundaries. Essentially, fuzzy sets allow the membership of elements to a set with different degrees, rather than the binary inclusion of traditional set theory. This methodology proves particularly useful for dealing with inaccurate and uncertain information, such as human language or opinions.

Manuscript received August 6, 2024; revised January 10, 2025. This work was supported in part by the Key Scientific Research Program of Universities Funded by Henan Province (No. 24B110002), Natural Science Foundation of Henan Province (No. 242300420239), Natural Science Basic Research Plan in Shaanxi Province of China (No.2023-JC-YB-027) and Natural Science Foundation of Hubei Province (No. 2024AFB846).

Yongwei Yang is an associate professor at the School of Mathematics and Computer Science, Tongling University, Tongling 244061, China. (corresponding author to provide e-mail: yangyw2010@126.com).

Hongwei Wang is a professor at the School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China. (e-mail: wanghwxxu@gmail.com).

Ting Qian is an associate professor at the College of Science, Xi'an Shiyou University, Xi'an, 710065, China. (e-mail: qiant2000@126.com).

However, in many real-world applications, it is also necessary to consider the degree of non-membership, which quantifies the extent to which an element does not belong to a set. To address this, Atanassov [2] proposed intuitionsitic fuzzy sets by introducing a second parameter, the non-membership degree, denoted as $\nu(x)$, which also ranges from 0 to 1. As an extension of traditional fuzzy sets, the intuitionistic fuzzy set defines membership and non-membership degrees for each element in the universe of discourse, such that for any element x, $0 \leq \mu(x) + \nu(x) \leq 1$. The difference $1 - \mu(x) - \nu(x)$ represents a measure of uncertainty or hesitancy about an element's membership in the set, often denoted as $\pi(x)$. This hesitancy index captures the ambiguity or lack of knowledge in the available information about the element. Intuitionistic fuzzy sets have been applied in various fields, such as decision making [3], [4], pattern recognition [5], and medical diagnosis [6], offering a more nuanced representation of uncertainty and vagueness compared to traditional fuzzy sets.

Traditional fuzzy sets and intuitionsitic fuzzy sets primarily consider membership and non-membership degrees, neglecting the significance of neutrality in various decision-making contexts, such as medical diagnosis, personnel selection, and voting. In order to address limitations of fuzzy sets and intuitionistic fuzzy sets, Cuong [7] introduced a new generalization of fuzzy sets called picture fuzzy sets. In a picture fuzzy set, each element x is associated with three degrees: membership $\mu(x)$, non-membership $\nu(x)$, and neutrality $\eta(x)$, all of which lie within the interval $[0, 1]$. The sum of these degrees is constrained to be less than or equal to 1 ($\mu(x)+\nu(x)+\eta(x) \leq 1$). This triadic representation allows picture fuzzy sets to capture more nuanced information, accommodating not only affirmative and negative stances but also neutral positions that are often present in real-world contexts. Basic operations, such as union, intersection, and complement, have been defined for picture fuzzy sets to broaden their applicability in various analytical contexts. Consequently, the versatility and expressiveness of picture fuzzy sets have attracted considerable attention from researchers across various disciplines, leading to a proliferation of studies exploring their theoretical foundations and practical applications. The capacity of picture fuzzy sets to represent neutral attitudes, alongside affirmative and negative stances, makes them particularly well-suited for handling complex decision-making problems that inherently involve such nuances. By proposing two correlation coefficients, Singh [8] enabled the analysis of the strength of association between different picture fuzzy sets, which is crucial in many decision making contexts. Similarly, the work of Ganie et al. [9] further refined two new correlation coefficients for picture fuzzy sets receiving their values in the closed interval $[-1, 1]$ to express the nature the nature of correlation more appropriately. Moreover, the research by He and Wang in 2023 demonstrates the practical applicability of picture fuzzy sets in evaluating new energy vehicles based on online reviews [10]. By leveraging the picture fuzzy number operation law proposed by Wang et al. in 2018 [11], they were able to develop an evaluation analysis method that accounts for the complexity and diversity of opinions expressed in online reviews. This work underscores the potential of picture fuzzy sets in extracting valuable insights from large and heterogeneous datasets, which is crucial in today's data-driven world. Collectively, these research efforts have significantly expanded the scope and impact of picture fuzzy sets. Traditional distances between fuzzy sets often have limitations such as not meeting the axiomatic definition of distance or producing counter-intuitive results, to overcome these limitations, Luo and Zhang [12] gave a new distance between picture fuzzy sets by aggregating three-dimensional divergence. The proposed distance measure satisfies the axiomatic definition of distance, ensuring that it behaves as expected in mathematical and logical terms. Additionally, it overcomes the counter-intuitive defects of some traditional distance measures, producing more intuitive and meaningful results. Chitra and Prabakaran [13] present an approach to order the q̃-rung picture fuzzy numbers, and further deduced q̃-rung picture fuzzy ordered Frank weighted arithmetic and geometric accumulation operators. These research results not only deepen our theoretical understanding of picture fuzzy sets, but also demonstrate their practical usefulness in a wide range of domains, including medical diagnosis [14], personnel selection [15], and many others [16]. As research in this area continues to evolve, we can expect to see more innovative applications of picture fuzzy sets, which will help us better cope with the complexity and uncertainty of the real world.

In the voting model, the presence of invalid votes or preferences is often an important factor that requires separate consideration from acceptance or rejection. Invalid votes may indicate errors in the voting process, voter misunderstanding, or other factors that affect the validity of the vote. Cuong's picture fuzzy set (or type-1 picture fuzzy sets) fails to capture this distinction by lumping invalid votes into the refusal membership degree, potentially leading to biased or inaccurate decision outcomes. To address this limitation, Yang [17] proposed the concept of refined picture fuzzy sets, referred to as type-2 picture fuzzy sets, which explicitly model the "invalid" degree as a separate dimension. By incorporating the invalid membership degree, type-2 picture fuzzy sets provide a more comprehensive framework for addressing decision making problems in voting models.

An aggregation operator is a mathematical function or a set of rules that are used to combine multiple values or evaluations into a single value or result. In the context of multi-attribute decision making, aggregation operators play a crucial role in combining the performance scores or evaluations of different attributes for various alternatives. The choice of an appropriate aggregation operator is important because it can significantly impact the final decision outcome. Different aggregation operators may emphasize different aspects of the evaluations and may result in different rankings or choices of alternatives. In recent years, the field of aggregation operators has seen a significant growth in research, with various types of aggregation operators being proposed and applied in diverse domains. Picture fuzzy sets, including both type-1 and type-2 picture fuzzy sets, offer a more nuanced and flexible way to represent uncertain and vague information compared to traditional fuzzy sets. However, the aggregation of picture fuzzy information is more complex and requires specialized aggregation operators ([18], [19], [20]).

Motivated by the concept of geometric aggregation operators for type-1 picture fuzzy numbers [21], Yang and Li extended this idea to type-2 picture fuzzy sets by defining new operations and proposing a series of arithmetic aggregation operators specifically tailored to this type of fuzzy sets [22]. The traditional aggregation operators used in type-1 picture fuzzy multi-attribute decision making methods often fail to capture the complex relationships between multiple attributes. This limitation can lead to suboptimal decision results because important interactions and dependencies between attributes are overlooked. Heronic mean provides a more sophisticated and comprehensive approach to aggregating situational fuzzy information by incorporating the correlation between input arguments and their relationships. This property makes it particularly suitable for decision making scenarios where the relationships between attributes are important. The extension of the geometric Heronian mean operator to the type-2 picture fuzzy environment is a significant advancement in the field of multi-attribute decision making, especially for handling complex and uncertain information.

The geometric Heronian mean operator, which inherently considers both the mutuality between attribute values and the correlation between input arguments and themselves, is well-suited for handling the inherent uncertainties and ambiguities of type-2 picture fuzzy sets. By incorporating the operations on type-2 picture fuzzy numbers, we have developed type-2 picture fuzzy geometric Heronian mean aggregation operators and the type-2 picture fuzzy weighted geometric Heronian mean operators that can capture the complex relationships between attributes and provide more reliable decision making outcomes. The basic properties of these operators, such as idempotency, monotonicity, and boundedness are obtained. These properties ensure that the aggregation results are consistent, predictable, and within a reasonable range, which is crucial for making informed decisions. The new multiple attribute decision making method based on the proposed type-2 picture fuzzy weighted geometric Heronian mean operator provides a practical and effective way to evaluate and choose the best financial products to reduce risks in the type-2 picture fuzzy environment. By comparing the performance of the proposed method with existing methods, we have demonstrated that the proposed method is able to handle the inherent uncertainties and ambiguities of type-2 picture fuzzy sets more effectively and provide more reliable decision outcomes.

II. Preliminaries

In the section, we recall some fundamental notions associated with picture fuzzy sets. By incorporating an extra membership function to intuitionistic fuzzy sets, namely the degree of neutral membership function, the type-1 picture fuzzy set is obtained.

Definition 2.1: [7] Consider X as the universe of discourse. A type-1 picture fuzzy set A on the universe X, is characterized by objects in the form

$$A = \{< x, \mu_A(x), \eta_A(x), \nu_A(x) > | x \in X\},$$

where $\mu_A(x)$ ranges from 0 to 1 and is termed as the positive membership degree of x, $\eta_A(x)$ ranges from 0 to 1 and represents the neutral membership degree, $\nu_A(x)$ lies within $[0, 1]$ and stands for the negative membership degree, and they satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1.$$

Yang proposed an enhanced version of Cuong's picture fuzzy sets by considering the "invalid" degree as a separate membership function.

Definition 2.2: [17] Let X be the universe of discourse. A type-2 picture fuzzy set, denoted as A in the universe X, can be represented as an object in the form:

$$A = \{< x, \mu_A(x), \eta_A(x), \nu_A(x), \vartheta_A(x) > | x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is termed as the neutral membership degree of x, $\eta_A(x) \in [0, 1]$ is called the neutral membership degree, $\nu_A(x) \in [0, 1]$ is referred to as the negative membership degree, $\vartheta_A(x)$ ranges from 0 to 1 and is known as the invalid membership degree. These membership degrees satisfy

$$\mu_A(x) + \eta_A(x) + \nu_A(x) + \vartheta_A(x) \leq 1.$$

For any $x \in X$,

$$\rho_A(x) = 1 - \big(\mu_A(x) + \eta_A(x) + \nu_A(x) + \vartheta_A(x)\big)$$

could be called the refusal membership degree of x in A. And

$$\tau_A(x) = \eta_A(x) + \vartheta_A(x)$$

could be called the degree of indefinitely determined membership of x in A.

If the invalid membership degree $\vartheta_A(x)$ is equal to 0 for any $x \in X$, then the type-2 picture fuzzy set is reduced to a type-1 picture fuzzy set. Thus, the type-2 picture fuzzy set can be viewed as an extension or generalization of the type-1 picture fuzzy set.

The quadruplet $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ is commonly referred as a type-2 picture fuzzy number for convenience. It is required that $\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha \in [0, 1]$ and that their sum does not exceed 1.

Definition 2.3: [22] Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ represent a type-2 picture fuzzy number.

1) The score function S of a type-2 picture fuzzy number, which quantifies the degree of fuzziness, is expressed as:
$$S(\alpha) = \mu_\alpha - \nu_\alpha.$$
2) The accuracy function H of a type-2 picture fuzzy number, measures the precision of the fuzzy number and is defined by:
$$H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha.$$
3) The participation function P reflects the overall involvement of the fuzzy number and is given by:
$$P(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha + \vartheta_\alpha.$$

In the context of type-2 picture fuzzy numbers, a robust comparison approach is indispensable for establishing rankings and facilitating decision-making processes. Based on score functions, accuracy functions, and participation functions, we provide a comprehensive framework for evaluating and comparing these complex fuzzy numbers.

Definition 2.4: [22] For any two type-2 picture fuzzy numbers α and β,

1) If $S(\alpha) < S(\beta)$, then β is considered superior to α, which we denote as $\alpha < \beta$.
2) If $S(\alpha) = S(\beta)$,
   a) $H(\alpha) < H(\beta)$, then $\alpha < \beta$.
   b) $H(\alpha) = H(\beta)$,
      i) $P(\alpha) < P(\beta)$, then $\alpha < \beta$.
      ii) $P(\alpha) = P(\beta)$, then $\alpha = \beta$.

Several operational rules for type-2 picture fuzzy numbers are provided as follows.

Definition 2.5: Ler $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \vartheta_\beta)$ be any two type-2 picture fuzzy numbers, and $\lambda > 0$.

1) The complement α^c of α is defined as: $\alpha^c = (\nu_\alpha, \eta_\alpha, \mu_\alpha, \vartheta_\alpha)$;
2) The partial order relation $\alpha \leq_p \beta$ is established if and only if the conditions $\mu_\alpha \leq \mu_\beta, \eta_\alpha \leq \eta_\beta, \nu_\alpha \geq \nu_\beta$ and $\vartheta_\alpha \leq \vartheta_\beta$ are simultaneously satisfied;
3) $$\alpha \oplus \beta = \begin{pmatrix} 1 - (1 - \mu_\alpha)(1 - \mu_\beta), \\ \eta_\alpha \eta_\beta, \\ \nu_\alpha \nu_\beta, \\ (\eta_\alpha + \vartheta_\alpha)(\eta_\beta + \vartheta_\beta) - \eta_\alpha \eta_\beta \end{pmatrix} = \begin{pmatrix} 1 - (1 - \mu_\alpha)(1 - \mu_\beta), \\ \eta_\alpha \eta_\beta, \\ \nu_\alpha \nu_\beta, \\ \tau_\alpha \tau_\beta - \eta_\alpha \eta_\beta \end{pmatrix};$$
4) $$\alpha \otimes \beta = \begin{pmatrix} \mu_\alpha \mu_\beta \\ \eta_\alpha \eta_\beta, \\ 1 - (1 - \nu_\alpha)(1 - \nu_\beta), \\ (\eta_\alpha + \vartheta_\alpha)(\eta_\beta + \vartheta_\beta) - \eta_\alpha \eta_\beta \end{pmatrix} = \begin{pmatrix} \mu_\alpha \mu_\beta, \\ \eta_\alpha \eta_\beta, \\ 1 - (1 - \nu_\alpha)(1 - \nu_\beta), \\ \tau_\alpha \tau_\beta - \eta_\alpha \eta_\beta \end{pmatrix};$$
5) $$\lambda \alpha = \begin{pmatrix} 1 - (1 - \mu_\alpha)^\lambda \\ \eta_\alpha^\lambda, \\ \nu_\alpha^\lambda, \\ (\eta_\alpha + \vartheta_\alpha)^\lambda - \eta_\alpha^\lambda \end{pmatrix} = \begin{pmatrix} 1 - (1 - \mu_\alpha)^\lambda \\ \eta_\alpha^\lambda, \\ \nu_\alpha^\lambda, \\ \tau_\alpha^\lambda - \eta_\alpha^\lambda \end{pmatrix};$$

6) $\alpha^{\lambda}=\begin{pmatrix}\mu_{\alpha}^{\lambda},\\ \eta_{\alpha}^{\lambda},\\ 1-(1-\nu_{\alpha})^{\lambda},\\ (\eta_{\alpha}+\vartheta_{\alpha})^{\lambda}-\eta_{\alpha}^{\lambda}\end{pmatrix}=\begin{pmatrix}\mu_{\alpha}^{\lambda},\\ \eta_{\alpha}^{\lambda},\\ 1-(1-\nu_{\alpha})^{\lambda},\\ \tau_{\alpha}^{\lambda}-\eta_{\alpha}^{\lambda}\end{pmatrix}$.

Theorem 2.6: Let $\alpha=(\mu_{\alpha},\eta_{\alpha},\nu_{\alpha},\vartheta_{\alpha})$ and $\beta=(\mu_{\beta},\eta_{\beta},\nu_{\beta},\vartheta_{\beta})$ be two type-2 picture fuzzy numbers and $\lambda,\lambda_1,\lambda_2>0$. Then

1) $\alpha\oplus\beta=\beta\oplus\alpha$,
2) $\alpha\otimes\beta=\beta\otimes\alpha$,
3) $\lambda\alpha\oplus\lambda\beta=\lambda(\alpha\oplus\beta)$,
4) $\lambda_1\alpha\oplus\lambda_2\alpha=(\lambda_1+\lambda_2)\alpha$,
5) $\lambda_1(\lambda_2\alpha)=(\lambda_1\lambda_2)\alpha$,
6) $(\alpha^{\lambda_1})^{\lambda_2}=\alpha^{\lambda_1\lambda_2}$.

Proof: 1) and 2) are obvious.

3) According to Definition 2.5, we get

$$\lambda\alpha=\begin{pmatrix}1-(1-\mu_{\alpha})^{\lambda}\\ \eta_{\alpha}^{\lambda},\\ \nu_{\alpha}^{\lambda},\\ \tau_{\alpha}^{\lambda}-\eta_{\alpha}^{\lambda}\end{pmatrix},\ \lambda\beta=\begin{pmatrix}1-(1-\mu_{\beta})^{\lambda}\\ \eta_{\alpha}^{\beta},\\ \nu_{\beta}^{\lambda},\\ \tau_{\beta}^{\lambda}-\eta_{\beta}^{\lambda}\end{pmatrix}$$

and

$$\alpha\oplus\beta=\begin{pmatrix}1-(1-\mu_{\alpha})(1-\mu_{\beta}),\\ \eta_{\alpha}\eta_{\beta},\\ \nu_{\alpha}\nu_{\beta},\\ (\eta_{\alpha}+\vartheta_{\alpha})(\eta_{\beta}+\vartheta_{\beta})-\eta_{\alpha}\eta_{\beta}\end{pmatrix}.$$

Thus, we can obtain

$$\lambda\alpha\oplus\lambda\beta=\begin{pmatrix}1-(1-\mu_{\alpha})^{\lambda}(1-\mu_{\beta})^{\lambda}\\ \eta_{\alpha}^{\lambda}\eta_{\beta}^{\lambda},\\ \nu_{\alpha}^{\lambda}\nu_{\beta}^{\lambda},\\ \tau_{\alpha}^{\lambda}\tau_{\beta}^{\lambda}-\eta_{\alpha}^{\lambda}\eta_{\beta}^{\lambda},\end{pmatrix}=\lambda(\alpha\oplus\beta).$$

4) First of all, we have

$$\lambda_1\alpha=\begin{pmatrix}1-(1-\mu_{\alpha})^{\lambda_1}\\ \eta_{\alpha}^{\lambda_1},\\ \nu_{\alpha}^{\lambda_1},\\ \tau_{\alpha}^{\lambda_1}-\eta_{\alpha}^{\lambda_1}\end{pmatrix},\quad \lambda_2\alpha=\begin{pmatrix}1-(1-\mu_{\alpha})^{\lambda_2}\\ \eta_{\alpha}^{\lambda_2},\\ \nu_{\alpha}^{\lambda_2},\\ \tau_{\alpha}^{\lambda_2}-\eta_{\alpha}^{\lambda_2}\end{pmatrix},$$

then by the definition of $\oplus$, we get that

$$\lambda_1\alpha\oplus\lambda_2\alpha=\begin{pmatrix}1-(1-\mu_{\alpha})^{\lambda_1+\lambda_2}\\ \eta_{\alpha}^{\lambda_1+\lambda_2},\\ \nu_{\alpha}^{\lambda_1+\lambda_2},\\ \tau_{\alpha}^{\lambda_1+\lambda_2}-\eta_{\alpha}^{\lambda_1+\lambda_2}\end{pmatrix}=(\lambda_1+\lambda_2)\alpha.$$

5) It is easy to get that $\lambda_2\alpha=\begin{pmatrix}1-(1-\mu_{\alpha})^{\lambda_2}\\ \eta_{\alpha}^{\lambda_2},\\ \nu_{\alpha}^{\lambda_2},\\ \tau_{\alpha}^{\lambda_2}-\eta_{\alpha}^{\lambda_2}\end{pmatrix}$. And so, $\lambda_1(\lambda_2\alpha)=\begin{pmatrix}1-(1-\mu_{\alpha})^{\lambda_1\lambda_2}\\ \eta_{\alpha}^{\lambda_1\lambda_2},\\ \nu_{\alpha}^{\lambda_1\lambda_2},\\ \tau_{\alpha}^{\lambda_1\lambda_2}-\eta_{\alpha}^{\lambda_1\lambda_2}\end{pmatrix}=(\lambda_1\lambda_2)\alpha$.

6) Since $\alpha^{\lambda_1}=\begin{pmatrix}\mu_{\alpha}^{\lambda_1},\\ \eta_{\alpha}^{\lambda_1},\\ 1-(1-\nu_{\alpha})^{\lambda_1},\\ \tau_{\alpha}^{\lambda_1}-\eta_{\alpha}^{\lambda_1}\end{pmatrix}$, then we have

$$(\alpha^{\lambda_1})^{\lambda_2}=\begin{pmatrix}\mu_{\alpha}^{\lambda_1\lambda_2},\\ \eta_{\alpha}^{\lambda_1\lambda_2},\\ 1-(1-\nu_{\alpha})^{\lambda_1\lambda_2},\\ \tau_{\alpha}^{\lambda_1\lambda_2}-\eta_{\alpha}^{\lambda_1\lambda_2}\end{pmatrix}=(\alpha^{\lambda_1}\lambda_2).$$

■

Definition 2.7: [23] Let x_i $(i=1,2,\cdots,n)$ be a set of non-negative real numbers and p,q be non-negative real numbers. Then the geometric Heronian mean between x_i is mathematically calculated using the following formula:

$$\mathrm{HM}(x_1,x_2,\cdots,x_n)=\frac{1}{p+q}\Big(\prod_{i=1,j=i}^{n}(px_i+qx_j)^{\frac{2}{n(n+1)}}\Big).$$

III. Heronian Mean Aggregation operators Based on Type-2 Picture Fuzzy Sets

In this section, we introduce the notions of type-2 picture fuzzy geometric Heronian mean aggregation operators and type-2 picture fuzzy weighted geometric Heronian mean operators, and study some fundamental properties associated with them.

Definition 3.1: Let $\{\alpha_i\}_{i=1}^{n}$ be a set of type-2 picture fuzzy numbers, with p,q being non-negative real numbers that do not both equal zero simultaneously. If

$$\mathrm{T2HM}\big(\alpha_1,\alpha_2,\cdots,\alpha_n\big)=\frac{1}{p+q}\Big(\bigotimes_{i=1,j=i}^{n}(p\alpha_i\oplus q\alpha_j)^{\frac{2}{n(n+1)}}\Big)$$

then T2HM is called a type-2 picture fuzzy geometric Heronian mean aggregation operator.

Based on the operations of type-2 picture fuzzy numbers in Definition 2.5, we can derive Theorem 3.2.

Theorem 3.2: Given a set of type-2 picture fuzzy numbers $\alpha_i=(\mu_{\alpha_i},\eta_{\alpha_i},\nu_{\alpha_i},\vartheta_{\alpha_i})$ $(i=1,2,\cdots,n)$, the result obtained through the application of the type-2 picture fuzzy geometric Heronian mean aggregation operator is also a type-2 picture fuzzy number. Specifically, the aggregated value has the following characteristics: $\mathrm{T2HM}\big(\alpha_1,\alpha_2,\cdots,\alpha_n\big)$

$$=\begin{pmatrix}1-\left(1-\prod_{i=1,j=i}^{n}\Big(1-(1-\mu_{\alpha_i})^{p}(1-\mu_{\alpha_j})^{q}\Big)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}},\\ \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^{p}\eta_{\alpha_j}^{q})^{\frac{2}{n(n+1)(p+q)}},\\ \Big(1-\prod_{i=1,j=i}^{n}(1-\nu_{\alpha_i}^{p}\nu_{\alpha_j}^{q})^{\frac{2}{n(n+1)}}\Big)^{\frac{1}{p+q}},\\ \prod_{i=1,j=i}^{n}\Big(\tau_{\alpha_i}^{p}\tau_{\alpha_j}^{q}\Big)^{\frac{2}{n(n+1)(p+q)}}-\prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^{p}\eta_{\alpha_j}^{q})^{\frac{2}{n(n+1)(p+q)}}\end{pmatrix}.$$

Proof: By the operational laws for type-2 picture fuzzy numbers, we have

$$p\alpha_i=\begin{pmatrix}1-(1-\mu_{\alpha_i})^{p},\\ \eta_{\alpha_i}^{p},\\ \nu_{\alpha_i}^{p},\\ \tau_{\alpha_i}^{p}-\eta_{\alpha_i}^{p}\end{pmatrix},$$

and

$$q\alpha_j = \begin{pmatrix} 1-(1-\mu_{\alpha_j})^q \\ \eta_{\alpha_j}^q, \\ \nu_{\alpha_j}^q, \\ \tau_{\alpha_j}^p - \eta_{\alpha_j}^p \end{pmatrix}.$$

Then

$$p\alpha_i \oplus q\alpha_j = \begin{pmatrix} 1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q, \\ \eta_{\alpha_i}^p\eta_{\alpha_j}^q, \\ \nu_{\alpha_i}^p\nu_{\alpha_j}^q, \\ \tau_{\alpha_i}^p\tau_{\alpha_j}^q - \eta_{\alpha_i}^p\eta_{\alpha_j}^q \end{pmatrix},$$

and so

$$(p\alpha_i \oplus q\alpha_j)^{\frac{2}{n(n+1)}} = \begin{pmatrix} \left(1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q\right)^{\frac{2}{n(n+1)}}, \\ (\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)}}, \\ 1-(1-\nu_{\alpha_i}^p\nu_{\alpha_j}^q)^{\frac{2}{n(n+1)}}, \\ (\tau_{\alpha_i}^p\tau_{\alpha_j}^q)^{\frac{2}{n(n+1)}} - (\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)}} \end{pmatrix},$$

$$\bigotimes_{i=1,j=i}^{n} (p\alpha_i \oplus q\alpha_j)^{\frac{2}{n(n+1)}} = \begin{pmatrix} \prod_{i=1,j=i}^{n} \left(1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q\right)^{\frac{2}{n(n+1)}}, \\ \prod_{i=1,j=i}^{n} (\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)}}, \\ 1-\prod_{i=1,j=i}^{n}(1-\nu_{\alpha_i}^p\nu_{\alpha_j}^q)^{\frac{2}{n(n+1)}}, \\ \prod_{i=1,j=i}^{n} (\tau_{\alpha_i}^p\tau_{\alpha_j}^q)^{\frac{2}{n(n+1)}} - \prod_{i=1,j=i}^{n} (\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)}} \end{pmatrix}.$$

Then we get that

$$\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n) = \frac{1}{p+q}\left(\bigotimes_{i=1,j=i}^{n}(p\alpha_i\oplus q\alpha_j)^{\frac{2}{n(n+1)}}\right) = \begin{pmatrix} 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}}, \\ \left(1-\prod_{i=1,j=i}^{n}(1-\nu_{\alpha_i}^p\nu_{\alpha_j}^q)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}\left(\tau_{\alpha_i}^p\tau_{\alpha_j}^q\right)^{\frac{2}{n(n+1)(p+q)}} - \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}} \end{pmatrix},$$

which completes the proof. ■

Theorem 3.3: (Idempotency) Consider a set of type-2 picture fuzzy numbers $\{\alpha_i\}_{i=1}^n$ such that all elements are identical, i.e., $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha=(\mu_\alpha,\eta_\alpha,\nu_\alpha,\vartheta_\alpha)$, then

$$\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\alpha.$$

Proof: By Theorem 3.2, we can conclude that

$$\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n) = \begin{pmatrix} 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}}, \\ \left(1-\prod_{i=1,j=i}^{n}(1-\nu_{\alpha_i}^p\nu_{\alpha_j}^q)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}\left(\tau_{\alpha_i}^p\tau_{\alpha_j}^q\right)^{\frac{2}{n(n+1)(p+q)}} - \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}} \end{pmatrix},$$

Given that $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha=(\mu_\alpha,\eta_\alpha,\nu_\alpha,\vartheta_\alpha)$, it follows that $\mu_{\alpha_1}=\mu_{\alpha_2}=\cdots=\mu_{\alpha_n}=\mu_\alpha$, and so

$$\begin{aligned} &1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \\ &=1-\left(1-\left(\left(1-(1-\mu_\alpha)^{p+q}\right)^{\frac{n(n+1)}{2}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \\ &=1-\left((1-\mu_\alpha)^{p+q}\right)^{\frac{1}{p+q}} \\ &=\mu_\alpha. \end{aligned}$$

Similarly,

$$\prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}}=\eta_\alpha,$$

$$\left(1-\prod_{i=1,j=i}^{n}(1-\nu_{\alpha_i}^p\nu_{\alpha_j}^q)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}=\nu_\alpha,$$

and

$$\prod_{i=1,j=i}^{n}\left(\tau_{\alpha_i}^p\tau_{\alpha_j}^q\right)^{\frac{2}{n(n+1)(p+q)}} - \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}}=\vartheta_\alpha.$$

Hence, $\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\alpha$. ■

Theorem 3.4: (Monotonicity) Let α_i $(i=1,2,\cdots,n)$ and $\beta_i(i=1,2,\cdots,n)$ be collections of type-2 picture fuzzy numbers. If $\alpha_i \leq_p \beta_i$ for all i, then

$$\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n) \leq \text{T2HM}(\beta_1,\beta_2,\cdots,\beta_n).$$

Proof: To facilitate our discussion, according to Theorem 3.2, we set $\alpha^*=\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\beta^*=\text{T2HM}(\beta_1,\beta_2,\cdots,\beta_n)$. Then

$$\alpha^*=(\mu_{\alpha^*},\eta_{\alpha^*},\nu_{\alpha^*},\vartheta_{\alpha^*})=\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n) = \begin{pmatrix} 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}}, \\ \left(1-\prod_{i=1,j=i}^{n}(1-\nu_{\alpha_i}^p\nu_{\alpha_j}^q)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}\left(\tau_{\alpha_i}^p\tau_{\alpha_j}^q\right)^{\frac{2}{n(n+1)(p+q)}} - \prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}} \end{pmatrix}$$

and

$$\beta^*=(\mu_{\beta^*},\eta_{\beta^*},\nu_{\beta^*},\vartheta_{\beta^*})=\text{T2HM}(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

$$=\begin{pmatrix} 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_{\beta_i})^p(1-\mu_{\beta_j})^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}(\eta_{\beta_i}^p\eta_{\beta_j}^q)^{\frac{2}{n(n+1)(p+q)}}, \\ \left(1-\prod_{i=1,j=i}^{n}(1-\nu_{\beta_i}^p\nu_{\beta_j}^q)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n}\left(\tau_{\beta_i}^p\tau_{\beta_j}^q\right)^{\frac{2}{n(n+1)(p+q)}}-\prod_{i=1,j=i}^{n}(\eta_{\beta_i}^p\eta_{\beta_j}^q)^{\frac{2}{n(n+1)(p+q)}} \end{pmatrix}.$$

Since $\alpha_j \leq_p \beta_j$ for all j, that is, $\mu_{\alpha_j} \leq \mu_\beta, \eta_{\alpha_j} \leq \eta_\beta, \nu_{\alpha_j} \geq \nu_\beta, \vartheta_{\alpha_j} \leq \vartheta_\beta$, we have

$$1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}$$
$$\leq 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_{\beta_i})^p(1-\mu_{\beta_j})^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}},$$

$$\prod_{i=1,j=i}^{n}(\eta_{\alpha_i}^p\eta_{\alpha_j}^q)^{\frac{2}{n(n+1)(p+q)}} \leq \prod_{i=1,j=i}^{n}(\eta_{\beta_i}^p\eta_{\beta_j}^q)^{\frac{2}{n(n+1)(p+q)}},$$

$$\left(1-\prod_{i=1,j=i}^{n}(1-\nu_{\alpha_i}^p\nu_{\alpha_j}^q)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}$$
$$\geq \left(1-\prod_{i=1,j=i}^{n}(1-\nu_{\beta_i}^p\nu_{\beta_j}^q)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}},$$

$$\prod_{i=1,j=i}^{n}\left(\tau_{\alpha_i}^p\tau_{\alpha_j}^q\right)^{\frac{2}{n(n+1)(p+q)}} \leq \prod_{i=1,j=i}^{n}\left(\tau_{\beta_i}^p\tau_{\beta_j}^q\right)^{\frac{2}{n(n+1)(p+q)}}.$$

That is, $\mu_{\alpha^*} \leq \mu_{\beta^*}, \eta_{\alpha^*} \leq \eta_{\beta^*}, \nu_{\alpha^*} \geq \nu_{\beta^*}$ and $\vartheta_{\alpha^*} + \eta_{\alpha^*} \leq \vartheta_{\beta^*} + \eta_{\beta^*}$

Therefore, by the score function of type-2 picture fuzzy numbers, we have $S(\alpha^*) \leq S(\beta^*)$

And we consider the following cases:

1) If $S(\alpha^*) < S(\beta^*)$, according to Definition 2.3, we conclude that
$$\alpha^* < \beta^*.$$
2) If $\mathrm{S}(\alpha^*) = S(\beta^*)$, then we get that
$$\mu_{\alpha^*} = \mu_{\beta^*},$$
and
$$\nu_{\alpha^*} = \nu_{\beta^*}.$$
Observing that $\eta_{\alpha^*} \leq \eta_{\beta^*}$, we get that $H(\alpha^*) \leq H(\beta^*)$ based on the accuracy function of type-2 picture fuzzy numbers. Now we consider the following two sub-cases:
   a) If $H(\alpha^*) = H(\beta^*)$,then it follows that
   $$\alpha^* < \beta^*.$$
   b) If $H(\alpha^*) < H(\beta^*)$, then it follows that
   $$\eta_{\alpha^*} = \eta_{\beta^*}.$$
   Since $\vartheta_{\alpha^*}+\eta_{\alpha^*} \leq \vartheta_{\beta^*}+\eta_{\beta^*}$, then $\vartheta_{\alpha^*} \leq \vartheta_{\beta^*}$. By the participation function of type-2 picture fuzzy numbers, we get
   $$P(\alpha^*) \leq P(\beta^*),$$
   and so
   $$\alpha^* \leq \beta^*.$$

Based on the above comprehensive analysis, we can ascertain that $\alpha^* \leq \beta^*$, that is, $\mathrm{T2HM}(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \mathrm{T2HM}(\beta_1, \beta_2, \cdots, \beta_n)$. ■

Theorem 3.5: (Boundedness) Given a collection of type-2 picture fuzzy numbers α_j $(j = 1, 2, \cdots, n)$, we can define two extreme type-2 picture fuzzy numbers

$$\alpha^+ = \left(\max_j\{\mu_{\alpha_j}\}, \max_j\{\eta_{\alpha_j}\}, \min_j\{\nu_{\alpha_j}\}, \max_j\{\vartheta_{\alpha_j}\}\right),$$

$$\alpha^- = \left(\min_j\{\mu_{\alpha_j}\}, \min_j\{\eta_{\alpha_j}\}, \max_j\{\nu_{\alpha_j}\}, \min_j\{\vartheta_{\alpha_j}\}\right),$$

where α^+ represents the "most positive" or "most optimistic" view among the given type-2 picture fuzzy numbers, and α^- represents the "most negative" or "most pessimistic" view. Then we have

$$\alpha^- \leq \mathrm{T2HM}(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \alpha^+.$$

Proof: According to Definition 2.5, we get that $\alpha^- \leq_p \alpha_j \leq_p \alpha^+$ for all j. By Theorem 3.3 and Theorem 3.4, we obtain that

$$\begin{aligned} \alpha^- &= \mathrm{T2HM}(\alpha^-, \alpha^-, \cdots, \alpha^-) \\ &\leq \mathrm{T2HM}(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &\leq \mathrm{T2HM}(\alpha^+, \alpha^+, \cdots, \alpha^+) \\ &= \alpha^+. \end{aligned}$$

Hence, the proof is completed. ■

Attribute weights are crucial in the aggregation of various attributes during multi-attribute decision-making processes. To account for the significance of the aggregated arguments and inspired by the work in reference [24], we define a type-2 picture fuzzy weighted geometric Heronian mean aggregation operator as follows.

Definition 3.6: Let α_i $(i = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers and their corresponding weights as $(w_1, w_2, \cdots, w_n)$, where $w_i \in [0, 1]$ represents the weight of the i-th attribute and $\sum_{i=1}^{n} w_i = 1$. If

$$\begin{aligned} &\mathrm{T2HM}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &= \frac{1}{p+q}\Big(\bigotimes_{i=1,j=i}^{n}\left(p\alpha_i \oplus q\alpha_j\right)^{\frac{w_iw_j}{\Lambda}}\Big) \end{aligned}$$

where $\Lambda = \sum_{i=1,j=i}^{n}(w_iw_j)$, then $\mathrm{T2HM}_w$ is called a type-2 picture fuzzy weighted geometric Heronian mean aggregation operator.

In particular, if $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$, then the proposed type-2 picture fuzzy weighted geometric Heronian mean aggregation operator is reduced to the proposed type-2 picture fuzzy geometric Heronian mean aggregation operator shown in Definition 3.1. The type-2 picture fuzzy weighted geometric Heronian mean aggregation operator can be applied in various multi-attribute decision-making problems, where type-2 picture fuzzy numbers serve as tools to capture the inherent uncertainty and variability of the attributes. Incorporating weights allows decision makers to better reflect the relative importance of different attributes in their evaluations.

Theorem 3.7: Let's denote the set of type-2 picture fuzzy numbers as $\alpha_i = (\mu_{\alpha_i}, \eta_{\alpha_i}, \nu_{\alpha_i}, \vartheta_{\alpha_i})$ $(i = 1, 2, \cdots, n)$. Then the aggregated value by using the type-2 picture fuzzy weighted geometric Heronian mean aggregation operator is also an type-2 picture fuzzy number, and

$$\text{T2HM}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \begin{pmatrix} 1 - \left(1 - \prod_{i=1,j=i}^{n} \left(1 - (1-\mu_{\alpha_i})^p (1-\mu_{\alpha_j})^q\right)^{\frac{w_i w_j}{\Lambda}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n} (\eta_{\alpha_i}^p \eta_{\alpha_j}^q)^{\frac{w_i w_j}{\Lambda(p+q)}}, \\ \left(1 - \prod_{i=1,j=i}^{n} (1 - \nu_{\alpha_i}^p \nu_{\alpha_j}^q)^{\frac{w_i w_j}{\Lambda}}\right)^{\frac{1}{p+q}}, \\ \prod_{i=1,j=i}^{n} \left(\tau_{\alpha_i}^p \tau_{\alpha_j}^q\right)^{\frac{w_i w_j}{\Lambda(p+q)}} - \prod_{i=1,j=i}^{n} (\eta_{\alpha_i}^p \eta_{\alpha_j}^q)^{\frac{w_i w_j}{\Lambda(p+q)}} \end{pmatrix}$$

Proof: The proof of Theorem 3.7 is similar to that of Theorem 3.2. ■

Next, we present several properties associated with the type-2 picture fuzzy weighted geometric Heronian mean aggregation operator. The proofs for these properties follow a similar approach to those utilized for the type-2 picture fuzzy geometric Heronian mean aggregation operator.

Theorem 3.8: Consider a collection of type-2 picture fuzzy numbers α_j $(j = 1, 2, \cdots, n)$ with the weight vector $w = (w_1, w_2, \cdots, w_n)$ satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, we have:

(1) (Idempotency) If all type-2 picture fuzzy numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are identical and equal to type-2 picture fuzzy number $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$, then

$$\text{T2HM}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha.$$

(2) (Monotonicity) Suppose $\beta_i (i = 1, 2, \cdots, n)$ is another collection of type-2 picture fuzzy numbers, if $\alpha_j \leq_p \beta_j$ for all j, then

$$\text{T2HM}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \text{T2HM}_w(\beta_1, \beta_2, \cdots, \beta_n).$$

(3) (Boundedness) Define two extreme type-2 picture fuzzy numbers α^+ and α^- as

$$\alpha^+ = \left(\max_j\{\mu_{\alpha_j}\}, \max_j\{\eta_{\alpha_j}\}, \min_j\{\nu_{\alpha_j}\}, \max_j\{\vartheta_{\alpha_j}\}\right),$$

$$\alpha^- = \left(\min_j\{\mu_{\alpha_j}\}, \min_j\{\eta_{\alpha_j}\}, \max_j\{\nu_{\alpha_j}\}, \min_j\{\vartheta_{\alpha_j}\}\right).$$

Then we have

$$\alpha^- \leq \text{T2HM}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \alpha^+.$$

IV. Decision Making Model Based on Type-2 Picture Fuzzy Weighted Geometric Heronian Mean Aggregation Operator

In this section, we use the decision-making method based on the proposed type-2 picture fuzzy weighted geometric Heronian mean aggregation operator to solve the decision-making problem. The specific decision-making frame diagram and procedures are described as Fig. 1.

Consider a collection of m alternatives denoted as $\{A_i, A_2, \cdots, A_m\}$, and a set of n attributes represented by $\{G_1, G_2, \cdots, G_n\}$. The weights assigned to these attributes form a vector $\{w_1, w_2, \cdots, w_n\}$, where $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

When evaluating alternatives, decision makers are required to utilize a type-2 picture fuzzy number, denoted as $\tilde{r}_{ij} = (\mu_{ij}, \eta_{ij}, \nu_{ij}, \vartheta_{ij})$, to represent their preferences concerning the attribute G_j of the alternative A_i according to specific criteria. As a result, a type-2 picture fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$ is constructed.

The following steps outline the solution to the type-2 picture fuzzy multi-attribute decision-making (MADM) problem using a type-2 picture fuzzy weighted geometric Heronian mean aggregation operator. The alternatives are then ranked in descending order, and the optimal choice is determined.

Step 1. The initial decision matrix $\tilde{R}$ should be normalized to a new type-2 picture fuzzy decision matrix $R = (r_{ij})_{m\times n}$.

In the context of a multi attribute decision making problem, there are generally two types of attributes: benefit type criterion and the cost type criteria. To ensure uniformity across all criteria, the following equation is employed to transform cost type criteria to benefit type criteria:

$$r_{ij} = (\mu_{ij}, \eta_{ij}, \nu_{ij}, \vartheta_{ij}) = \begin{cases} \tilde{r}_{ij}, & G_j \in I_1, \\ \tilde{r}_{ij}^c, & G_j \in I_2, \end{cases} \tag{IV.1}$$

where I_1 and I_2 represent the benefit type criteria and the cost type criteria, respectively.

Step 2. Aggregate the evaluation information of each attribute into the comprehensive evaluation value of each alternative.

The decision expert or manager, who possesses extensive experience in the respective field, assigns the weights $\{w_1, w_2, \cdots, w_n\}$ to the criteria.

For the alternatives A_i $(i = 1, 2, \cdots, m)$, we utilize the decision information provided in the normalized type-2 picture fuzzy decision matrix R. We then select the values of the parameters p and q, and apply the type-2 picture fuzzy weighted geometric Heronian mean aggregation operator

$$\text{T2HM}_w(r_{i1}, r_{i2}, \cdots, r_{in}) = \frac{1}{p+q}\Big(\bigotimes_{j=1,k=j}^{n} (pr_{ij} \oplus qr_{ik})^{\frac{w_i w_j}{\Lambda}}\Big) \tag{IV.2}$$

to derive the overall preference values α_i corresponding to alternatives A_i.

Step 3. Calculate score values to explore appropriate options or candidates.

According to Definition 2.3, we calculate the scores $S(\alpha_i)$ $(i = 1, 2, \cdots, m)$ of the overall type-2 picture fuzzy numbers α_i $(i = 1, 2, \cdots, m)$. These scores are utilized to establish a ranking for all the alternatives A_i $(i = 1, 2, \cdots, m)$. If the score values $S(\alpha_i)$ and $S(\alpha_j)$ are identical, then we need to calculate the accuracy degrees $H(\alpha_i)$ and $H(\alpha_j)$, respectively. Furthermore, if the accuracy degrees $H(\alpha_i)$ and $H(\alpha_j)$ happen to be the same, then the participation degrees $P(\alpha_i)$ and $P(\alpha_j)$ are needed to calculate, respectively.

Step 4. Rank all the alternatives A_i $(i = 1, 2, \cdots, m)$ in a decreasing order and derive the priority of each alternative A_i $(i = 1, 2, \cdots, m)$ based on the score value $S(\alpha_i)$. A more considerable score value indicates a more favorable alternative.

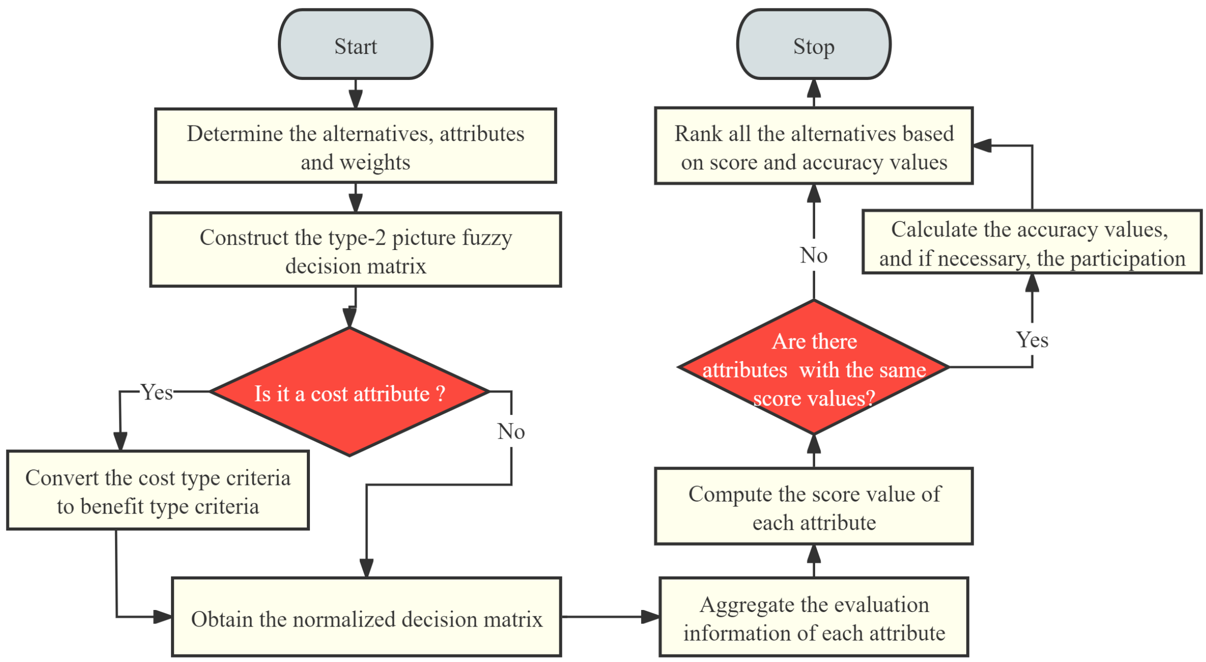

Fig. 1: Frame Diagram for Multi-attribute Decision Making Algorithm.

V. Numerical Example of Financial Products Based on Type-2 Picture Fuzzy Sets

To confirm the viability and applicability of the suggested decision framework, this section makes use of the newly introduced decision making method based on the type-2 picture fuzzy weighted geometric Heronian mean aggregation operator for evaluating and selecting a suitable financial product. Furthermore, to establish the reliability and superiority of this framework, sensitivity analysis and comparative studies are incorporated.

A. An Application Example

Financial products, as capital investment and management plans tailored by commercial banks for specific target clientele based on thorough analysis and research, have become increasingly complex, particularly in the context of Internet financing. These products, while offering diverse investment opportunities, also carry inherent risks and challenges. Therefore, guiding investors in selecting optimal financial products to reduce risks has become an urgent problem to be solved. In the process of evaluating financial products, decision makers often encounter evaluation indicators that are uncertain and random, rendering precise quantitative analysis using real numbers difficult. Traditional evaluation methods may fall short in capturing the nuanced and complex nature of these indicators. Type-2 picture fuzzy sets emerge as a powerful tool to address this challenge. By allowing for the representation of qualitative indicators with greater flexibility and comprehensiveness, Type-2 picture fuzzy sets can better handle the uncertainties and randomness inherent in financial product evaluations. This capability stems from the ability of type-2 picture fuzzy sets to encapsulate membership degrees and their associated uncertainties, thereby providing a more nuanced view of the evaluation process.

An investor wants to invest some money into a financial product, where there are five possible financial products A_1, A_2, A_3, A_4 and A_5 as alternatives. To make a rational decision, the investor solicited the expertise of a seasoned professional in Internet finance to evaluate potential alternatives based on six key attributes: G_1, G_2, G_3, G_4, G_5 and G_6. Here, G_1 denotes the rate of return on investment, G_2 denotes the platform security analysis, G_3 denotes the product liquidity, G_4 denotes the information transparency, G_5 denotes the product usability, G_6 denotes the product innovation strength.

Given the type-2 picture fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{5\times 6}$ that contains evaluation information for five alternatives A_1, A_2, A_3, A_4 and A_5 under six factors $G_i(i = 1, 2, \cdots, 6)$, where each r_{ij} is represented as a type-2 picture fuzzy number, we can proceed with the aggregation process using the weights provided. The weight vector of the six attributes $G_i(i = 1, 2, \cdots, 6)$ is $(0.21, 0.18, 0.15, 0.17, 0.16, 0.13)$.

Here's a summary of the approach we will apply to evaluate the best alternative(s).

Step 1. Apply Eq. IV.1 to normalize the cost type attribute. Since all attributes belong to the beneficial category, normalization is unnecessary, so we get the normalized type-2 picture fuzzy decision matrix $R = (r_{ij})_{5\times 6} = \tilde{R}$.

Step 2. Taking into account the data provided in Eq. IV.2 with the parameters set to $p = q = 0.5$, the aggregation values α_i $(i = 1, 2, \cdots, 5)$ are listed below:

$$\begin{aligned}
\alpha_1 &= \text{T2HM}_w(r_{11}, r_{12}, \cdots, r_{16}) \\
&= (0.4784, 0.1754, 0.1717, 0.1326), \\
\alpha_2 &= \text{T2HM}_w(r_{21}, r_{22}, \cdots, r_{26}) \\
&= (0.3721, 0.1928, 0.1446, 0.1723), \\
\alpha_3 &= \text{T2HM}_w(r_{31}, r_{32}, \cdots, r_{36}) \\
&= (0.4881, 0.1733, 0.1678, 0.1153), \\
\alpha_4 &= \text{T2HM}_w(r_{41}, r_{42}, \cdots, r_{46}) \\
&= (0.5159, 0.1541, 0.1548, 0.1392),
\end{aligned}$$

TABLE I: The Intuitionistic Fuzzy Decision Matrix

	A_1	A_2	A_3	A_4	A_5
G_1	(0.54, 0.18, 0.13, 0.12)	(0.23, 0.38, 0.11, 0.23)	(0.52, 0.15, 0.17, 0.11)	(0.64, 0.10, 0.12, 0.13)	(0.58, 0.11, 0.18, 0.11)
G_2	(0.42, 0.15, 0.18, 0.13)	(0.38, 0.23, 0.13, 0.14)	(0.45, 0.19, 0.20, 0.11)	(0.52, 0.15, 0.15, 0.14)	(0.57, 0.13, 0.14, 0.12)
G_3	(0.48, 0.17, 0.21, 0.12)	(0.31, 0.25, 0.15, 0.14)	(0.48, 0.26, 0.13, 0.12)	(0.54, 0.16, 0.18, 0.11)	(0.52, 0.21, 0.13, 0.12)
G_4	(0.51, 0.18, 0.15, 0.15)	(0.47, 0.11, 0.23, 0.12)	(0.53, 0.12, 0.14, 0.11)	(0.45, 0.14, 0.16, 0.17)	(0.55, 0.12, 0.15, 0.13)
G_5	(0.55, 0.14, 0.14, 0.15)	(0.43, 0.13, 0.15, 0.23)	(0.45, 0.16, 0.24, 0.11)	(0.51, 0.21, 0.14, 0.12)	(0.36, 0.14, 0.31, 0.14)
G_6	(0.33, 0.28, 0.27, 0.11)	(0.51, 0.12, 0.11, 0.14)	(0.49, 0.22, 0.12, 0.13)	(0.37, 0.25, 0.21, 0.15)	(0.51, 0.16, 0.18, 0.13)

$$\begin{aligned}\alpha_5 &= \mathrm{T2HM}_w(r_{51}, r_{52}, \cdots, r_{56})\\ &= (0.5197, 0.1378, 0.1785, 0.1249).\end{aligned}$$

Step 3. The scores $S(\alpha_i)$ $(i = 1, 2, \cdots, 5)$ of the overall type-2 picture fuzzy numbers α_i $(i = 1, 2, \cdots, 5)$ are calculated by Definition 2.3 as:

$$\begin{aligned}S(\alpha_1) &= 0.3066,\\ S(\alpha_2) &= 0.2275,\\ S(\alpha_3) &= 0.3203,\\ S(\alpha_4) &= 0.3611,\\ S(\alpha_5) &= 0.3412.\end{aligned}$$

Step 4. Rank all the alternatives and select the best one(s).

The ordering of the score values $S(\alpha_i)$ $(i = 1, 2, \cdots, 5)$ is

$$S(\alpha_4) > S(\alpha_5) > S(\alpha_3) > S(\alpha_1) > S(\alpha_2).$$

Thus, the ranking of alternatives A_i $(i = 1, 2, \cdots, 5)$ is

$$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2,$$

where $\succ$ means "preferred to". Thus, the best financial product is A_4.

B. Sensitive Analysis

In this section, we explore the impact of the parameters p and q on the aggregation results when employing the type-2 picture fuzzy weighted geometric Heron mean operator. Consequently, we undertake a sensitivity analysis to assess the influence of these generalized parameters on the ranking outcomes from the previous example. In other words, we assign various values to the parameters p and q to rank all the options and to analyze how changes in these parameter values affect the ordering results.

As shown in Table II, the ranking results for the schemes remain relatively consistent despite variations in the values of the parameters p and q.

Next, we fix the value of the parameter $p = 0$ and let $q \in (0, 12]$. The changes in scores for each scheme is recorded and are illustrated in Figure 2.

From Fig. 2, it can be seen that when $p = 0$ and $q \in (0, 12]$, the alternative A_3 is the best financial product:

1) If $p = 0$ and $q \in (0, 6.16]$, the ranking of alternatives A_i $(i = 1, 2, \cdots, 5)$ is
$$A_3 \succ A_4 \succ A_5 \succ A_1 \succ A_2;$$
2) If $p = 0$ and $q \in [6.17, 12]$, the ranking of alternatives A_i $(i = 1, 2, \cdots, 5)$ is
$$A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2.$$

From Fig. 3, we can find that if $q = 0$ and

1) $p \in (0, 3.76]$, the ranking of the four alternatives is
$$A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2;$$
2) $p \in [3.77, 6.56]$, the ranking of the four alternatives is
$$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2;$$
3) $p \in [6.57, 7.90]$, the ranking of the four alternatives is
$$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2.$$
4) $p \in [7.91, 12]$, the ranking of the four alternatives is
$$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2.$$

As the parameters p and q vary, different scores are obtained for the five alternatives. Figs. 4–8 provide a detailed depiction of the scores for these alternatives, as calculated using the type-2 picture fuzzy weighted geometric Heronian mean aggregation operator.

In conclusion, it is evident that the score values change in accordance with the change of p and q, demonstrating the influence of the decision-maker's choice of parameters on the resulting scores. Therefore, during the decision making process, the parameters can be appropriately chosen based on the decision-maker's risk preferences. The novel method proposed in this paper can offer more flexible or reliable decision-making resolutions. Moreover, the reasonable and best alternative can be properly obtained on the basis of the practical MADM problems, namely, the new method can offer a powerful and effective mathematic tool for the MADM under uncertainty.

C. Comparison Study

In this subsection, a comparative analysis is conducted to validate the effectiveness of the proposed method. We utilize specific existing operators to tackle and contrast the previously discussed numerical example with the introduced approach. To assess the invented techniques against existing ones, we take into account several valuable and prominent techniques that can enhance the value of the proposed theory. Specifically, we consider existing techniques such as the picture fuzzy Heronian mean aggregation operators developed by Wei et al. [25], as well as type-2 picture fuzzy weighted average operators [22]. To consider the data in Table I, the comparative analysis is listed in Table III.

From Table III, it is evident that when the parameters p and q are set to 0.5, the optimal scheme is achieved through the method introduced in this paper, which aligns with the scheme obtained by Wei et al.'s method [25] and the approach based on type-2 picture fuzzy weighted average operators [22]. However, as the parameters p and q change, the schemes derived from the proposed method diverge from those obtained by other methods. This divergence primarily stems from the fact that both the proposed method and the one in [25] consider the correlations between attributes, whereas the method in [22] assumes that the attributes are independent of each other. Consequently, for MADM problems

TABLE II: Sorting Results for Different Parameters p and q

p	q	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	$S(\alpha_5)$	Ranking Values
0.5	0.5	0.3066	0.2275	0.3203	0.3611	0.3412	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
0.5	1	0.2945	0.2357	0.3176	0.3438	0.3240	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
1	0.5	0.3125	0.2117	0.3194	0.3731	0.3499	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
1	1	0.3021	0.2220	0.3176	0.3571	0.3348	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
1	2	0.2850	0.2256	0.3120	0.3371	0.3108	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$
2	1	0.3054	0.2023	0.3153	0.3660	0.3389	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
2	2	0.2909	0.2089	0.3113	0.3479	0.3180	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
2	3	0.2768	0.2074	0.3061	0.3340	0.2983	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$
3	2	0.2907	0.1940	0.3093	0.3508	0.3155	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
3	3	0.2781	0.1941	0.3049	0.3380	0.2978	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$
10	10	0.1955	0.1110	0.2707	0.2775	0.1824	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$

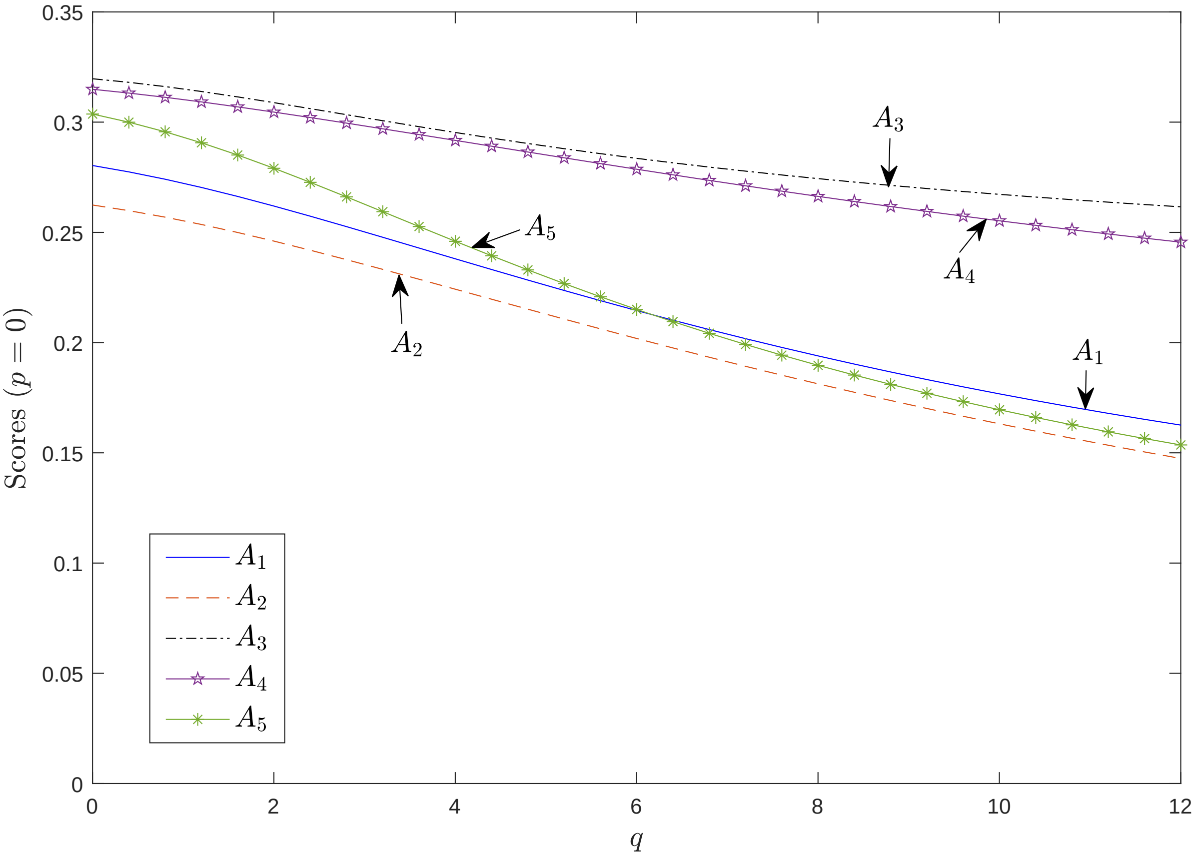

Fig. 2: Scores of Alternatives ($p = 0$, $q \in (0, 12]$)

TABLE III: Comparison of the Existing Methods with the Proposed Method

Methods	Score Values	Ranking Values
The proposed method ($p = q = 0.5$)	0.3066, 0.2275, 0.3203, 0.3611, 0.3412	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
Wei et al. [25] ($p = q = 0.5$)	0.8521, 0.8266, 0.8582, 0.8661, 0.8643	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
The proposed method ($p = 0$, $q = 3$)	0.2502, 0.2354, 0.3020, 0.2983, 0.2628	$A_3 \succ A_4 \succ A_5 \succ A_1 \succ A_2$
Wei et al. [25] ($p = 0$, $q = 3$)	0.8553, 0.8548, 0.8679, 0.8625, 0.8660	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$
Type-2 picture fuzzy weighted average operators [22]	0.3144, 0.2442, 0.3247, 0.3687, 0.3515	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$

involving attribute correlations, the proposed method and the one in [25] are more appropriate. Furthermore, a comparison reveals that while the schemes from the proposed method and [25] differ as parameters change, the outcomes of the method presented here exhibit greater stability.

To sum up, the proposed methods are effective and feasible and are sufficient to deal with practical MADM problems. However, this is only a case study. It is important to recognize that these findings are based on a single case study. This implies that the observed stability of the method in this particular instance does not necessarily indicate its superiority over alternative methods in different scenarios. Indeed, each method is tailored to specific contextual requirements and may perform differently depending on the application

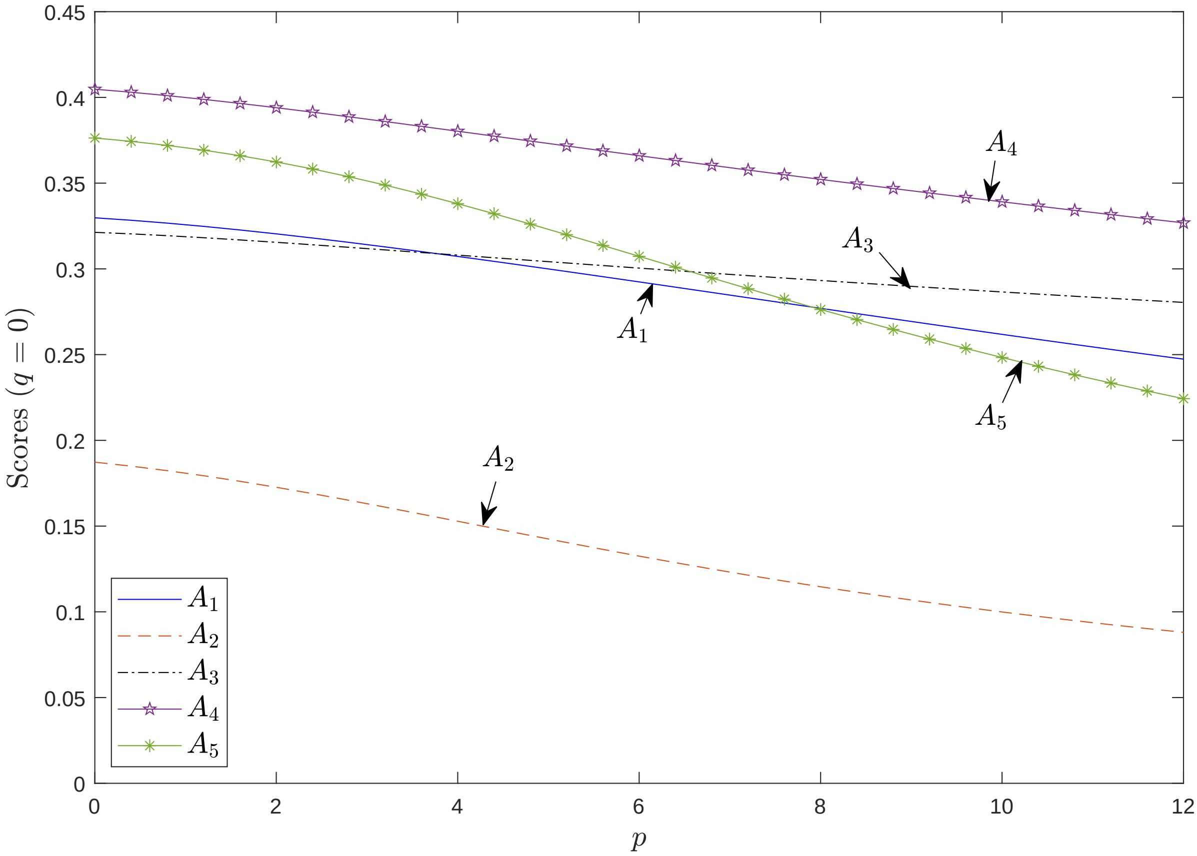

Fig. 3: Scores of Alternatives ($p \in (0, 12]$, $q = 0$)

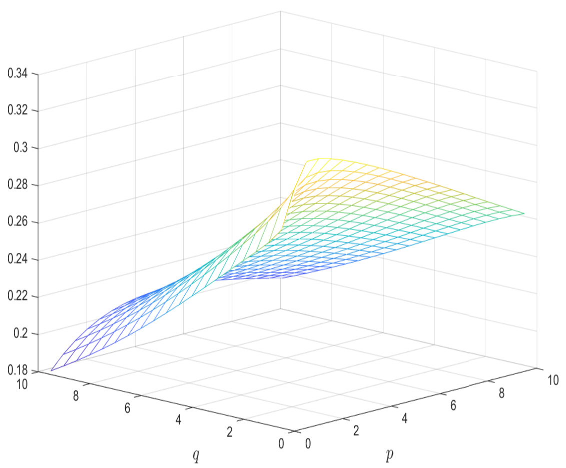

Fig. 4: Scores of Alternative A_1 ($p \in (0, 10]$, $q \in (0, 10]$)

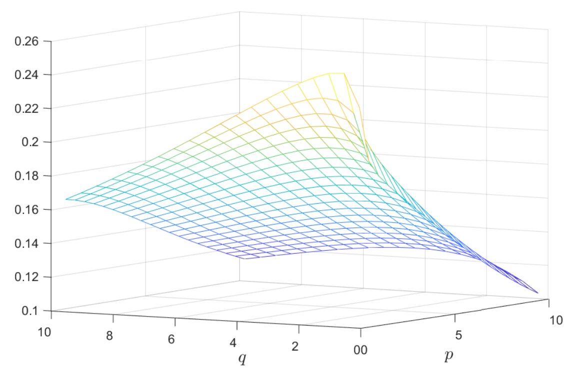

Fig. 5: Scores of Alternative A_2 ($p \in (0, 10]$, $q \in (0, 10]$)

environment.

VI. Conclusion

In this paper, we have broadened the scope of the geometric Heronian mean operator to the realm of type-2 picture fuzzy sets, recognizing its inherent ability to consider both the mutuality between attribute values and the correlation between input arguments. This extension is particularly well-suited for handling the inherent uncertainties and ambiguities of type-2 picture fuzzy sets, which are common in complex decision making scenarios. By integrating operations on type-2 picture fuzzy numbers, we have developed type-2 picture fuzzy geometric Heronian mean aggregation operators and type-2 picture fuzzy weighted geometric Heronian mean operators. Furthermore, we have proposed a new multiple attribute decision making method based on the type-2 picture fuzzy weighted geometric Heronian mean operator. Through a comparative analysis with existing methods, we have demonstrated that the proposed method is able to handle the inherent uncertainties and ambiguities of type-2 picture fuzzy sets more effectively and provide more reliable decision outcomes.

In summary, this research contributes to the advancement of decision making methods in uncertain environments by

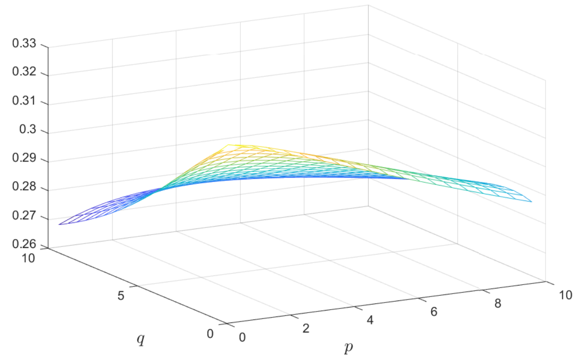

Fig. 6: Scores of Alternative A_3 ($p \in (0, 10]$, $q \in (0, 10]$)

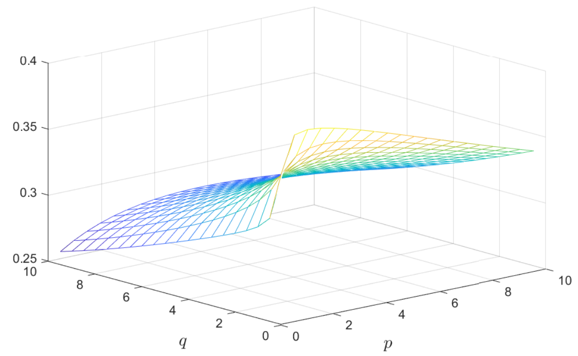

Fig. 7: Scores of Alternative A_4 ($p \in (0, 10]$, $q \in (0, 10]$)

developing and analyzing new aggregation operators based on the geometric Heronian mean in the type-2 picture fuzzy setting. The proposed method offers decision-makers a powerful tool to navigate complex decision making scenarios and make more informed choices. Future research can further explore the applications of these operators in other domains and refine the method to handle even more complex uncertainties and ambiguities.

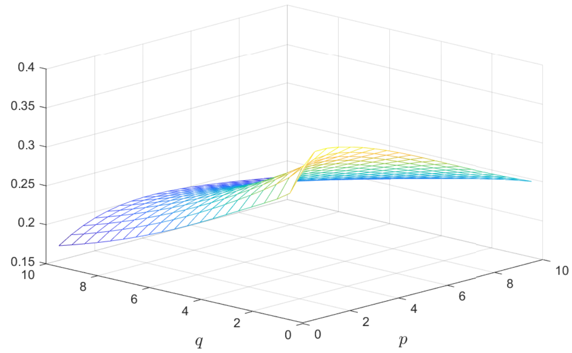

Fig. 8: Scores of Alternative A_5 ($p \in (0, 10]$, $q \in (0, 10]$)

REFERENCES

[1] L. A. Zadeh, “Fuzzy sets,” *Information & Control*, vol. 8, no. 3, pp. 338–353, 1965.
[2] K. T. Atanassov, “Intuitionistic fuzzy sets,” *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986.
[3] M. Qiyas, N. Khan, M. Naeem and S. Abdullah, “Intuitionistic fuzzy credibility Dombi aggregation operators and their application of railway train selection in Pakistan,” *AIMS Mathematics*, vol. 8, no. 3, pp. 6520–6542, 2023.
[4] Z. Yang, L. Zhang and C. Liang, “A method for group decision making with multiplicative consistent interval-valued intuitionistic fuzzy preference relation,” *IAENG International Journal of Applied Mathematics*, vol. 51, no. 1, pp. 152–163, 2021.
[5] W. Y. Zeng, H. S. Cui, Y. Q. Liu and Z. S. Xu, “Novel distance measure between intuitionistic fuzzy sets and its application in pattern recognition,” *Iranian Journal of Fuzzy Systems*, vol. 19, no. 3, pp. 127–137, 2022.
[6] N. X. Thao, M. Ali and F. Smarandache, “An intuitionistic fuzzy clustering algorithm based on a new correlation coefficient with application in medical diagnosis,” *Journal of intelligent & fuzzy systems*, vol. 36, no. 1, pp. 189–198, 2019.
[7] B. C. Cuong, “Picture fuzzy sets,” *Journal of Computer Science and Cybernetics*, vol. 30, no. 4, pp. 409–420, 2014.
[8] P. Singh, “Correlation coefficients for picture fuzzy sets,” *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 2, pp. 591–604, 2015.
[9] A. H. Ganie, S. Singh and P. K. Bhatia, “Some new correlation coefficients of picture fuzzy sets with applications,” *Neural Computing and Applications*, vol. 32, no. 1, pp. 12609–12625, 2020.
[10] S. He and Y. Wang, “Evaluating new energy vehicles by picture fuzzy sets based on sentiment analysis from online reviews,” *Artificial Intelligence Review*, vol. 56, no. 3, pp. 2171–2192, 2023.
[11] L. Wang, J. Peng and J. Wang, “A multi-criteria decision making framework for risk ranking of energy performance contracting project under picture fuzzy environment,” *Journal of Cleaner Production*, vol. 191, pp. 105–118, 2018.
[12] M. Luo and G. Zhang, “Divergence-based distance for picture fuzzy sets and its application to multi-attribute decision making,” *Soft Computing*, vol. 28, pp. 253–269, 2024.
[13] R. Chitra and K. Prabakaran, “Ordering q-rung picture fuzzy numbers by possible grading technique and its utilization in decision-making problem,” *IAENG International Journal of Applied Mathematics*, vol. 53, no. 4, pp. 1510–1519, 2023.
[14] M. Bani-Doumi, J. Serrano-Guerrero, F. Chiclana, F. P. Romero and J. A. Olivas, “A picture fuzzy set multi criteria decision making approach to customize hospital recommendations based on patient feedback,” *Applied Soft Computing*, vol. 153, pp. 1–12, 2024.
[15] M. Sarfraz and D. Pamucar, “A parametric similarity measure for spherical fuzzy sets and its applications in medical equipment selection,” *Journal of Engineering Management and Systems Engineering*, vol. 3, pp. 38–52, 2024.
[16] J. J. Peng, X. G. Chen, X. K. Wang, J. Q. Wang, Q. Q. Long and L. J. Yin, “Picture fuzzy decision making theories and methodologies: a systematic review,” *International Journal of Systems Science*, vol. 54, no. 13, pp. 2663–2675, 2023.
[17] Y. Yang, “Methods for multi-attribute decision making with distances on refined picture fuzzy sets,” *Fuzzy Systems and Mathematics*, vol. 35, no. 4, pp. 107–114, 2021.
[18] S. Khan, S. Abdullah and S. Ashraf, “Picture fuzzy aggregation information based on Einstein operations and their application in decision making,” *Mathematical Sciences*, vol. 13, pp. 213–229, 2019.
[19] C. Tian, J. Peng, Z. Zhang, J. Wang and M. Goh, “An extended picture fuzzy multimoora method based on Schweizer-Sklar aggregation operators,” *Soft Computing*, vol. 26, pp. 3435–3454, 2022.
[20] L. Yang and B. Li, “Picture hesitant fuzzy normalized weighted bonferroni mean operator based on einstein operations and its application,” *Engineering Letters*, vol. 31, no. 4, pp. 1598–1604, 2022.
[21] C. Wang, X. Zhou, H. Tu and S. Tao, “Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making,” *Italian Journal of Pure and Applied Mathematics*, vol. 37, pp. 477–492, 2017.
[22] Y. Yang and J. Li, “Arithmetic aggregation operators on type-2 picture fuzzy sets and their application in decision making,” *Engineering Letters*, vol. 31, no. 4, pp. 1464–1473, 2023.
[23] P. Rani, A. R. Mishra, A. Saha, I. M. Hezam and D. Pamucar, “Fermatean fuzzy Heronian mean operators and MEREC-based additive ratio assessment method: an application to food waste treatment technology selection,” *International Journal of Intelligent Systems*, vol. 37, no. 3, pp. 2612–2647, 2022.
[24] R. Verma and B. D. Sharma, “Hesitant fuzzy geometric heronian mean operators and their application to multi-criteria decision making,” *Scientiae Mathematicae Japonicae*, vol. 78, no. 4, pp. 23–39, 2015.
[25] G. Wei, M. Lu and H. Gao, “Picture fuzzy heronian mean aggregation operators in multiple attribute decision making,” *International Journal of Knowledge-Based and Intelligent Engineering Systems*, vol. 22, no. 3, pp. 167–175, 2018.