Finite Time Prescribed Performance Fuzzy Control for Switched Stochastic Nonlinear Systems with Input Saturation

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Abstract—**This manuscript studies the finite time prescribed performance control problem for a class of switched stochastic nonlinear systems with input saturation. Firstly, utilizing the prescribed performance control, the tracking error can be constrained within the predefined bound to improve the transient characteristics of systems. And then, by introducing an auxiliary system, the input saturation problem is settled. Via combining the common Lyapunov function, fuzzy logic systems, and back-stepping technique, a novel finite time adaptive fuzzy tracking controller is developed to tackle unknown nonlinear functions and stochastic disturbances. Furthermore, using comparison theorem and the mean value theorem of integrals removes the linear growth conditions for nonlinear term and stochastic disturbance term in stochastic nonlinear systems. The proposed controller guarantees the tracking error can converge to an arbitrarily small neighborhood near the origin in finite time, and all the signals in closed-loop systems are semi-globally uniformly bounded. Finally, two simulation examples are presented to illustrate the effectiveness of the presented control strategy.**

Index Terms—**Finite-time stability, prescribed performance control, input saturation, fuzzy logic systems, switched stochastic nonlinear systems.**

I. Introduction

NONLINEAR control, especially switched stochastic nonlinear control, has attracted the attention of more and more scholars and experts over the past decades [1]-[3]. In [4], the stability of switched stochastic nonlinear systems with strict-feedback form is introduced. In [5], based on the common Lyapunov stability theory and stochastic small-gain theorem, a new robust adaptive fuzzy back-stepping stabilization control strategy is developed. In [6], by using back-stepping technique and dead-zone compensation function, a novel adaptive fuzzy control scheme is proposed. By utilizing the structural characteristics of fuzzy systems and common Lyapunov function, the trajectory tracking controller is designed for a class of non-affine stochastic nonlinear switched systems with the non-lower triangular in [7].To mitigate data transmission and deal with system uncertainties, the event-triggered mechanism and fuzzy systems are exploited to generate the control signal for switched stochastic nonlinear systems with state constraints in [8]. To reduce the effects of unknown homogeneous growth rate and time-varying delays, a new homogeneous output feedback controller is designed by using two dynamic gains and the Lyapunov–Krasovskii function in [9]. Literature [10] utilizes the multiple Lyapunov–Krasovskii functions (MLKFs) method to investigates the stability of switched stochastic delay systems with unstable subsystems. For the adaptive tracking control problem of switched stochastic nonlinear systems with non-symmetric dead-zone input, literature [11] combines fuzzy logic system, multiple stochastic Lyapunov functions and adaptive back-stepping methods to design a novel adaptive state feedback controller.

It is worth noting that the above results are concerned with the infinite-time stability problems, which means the performances of systems can be obtained only when the time tends to infinity. However, in many real applications, the control performances are expected to be realized in finite time[12]. To satisfy the demand, finite-time control (FTC) is first proposed in [13] for a class of double-integral systems. Literature [14] uses a power integrator technique to address the finite-time stability (FTS) problem for continuous nonlinear systems. After the success of FTS for deterministic systems, scholars began to solve FTS problem of stochastic systems. In [15], based on stability in probability, a stochastic finite-time stability theorem is proposed. Based on the almost surely finite-time stability theorem , literature [16] proves that almost surely global finite-time stability of stochastic nonlinear systems in strict-feedback form can be guaranteed by a continuous control law. In [17], by utilizing the comparison theorem and mean value theorem of integrals, a finite-time stability criterion for switched stochastic nonlinear systems is presented.

Since the input saturation can degrade the system performance and even destabilize the system, it is necessary for us to consider this subject[18]-[19]. Wang H Q *et al.* applied a smooth non-affine function to approximate the input saturation function and presented a novel adaptive neural control scheme without requiring the prior knowledge for bound of input saturation [20]. In [21], Li H *et al.* introduced an auxiliary function, which comprises all state variables, to solve the input saturation problem. With the aid of efficient dynamical systems, a novel adaptive neural tracking controller is designed to address the input saturation problem of nonlinear stochastic switched non-lower triangular systems [22]. By using the Gaussian error function based continuous differentiable switching model and some special techniques, the switching asymmetric saturation nonlinearity is overcome in [23]. By combining neural network approx-

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imation ability and the back-stepping technique, literature [24] settles the adaptive output-feedback tracking control problem of switched non-strict feedback nonlinear systems with unknown control direction and asymmetric saturation.

Recently, a novel control methodology called prescribed performance control (PPC) has been proposed in [25]. Since the PPC can ensure that the tracking error is constrained within the pre-specified bound, and the error convergence rate is not less than the pre-specified value, it has attracted considerable attention [26]-[27]. In [28], synthesizing PPC and back-stepping technique develops an adaptive NN output feedback tracking control scheme under deterministic switching signal.Based on the finite-time performance function (FTPF), literature [29] proposes a modified finite-time adaptive NN control design strategy to simplify the controller design.By adopting a piece-wise function to characterize finite-time prescribed performance, literature [30] develops fuzzy adaptive switching control for stochastic systems with arbitrary switching signal.Based on the finite time prescribed performance control, literature [31] designs an adaptive fuzzy controller for a class of strict-feedback systems in the presence of actuator faults and dynamic disturbances.By utilizing the multiple Lyapunov function method and the back-stepping technique together with the prescribed performance bounds, an adaptive NN controller is established in [32].

Motivated by above discussions, we will develop an adaptive fuzzy finite-time controller for a class of switched stochastic nonlinear systems with prescribed performance and input saturation. Comparing with the existing documents, the main contributions in this paper are summarized as follows:

(1) Combining the finite time control, prescribed performance control and fuzzy inference system to design the adaptive controller for switching stochastic nonlinear systems for the first time. The proposed controller takes into account the transient and steady state characteristics of the closed-loop system at the same time, ensures that the system is finite time stable at the origin, and the tracking error is limited to the constraint space specified by the prescribed performance functions, which effectively improves the convergence speed of the system and the control accuracy.

(2) Using fuzzy logic systems to approximate complex nonlinear functions and unknown perturbations in the controlled system greatly reduces the online computation amount of the controller.

(3) Using auxiliary dynamic systems to compensate for the nonlinearity introduced by input saturation improves the robustness of switched stochastic nonlinear systems.

(4) Using comparison theorem and the mean value theorem of integrals removes the linear growth conditions for nonlinear term and stochastic disturbance term in stochastic nonlinear systems to broaden the application field of the controller.

II. System statements and Preliminaries

A. Preliminaries

Consider the following stochastic non-linear system

$$dx = f(x,u)dt + g^T(x,u)d\omega, \tag{1}$$

where $x \in R^n$ denotes the state variable; $u \in R^m$ denotes the input of the system; $f : R^{n+m} \to R^n$ and $g : R^{n+r} \to R^r$ are the continuous Borel measurable functions, and satisfy $f(0,0) = g(0,0) = 0$; ω represents an r-dimensional Brownian motion defined on complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ with Ω representing a sample space, F representing a σ-field, $\{F_t\}_{t\geq 0}$ representing a filtration, and P representing a probability measure.

Definition 1[17]: The solution $x(t)$ of system (1) is practical finite-time stable in mean square, if for all $x(t_0) = x_0$, there exist a constant $\varepsilon > 0$ and a settling time $T(\varepsilon, x_0) < \infty$ such that $E(|x(t)|^2) < \varepsilon$ for $\forall t > t_0 + T$.

Definition 2 ($It\hat{o}$ formula) [9]: For any given positive function $V(x,t) \in C^{2,1}$, define the differential operator L related with $dx = f(x,t)dt + g(x)d\omega$ as follows:

$$LV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f + \frac{1}{2} Tr\left[g^T \frac{\partial^2 V}{\partial x^2} g\right], \tag{2}$$

where $Tr[\cdot]$ represents the trace of matrix.

Lemma 1 [6]: For $z_j \in R$, $j = 1, \cdots, m$, $0 < p \leq 1$, the following inequalities holds:

$$\left(\sum_{j=1}^{m} |z_j|\right)^p \leq \sum_{j=1}^{m} |z_j|^p \leq m^{1-p} \left(\sum_{j=1}^{m} |z_j|\right)^p. \tag{3}$$

Lemma 2 [33]: If $\dot{\hat{\theta}}(t) = -\gamma\hat{\theta}(t) + \kappa\nu(t)$ and $\hat{\theta}(t_0) \geq 0$, then $\theta(t) \geq 0$ for $\forall t \geqslant t_0$, where constants $\gamma > 0, \kappa > 0$, and function $\nu(t)$ is positive.

Lemma 3 [34]: For $\forall (z,x) \in R$, such that

$$|z|^{p_1}|x|^{p_2} \leq \frac{p_1 p_3}{p_1 + p_2}|z|^{p_1+p_2} + \frac{p_2 p_3^{-\frac{p_1}{p_2}}}{p_1 + p_2}|x|^{p_1+p_2}, \tag{4}$$

where $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$.

Lemma 4 [17]: Consider the systems $\dot{x} = f(x,u)$, for $\forall 0 \leq \tau \leq t$, if there exist a function $\varpi(x(t)) \in C^2$, constants $c > 0$, $0 < \upsilon < 1$ and $\tau > 0$, κ_∞-function α_1 and α_2 satisfying

$$\begin{cases} \alpha_1(\|x\|) \leq \varpi(x(t)) \leq \alpha_2(\|x\|) \\ \varpi(x(t)) - \varpi(x(\tau)) \leq -c\int_\tau^t \varpi^\upsilon(x(s))ds \\ \qquad +d(t-\tau). \end{cases} \tag{5}$$

Then, there are a real number

$$T^* = \frac{1}{(1-\upsilon)\beta c}\left[\varpi^{1-\upsilon}(x(0)) - \left(\frac{d}{(1-\beta)c}\right)^{(1-\upsilon)/\upsilon}\right] > 0$$

and a constant $\varepsilon = \alpha_1^{-1}\left[\left(\frac{d}{(1-\beta)c}\right)^{1/\upsilon}\right]$, such that $\|x\| < \varepsilon$, for $\forall t \geq T^*$.

Lemma 5 (Young's inequality)[35]: For $\forall x, y \in R$, there exists

$$xy \leq \frac{\beta^p}{p}|x|^p + \frac{1}{q\beta^q}|y|^q, \tag{6}$$

where $\beta \geq 0$, $(p-1)(q-1) = 1$, $p > 1$, and $q > 1$.

B. System statement

Consider the following switched stochastic nonlinear system with input saturation

$$\begin{cases} dx_i = (h_{i,\eta(t)}(\bar{x}_i)x_{i+1} + f_{i,\eta(t)}(\bar{x}_i))dt \\ \qquad +g^T_{i,\eta(t)}(\bar{x}_i)d\omega, \qquad 1 \le i \le n-1 \\ dx_n = (h_{n,\eta(t)}(\bar{x}_n)u(v) + f_{n,\eta(t)}(\bar{x}_n))dt \\ \qquad +g^T_{n,\eta(t)}(\bar{x}_n)d\omega \\ y = x_1 \end{cases} \tag{7}$$

where $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T \in R^i$ and $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ are the state variables; $y \in R$ denotes the system output; ω is r-dimensional independent Wiener process with $E\{dw(t)\} = 0$; $\eta(t) : [0, \infty) \to M = \{1, 2, \cdots, m\}$ is the switching law. For $i = 1, 2, \cdots n$ and $k = 1, 2, \cdots m$, the functions $h_{i,k}(\cdot) : R^n \to R$, $f_{i,k}(\cdot) : R^n \to R$ and $g^k_{i,k}(\cdot) : R^n \to R^r$ are smooth unknown nonlinear functions. $u(v)$ is the control input with saturation non-linearity, which is defined as :

$$\begin{aligned} u(v(t)) &= \mathrm{sat}(v(t)) \\ &= \begin{cases} \mathrm{sign}(v(t))\,\omega_0, & |v(t)| \ge \omega_0, \\ v(t)\,, & |v(t)| < \omega_0. \end{cases} \end{aligned} \tag{8}$$

Remark 1: Most of the existing finite-time control schemes, such as [36]-[37], were designed for deterministic systems. However, system (7) is the stochastic system, which is more ubiquitous in practical plants. The finite-time control methodology for the non-linear stochastic system in [18], [22] requires that the nonlinear terms and the stochastic disturbance terms of the system are unknown and satisfy some linear growth conditions. However, the nonlinear term $f_{i,k}(\bar{x}_i)$ and stochastic disturbance term $g_{i,k}(\bar{x}_i)$ of system (7) do not satisfy the linear growth conditions. By using the comparison theorem and mean value of integrals in [19], the linear growth constraint can be removed. Since input saturation can degrade or even deteriorate system performance, the effects of input saturation on system performance are considered here.

The objective of this paper is to design an adaptive fuzzy tracking controller for the system (7) such that :

(1) the system output y can track the desired signal y_d for $t > T^*$ with T^* being the settling time;

(2) the tracking error evolves strictly within a prescribed decreasing bound;

(3) All signals in a closed-loop system are bounded in probability.

Assumption 1: The desired trajectory y_d and its i-order derivatives are continuous and bounded for $i = 1, \cdots, n$.

Assumption 2: For $j \in I$, there exist unknown constants $\underline{h}$ and $\overline{h}$ such that $0 < \underline{h} \le |h_{j,r}(\bar{x}_i)| \le \overline{h} < \infty, \ \forall \bar{x}_i \in R^i, 1 \le i \le n.$

C. Fuzzy logic systems

According to universal approximation theorem, fuzzy logic systems can approximate any Borel measurable function with arbitrary precision. Therefore, in the subsequent adaptive controller design, fuzzy logic systems will be used to approximate all unknown functions in the switched stochastic nonlinear systems.

Fuzzy logic systems consist of a series of fuzzy rules, and the fuzzy rules can be written as:

R^l: If x_1 is F^l_1 and $\cdots$ and x_n is F^l_n,
then y is G^l,

where $l = 1, 2, \cdots, N$; F^l_i and G^l represent fuzzy sets; N stands for the number of the fuzzy rules; $x = [x_1, x_2, \ldots, x_N]^T \in R^N$ and $y \in R$ represent the input and output of the fuzzy system respectively. By applying the singleton fuzzifier, product inference, and center-average defuzzification, the output of the fuzzy logic system can be described as:

$$y(x) = \frac{\sum_{l=1}^N \Phi_l \prod_{i=1}^n \mu_{F_i{}^l}(x_i)}{\sum_{l=1}^N \left[\prod_{i=1}^n \mu_{F_i{}^l}(x_i)\right]}, \tag{9}$$

where $\mu_{F^l_i}$ is the membership function of the fuzzy set F^l_i; $\Phi_l = \max_{y\in R} \mu_{G^l}(y)$, and $\mu_{G^l}(y)$ is the membership function of the fuzzy set G^l.

Let

$$\xi_l(x) = \frac{\prod_{i=1}^n \mu_{F_i{}^l}(x_i)}{\sum_{l=1}^N \left[\prod_{i=1}^n \mu_{F_i{}^l}(x_i)\right]}$$

, $\xi(x) = [\xi_1(x), \xi_2(x), \cdots, \xi_N(x)]^T$, and $\Phi = [\Phi_1, \Phi_2, \cdots, \Phi_N]^T$. So, (9) can be rewritten as:

$$y(x) = \Phi^T \xi(x) \tag{10}$$

Choosing $\mu_{F_i{}^l}(x_i)$ is the Gaussian function, it has

$$\mu_{F_i{}^l}(x_i) = \exp\left(\frac{\left(x_i - x_i{}^l\right)^2}{\left(\sigma_i{}^l\right)^2}\right), \tag{11}$$

where $x^l_i = \left[x_{i1}{}^l, x_{i2}{}^l, \cdots x_{in}{}^l\right]^T$ is the central function, and σ^l_i is the width of Gaussian function.

D. Prescribed tracking performance

A practical control system should not only be stable, but also satisfy some transient and steady-state performances. Therefore, it is practical to require that the tracking error evolves strictly within a prescribed decreasing bound. That is

$$-\underline{\delta}\,\rho(t) < e_1(t) < \bar{\delta}\rho(t), \quad \forall t \ge 0, \tag{12}$$

where $\underline{\delta}$ and $\bar{\delta}$ are positive constants. $e_1 = y - y_d$ is tracking error. The smooth function $\rho(t)$ is the performance function, which satisfies three properties: (1) $\rho(t) > 0$; (2) $\dot{\rho}(t) \le 0$; (3) $\lim_{t\to T_f} \rho(t) = \rho_{T_f} > 0$ and $\rho(t) = \rho_{T_f}$ for $t \ge T_f$ with ρ_{T_f} and T_f being the arbitrarily small constant and the settling time, respectively. In this paper, the following performance function is chosen.

$$\rho(t) = (\rho_0 - \rho_\infty)e^{-\kappa t} + \rho_\infty, \tag{13}$$

where κ, ρ_0 and ρ_∞ are positive design parameters.

Remark 2: It can be deduced from (13) that the initial error and steady-state error satisfy $-\underline{\delta}\rho_0 < e_1(0) < +\bar{\delta}\,\rho_0$ and $-\underline{\delta}\rho_\infty < e_1(\infty) < +\bar{\delta}\rho_\infty$ respectively. So, by reasonably choosing ρ_0, ρ_∞ and κ, the steady-state error and convergence speed can satisfy the prescribed performance.

To transform the constrained tracking error condition (12) to an unconstrained form, the following error transform is used.

$$e_1(t) = \rho(t)S(\varsigma_1), \tag{14}$$

where ς_1 is the transformed error, and the error transformation function $S(\varsigma_1)$ is defined as

$$S(\varsigma_1)=\frac{\overline{\delta}e^{\varsigma_1}-\underline{\delta}e^{-\varsigma_1}}{e^{\varsigma_1}+e^{-\varsigma_1}}. \tag{15}$$

Remark 3: It follows from (15) that (1) $S(\varsigma_1)$ is a smooth and strictly increasing function; (2) $S(\varsigma_1)\in(-\underline{\delta},+\overline{\delta})$; (3) $\lim_{\varsigma_1\to+\infty}S(\varsigma_1)=+\overline{\delta}$ and $\lim_{\varsigma_1\to-\infty}S(\varsigma_1)=-\underline{\delta}$.

Remark 4: It follows from $-\underline{\delta}<S(\varsigma)<+\overline{\delta}$ and $\rho(t)>0$ that $-\underline{\delta}\rho(t)<e_1(t)<+\overline{\delta}\,\rho(t)$ holds.

Furthermore, it can be deduced from (14) and (15) that

$$\varsigma_1=S^{-1}\left(\frac{e_1}{\rho}\right)=\frac{1}{2}\ln\left(\frac{e_1/\rho+\underline{\delta}}{\overline{\delta}-e_1/\rho}\right), \tag{16}$$

and

$$\dot{\varsigma}_1=\beta\left(\dot{e}_1-\frac{\dot{\rho}}{\rho}e_1\right), \tag{17}$$

where $\beta=(1/(e_1/\rho+\underline{\delta})-1/(e_1/\rho-\overline{\delta}))/(2\rho)>0$, and $\dot{\rho}=-k(\rho_0-\rho_\infty)e^{-\kappa t}$.

For the convenience of derivation, let $h_{i,\eta(t)}(\bar{x}_i)=h_{i,\eta(t)}$, $f_{i,\eta(t)}(\bar{x}_i)=f_{i,\eta(t)}$, and $g_{i,\eta(t)}(\bar{x}_i)=g_{i,\eta(t)}$.

Substituting $e_1=x_1-y_d$ and $dx_1=(h_{i,\eta(t)}x_2+f_{1,\eta(t)})dt+g_{1,\eta(t)}^Td\omega$ into (17), (7) can be rewritten as

$$\begin{cases} d\varsigma_1=\beta(h_{1,\eta(t)}x_2+f_{1,\eta(t)}-\dot{y}_d-\frac{\dot{\rho}}{\rho}e_1)dt \\ \qquad +\beta g_{1,\eta(t)}^T d\omega \\ dx_i=(h_{i,\eta(t)}x_{i+1}+f_{i,\eta(t)})dt+g_{i,\eta(t)}^T d\omega. \\ \qquad\qquad 2\le i\le n-1 \\ dx_n=(h_{n,\eta(t)}u(v)+f_{n,\eta(t)})dt+g_{n,\eta(t)}^T d\omega \\ \quad y=x_1 \end{cases} \tag{18}$$

III. Controller Design

An adaptive fuzzy finite-time controller is designed for switched stochastic nonlinear systems (18) in this section. First, the following coordinate transformations are developed.

$$z_1=\varsigma_1, \tag{19}$$

$$z_i=x_i-\alpha_{i-1}, \tag{20}$$

$$z_n=x_n-\alpha_{n-1}-\tilde{l}, \tag{21}$$

where $\alpha_{i-1},\ i=2,\cdots,n-1$ is the virtual control variable; $\tilde{l}$ is an auxiliary design signal, which will be designed later; $\bar{y}_d^{(i)}=[y_d,y_d^{(1)},\cdots,y_d^{(i)}]^T$ with $y_d^{(i)}$ representing the i-order derivative of y_d.

Step 1:

Choose Lyapunov function as:

$$V_1=\frac{{z_1}^4}{4}+\frac{\underline{h}\tilde{\theta}_1^2}{2r_1}, \tag{22}$$

where $\tilde{\theta}_1=\theta_1-\hat{\theta}_1$ is the parameter error, $\hat{\theta}_1$ is the estimation of θ_1, and r_1 is a positive design parameter.

Applying (2), (18) and (19), one can get

$$\begin{aligned} LV_1=&\,{z_1}^3\beta h_{1,k}z_2+\tfrac{3}{2}{z_1}^2(\beta g_{1,k})^T(\beta g_{1,k}) \\ &+{z_1}^3\beta(h_{1,k}\alpha_1+f_{1,k}(x_1)-\dot{y}_d-\tfrac{\dot{\rho}}{\rho}e_1) \\ &-\tfrac{\underline{h}\tilde{\theta}_1\dot{\hat{\theta}}_1}{r_1}. \end{aligned} \tag{23}$$

According to Lemma 5 and Assumption 2, the following inequality holds:

$$\frac{3}{2}z_1^2(\beta g_{1,k})^T(\beta g_{1,k})\le\frac{3}{4}l_1^{-2}z_1^4\left\|\beta g_{1,k}\right\|^4+\frac{3}{4}l_1^2, \tag{24}$$

$$z_1^3z_2\beta h_{1,k}\le\frac{3}{4}z_1^4\beta^{\frac{4}{3}}\overline{h}+\frac{\overline{h}}{4}z_2^4, \tag{25}$$

where $l_1>0$ denotes a design parameter.

Substituting (24) and (25) into (23), one obtains

$$\begin{aligned} LV_1\le&-k_1z_1^{4\upsilon}-\tfrac{3}{4}z_1^4+{z_1}^3\hat{f}_{1,k}+{z_1}^3\beta h_{1,k}\alpha_1 \\ &+\tfrac{3}{4}{l_1}^2+\tfrac{\overline{h}}{4}{z_2}^4-\tfrac{\underline{h}\tilde{\theta}_1\dot{\hat{\theta}}_1}{r_1}, \end{aligned} \tag{26}$$

where $\hat{f}_{1,k}=\beta f_{1,k}+\frac{3}{4}\overline{h}\beta^{\frac{4}{3}}z_1-\beta\dot{y}_d-\frac{\beta\dot{\rho}}{\rho}e_1+\frac{3}{4}l_1^{-2}z_1\left\|\beta g_{1,k}\right\|^4+k_1z_1^{4\upsilon-3}+\frac{3}{4}z_1$ and $\upsilon=4q-1/4q+1$ $(q\ge2,q\in N)$.

Since $f_{1,k}$ are unknown functions, we can employ the fuzzy logic systems to approximate. That is

$$\hat{f}_{1,k}=\Phi_{1,k}^T\xi_{1,k}(X_1)+\delta_{1,k}(X_1), \tag{27}$$

where $|\delta_{1,k}(X_1)|\le\varepsilon_{1,k}$, $\varepsilon_{1,k}>0$ represents any constant and $X_1=[z_1,y_d,\dot{y}_d]^T$.

By Lemma 5 and $\xi_{1,k}^T\xi_{1,k}\le1$, we have

$$\begin{aligned} {z_1}^3\ \hat{f}_{1,k}\ \le&\ \frac{{z_1}^6\|\Phi_{1,k}\|^2\|\xi_{1,k}\|^2}{2a_{1,k}^2}+\frac{a_{1,k}^2}{2}+\frac{3}{4}{z_1}^4 \\ &+\frac{1}{4}\varepsilon_{1,k}^4 \\ \le&\ \frac{\underline{h}{z_1}^6\theta_1}{2a_{1,\min}^2}+\frac{a_{1,\max}^2}{2}+\frac{3}{4}{z_1}^4+\frac{1}{4}\varepsilon_{1,\max}^4, \end{aligned} \tag{28}$$

where $\theta_1=\|\Phi_{1,k}\|^2/\underline{h}$, $a_{1,k}>0$ is a design parameter, $a_{1,\min}=\min\{a_{1,k}:k\in M\}$, $a_{1,\max}=\max\{a_{1,k}:k\in M\}$, and $\varepsilon_{1,\max}=\max\{\varepsilon_{1,k}:k\in M\}$.

Substituting (28) into (26) obtains

$$\begin{aligned} LV_1\ \le&\ -k_1z_1^{4\upsilon}+z_1^3\left(\beta h_{1,k}\alpha_1+\frac{\underline{h}{z_1}^3\theta_1}{2a_{1,\min}^2}\right) \\ &+\frac{\overline{h}}{4}z_2^4+\frac{a_{1,\max}^2}{2}+\frac{1}{4}\varepsilon_{1,\max}^4+\frac{3}{4}l_1^2-\frac{\underline{h}\tilde{\theta}_1\dot{\hat{\theta}}_1}{r_1}. \end{aligned} \tag{29}$$

Subsequently, choose the virtual control signal and the adaptive law as follows:

$$\alpha_1=-\frac{k_1}{\beta}{z_1}^{4\upsilon-3}-\frac{{z_1}^3\hat{\theta}_1}{2\beta a_{1,\min}^2}, \tag{30}$$

$$\dot{\hat{\theta}}_1=\frac{r_1{z_1}^6}{2a_{1,\min}^2}-\gamma_1\hat{\theta}_1, \tag{31}$$

where $\hat{\theta}_1$ is estimation of unknown parameter θ_1; k_1, r_1 and γ_1 are the positive design parameters.

Substituting (30)-(31) into (29), we obtain

$$\begin{aligned} LV_1\ \le&\ -k_1(1+h_{1,k})z_1^{4\upsilon}+\frac{\underline{h}\gamma_1}{r_1}\tilde{\theta}_1\hat{\theta}+\frac{\overline{h}}{4}z_2^4 \\ &+\frac{a_{1,\max}^2}{2}+\frac{1}{4}\varepsilon_{1,\max}^4+\frac{3}{4}l_1^2. \end{aligned} \tag{32}$$

It is noted that

$$\frac{\underline{h}\gamma_1}{r_1}\tilde{\theta}_1\hat{\theta}_1\le-\frac{\underline{h}\gamma_1}{2r_1}\tilde{\theta}_1^2+\frac{\underline{h}\gamma_1}{2r_1}\theta_1^2. \tag{33}$$

So, (32) can be rewritten as

$$LV_1\le-c_1z_1^{4\upsilon}-\frac{\underline{h}\gamma_1}{2r_1}\tilde{\theta}_1^2+\frac{\overline{h}}{4}z_2^4+\upsilon_1, \tag{34}$$

where $c_1=k_1(1+\underline{h})$ and $\upsilon_1=\underline{h}\gamma_1\theta_1^2/(2r_1)+a_{1,\max}^2/2+\varepsilon_{1,\max}^4/4+3l_1^2/4$.

Step *i* ($2 \leqslant i \leqslant n-1$)**:**

By the coordinate transformation $z_i = x_i - \alpha_{i-1}$ and $It\hat{o}$ formula, it has

$$dz_i = [h_{i,k}x_{i+1} + f_{i,k} - L\alpha_{i-1}]\,dt + \left(g_{i,k} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}g_{j,k}\right)^T d\omega, \tag{35}$$

with

$$\begin{aligned} L\alpha_{i-1} = &\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}[h_{j,k}x_{j+1} + f_{j,k})] \\ &+\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}g_{p,k}^T g_{q,k} + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial {y_d}^{(j)}}y_d^{(j+1)} \\ &+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j. \end{aligned} \tag{36}$$

Similar to Step 1, choose Lyapunov function as:

$$V_i = V_{i-1} + \frac{1}{4}z_i^4 + \frac{\underline{h}\tilde{\theta}_i^2}{2r_i}, \tag{37}$$

where $r_i > 0$ is a design parameter.

According to $It\hat{o}$ formula, one has

$$\begin{aligned} LV_i \;=\; & LV_{i-1} + \frac{3}{2}z_i^2\left\|g_{i,k} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\right\|^2 \\ &+z_i^3\left(h_{i,k}(z_{i+1}+\alpha_i) + f_{i,k} - L\alpha_{i-1}\right) \\ &-\frac{\underline{h}\tilde{\theta}_i\dot{\hat{\theta}}_i}{r_i}. \end{aligned} \tag{38}$$

By Lemma 5 and Assumption 2 , one has

$$\begin{aligned} L_i \;=\; & \frac{3}{2}z_i^2\left\|g_{i,k} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}g_{j,k}\right\|^2 \\ &\leq \frac{3}{4}l_i^2 + \frac{3}{4}l_i^{-2}z_i^4\left\|g_{i,k} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}g_{j,k}\right\|^4, \end{aligned} \tag{39}$$

$$h_{i,k}z_i^3 z_{i+1} \leq \frac{3}{4}\overline{h}z_i^4 + \frac{\overline{h}}{4}z_{i+1}^4, \tag{40}$$

where $l_i > 0$ is a design parameter. Substituting (39) and (40) into (38), one gets

$$\begin{aligned} LV_i \;\leq\; & -\sum_{j=1}^{i-1}\left(c_j {z_j}^{4\upsilon} + \frac{\underline{h}\gamma_j}{2r_j}\tilde{\theta}_j^2\right) + \sum_{j=1}^{i-1}\upsilon_j \\ &+z_i^3\beta h_{i,k}\alpha_i + \frac{\overline{h}}{4}z_{i+1}^4 + z_i^3\hat{f}_{i,k} \\ &-\frac{3}{4}z_i^4 + \frac{3}{4}l_i^2 - \frac{\underline{h}\tilde{\theta}_i\dot{\hat{\theta}}_i}{r_i} - k_i z_i^4, \end{aligned} \tag{41}$$

where

$$\begin{aligned} \hat{f}_{i,k} \;=\; & \frac{3}{4}l_i^{-2}z_i\left\|g_{i,k} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}g_{j,k}\right\|^4 \\ &+f_{i,k} - L\alpha_{i-1} + +\overline{h}z_i + (k_i + \frac{3}{4})z_i. \end{aligned} \tag{42}$$

As the function $\hat{f}_{i,k}$ is unknown, we use the fuzzy logic systems to model, such that

$$\hat{f}_{i,k} = \Phi_{i,k}^T\xi_{i,k}(X_i) + \delta_{i,k}(X_i), \tag{43}$$

where $X_i = [x_1, \cdots, x_i, \hat{\theta}_1, \cdots, \hat{\theta}_{i-1}, y_d, \dot{y}_d, \cdots, y_d^{(i)}]^T$, $|\delta_{i,k}(X_i)| \leq \varepsilon_{i,k}$, $\varepsilon_{i,k} > 0$ is any constant.

Then, similar to (28), one has

$$z_i^3\hat{f}_{i,k} \leq \frac{\underline{h}{z_i}^6\theta_i}{2a_{i,\min}^2} + \frac{a_{i,\max}^2}{2} + \frac{3}{4}{z_i}^4 + \frac{1}{4}\varepsilon_{i,\max}^4, \tag{44}$$

where $\theta_i = \|\Phi_{i,k}\|^2/\underline{h}$, $a_{i,\min} = \min\{a_{i,k} : k \in M\}$ and $a_{i,\max} = \max\{a_{i,k} : k \in M\}$ with $a_{i,k}$ being a design parameter; $\varepsilon_{i,\max} = \max\{\varepsilon_{i,k} : k \in M\}$.

Substituting (44) into (41) can obtain

$$\begin{aligned} LV_i \;\leq\; & -\sum_{j=1}^{i-1}\left(c_j {z_j}^{4\upsilon} + \frac{\underline{h}\gamma_j}{2r_j}\tilde{\theta}_j^2\right) + \sum_{j=1}^{i-1}\upsilon_j \\ &+z_i^3 h_{i,k}\alpha_i + \frac{\underline{h}{z_i}^6\theta_i}{2a_{i,\min}^2} + \frac{\overline{h}}{4}z_{i+1}^4 + \frac{a_{i,\max}^2}{2} \\ &+\frac{1}{4}\varepsilon_{i,\max}^4 + \frac{3}{4}l_i^2 - \frac{\underline{h}\tilde{\theta}_i\dot{\hat{\theta}}_i}{r_i} - k_i z_i^{4\upsilon}. \end{aligned} \tag{45}$$

Choose the virtual control signal α_i and the adaptive law $\dot{\hat{\theta}}_i$ as:

$$\alpha_i = -k_i z_i^{4\upsilon-3} - \frac{{z_i}^3\hat{\theta}_i}{2a_{i,\min}^2}, \tag{46}$$

$$\dot{\hat{\theta}}_i = \frac{r_i {z_i}^6}{2a_{i,\min}^2} - \gamma_i\hat{\theta}_i, \tag{47}$$

where $\hat{\theta}_i$ is estimated value of the unknown parameter θ_i, $\hat{\theta}_i(0) \geq 0$; λ_i, r_i and γ_i are the positive design parameters.

Substituting (46) and (47) into (45), and using

$$\frac{\underline{h}\gamma_i}{r_i}\widetilde{\theta}_i\widehat{\theta}_i \leq -\frac{\underline{h}\gamma_i}{2r_i}\widetilde{\theta}_i^2 + \frac{\underline{h}\gamma_i}{2r_i}\theta_i^2, \tag{48}$$

(45) can be rewritten as

$$LV_i \leq -\sum_{j=1}^{i}\left(c_j {z_j}^{4\upsilon} + \frac{\underline{h}\gamma_j}{2r_j}\tilde{\theta}_j^2\right) + \sum_{j=1}^{i}\upsilon_j + \frac{\overline{h}}{4}z_{i+1}^4, \tag{49}$$

where $i \in (2, \cdots, n-1)$, $\upsilon_j = 3l_j^2/4 + a_{j,\max}^2/2 + \varepsilon_{j,\max}^4/4 + \underline{h}\gamma_j\theta_j^2/(2r_j)$.

Step *n*:

Take the following Lyapunov function:

$$V_n = V_{n-1} + \frac{1}{4}z_n^4 + \frac{\underline{h}\tilde{\theta}_n^2}{2r_n}. \tag{50}$$

By (21) and $It\hat{o}$ formula, one has

$$\begin{aligned} dz_n \;=\; & \left[h_{n,k}u(v) + f_{n,k} - L\alpha_{n-1} - \tilde{\dot{l}}\right]dt \\ &+\left(g_{n,k} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{i-1}}{\partial x_j}g_{j,k}\right)^T d\omega, \end{aligned} \tag{51}$$

where

$$\begin{aligned} L\alpha_{n-1} = &\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}[h_{j,k}x_{j+1} + f_{n-1,k}] \\ &+\frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}g_{p,k}^T g_{q,k} \\ &+\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial {y_d}^{(j)}}y_d^{(j+1)} + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j \end{aligned} \tag{52}$$

According to Lemma 2, it has

$$\begin{aligned}LV_n = LV_{n-1} + \tfrac{3}{2}z_n^2\left\|g_{n,k} - \sum_{j=1}^{n-1}\tfrac{\partial\alpha_{n-1}}{\partial x_j}g_{n,k}\right\|^2 \\ + z_n^3(h_{n,k}u(v) + f_{n,k}(\bar{x}_n) - L\alpha_{n-1} - \dot{\tilde{l}}) \\ - \tfrac{\underline{h}\tilde{\theta}_n\dot{\hat{\theta}}_n}{r_n}\end{aligned} \quad . \tag{53}$$

In order to implement the controller design, the following dynamic system is introduced [21]:

$$\dot{\tilde{l}} = -\tilde{l} + u(v) - v. \tag{54}$$

It follows from (49) and (54) that (53) can be expressed as:

$$\begin{aligned}LV_n = -\sum_{j=1}^{n-1}(c_j {z_j}^{4v} + \tfrac{\underline{h}\gamma_j}{2r_j}\tilde{\theta}_j^2) + \sum_{j=1}^{n-1}\upsilon_j + \tfrac{\overline{h}}{4}z_n^4 \\ + z_n^3(h_{n,k}(\tilde{l} + v) + f_{n,k}(\bar{x}_n) - L\alpha_{n-1}) \\ + \tfrac{3}{2}z_n^2\left\|g_{n,k} - \sum_{j=1}^{n-1}\tfrac{\partial\alpha_{n-1}}{\partial x_j}g_{n,k}\right\|^2 - \tfrac{\underline{h}\tilde{\theta}_n\dot{\hat{\theta}}_n}{r_n}.\end{aligned} \tag{55}$$

By Young's inequalities, we can get

$$\begin{aligned}L_n &= \frac{3}{2}z_n^2\left\|g_{n,k} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}g_{n,k}\right\|^2 \\ &\le \frac{3}{4}l_n^{-2}z_n^4\left\|g_{n,k} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}g_{n,k}\right\|^4 + \frac{3}{4}l_n^2.\end{aligned} \tag{56}$$

where l_n represents a positive design parameter. Substituting (56) into (55), it gives

$$\begin{aligned}LV_n \le -\sum_{j=1}^{n-1}\left(c_j {z_j}^{4v} + \tfrac{\underline{h}\gamma_j}{2r_j}\tilde{\theta}_j^2\right) + \sum_{j=1}^{n-1}\upsilon_j + \tfrac{3}{4}l_n^2 \\ + z_n^3\left(h_{n,k}(\tilde{l} + v) + \hat{f}_{n,k}(\bar{x}_n)\right) - \tfrac{3}{4}z_n^4 \\ - k_n z_n^4 - \tfrac{\underline{h}\tilde{\theta}_n\dot{\hat{\theta}}_n}{r_n},\end{aligned} \tag{57}$$

where $\hat{f}_{n,k} = f_{n,k}(\bar{x}_n) - L\alpha_{n-1} + \frac{\overline{h}}{4}z_n + (k_n + \frac{3}{4})z_n + \frac{3}{4}l_n^{-2}z_n\left\|g_{n,k} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}g_{n,}\right\|^4$.

Using fuzzy logic systems to estimate the unknown function $\hat{f}_{n,k}$, that is

$$\hat{f}_{n,k} = \Phi_{n,k}^T\xi_{n,k}(X_n) + \delta_{n,k}(X_n), \tag{58}$$

where $X_n = [x_1, \cdots, x_n, \hat{\theta}_1, \cdots, \hat{\theta}_{n-1}, y_d, \dot{y}_d, \cdots, y_d^{(n)}]^T$, $|\delta_{n,k}(X_n)| \le \varepsilon_{n,k}$, $\varepsilon_{n,k} > 0$ is any constant.

By Lemma 5 and $\xi_{n,k}^T\xi_{n,k} \le 1$, one has

$$z_n^3\hat{f}_{n,k} \le \frac{{z_n}^6\theta_n}{2a_{n,\min}^2} + \frac{a_{n,\max}^2}{2} + \frac{3}{4}z_n^4 + \frac{1}{4}\varepsilon_{n,\max}^4, \tag{59}$$

where $\theta_n = \|\Phi_{n,k}\|^2/\underline{h}$, $a_{n,\min} = \min\{a_{n,k} : k \in M\}$ and $a_{n,\max} = \max\{a_{n,k} : k \in M\}$ with $a_{n,k}$ being a design parameter; $\varepsilon_{n,\max} = \max\{\varepsilon_{n,k} : k \in M\}$.

Substituting (59) into (57) gets

$$\begin{aligned}LV_n \le -\sum_{j=1}^{n-1}\left(c_j {z_j}^{4v} + \tfrac{\underline{h}\gamma_j}{2r_j}\tilde{\theta}_j^2\right) + \sum_{j=1}^{n-1}\upsilon_j + \tfrac{3}{4}l_n^2 \\ + z_n^3\left(h_{n,k}(\tilde{l} + v)\right) - k_n z_n^4 - \tfrac{\underline{h}\tilde{\theta}_n\dot{\hat{\theta}}_n}{r_n} \\ + \tfrac{{z_n}^6\theta_n}{2a_{n,\min}^2} + \tfrac{a_{n,\max}^2}{2} + \tfrac{1}{4}\varepsilon_{n,\max}^4.\end{aligned} \tag{60}$$

Design the control signal ν and the adaptation law $\hat{\theta}_n$ as:

$$v = -c_n z_n^{4v-3} - \frac{{z_n}^3\hat{\theta}_n}{2a_{n,\sigma(t)}^2} - \tilde{l}, \tag{61}$$

$$\dot{\hat{\theta}}_n = \frac{r_n {z_n}^6}{2a_{n,\sigma(t)}^2} - \gamma_n\hat{\theta}_n \quad , \tag{62}$$

where $\hat{\theta}_n(0) \ge 0$; and $\lambda_n > 0$, $r_n > 0$ and $\gamma_n > 0$ are the design parameters.

Substituting (61) and (62) into (60), and using

$$\frac{\underline{h}\gamma_n}{r_n}\widetilde{\theta}_n\widehat{\theta}_n \le -\frac{\underline{h}\gamma_n}{2r_n}\widetilde{\theta}_n^2 + \frac{\underline{h}\gamma_n}{2r_n}\theta_n^2, \tag{63}$$

one can obtain

$$LV_n \le -\sum_{j=1}^{n}\left(c_j {z_j}^{4v} + \frac{\underline{h}\gamma_j}{2r_j}\tilde{\theta}_j^2\right) + \sum_{j=1}^{n}\upsilon_j, \tag{64}$$

where $\upsilon_j = 3l_j^2/4 + a_{j,\max}^2/2 + \varepsilon_{j,\max}^4/4 + \underline{h}\gamma_j\theta_j^2/(2r_j)$. By far, the back-stepping design process is accomplished. Then, the main result can be summarized as the following theorem.

Theorem 1: For the switched stochastic nonlinear system with input saturation (7), if the initial condition satisfies $-\underline{\delta}\rho_0 < e_1(0) < +\bar{\delta}\ \rho_0$, then the tracking error satisfies the prescribed performance and all the signals of the closed-loop system are practical finite-time stable in mean square with virtual controller in (30), (46), adaptive laws in (31), (47), (62), and control input in (61).

Proof : Let Lyapunov function $V(X_n(t)) = V_n(t)$ and $\lambda = \min\{c_j, \gamma_j\}$ with $j = [1, 2, \cdots, n]$, (64) can be rewritten as:

$$\begin{aligned}LV_n(t) \le & -4^\nu\lambda\sum_{j=1}^{n}\left(\frac{z_j^4}{4}\right)^\nu \\ & -\lambda\sum_{j=1}^{n}\frac{\underline{h}}{2r_j}\tilde{\theta}_j^2 + \sum_{j=1}^{n}\upsilon_j.\end{aligned} \tag{65}$$

Substituting $p_1 = 1 - \upsilon$, $p_2 = \upsilon$, $p_3 = \upsilon^{\frac{\upsilon}{1-\upsilon}}$, $z = 1$, and $x = \sum_{j=1}^{n}\frac{\underline{h}}{2r_j}\tilde{\theta}_j^2$ into lemma 3 can get

$$\left(\sum_{j=1}^{n}\frac{\underline{h}}{2r_j}\tilde{\theta}_j^2\right)^\upsilon \le \sum_{j=1}^{n}\frac{\underline{h}}{2r_j}\tilde{\theta}_j^2 + \mu, \tag{66}$$

where $\mu = (1 - \upsilon)\upsilon^{\upsilon/1-\upsilon}$.

According to Lemma 1, combining (65) and (66), the following inequality holds.

$$\begin{aligned}LV_n(t) \le & -4^\nu\lambda\left(\sum_{j=1}^{n}\frac{{z_j}^4}{4}\right)^\upsilon \\ & -\lambda\left(\sum_{j=1}^{n}\frac{\underline{h}}{2r_j}\tilde{\theta}_j^2\right)^\upsilon + d,\end{aligned} \tag{67}$$

where $d = \sum_{j=1}^{n}\upsilon_j + \lambda\mu$. And then, according to Lemma 1 and $4^\nu > 1$ with $\nu \in (0, 1)$, one can get

$$LV_n(t) \le -\lambda V_n^\upsilon(t) + d. \tag{68}$$

According to $It\hat{o}$ formula, for $0 \leq \tau < t$, it obtains

$$E\left[V_n(t)\right] - E\left[V_n(\tau)\right] = \int_{\tau}^{t} E\left[LV_n(s)\right] ds, \tag{69}$$

where $E\left[\cdot\right]$ denotes mathematical expectation.

Taking the mathematical expectation of (68), we can get

$$\begin{aligned} E\left[LV_n(s)\right] &\leq -\lambda\, E\left[V_n^{\upsilon}(s)\right] + d \\ &\leq -\lambda\, \left(E[V_n(s)]\right)^{\upsilon} + d. \end{aligned} \tag{70}$$

Substituting (70) into (69) leads to

$$\begin{aligned} E\left[V_n(t)\right] - E\left[V_n(\tau)\right] \leq -\lambda \int_{\tau}^{t} \left(E[V_n(s)]\right)^{\upsilon} ds \\ + d(t-\tau). \end{aligned} \tag{71}$$

It follows from Lemma 4 with $\varpi(x(t) = E\left[V_n(t)\right\}]$ that there exists a setting time

$$T^{*} = \frac{\left[\left(E\left[V_n(0)\right]\right)^{1-\upsilon} - \left(\frac{d}{(1-\beta)c}\right)^{(1-\upsilon)/\upsilon}\right]}{(1-\upsilon)\beta c}, \tag{72}$$

such that

$$E[V_n(t)] \leq \varepsilon, t \geq T^{*}, \tag{73}$$

where $\varepsilon = \left(\frac{d}{(1-\beta)c}\right)^{1/\upsilon}$.

According to the definition of $V_n(t)$, it produces

$$E\left(\sum_{j=1}^{n} {z_j}^4\right) \leq 4E\left[V_n(t)\right] \leq 4\varepsilon,\ t \geq T^{*}. \tag{74}$$

According to the property of mathematical expectation, it is easy to verify that

$$\left[E\left({z_j}^2\right)\right]^2 \leq E\left({z_j}^4\right) \leq E\left(\sum_{j=1}^{n} {z_j}^4\right) \leq 4\varepsilon, \tag{75}$$

where $t \geq T^{*}$.

Thus

$$E\left({z_j}^2\right) \leq 2\sqrt{\varepsilon},\ t \geq T^{*}. \tag{76}$$

It can be deduced from (14) (19) and (76) that (77) holds.

$$E\left[e_1\right] = E\left[\left|y - y_d\right|^2\right] \leq 2\sqrt{\varepsilon},\ t \geq T^{*}. \tag{77}$$

In the similar way, we can get

$$E\left(\tilde{\theta}_j^2\right) \leq 4\, r_{\max}\, \varepsilon,\ t \geq T^{*}, \tag{78}$$

where $r_{\max} = \max\left\{\underline{h} r_j, 1 \leq j \leq n\right\}$.

Remark 5: It can be deduced from (76) , (77) and (78) that all the signals z_j, $\tilde{\theta}_j$, $j = 1, 2, \cdots, n$, and the tracking error e_1 in the closed system are practical finite-time stable in mean square. Furthermore, it can also be concluded that the tracking error e_1 can converges to the prescribed domain defined by (12) .

The proof is complete.

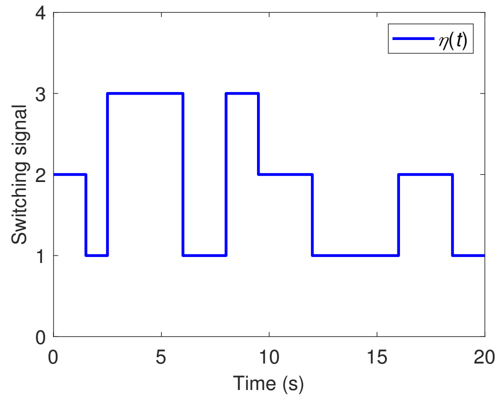

Fig. 1. Switching signal $\eta(t)$ for Example 1

IV. Simulation Example

In this section, two examples and two other controllers are given to illustrate the effectiveness of the proposed adaptive controller (FTPPFCA) in this paper. The first example is a numerical simulation, and the second one is Brusselator model. The two controllers used as a comparison are finite time fuzzy controller (FTFC) and finite time fuzzy controller with auxiliary system (FTFCA).

Example 1. Consider the second-order switched stochastic nonlinear systems as follows [17]:

$$\begin{cases} dx_1 = (x_2 + f_{1,\eta(t)}(\bar{x}_1))dt + g_{1,\eta(t)}(\bar{x}_1)d\omega, \\ dx_2 = (u(v) + f_{2,\eta(t)}(\bar{x}_2))dt + g_{2,\eta(t)}(\bar{x}_2)d\omega, \\ y = x_1, \end{cases} \tag{79}$$

where $\eta(t)\ :\ [0,\infty] \ \rightarrow\ \{1,2,3\}$, $f_{11} = x_1$, $f_{12} = x_1^2/(1+x_1^2)$, $f_{13} = 2x_1\cos(x_1)$, $f_{21} = (x_1)^2\cos^2(x_2)$, $f_{22} = x_2^2/(1+x_1^2+x_2^2)$, $f_{23} = 1.5\sin^2(x_1)x_2^2$, $g_{11} = 0.1x_1^2/(1+x_1^2)$, $g_{12} = 0.05\cos(x_1)$, $g_{13} = 0.03x_1^2/(1+x_1^2)$, $g_{21} = 0.05x_1^2/(1+x_1^2+x_2^2)$, $g_{22} = 0.1/(1+x_2^2)$, $g_{23} = 0.05\sin x_2$.The fuzzy membership functions are chosen as $\mu_{F_i^j} = \exp(-0.5(x_i-j)^2)$, $i = 1,2,3$, $j = 0, \pm1, \pm3, \pm4, \pm7, \pm9$. The input saturation is defined as:

$$u(v(t)) = \text{sat}(v(t)) = \begin{cases} 20\text{sign}(v(t)), & |v(t)| \geqslant 20 \\ v(t), & |v(t)| < 20 \end{cases}. \tag{80}$$

The design parameters are selected as follow: $\upsilon = 99/101$, $\lambda_1 = 20$, $\lambda_2 = 22$, $a_{1,\min} = 2$, $a_{2,\min} = 2$, $r_1 = 2$, $r_2 = 2$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$. The initial conditions are chosen as $[x_1(0), x_2(0)]^T = [0.15, 0.1]^T$ and $[\hat{\theta}_1(0), \hat{\theta}_2(0)]^T = [0.3, 0.4]^T$. $y_d = \sin(2t)$, $\overline{\delta} = \underline{\delta} = 1, \rho_\infty = 0.15, \rho_0 = 3, K = 1$. The switching signal $\eta(t)$ with time is shown in Fig. 1.

The simulation results are shown in Figs. 2-8. Figures 2 and 3 show the curves of the system output y and desired output y_d over time for the FTFC and FTPPFCA respectively. It can be seen from Fig.2 that the output of the FTFC increases with time and the closed-loop system is unstable. This can be due to the fact that input saturation deteriorates the performance of the system and leads to instability. However, the output of the FTPPFCA proposed in this paper can track the desired signal as shown in Fig.3. It indicates that the dynamic nonlinear system introduced in this paper overcomes the effect of input saturation to some extent and improves the stability of the system.

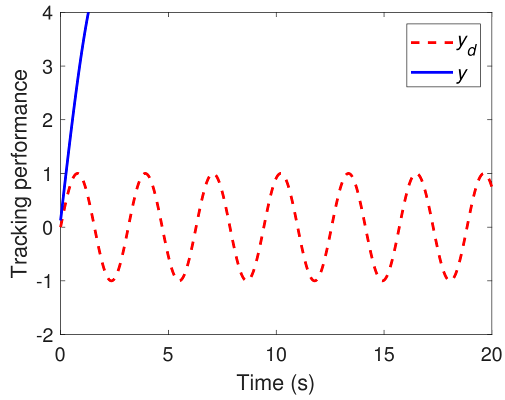

Fig. 2. System output y and desired signal y_d of FTFC for Example 1

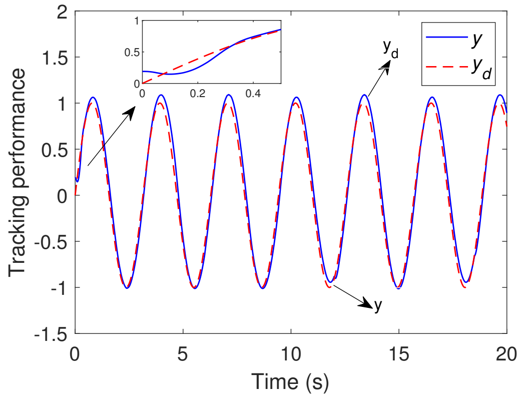

Fig. 3. System output y and desired signal y_d of FTPFCA for Example 1

To illustrate the effectiveness of prescribed performance control, the tracking error curves of the FTPFCA, FTFCA and the upper and lower bounds of the prescribed performance function are shown in Fig. 4. Obviously, the tracking error of the FTPFCA is limited to the range predefined by the prescribed performance function and owns faster convergence rate and smaller steady-state error than the FTFCA.

The control input v, actual control u, state variable x_2, and adaptive parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ are shown in Figs.5-8, respectively. It can be seen from Figs.5-8 that all signals in the closed-loop system are bounded and physically realizable.

Example 2. Consider the following Brusselator model [20]:

$$\begin{cases} dx_1 = C_1 - (D_1+1)x_1 + x_1^2 x_2, \\ dx_2 = D_1 x_1 + u - x_1^2 x_2, \\ y = x_1, \end{cases} \tag{81}$$

where x_1 and x_2 denote the concentration of intermediate reactants. C_1 and D_1 represent the initial and final products, which are usually selected as positive parameters. In addition, as a practical reaction process, stochastic disturbance and input saturation are unavoidable and suppose that jumping parameters reside in the model. Therefore, the revised Brus-

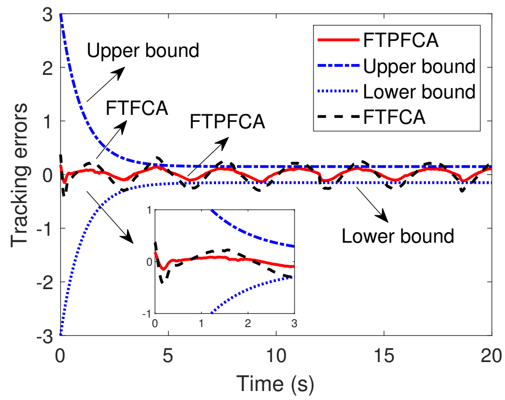

Fig. 4. Tracking errors e for Example 1

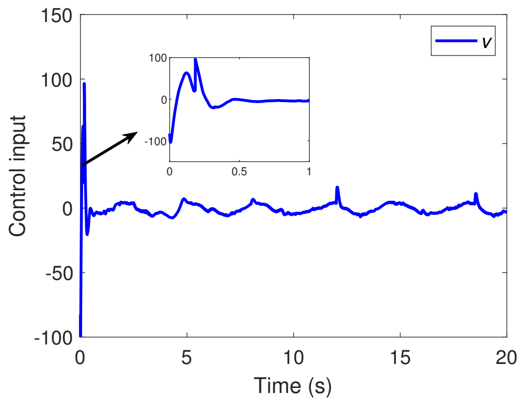

Fig. 5. Control input v for Example 1

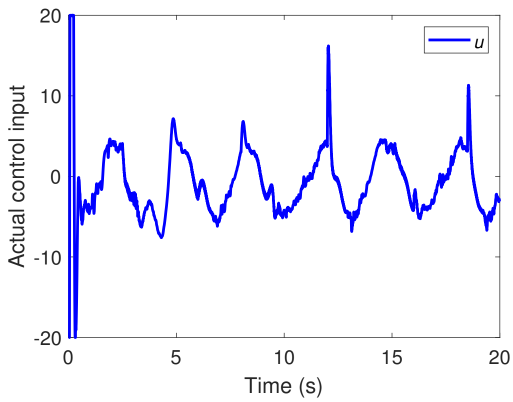

Fig. 6. Actual control u for Example 1

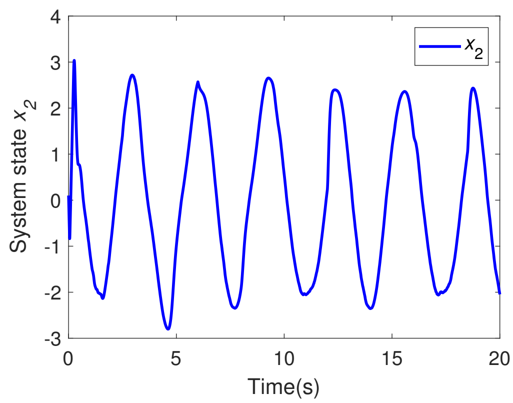

Fig. 7. System state of x_2 for Example 1

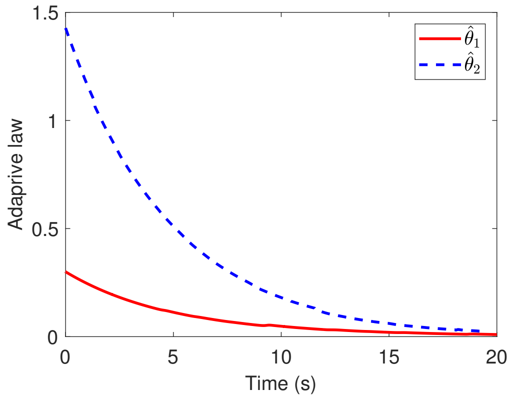

Fig. 8. Adaptive parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ for Example 1

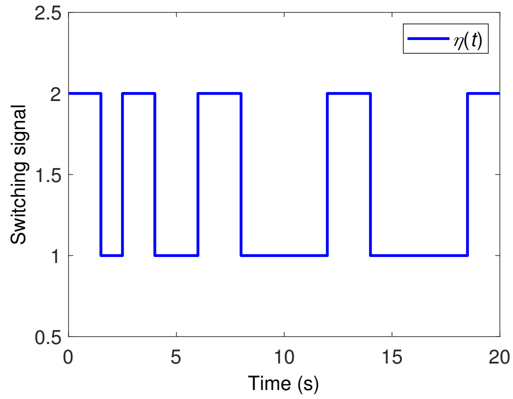

Fig. 9. Switching signal $\eta(t)$ for Example 2

selator model can be described as follows:

$$\begin{cases} dx_1 = (C_{1,\eta(t)} - (D_{1,\eta(t)} + 1)x_1 + x_1^2 x_2 \\ \qquad +f_{1,\eta(t)}(x_1))dt + (g_{1,\eta(t)})^T(x_1)d\omega, \\ dx_2 = (D_{1,\eta(t)}x_1 + u - x_1^2 x_2 + f_{2,\eta(t)}(x_2))dt \\ \qquad +(g_{2,\eta(t)})^T(x_2)d\omega, \\ y = x_1, \end{cases} \tag{82}$$

where $C_{11} = D_{11} = 4/5$, $C_{12} = D_{12} = 9/10$, $f_{11}(x) = f_{12}(x) = -x_1/6$, $g_{11} = g_{12} = 0.5D(\sin(x_1^2))$, $f_{21}(x) = -x_2^2$, $f_{22}(x) = 5x_2/6$, $g_{21} = g_{22} = Dx_2^2$, and $y_d = 0.25\sin(2t)$. The fuzzy membership functions are chosen as $\mu_{F_i^j} = \exp(-0.5(x_i - j)^2)$, $i = 1,2,3$, $j = 0, \pm 1, \pm 3, \pm 4, \pm 7, \pm 9$. The input saturation is defined as:

$$u(v(t)) = \begin{cases} 4\mathrm{sign}(v(t)), |v(t)| \geqslant 4 \\ v(t), \qquad\qquad |v(t)| < 4 \end{cases}. \tag{83}$$

The prescribed performance function is given as:

$$\rho(t) = (\rho_0 - \rho_\infty)e^{-Kt} + \rho_\infty, \tag{84}$$

where $\overline{\delta} = \underline{\delta} = 1, \rho_\infty = 0.05, \rho_0 = 0.5, K = 1$. The design parameters are chosen as: $c_1 = c_2 = 10.5$, $a_1 = a_2 = 3.5$, $r_1 = r_2 = 0.55$, $\gamma_1 = \gamma_2 = 0.55$, $D = 0.01$. The initial conditions are $[x_1(0), x_2(0)] = [0.02, 0.1]$, $\left[\hat{\theta}_1(0), \hat{\theta}_2(0)\right] = [0.02, 0.03]$. The switching signal $\eta(t)$ with time is shown in Fig. 9.

The system output versus desired output curves for the FTPFCA controller are shown in Fig. 10 and the output error curves are shown in Fig. 11. For comparison, the output error curves of both FTFCA and FTFC controllers are also given in Fig. 11. From these two figures, it can be seen that:

(1) the maximum steady state error (MSSE) of the FTPFCA controller is 0.045 mol/L, while the MSSE of the FTFCA and FTFC controllers are 0.059 mol/L and 0.12 mol/L, respectively. The MSSEs of the FTPFCA and FTFCA controllers are less than half of that of the FTFC, which can be attributed to the fact that the auxiliary system introduced in FTPFCA and FTFCA reduces the effect of input saturation on the MSSE.

(2) The overshoots of the FTPFCA, FTFCA, and FTFC are respectively 0.89 mol/L, 0.352 mol/L, and 0.378 mol/L. The FTPFCA controller has the smallest overshoot, which is mainly due to the prescribed performance control that improves the transient characteristics of the closed loop system.

The control input v and actual control u are presented in Figs. 12 and 13, respectively. The state variable x_2, adaptive parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ are shown in Figs. 14-16. From these figures, it is clear that all signals of the closed-loop system are bounded.

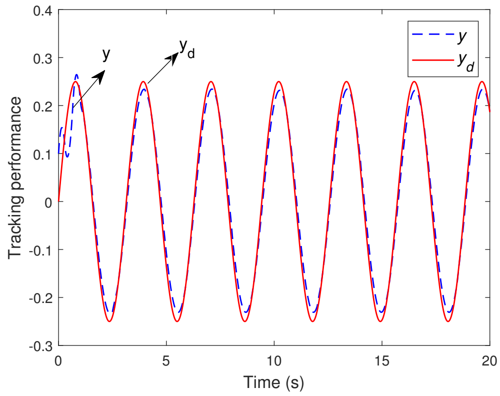

Fig. 10. System output y and desired signal y_d of FTPFCA for Example 2

V. Conclusion

A finite time prescribed performance controller is designed for the switched stochastic nonlinear systems with input saturation using the back-stepping method. The finite time control theory and prescribed performance control are used

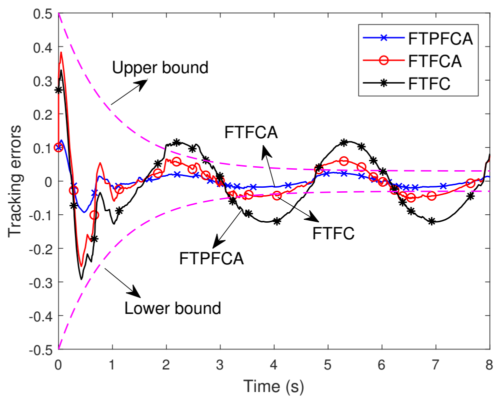

Fig. 11. Tracking errors e for Example 2

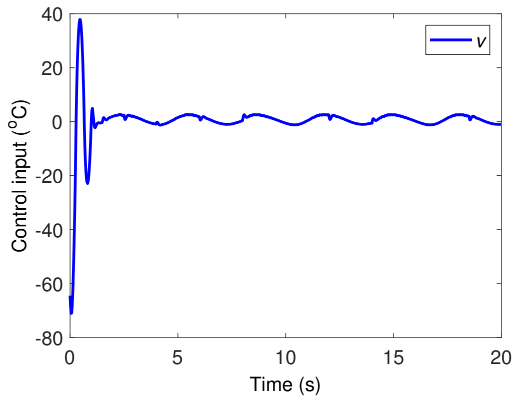

Fig. 12. Control input v of FTPFCA for Example 2

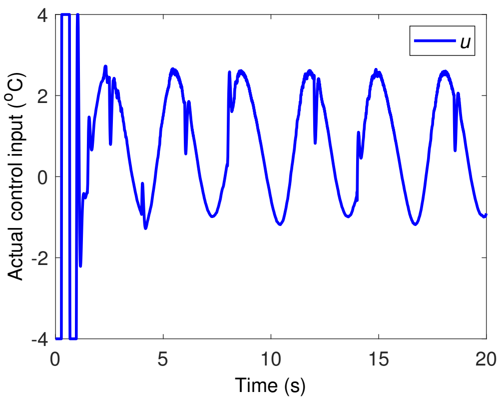

Fig. 13. Actual control u of FTPFCA for Example 2

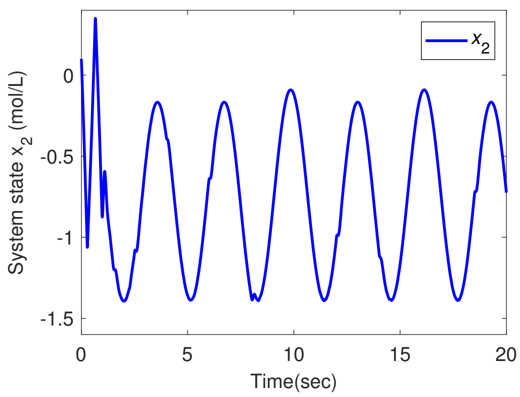

Fig. 14. System state x_2 of FTPFCA for Example 2

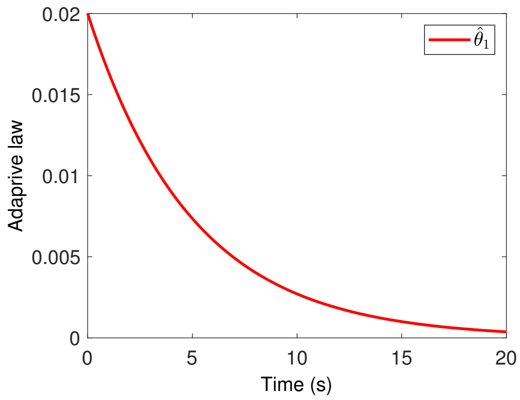

Fig. 15. Adaptive parameters $\hat{\theta}_1$ of FTPFCA for Example 2

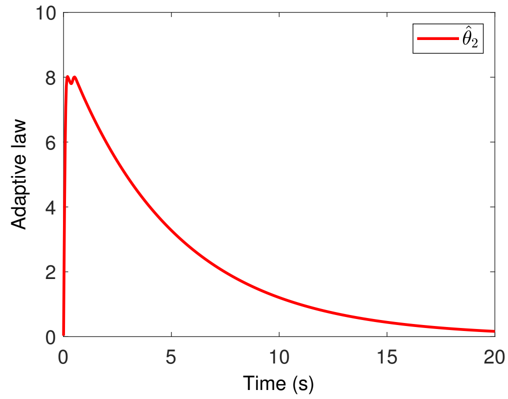

Fig. 16. Adaptive parameters $\hat{\theta}_2$ of FTPFCA for Example 2

to improve the transient and steady-state characteristics of the system, ensure that the system can converge to the steady-state value in a finite time, and improve the control speed and reduce the overshoot of the system. An auxiliary system is introduced to solve the input saturation problem, which improves the robustness of the system. The unknown function in the system is approximated by the fuzzy inference system and an adaptive algorithm. The designed controller ensures that the system can track the desired signal quickly and the tracking error can converge to a small neighborhood near the origin of the coordinates in finite time.

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